Mathematics III
Prof. Tanuja Srivastava
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture - 24
Solution of Drichlet Problem

Welcome to the lecture series on differential equations for under graduate students. Today's topic is Solution of Drichlet Problem, we are discussing about three dimensional Laplace equation. We had seen that if our boundary conditions are given on cylindrical surfaces or spherical surfaces, we could change our problem from the Cartesian coordinates to the cylindrical coordinates or spherical coordinates. The last lecture we have changed our problem that is the three dimensional Laplace equation into spherical coordinates.

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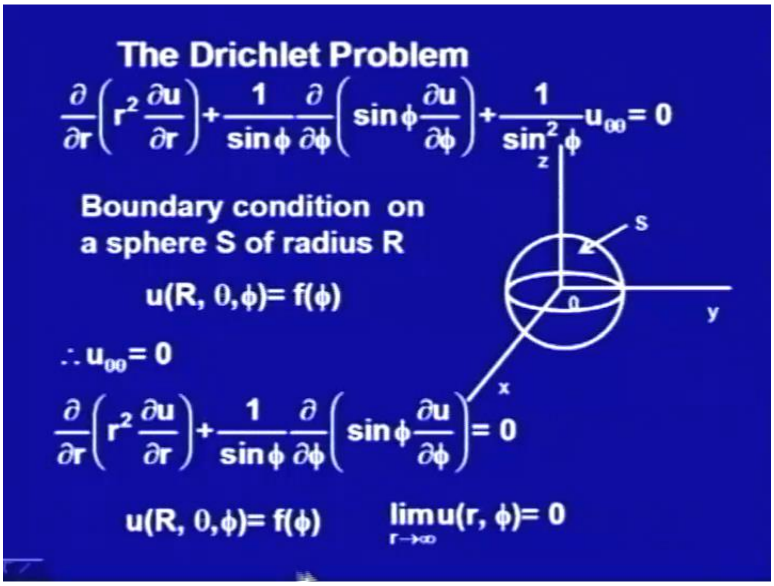

And we had seen that if our boundary condition is given on this sphere of radius are this s, and then we had change our Laplace equation into spherical coordinates as del by del r of r square del u by del r plus 1 upon sin phi del over del phi sin phi del u over del phi plus 1 upon sin square phi u theta theta is equal to 0.

Now, if the boundary condition on this that is the Drichlet problem says is the boundary condition is defined on the boundary, that is the sphere s only. If the boundary condition defined on this sphere s is independent of theta, that is let us say if it is u r theta phi is equal to f phi, which is independent of theta. Then we do expect that the solution of this boundary this equation would be also independent of theta, if that is independent of theta that is it would not be u r theta phi, but only u r phi that says is that the term u theta, theta would be 0.

So, we will change our partial differential equation as del by del r of r square del u over del r plus 1 upon sin phi del over del phi sin phi del u over del phi is equal to 0; with boundary condition as u r theta phi is equal to f phi and as or it is this point r that is r is approaching to infinite u r phi is approaching to 0. We have called this problem that is this partial differential equation with this boundary condition and this initial condition we could say or boundary condition, we have got that this is called the Drichlet problem. Today we will learn how to solve this Drichlet problem in spherical coordinates.

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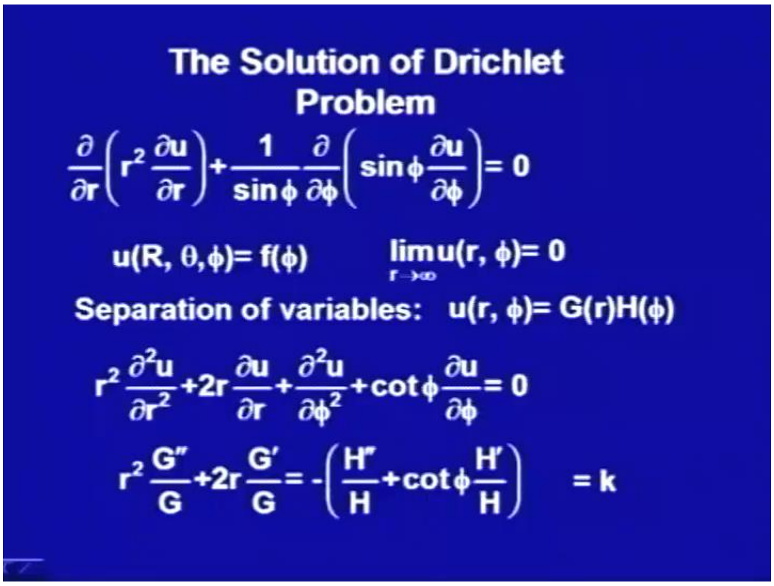

So, let us see is that is moved to the solution of Drichlet problem, we do have the partial differential equation in r and phi. The boundary condition is given on a sphere of the radius R u R theta phi is equal f phi, which is independent of theta. So, we do expect the solution to be on another condition is u r phi is approaching to 0 as r is approaching to infinity.

So, we will take the solution of this using the separation of variables or the product method. Since, we are seeing is that the solution would be depending upon r and phi only, so you will take the u r phi as the product of two functions G r and H phi, then the partial derivative of u with respect to r would turn out to be the simple ordinary derivative of G with respect to r and H as a constant. There it would remain as such and partial derivative of phi would turn out to be the ordinary derivative of H and G are remaining as same.

So, now if I take the first partial derivative with respect to r we would get d G upon d r times H phi. The second derivative with respect to r partial derivative of u, we would get d 2 G over d i 2 into H phi. Similarly for phi, we would get G r multiplied with d H over d phi and G r multiplied with d 2 H over d phi 2. So, now if I substitute these derivatives in this equation and divided by G H that is u r phi.

So, let us first write this equation in more simpler form as it would be r square del 2 u over del r 2 plus 2 r times del u over del r plus del by del phi sin phi that is it would be 1 upon sin phi sin phi cancel it out; and we will get del 2 u over del phi 2 plus cot phi derivative of sin phi is cos phi, so cos phi upon sin phi cot phi times del u over del phi is equal to 0. Now, substitute del 2 u over del r 2 del u over del r from this u r is equal to G r into H phi, we would get r square G double dash, because what we would be getting is del 2 u over del r 2 as G double dash r that is the derivative of G with respect to r into H phi.

