Mathematics III

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Lecture - 23
Three Dimensional Laplace Equations

Welcome to the lecture series and differential equations for under graduating students, today's topic is Three Dimensional Laplace Equations. We are learning about some important partial differential equations, which are very important in application engineering and physics. In that series today we will learn about three dimensional Laplace equations.

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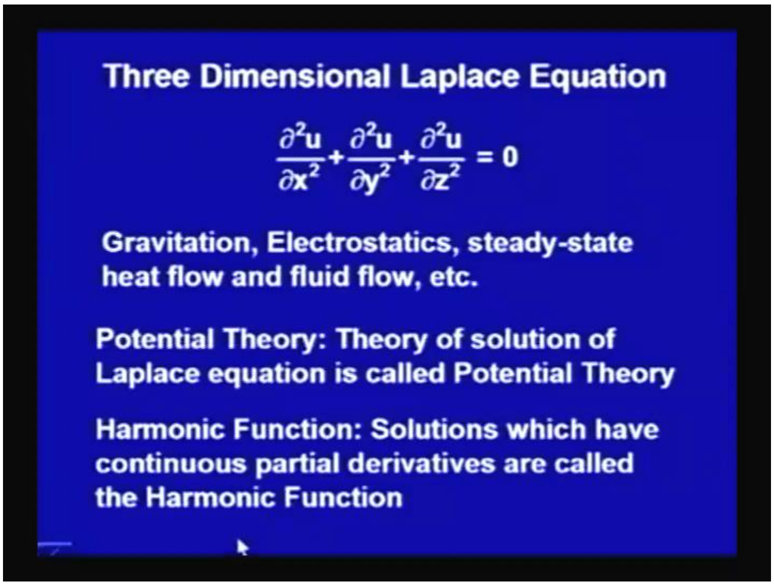

You have seen an example, that second order linear equation del 2 u over del x 2 plus del 2 u over del y 2 plus del 2 u over del z 2 is equal to 0. This is three dimensional Laplace equation, it is being called in three dimensions since we are having three co ordinates x y and z and here is the function of x y and z only, we are not having it is independent of t. This left hand side of this equation is called the Laplacian, this is very important equation in physics as well as in engineering applications. It is arising mainly gravitation electrostatics steady state heat flow and steady state fluid flow all the steady state it is coming.

The solution that is how to find out the solution of this equation theory of solution of this Laplace equation is called potential theory. So, we are also calling it potential theory that is understanding the solution as three dimensional Laplace equations. The solution if it is it is partial derivatives are continuous that is solutions, which have continuous partial derivatives they are called harmonic function.

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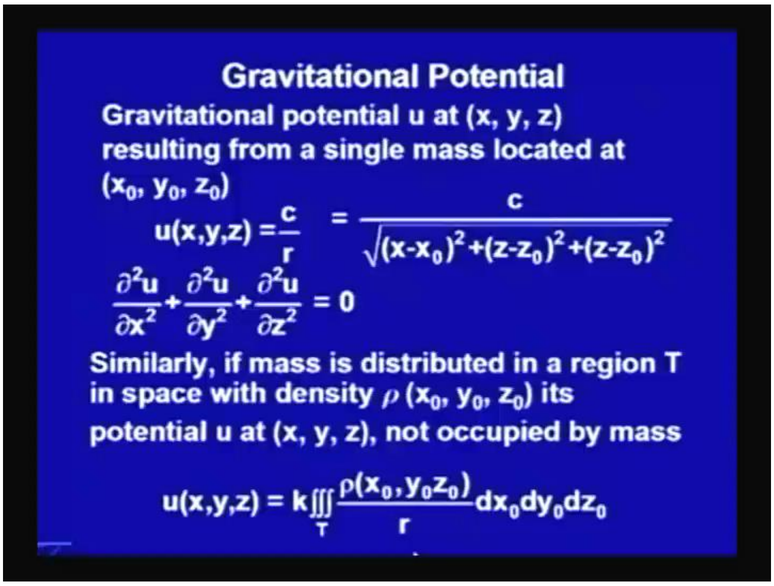

Let us, come to the one application that is gravitational potential we have learnt in physics that gravitational potential u at x y z resulting from a single mass located at the point x naught y naught z naught can be given as the function as u x y z as c by r. Where c is a constant and r is the distance between the point where this mass is located at x naught y naught z naught at the point at which we are calculating this potential x y z.

So, we do get is that we could write the distance as c upon square root of x minus x naught square plus z minus z naught whole square plus y minus y naught whole square. If we just apply this del 2 u over del x and del 2 u over del y del 2 u over del z 2 we find it out that Laplacian of this is 0, that is it is satisfying this Laplace equation.

Now, if similarly we define, the mass is distributed in the region t in the space with density rho x naught y naught z naught, it is potential u at the point x y z not occupied by the mass is given as k times integral on the region t, rho of x naught y naught z naught upon r d x naught d y naught d z naught.

Again, this will satisfy our Laplace equation, since this integral is with respect to x naught y naught z naught and rho is not depending upon the point x y and z. And again 1 upon r has the Laplacian 0, so we do have that this will also satisfy this one.

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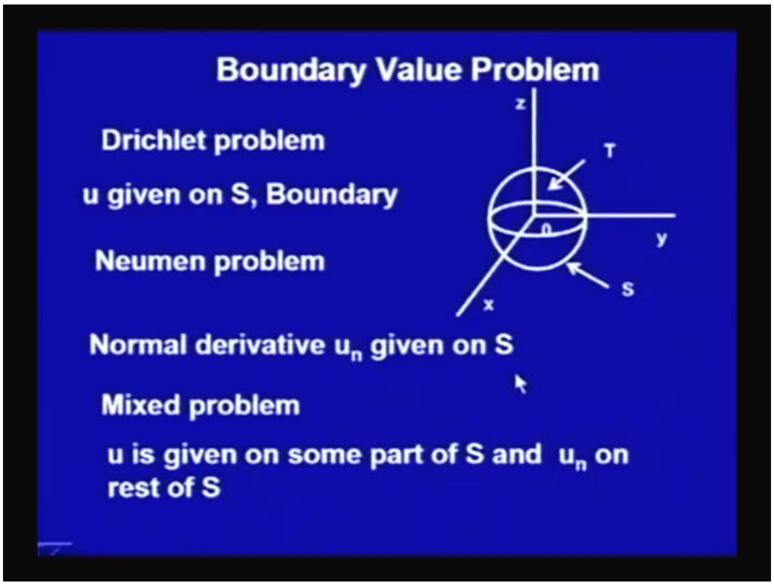

So from here, let us move to the theory of solving this Laplace equation, for solving of the Laplace equation practically it is resulting in the boundary value problems. We define the boundary value problems, because we are having is that the reason is three dimensional, so it is three dimensional one. So, let us here is the example for a sphere you could any three dimensional mass in which we are interested the reason is the three dimensional solid cube, and we do have it is been covered by a surface that is we are calling S.

So, we define the three kind of boundary value problem, first boundary value problem or Drichlet problem that we are calling in which the boundary condition is given on the surface that is the boundary of the region, that is u is given on s that is boundary. Second kind of boundary value problem or that we are calling Neumen problem, in which the boundary condition for the rather than for given for the u, we have been given that is normal derivative of u is been defined on the or has been prescribed on the boundary S.

And, in the third coming of boundary value problems are called the mixed problems in which any portion of boundary S u is been prescribed and the rest of the portion of S the normal derivative, u is prescribed. We have seen that in two dimensional problems or two dimensional Laplace differential equations also, we have seen that there reason was our applinary.

Here, we are having the reason is three dimensional solid, there we used to have the reason was the preliminary theorem and the boundary was the curve here boundary would be the surface. In the similar manner we have defined Drichlet problem normal derivative this Neumen problem and mixed problem. So, let us say we would start with the Drichlet problem first that is when the boundary condition is the u is defined at the boundary S surface S. So, solving it we do require is that we first actually make the change of co-ordinates or change of variables, the co-ordinates x y z they are the variables. So, we change the variables in such a manner that the region S that is the surface S becomes more simplified.