Now we are dividing it by g into H, so we would be getting it G double dash upon G plus 2 r g dash by G that is where this G double dash and G dash means they are derivative of the G r with respect to r the second derivative and the first derivative. Then here we would get it H double dash upon H and here, we would get cot phi times H dash upon H that we are taking on the right hand side.

So, we do get it r square g double dash upon G plus 2 r times G dash upon G is equal to minus of H double dash upon H plus cot phi times H dash upon H. Of course this g is function of r only and H is the function of phi only and here also we are having the coefficients also as r and phi the r function of phi only. So, we do get this left hand side is a function of r and the right hand side is the function of phi.

So, if they are remaining to be equal if you according to this equation that says is they must be equal to some constant say k now equating these two we do get as usual method we have done in many partial differential equations, that we do get two ordinary differential equation one is from, here r square G double dash plus 2 r G dash minus kg that is G, we are taking this side and the other one would be H double dash plus cot phi H dash plus k H.

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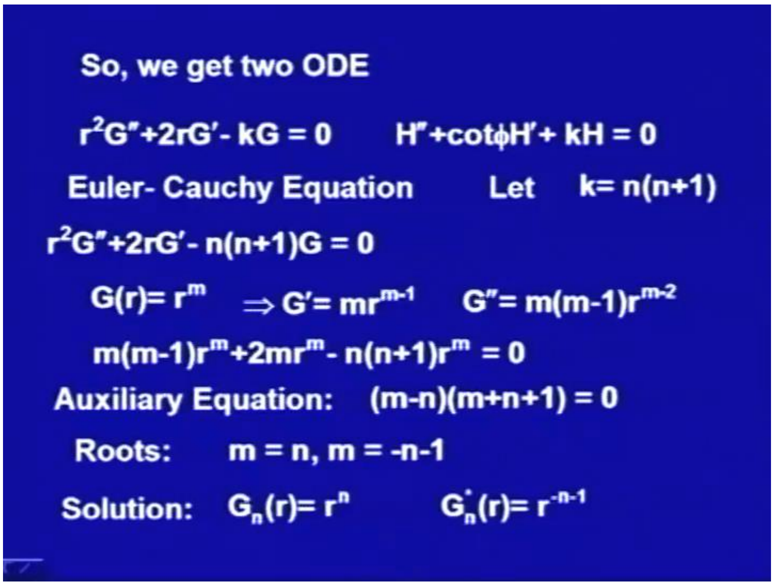

So, we are getting two ordinary differential equations r square G double dash plus 2 r G dash minus k G is equal to 0 and H double dash cot phi H dash plus k H is equal to 0. Now see the first equation, here the unknown variable is that variable is r and unknown function is G, so this is equation, which is in G and r if you do try to remember we are getting is this equation, where the second order linear equation, where the coefficients are r to the power r square and r so on.

So, this is we do know there is a must Euler-Cauchy equation, now the solution of this equation would be simpler if I take this constant k to be of a special form. So, let us say k is n into n plus 1, then what this equation will become this equation will become r square G double dash plus 2 r G dash minus n into n plus 1 times G is equal to 0.

We do know that if you have to remember it that is we have done this Euler-Cauchy equation. The solutions of this equation are assumed to be of the form r to the power m, so we take the solution of the form r to the power m that says is G dash would be m times r to the power m minus 1 and G double dash would be m into m minus 1 times r to the power m minus 2. If I substitute this that is if this is the solution they must satisfy this equation.

So, if I substitute these in this given equation what we would get it simply m into m minus 1 times r to the power m minus 2 into r square that is r to the power m plus 2 times m into r to the power m again r and r to the power m minus 1 minus n times n plus 1 r to the power m is equal to 0. Since r to the power m we are saying is this solution.

So, it would not be 0 for all r, so what we say is that the coefficient that is r to the power m if I do take common the coefficient must be equal to 0 and that gives me auxiliary equation, what would be that coefficient m square minus m plus 2 m minus n square minus n that is we would be getting is m square plus m minus n square plus n.

So, making it more simplified form the auxiliary equation is coming out as m minus n times m plus n plus 1 is equal to 0. The root of this auxiliary equation from here is clear that is one would be n another root would be minus n minus 1. So, these are the roots of this auxiliary equation that says is, now we are getting the two solutions of this second order ordinary differential equation, which is Euler-Cauchy equation, one is as given as r to the power n and another would be r to the power minus n minus 1 r 1 upon r to the power n plus 1.

Now, we are using two different solutions we are taking one solution as denoting it as G n r and whereas, G n star r. Here, n is arbitrary, what we have taken, we have taken this n to be this k to be a special form as n into n plus 1, where n we are taking as integer, so that we are getting it as a solution in a nice form that is how we are doing it, so we have got the two solutions for this first equation. Now, let us move to the second equation, since, we have got the solution of the first equation taking k as n into n plus 1 we will move to the second equation in the same manner taking k is equal to, so we will take k is equal to n into n plus 1.

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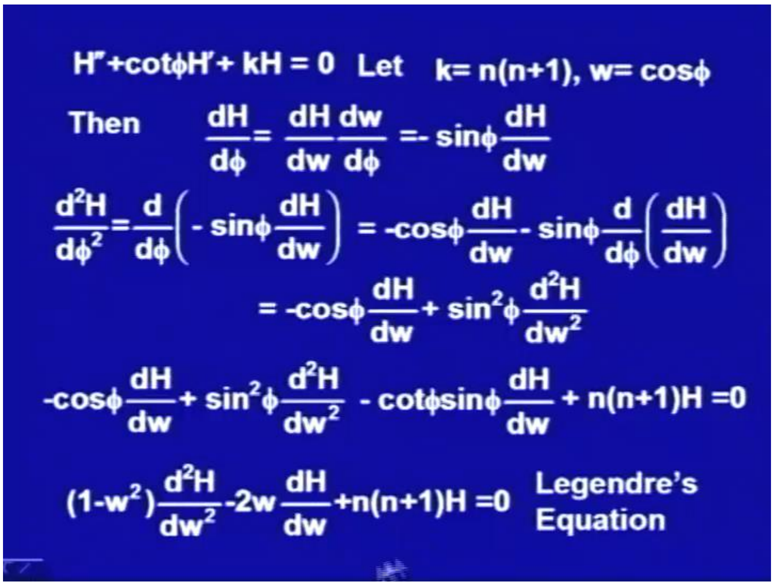

Now, this equation is in phi and we are having a coefficient as cot phi, what we would actually make the change of variable over here, that is rather than phi, now we would use another variable w, which is cos phi. So, what we say is that H of phi, we would be changing it to say some other function H of cos of phi, so we would take it that is function H of cos of phi in that if I change it. So, here the derivative H dash with respect to w with respect to phi we have to change it to with respective w, so we will go with the our chain rule.