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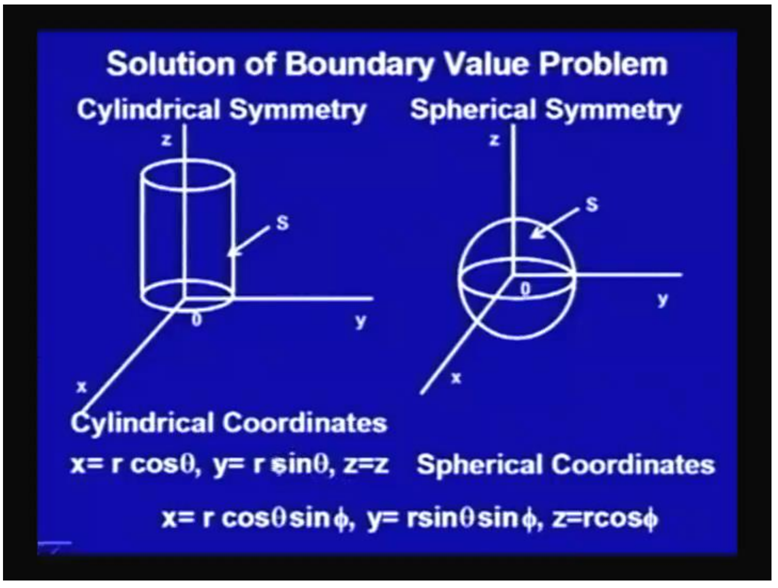

So, in that one, first we see if my surface is having cylindrical symmetry, so the solution of boundary value problem we will first see if the surface is having cylindrical symmetry that says is suppose let us say for example, my surface is that the my region is this cylinder and surface is that is for boundary condition we are defining the surface of this cylinder S.

Then, we would change my Laplace equation into the cylindrical co-ordinates, what are the cylindrical co-ordinates? They have been defined that is we are transforming x y and z the Cartesian co-ordinates into the co-ordinates r theta and z, where the relationship between them is x is r cos theta y is r sine theta and z is equal to z.

We will learn little bit later, that is how they are been acting, similarly, suppose my problem says that the boundaries is spherical symmetry that is we are having the reason is some ball and the surface is itself sphere. So, let us say this is the reason is this the ball and in the surface is of this one, so we do have that surface is boundary is this sphere.

Then, we would like to change my Laplace equation from this Cartesian co-ordinates that is x y z to the spherical co-ordinates, what are the spherical co-ordinates? They are been given in the terms of r theta and phi and the relationship between x y z and r theta phi are been defined in this relation, that is x is r cos theta sine phi y as r sine theta sine phi and z as r cos phi. Let us see first that is what these co-ordinates and that is, what are the cylindrical co-ordinates, what are this spherical co-ordinates and how we are transforming our Laplace equation into these new variables.

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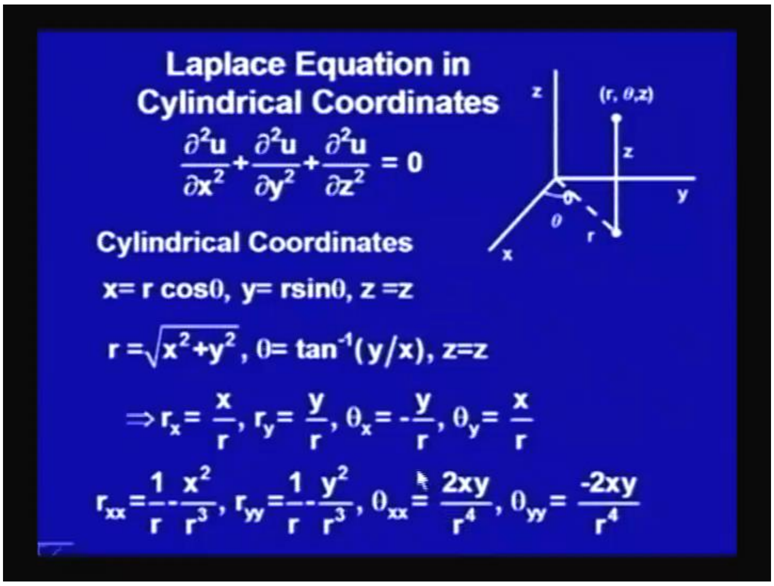