So, d H over d phi is we could write as dH over d w into d w over d phi from here, d w over d phi would be minus sin phi, so what we get it minus sin phi d H over d w. Similarly, if I go for the second derivative d 2 H over d phi 2 this is nothing but, d by d phi of d H over d phi d H over d phi is minus sin phi dh over d w. So, let us first differentiate it with respect to phi using the product rule, so the derivative of minus sin phi is s minus cos phi, so we do get minus cos phi d H upon d w minus sin phi times the derivative of d H upon d w with respect to phi.

Now, for this we will use the chain rule, so the first term is as such for the second term we would be getting is d by d Hof dh by d w that is d 2 H over d w 2 into d w over d phi d w over d phi is minus sin phi. So, will multiply it we get plus sin square phi d 2 h over d w 2, now substitute this d 2 H over d phi 2 and d H over d phi in this given equation. So, we change this equation in the variable phi 2 variable w, we do get minus cos phi d h over d w plus sin square phi d 2 H over d w 2 plus cot 5 times H dash that is minus sin phi cot phi d H over d w plus k is n into n plus one times H is equal to 0.

Now you see, equation we have to get in H and w we have to remove the terms of phi cot phi into sin phi cot phi is nothing but, cosine phi divided by sin phi. So, we would get it, here cosine phi, so we will get minus two times cosine phi cosine phi we are saying is w, so if cosine phi is w, then sin square phi is nothing but 1 minus w square. So, let us rewrite this equation in H and w only sin square phi would be writing as 1 minus w square d 2 H over d w 2 minus 2 cosine phi cosine phi would be w minus 2 w times d H over d w plus n into n plus one times H is equal to 0.

Now, let us see this equation also we do recognise this we had already done this is a famous Legendre’s equation, where we are having is one, if we do remember that there we have done it 1 minus x is called y of y dash minus 2 x y dash plus n plus into n plus 1 times y is equal to 0, where n has to be integer.

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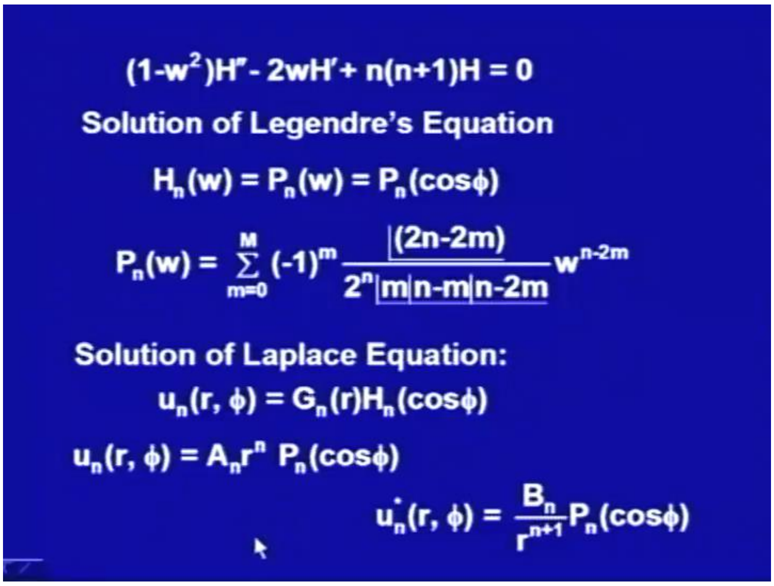

So, now, we will treat n as integer we do know the solution of this Legendre’s equation. is nothing but the Legendre polynomial P n w, now w we are saying is cosine phi, so will write P n cos phi. So, what we have got ultimately the solution for our partial differential equation u r phi, which is H phi into H phi G r into H phi, we do get this Legendre polynomial, if you do remember that is for your remembrance I am rewriting it over here, it is summation m running from 0 to capital M, capital M has to be either n by 2 or you could say is that integer value of n by 2 that is if n is even it could be simply n by 2 if n is odd it would be n minus 1 by 2.

From minus 1 to minus 1 to the power m into factorial 2 n minus 2 m upon 2 to the power n factorial m factorial n minus m into factorial n minus 2 m into w to the power n minus 2 m, you see this is the polynomial of degree n. We had, when m is equal to 0 the degree would be n, when m is n by 2 or up n minus 1 by 2 we would be getting it running over there that is the degree that power of w would be 0, so this is a polynomial of degree n.

So, the solution finally, we have got the solution of the Laplace equation as G n r into H n cos phi, which is H n cos phi is actually P n cos phi. So, we have got u n r phi as an times r to the power n P n cos phi, where I am using the first solution G n r as r to the power n and for that we are using the coefficient arbitrary constant A n.

Now, because we do know for the Legendre polynomial the arbitrary constant we already choose as one, so we do choose it here, as A n or we can say another solution, if I use instead of G n if I do use G n star I would get, let say another solution un star r phi as B n upon r to the power n plus one times P n cos phi, so we have got now two solutions for our Laplace equation.

So now, we have to find out the solution of digital problem that is we have to satisfy the boundary condition, boundary condition says is u r phi is f of phi at r is equal to capital R of x radius. Now, you see is that is here this solution un or un star they are depending upon an arbitrary integer n. So, to get that satisfy the boundary condition we would use this Fourier method for solving Drichlet problem using the boundary condition.