So, let us see 1 by 1 Laplace equation in cylindrical co-ordinates, so this is our given Laplace equation del 2 u over del x 2 plus del 2 u over del y 2 plus del 2 u over del z 2 is equal to 0, which is defined in the variables x y and z. Now, we want to define it in the terms of new variables, which we are calling cylindrical co-ordinates. Let us see, what are what we mean by cylindrical co-ordinates, as I said is that if my surface is having cylindrical symmetry. It will remember that is we are having is that cylinder is circular and base points are you says that any cross section is circle, while you do have that height is straight line. So, that is how we are defining, that is in this three dimensional space any point as in the Cartesian co-ordinates we are defining this point as x y z

Now, I would represent these point with three co-ordinates now r theta and z, z is same thing that is if I take, what is the distance along the z axis of this point from the x y plane. So, we do have this simple Cartesian, in from the x y plane whatever the distance from this x y plane to this point z we are having that is if I drop a perpendicular from this point to the x y plane this distance this height would be called z.

So, this is what limit, but what we are having is x y plane when we talk about that is this one would be this part. So here, if you do remember in the single dimensional one, we are defining polar co-ordinates. So we define, here that is because of circle we are defining the polar co-ordinates that is the point which is coming under x y plane for that the distance from the origin r and the angle it is making with the x axis that is theta.

So, what now this point we are defining as this point has been transported to the x y plane and in x y plane we are using the co-ordinate system r and theta and this is z. So, this is what we are having is cylindrical co-ordinates from the polar angle we do know if I am denoting this r as the distance from origin to this point if this point in x y plane is x into y.

Then, we do know that x is nothing, but r cos theta and y is nothing but r sine theta z is same as z, since we want to change my Laplace equation into these new variables r theta and z that is says is this derivatives of the function that is the function u x y z, now we would write it as u r theta z. So, when we are changing it so this derivative with respect to x y and z that we have to change the derivative with respect to r y r theta and z. So, z is same as z and that is the third co-ordinate itself.

So, we would first concentrate on this change of x and y into r and theta; that means, rather than talking about this three dimensional one we will first talk about the two dimensional one. And then we will add up the third one as such, so if that if says is now we would treat u as a function of x and y and then we want to change the variables from x and y to r and theta using this relationship.

So, what the things you we do know that we will use the chain rule and accordingly we would find it out, so let us write this r and theta in the terms of x and y. We do know that r would be nothing but square root of x square plus y square theta is nothing but tan inverse y upon x and z is equal to z, since we are concentrating on x and y. Then we would try to use the derivatives of r and theta with respect to x and y, because both r and theta are just the function of x and y only.

So, from here we do get the derivative of r with respect to x, so now, we are writing the partial derivative and we are using this notation rather than using del r by del x just for the space place we using this kind of notation, where r subscript x means it is the derivative of r partial derivative of r with respect to x. Similarly, if I do have the subscript y it says is that the partial derivative of r with respect to y and so on.

So, if I partially differentiate this one we do get is that we would get 1 upon 2 square root of x square plus y square downside, that is in the denominator. And the derivative of this with respect to x would be 2 x, so we would be getting is x upon r, similarly the partial derivative of r with respect to y we would get y upon r partial derivative of theta with respect to x.

The derivative of tan inverse x we do know 1 upon 1 plus x square, so using that 1, we do get it as 1 upon 1 plus y upon x square then the derivative of y by x with respect to x would be minus y upon x square. So, we just get it simplifying it we get it as minus y upon r, similarly the derivative of theta with respect to y would be x by r.

Now, we move to the second derivatives, that is r x x this is same as del 2 r over del x 2. Differentiating it again with respect to x, so what we would get 1 upon r then we would be getting is x times the derivative of 1 upon r that is minus 1 upon r square into the derivative of r with respect to x is x upon r, so we would be getting is 1 upon r minus x square upon r cube.

Similarly, the second derivative with respect to y that is differentiating r y with respect to y ones more we will get in a similar manner as 1 upon r minus y square upon r cube. Then differentiating theta x with respect to x 1 once more y is constant with respect to x 1 upon r its derivative is x by r square by r cube. So, we would x by r square, so we would get theta xx as 2 x y upon r to the power 4 in similar manner theta y y is equal to minus 2 x y upon r to the power 4.

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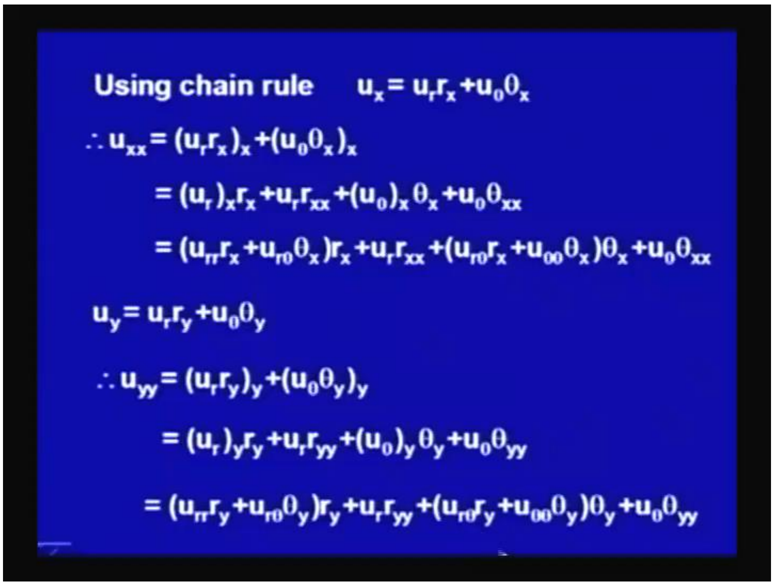

So now, we will find out the derivatives with respect to x and y and z change it to the derivative with respect to r and theta using the chain rule. We do know that u x can be written as u r into r x plus u theta into theta x.

Now, again differentiating it with respect to x, we can write u x x as u r r x differentiated with respect to x plus u theta x u theta theta x differentiated with respect to x. So, this subscript I am using for the derivative other than writing this del by del x I am using this subscript x.

So, now, if I differentiate this is the product of two functions one is the del u over del r and the other is del r over del x. So, we do get it this one as the derivative of u r with respect to x into r x plus u r into the derivative of r x with respect to x that is the second derivative of r with respect to x r x x, in a similar manner u theta the derivative of this with respect to x times theta x plus u theta times theta x x. Now, for u r differentiating it with respect to x, we do again use the chain rule what we would get, the derivative of u r with respect to r that is the second derivative of u with respect to r u r r into r x plus u r theta into theta x this r x is as such this term is as such u r times r x x.

Then, the derivative of u theta with respect to x again use the chain rule over here, the derivative with respect to r of u theta that we would say is del 2 u over del r del theta. So, u r theta times r x plus u theta theta times theta x that is the second derivative of u with respect to theta into theta x and then this term u theta plus theta x x is as such.

Similarly, we could get the u y also as using the chain rule u r times r y plus u theta times theta y, again differentiated with respect to y in a similar manner. We could write u r r y times derivative with respect to y and the derivative of u theta and theta y with respect to y.

Again, use the multiplication rule of the differentiation we do get it the derivative of u r with respect to y into r y plus u r times the derivative of r y with respect to y that is r y y this similar manner the derivative of u theta with respect to y times theta y plus u theta times theta y y.

Again, use this chain rule using the derivative of u r with respect to y, we do get it as u r r, r y plus u r theta theta y into r y plus this term as such u r r y y. And for this again using the chain rule u r theta r y plus u theta theta theta y times theta y and this term u theta theta y y as such.

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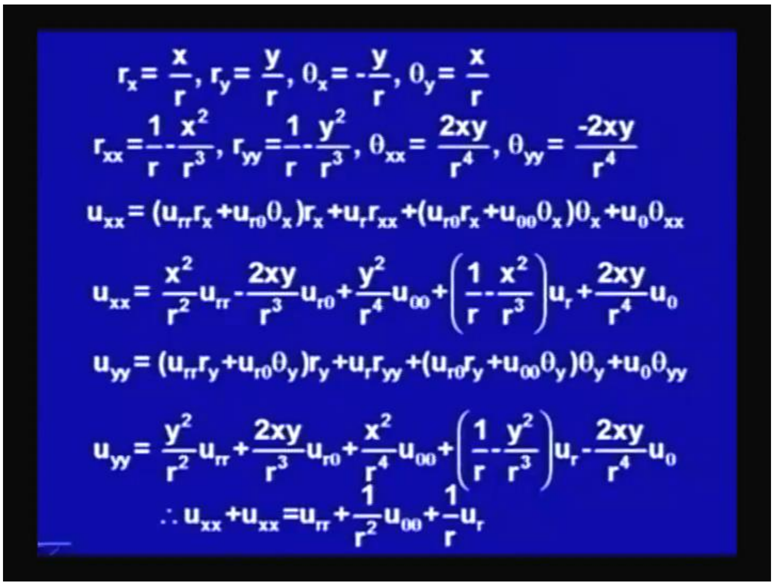