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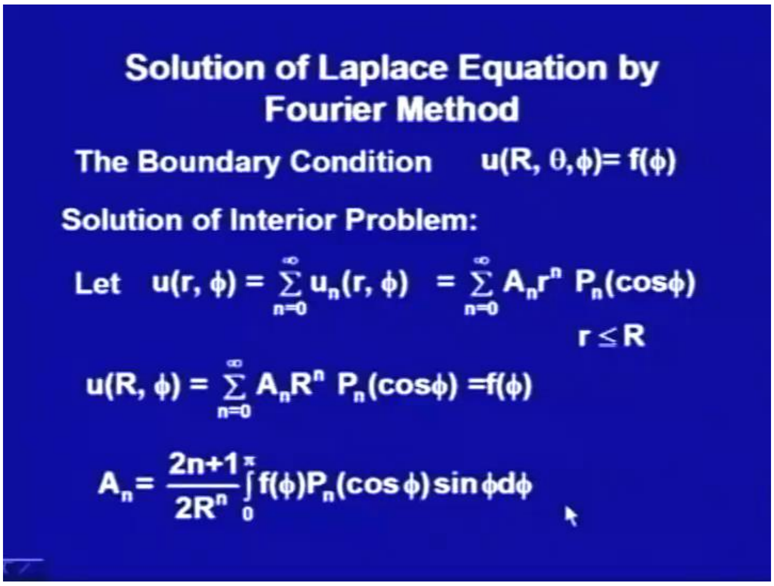

So, we will use the solution of Laplace equation by Fourier method, the boundary condition first we have been given is u R theta is u R theta phi is equal to f phi. Now, since if we do remember we are talking about that is our boundary is a surface of the sphere, surface of the sphere that says is that, we have to find out we have got that drichlet problem in the last lecture, we have got that it is example is that is calculation of potential or the temperature inside the boundary and outside the boundary.

So, let us first talk about the inside the boundary that is, what we are calling interior problem. So, let us first find out the solution of interior problem that is we will get u R phi at all those R, which are inside the sphere or that r is a small less than or equal to capital R, for that case we would use our solution from the first solution u and R phi.

Since my Laplace equation is again a second order linear equation homogeneous equation, we can use the principle of super position principle and that is fundamental result and we can have our solution as summation n is running from 0 to infinity un r phi, where un r phi is nothing but A n times r to the power n into P n cos phi. So, we have got is that lets says the solution is u r phi n is equal to 0 to infinity an r to the power n P n cos phi.

Now, for this we would satisfy our this boundary condition this solution we are saying is that is for r less than or equal to R. So, boundary condition means at r is equal to R, if i write it out the solution that this r has to change to this capital R this is a fixed value the radius of the sphere n is running from 0 to infinity A n R to the power n P n cos phi this must be equal to f phi.

Now, what it says is we have to find out this A n this constant such that this is satisfied, what is this, what we are saying is this function, f phi is represented by an infinite series, where this infinite series is actually consisting of this is a constant this is a fixed value r to the power n P n cos phi, P n cos phi this is the Legendre polynomial that is, what we are having polynomials of degree n. So, as in is varying we would be having polynomials in cos to the power n again we are getting is Fourier kind of polynomial.

So, what we are calling they are Fourier series of course, we would get the series in the terms of cosine terms powers of cosine phi. But, we have those special form that is rather than having only as the powers of cos and sin, we are having them as P n cos phi that is the Legendre polynomial in cos phi this we would call Legendre Fourier series. This has been a little bit you could obtain if you just say that is, since my a in the Legendre polynomials you have done that Legendre polynomials are orthogonal that says is in this way the representation can be done.

If you, do remember that in problems we have done that is those Eigen functions who all orthogonal, then we can present any function in the series form of the orthogonal functions. So, in that manner we could actually represented and using those kind of results here, I would not going the detail I would simply says that is A n times R to the power n from here, can be given as the coefficient 2 n plus 1 upon 2, so this r to the power n from here, I am taking is this 1 integral actually this is orthogonal and the minus 1 to plus 1, if you do remember that is P n w.

So, I could say is integral minus 1 to plus 1 P n w into f dash w f dash w, what we are saying, is f phi if I change my variable from phi to cosine phi sw. So, let us says that function f phi can be given as f dash of w, then f dash of w P n w d w integral minus 1 to plus 1 that should be ideally my coefficient.

Now since, w is we are taking as cosine phi, so if I change the variable my t w would change to minus sin phi cosine phi and the range that is the limit minus 1 to plus 1 minus 1 will take phi that is w as minus 1 will take phi as pi and w as plus 1 will take phi as 0, and since minus sin phi d phi that would be getting as t w using the properties of definite integral we could write it as 2 n plus 1 upon 2 to the power 2 into R to the power n integral 0 to pi f phi P n cos phi sin phi d phi.

So, what we have got we have got the solution of the interior problem or u r phi at for all r less than or equal to r that inside the boundary as a series summation n is running from 0 to infinity A n R to the power n P n cos phi, where an has been given by 2 n plus 1 upon 2 into R to the power n into integral 0 to pi f phi P n cos phi sin phi d phi, where f phi is actually the boundary condition, which is independent of theta.

Now, we see from here if my r is greater than R, if I take r approaching to infinity certainly this will never approach to 0, because we would be we are having is R to the power n and n is an integer, which is greater than 0. So, this will never approach to 0 that is the condition the another boundary condition, which says is u r phi is approaching to 0 as r is approaching to infinity cannot be obtained from this solution that is if r is greater than or equal to R outside the boundary that is outside this sphere this cannot be treated as a solution. So, that problem that is getting the solution u r phi at all the points, which are outside the sphere that we would call exterior problem and for that this solution would not work. So, let us see the solution of exterior problem.

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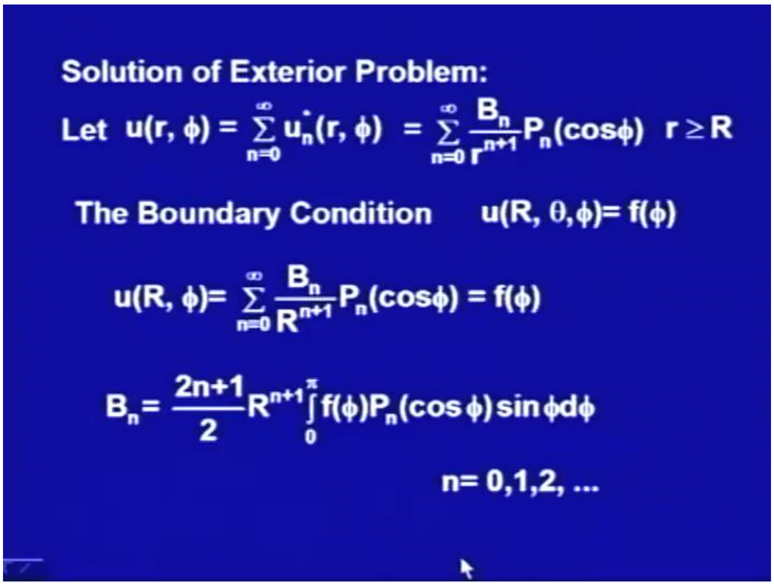

Let, u r phi is now another solution we are taking if you do remember the G n star, we are taking 1 upon r to the power n plus 1. So, there we would get if r is approaching to infinity that 1 upon r, r 1 upon r to the power n would approach to 0. So, we will treat take this G n star, that is now we assuming our solution u r phi as taking summation n is running from 0 to infinity B n upon r 2 to the power n plus 1 P n cos phi.

Now, satisfy the boundary condition that at r is equal to capital R that is at the this 1 the boundary condition if we do satisfy. So, this solution we are taking for r greater than or equal to R, now the boundary condition is that r is equal to r u capital R theta phi should be f phi try to satisfy this boundary condition for this solution we do get it as 0 to infinity B n upon capital R to the power n plus 1 P n cos phi is equal to f phi that is again we are getting f as a series of Fourier Legendre series we could say.

So, again using the same similar manner we could get B n upon r to the power n plus 1 as integral minus 1 to plus 1 P n w d w and f P n w into f dash w d w, where my w is cosine phi and the constant we would get 2 n, 2 n plus 1 upon 2. So, again changing it in the similar manner in the phi we do get 2 n plus 1 upon 2 into r to the power n plus 1. Because, now we would be getting B n upon r to the power n plus 1, so this would go this side into R to the power n plus 1 integral 0 to pi f phi P n cos phi sin phi d phi.

So, we have got this solution for all in 0, 1, 2, and so on, we have got now the solution of the exterior problem that is we could say potential at all the points, which are outside the boundary that is outside this sphere, we do get it as B n upon R to the power n plus 1 P n cos phi, where B n is given by this constant as 2 n plus 1 upon 2 R to the power n plus 1 integral 0 to pi f phi P n cos phi sin phi d phi, where f phi is nothing but the boundary condition that is the function, which is been given as the boundary condition.

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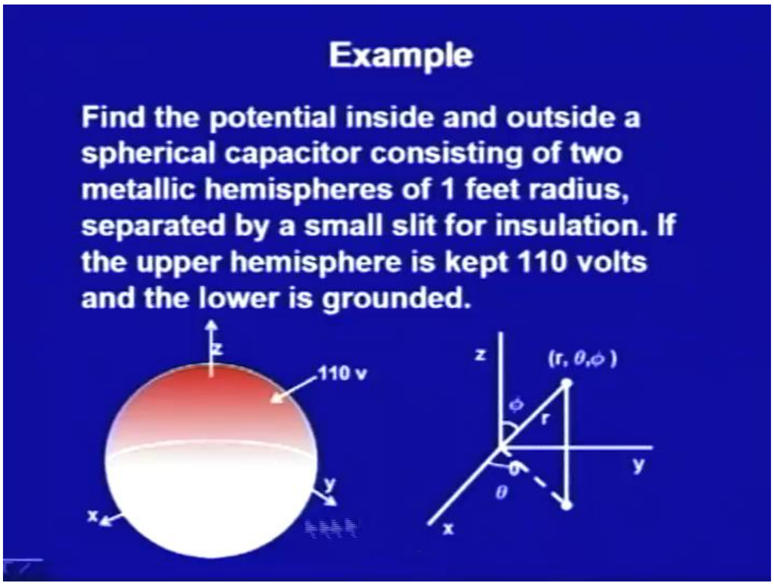

Now, let us see one example that is how this thing can be applied, find the potential inside and outside a spherical capacitor consisting of two metallic hemispheres of 1 feet radius, separated by a small slit for insulation. If the upper hemisphere is kept at 110 volts and the lower is grounded. So, now we have to find out the potential inside and outside this spherical capacitor while we have been given the temperature has been kept at this volt has been kept as 110 volts and the upper hemisphere and the lower hemisphere we have been it is been grounded.

So, let us see in the figure that is how this would this boundary condition would look like this says is there is a sphere say this upper sphere is are the 110 volts and the lower sphere we are having is it is 0 and there is a slit between these two spheres. Now, what this it says simply that I, because our boundary condition is given on the surface of the sphere.

So, we can and we have to find out the potential inside and outside this sphere that is we have to solve the interior problem as well as the exterior problem, we have to find out any point xyz inside this sphere and what is potential at that point and any point outside this sphere, what is the potential over there.

So, this we do know that this potential problem is nothing, but dimensional Laplace equation get the boundary conditions. We do know that since my boundary conditions have been given on the or the boundary is given as surface of this sphere we would look move to the spherical coordinates, if you are moving to this spherical coordinates you just see reminding that is, what we mean by spherical coordinates these xyz are our Cartesian coordinates.

If you are moving to this spherical coordinates, then we do have r is that the distance from the origin to any point whether it is inside or outside would wherever this any point in this space that is we are having is that is r and theta is the angle, which we would say this is in the x y plane that is whatever the cross section this circle it is making this is theta. Say if I take r is running from minus 1 to plus 1 that is the minus r to plus r or minus 1 to plus 1, because here the radius has been taken as one feet. So, it should be 0 to 1 here and minus 1 to plus 1 and theta is over her, then we see is that this phi we would be saying is that is to be going only for this sphere as 0 to pi.