So, now, we substitute this r theta r x r y theta x theta y in this what we have obtained, we had obtained this r x r y theta x and theta y, moreover we had also obtained r x x r y y theta xx and theta y y.

So, let us first find out u x x where u x x just now we had obtained as this formulation u r r r x plus u r theta, theta x times r x plus u r r x x plus u r theta r x plus u theta theta theta x times theta x plus u theta theta x x now substitute all these. Here, what we would get is, r since x by r, so I would be getting is the co-efficient of u r r is r x square that is x square upon r square. So, we would get x square upon r square times u r r co-efficient of u r theta.

We have used one thing that is the partial derivatives that is the function is continuous in such a manner that where we take the derivative u r theta or u theta r they are same. So, the multiple the co-efficient of u r theta is theta x r x here and from here it is r x theta x r x is x by r and theta x is minus y by r. So, we would be getting is minus x y upon r square and we would be getting it 2 times. So, we are getting it minus 2 xy upon r square ur theta plus y square upon r to the power 4 u theta theta plus 1 by r minus x square upon r cube times u r that is u r r x x and then u theta theta x x theta xx is 2 x y upon r to the power 4 u theta.

Similarly, we can get u y y again substituting this r y r y is y upon r multiplication of this we would be getting is y square upon r square u r r. And then the co-efficient of u r theta is theta y r y and theta y r y from here theta y and r y we would be getting as again x y upon r cube that is u r theta twice with the plus sign now. And, the co-efficient of u theta theta here would be x square upon r square that is we are getting is theta y whole square. And then u r as 1 upon co-efficient of u r as 1 upon r minus y square upon r cube minus the co-efficient of u theta is minus 2 x y upon r 4 that is theta y y is minus 2 x y upon r 4.

So, now u xx plus u y y, what we would get from here? We would get x square upon r square plus y square upon r square and if you remember r was nothing, but square root of x square plus y square. So, x square plus y square upon r square that is 1, similarly this term would get cancel it out here what we would be getting x square plus y square r is square, so 1 upon r square.

And, this term we would be getting as 1 upon r 1 upon r 2 upon r and then with the minus sign x square plus y square upon r cube. Again, what we would be getting a is x square plus y square is r square, so again you would be getting is 1 upon r and this term will get cancel it out. So, finally, we would get u r r plus 1 upon r square times u theta theta plus 1 upon r u r.

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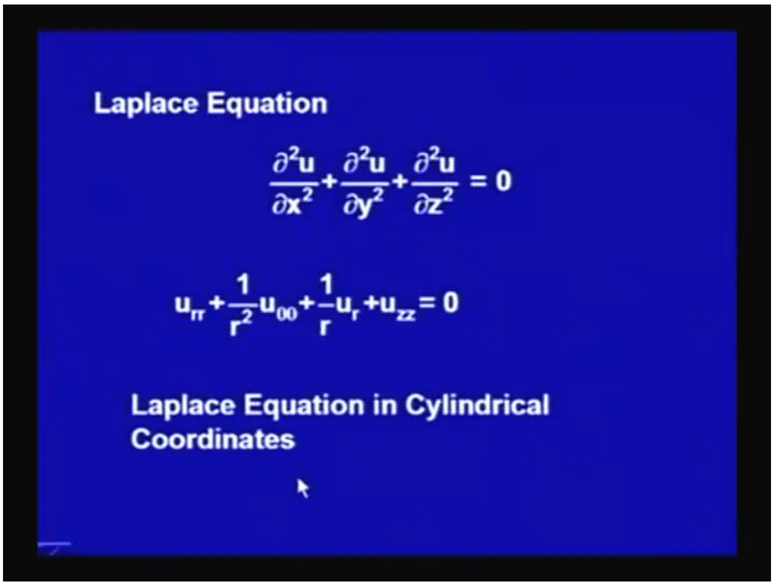

Now, so what will be our Laplace equation, in the cylindrical co-ordinate Laplace equation is this one we have changed only using this x and y, since z was totally independent same as the third co-ordinate over here. So, we would use simply u z z for this 1, so what we would be getting is the changed co-ordinate for x and y we have changed to r and theta we do get it u r r plus 1 upon r square u theta theta plus 1 upon r u r plus u z z is equal to 0, so this is our Laplace equation in cylindrical co-ordinates.