So, first hemisphere we would get this phi is from 0 to pi by 2 only and for the another hemisphere we would have that is the upper hemisphere 0 to pi by 2 and the lower hemisphere phi would range from pi by 2 to pi, what it says is that my boundary condition, which is been given that is upper hemisphere is kept at 110 volts and the lower is grounded that says is, now my boundary condition is not depending on theta we would be getting is it is a constant for phi ranging from 0 to pi by 2 and it is another constant from pi by 2 to pi. So, we can very well assume that our boundary condition is free of theta. So, we could say is u r theta phi is f of phi and then the solution we would assume is that would be also independent of theta that is we could move to the problem just now, we had solved.

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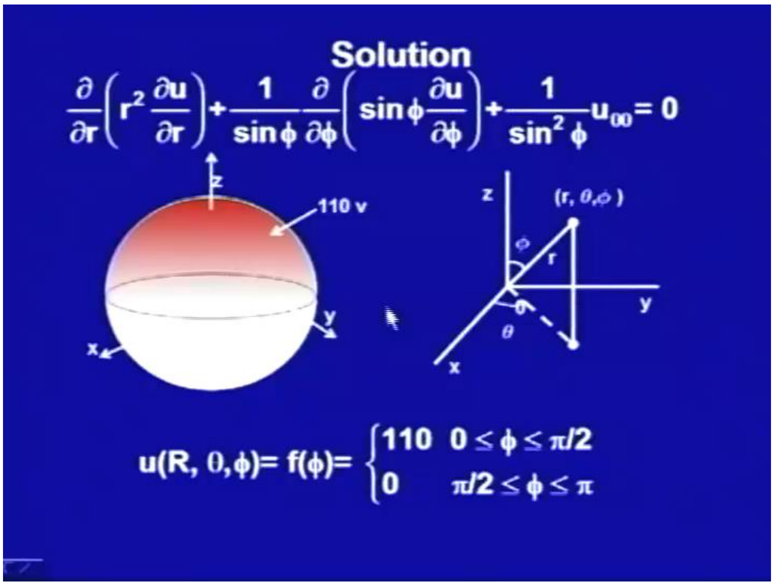

So, first we will change the problem that is three dimensional Laplace equation into spherical one as usual we have done it del upon del r r square del u upon del r plus 1 upon sin phi del upon del phi sin phi del u over del phi plus 1 upon sin square phi u theta, theta is equal to 0. Since, the boundary condition is given on this sphere as we have just now, seen is that it says is that it should be independent of theta it should depend only on phi, so we says that boundary condition is u r theta phi is equal to f phi, which is a function has been given as 110 for the phi ranging from 0 to pi by 2 and 0 for pi by 2 to pi.

That says is my solution would be independent of theta that says u theta, theta would be 0, so we will come back to this equation r squared d 2 over d r 2 plus 2 r times du over d r plus del 2 u u upon del phi 2 plus cot phi times del u over del phi is equal to 0, with the boundary condition u r theta phi is equal to f phi, which has been given by this function 1 10at a phi ranging from 0 to pi by 2 and 0 for phi ranging from pi by 2 to pi. And, this would lead to the and the other condition we would be requiring is that is the potential outside this one, so if r is very far away certainly u r phi must be 0.

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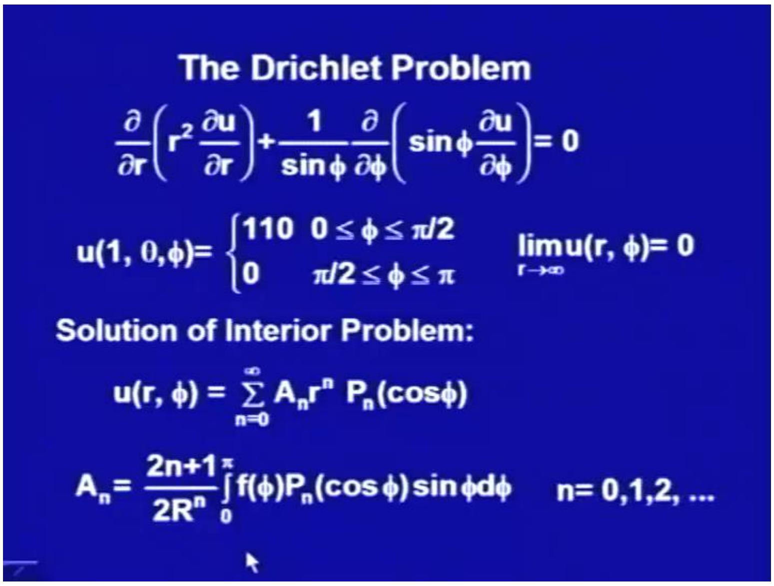

So, we say is that the other condition we would assume as, so we will come to the our Drichlet problem del upon del r r square del u over del r plus 1 upon sin phi del over del phi sin phi del u over del phi is equal to 0, with u 1 comma theta comma phi as hundred ten for phi ranging 0 to pi by 2 and 0 for phi ranging from pi by 2 to pi and u r phi is approaching to 0 as r is approaching to infinity. So, this is our known Drichlet problem for, which just now we had got the solution.

The solution we have got as the 2 that is since here, also in this example we have to find it out inside the sphere and outside this sphere, so inside this sphere. So, solution of interior problem we had find it out was u r phi summation n is running from 0 to infinity an r to the power n P n cos phi, where an was decided by the boundary condition on the constant was defined as 2 n plus 1 upon 2 to the power 2 into R to the power n integral 0 to pi f phi P n cos phi sin phi d phi. Now, here f phi is nothing but this constant between the range 0 to pi by 2 and 0 in the range 0 pi by 2 pi, so for evaluation of this A n we would have to evaluate the integral from 0 to pi by 2 only.

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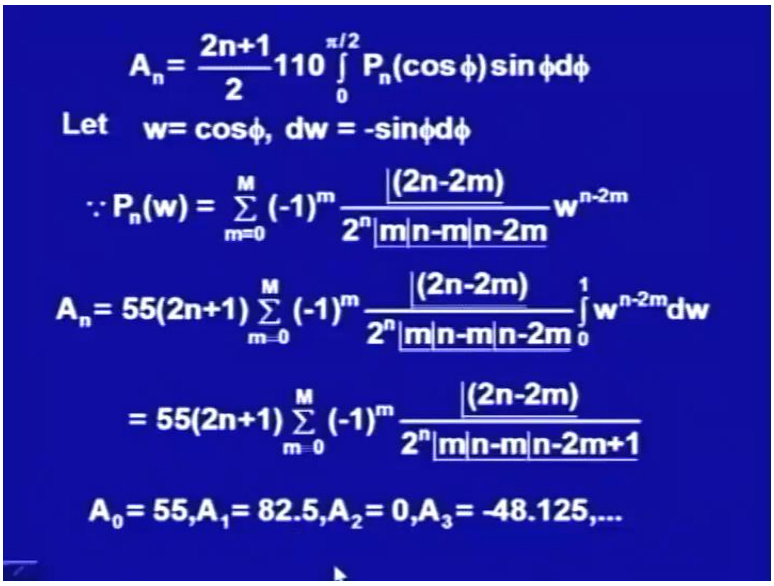

So, let us first evaluate this integral n 2 n plus 1 upon 2 r was 1, so r to the power n would vanish it is would come as one function f phi is constant in the range 0 to pi by 2 that is f phi is 110 integral is, now 0 to pi by 2 since 110 is constant, we are taking it outside the integral sign P n cos phi sin phi d phi. Now, let us move to this what is P n cos phi we would evaluate this integral 0 to pi by 2 P n cos phi sin phi d phi, again we will assume cos phi as w. So, minus sin phi would minus sin phi d phi would be d w and at 0 the cos 0 would be 1 and cos pi by 2 is 0, so we would get it 0 to 1. So, again in that manner we would write all these things and P n w we would if, we do remember we have got this polynomial of in a degree if m is running from 0 to m this summation minus 1 to the power m factorial 2 n minus 2 m upon 2 to the power n factorial m factorial n minus m factorial n minus 2 m into w to the power n minus 2 m.

So, now will substitute P n w in a state of P n w we would substitute this one and instead of phi we would change the variable to w. So, the ranges 1 to 0 and since, it is minus sin phi, so we would get 0 to 1 and what we would be getting now, 2 n plus 100 by 110 by 2 that is 55, so 55 into 2 n plus 1 we are again making is that the summation sign and the integral sign can be interchanged that is we could say is that the this interchange is possible if we says the series is uniform convergence and all those things.

So, we are assuming all those nice tease of the mathematics have been hold in 2 over here, so we can make it out. So, summation m is running from 0 to capital M minus 1 to the power m all these are constants, so we can take outside the integral sign factorial 2 n minus 2 m upon 2 to the power n factorial m factorial n minus m factorial n minus 2 m integral 0 to 1 W to the power n minus 2 m d w.

Now, this integral would be certainly W to the power n minus 2 m plus 1 upon n minus 2 m plus 1, which is evaluated from 0 to 1 will dissert only 1 upon n minus 2 m plus 1. So, if I am adding it 1 minus 1 upon n minus 2 m plus 1 that we will add it n minus 2 m into n minus 2 m plus 1 we would simply say it would be factorial n minus 2 m plus 1.

So finally, what if I substitute these things what I would get A n I would be getting is as 55 into 2 n plus 1 as such summation m is running from 0 to capital M, capital M means is the integral value consisting in m by 2 minus 1 to the power m 2 n minus 2 m factorial upon 2 to the power n factorial m factorial n minus m factorial n minus 2 m plus 1 this is the constant A n we have got now, for m is equal to 0 1 to 1, so on this is what is our constant.

So, let us try to find out certain values if I take n is equal to 0 n is equal to 0 says is that is capital M will also come out to be 0 that is this summation is only on the single value when m is equal to 0. M is equal to 0 means is here n is 0, so this is factorial 0 factorial 0 we do know is that we take it as 1 2 to the power 0 is 1 this is 0 this is 0 this is 1, so what we do get it simply here, 1 n is 0, so here also.

So, we will get simply A naught as 55 similarly if I go for n is equal to 1 n is equal to 1 means is 1 is odd number. So, this m will not go to 0 that is again we would be having only single point at m is equal to 0. Here n is equal to 1, so it would give me 3 into 55 and what we will get from here we would be getting is this minus 1 to the power 0 that will give me 1, here 2 n minus 2 m m is 0 n is 1, so it is factorial 2 here it is fact 2, so factorial 2 is 2 upon 2, then this is 0 this is factorial 1 this would be factorial 1 plus 1 2.

So, what we would be getting it 1 by 2, so we would get finally, it has up 165 by 2 or 82.5, because 55 into 3 would be 165 upon 2. So, 85.2 similarly if I go for n is equal to 2, then we have two terms 0 and 1 and we would get the coefficient from both the places, you just calculate it you would get as 2 minus 2 act with, so A 2 would be 0.

Similarly, for a three if I do go I will again get the two terms m is corresponding m is equal to 0 and m is equal to 1, again calculate all these putting this n is equal to 3 and m is 0 and 1 and that is this sum of the two terms finally, we would get like that we can calculate these coefficients.

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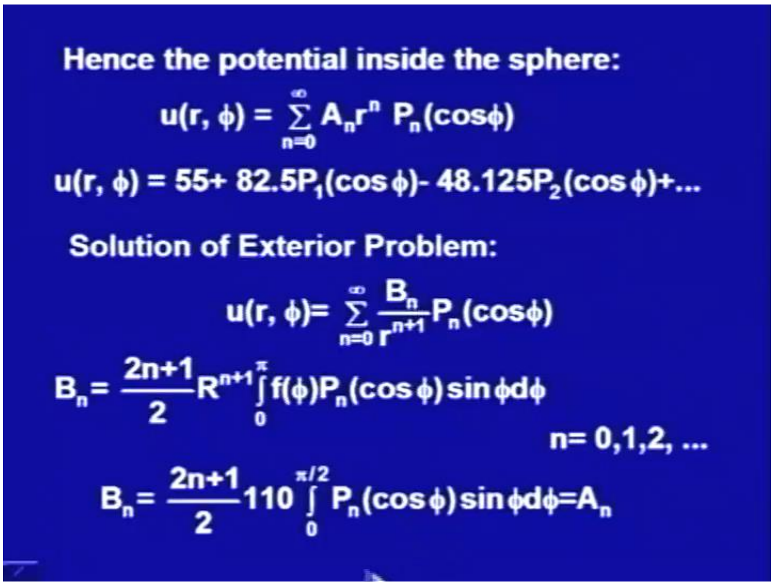

So finally, what will be the solution that is what is the potential inside this the sphere u r phi is n is equal to 0 to infinity an r to the power n P n cos phi. Now, A n just now we had calculated as for some values, so let us just try to write it out 55 that was the A naught r to the power 0 that is 1 p naught cos phi if we do remember in Legendre polynomial P naught x is 1 whatever be the x, so we would get 55. Then p 1 cos phi that is cos phi actually, you could do find it out using those ones, so this is 82.5 times p 1 cos phi minus 48.125 p 2 times cos phi and so on.