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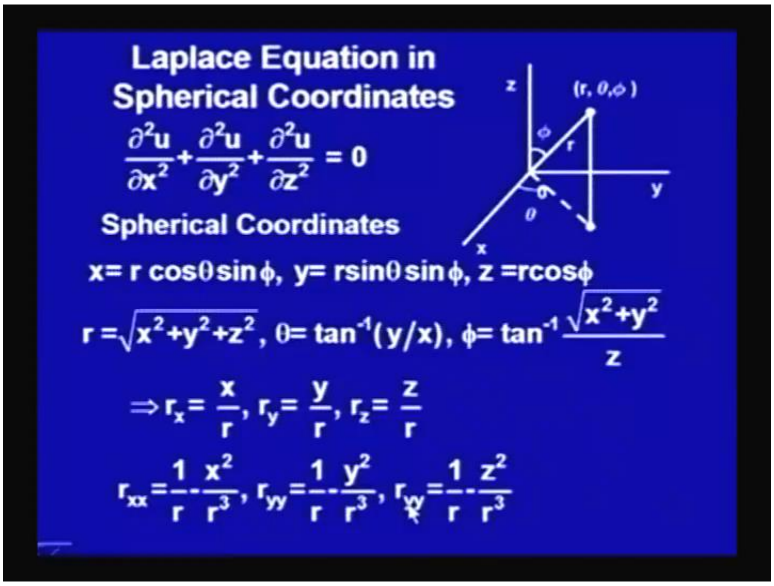

Will learn about how to solve these equations, but before that will move to the another co-ordinates that is if my surface is having spherical symmetry that is we change the Laplace equation in spherical co-ordinates, what are these spherical co-ordinates? We do have this is the Laplace equation, this we have to change into the spherical co-ordinates.

Now, what are the spherical co-ordinates if you do remember sphere, we are having all the around in the cylinder we were have is one was one dimensional was exactly same as the Cartesian one, but here we are having is all the round. So, we would talk about these co-ordinates in the all round shape. So, what we are having is in Cartesian co-ordinates this point would be simply called as x y z plane, we are having is that the distance z of this point from the origin in the from parallel to the z axis parallel to the x axis and parallel to y axis that is what we are calling x y z.

Now see, because it is a sphere, so what we would say is we trying the origin? Origin is being trying to this point. So, now, this is do not take it this is not in the x y plane this is now in the three dimensional one. So, origin is been trying to this one, this is what we are saying as the distance between the origin and this point that we are calling r, so this is r.

Then, this distance the line which is joining the origin to this one this is making one angle phi with respect to the z axis that is, what we are calling phi and as usual if I make it is a perpendicular drop on the x y plane I will get this point. So, this point in the polar co ordinates we do have that this distance we are calling r and this angle we are calling theta.

So, now, r I am not taking and r we are taking is the distance of this point from the in this three dimensional displace from the origin and theta is that angle which would make a phi drop apart from the that is phi takes its image in the x y plane. Then this angle with this line would be making that is the line joining the origin to this point in the x y plane, this line with the angle it is making this is called theta.

Because, while we are taking it one since we are talking about the sphere its cross section would be any circle,m whatever be the polar co-ordinates for the circle we do have r and theta, but that r we are leaving because that r we could find it out using this r that is in the three dimensional distance from origin to the point and the surface of the sphere.

So, if I just make it out we do know that from this point if we are talking about then x is the distance from here to here. So, first let us try to make that is that we would be getting it in the terms of r and phi, so we would get it as r this distance would be r cos phi this would be r sine phi, because phi is the angle on this side of the z axis. So, this distance this distance would be r sine phi and this distance would be r cos phi.