Now, from those functions the Legendre polynomials you had already done you have them that is, what is P n x P 1 x P 2 x P 3 x P 4 x like that you had already calculated you can, where is here write the series in the terms of cos phi are the powers of cos phi and from there, what we do have to we can calculate the potential at any point half the series sum. Series sum certainly, is not going to be very easy to find it out unless until we do know, what is the convergence and particularly, what will be actually this particular series some, but certainly using certain functions that is nth partial sum that is for some functions of first sum in as 5 6 or something we could find out we can approximate the potential at that point at any particular point are.

Now, move to the potential it outside the sphere that is solution of exterior problem we do remember that we had find out the solution of t exterior problem not using this an into R to the power n, but B n upon r to the upon R to the power n plus 1.

So, for that the solution was n is running from 0 to infinity B n upon r to the power n plus 1 P n cos phi here B n, if you do remember we were having 2 n plus 1 upon 2 r to the power n plus 1 0 to pi f phi P n cos phi sin phi d phi. Now, r is in our given problem the radius is one feet that is r is 1, so what I would get is 2 n plus 1 upon 2 times integral 0 to pi f phi P n cos phi sin phi d phi that is same as our A n, because why f phi is your constant in the range 0 to pi by 2 and 0 in the range pi by 2 to pi. So, what we do get is 2 n plus 1 upon 2 into 110 0 to pi by 2 P n cos phi sin phi d phi this is same as A n.

So, in this our particular problem, what we have got that B n and an are coming as same just now we have calculated that a naught as 55 a 2 a 1 as 82.5 a 2 as it should be P 3 a 2 as 0 a 3 as minus 48 point something.

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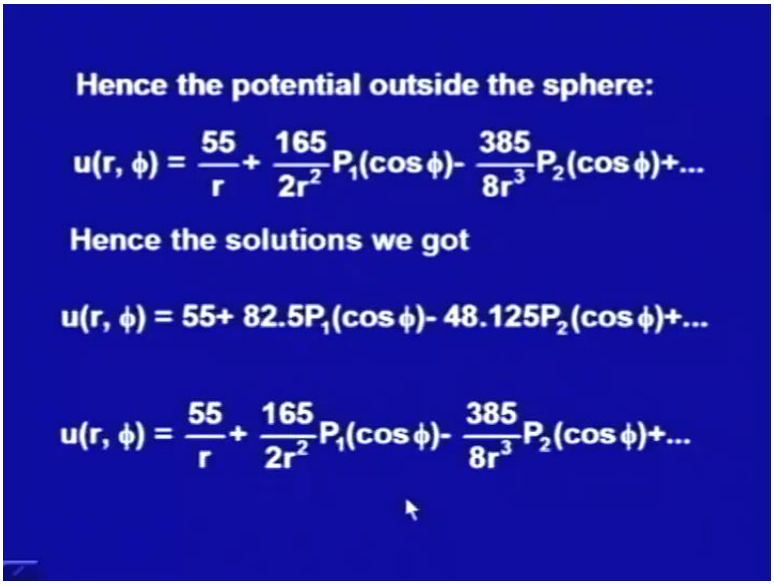

So we do get now, the solution outside the sphere that is exterior problem B naught is same as A naught, so what we are getting is B n upon R to the power n, so R to the power n plus 1. So, we would get 55 upon r plus a 1 we have got as 165 by 2 upon r square P 1 cos phi minus 385 upon 8 r cube P 3 cos phi and so on.

So, what we have got finally actually, we have to solve we have to get the potential inside the sphere and outside the sphere, what we have got for the inside we have got this series for outside we have got this series. Using this Legendre polynomials P 1 x is x P 2 x is x square minus and so on P 3 x is x cube and all those terms we can find out the approximate values at different points r over here and then we can get the approximate solutions. You, can check that is with the some as we have done in many our solutions that is we have took the in the Fourier series also we have done is that is calculated the partial sums for n is equal to 3, 4, 5, and so on.

So, you can try here with the n is equal to say phi that is we would take the first term second term this third, because the a 2 is 0, so this would be a 3 and so on, you could go till for first six terms and you can check for different values of r first thing is you could check very easily for if I take r is equal to capital R that is one itself by both these solutions would match that is at the boundary r is equal to 1 both these solutions would not and there you could check that it would actually approximate our boundary condition and for different values that is when r is less than 1 we have to use this solution for r greater than 1 we have to use this solution.

So, here r has also to also come, so we do get that different solutions that is potential inside and outside. So, today we had learnt that is how to solve the Drichlet problem when the boundary condition is given on a sphere, similarly if the boundary condition is go when on the cylinder we can change our problem to the cylindrical problem and there using the boundary condition with the rate is depending upon z or not. And all those things we have to either we can change it to the two variables that is r theta and z either we change it to the r n theta or we change to theta and z.

And accordingly we can use again the product method that is the method of separation of variable and we can go ahead with the solution. So, the method for solving this partial differential equation that is three dimensional Laplace equation first we are trying to see is that is, where the boundary condition has been given if the boundary condition is given on a some known surfaces that is as a cylinder or this sphere, we will try to change our problem to the spherical coordinates or cylindrical coordinates. If the boundary is given on the cubes or this cuboids we could move to the and directly to the Cartesian coordinates itself we could solve.

We could use this problem we could solve this using this product method. So, this spherical and cylindrical coordinates we have taken because of its practical applicable team in engineering problems. So, we have learnt how to solve the three dimensional Laplace equation, so that is all for this today is lecture.

Thank you.