So, we have got that z is r cos phi, now what would be x and y x and y, now we are getting as this distance that is what I said is other requirement this is because this distance is nothing but r sine phi if this is r sine phi and this is the angle theta. We do know this next called this distance as rho.

Then, we do know that x is rho cos theta and y is rho sine theta now what is rho, rho is r sine phi. So, we have got x as r cos theta sine phi or rather we could say is now my this rho is r cos r sine phi multiply with cos theta, but I have written in this manner because we are writing it r theta phi. So, r cos theta sine phi y is r sine theta sine phi and z is r cos phi.

Now, again if you have to change it, to these new co-ordinates that is in the new variables r theta and phi we would use this relationship. So, again we would require to change it to in the r theta and phi, so we require those relations. So, let see r would be now square root of x square plus y square plus z square that is the distance from origin to this point x y z theta as I said would be tan inverse y upon x phi is.

Now, the angle from here we would find it out that is we do get it x square plus y square would be r square, square root of x square plus y square would be r sine phi and z is r cos phi. So, we would get phi as tan inverse of square root of x square plus y square upon z, find out the derivative of r theta and phi with respect to x y and z and the second derivative that we would be requiring.

So, first the derivative of r with respect to x y and z, if I differentiate it as in the first manner we had seen here also we would get it as x by r, r y that is the derivative with respect to y as y upon r, r z as the derivative with respect to z as z upon r. From here, if I just go with the second derivative again with respect to x. So, I will get 1 upon r into x times the derivative of 1 upon r that is minus 1 upon r square and the derivative of r with respect to x is x by r, so again we would get 1 by r minus x square upon r cube.

Similarly, the second derivative of r with respect to y is 1 upon r minus y square upon r cube and the third derivative second derivative with respect to z that is r z z would be 1 upon r minus z square upon r cube.

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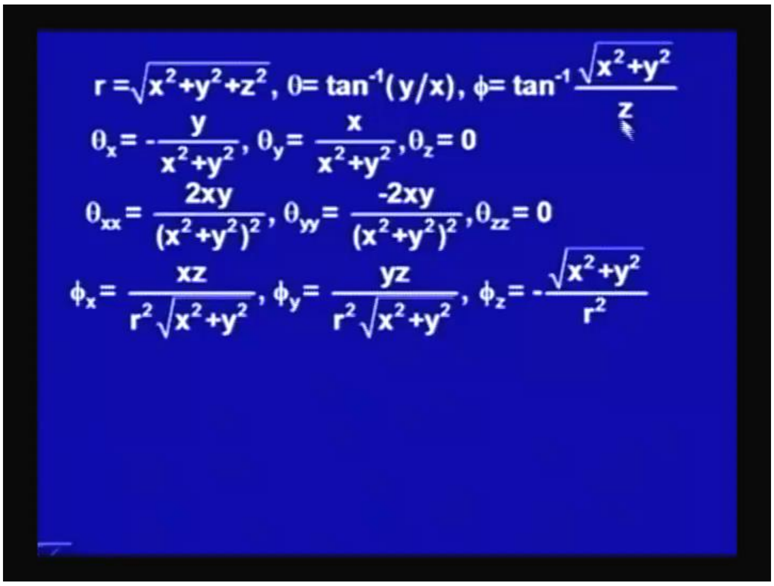

Now, since this r is this one now let us come to the derivative of theta with respect to x y and z. Now, we see is that theta is a function of y and x only its derivative with respect to z would be 0.

So, theta x that is the derivative of x with respect to theta with respect to x as in the previous one we had obtained, there we had written alpha, but now this r is having x square plus y square plus z square. We would get minus y upon x square plus y square theta y as x upon x square plus y square and since it is not containing any term of z, we will get theta z is equal to 0.

So, the second derivatives theta x x we would get it as 2 x y upon x square plus y square the whole square because its derivative would be minus 1 upon x square plus y square and the derivative of this would be 2 x. So, we would be getting like this one, similarly for theta y we would be getting it minus 2 x y upon x square plus y square whole square and the second derivative with respect to z again it would be 0.

Now come to this phi, its derivative with respect to x tan inverse x does have its derivative 1 upon x square 1 upon 1 plus x square, and then this is the function of x y and z that we will define.

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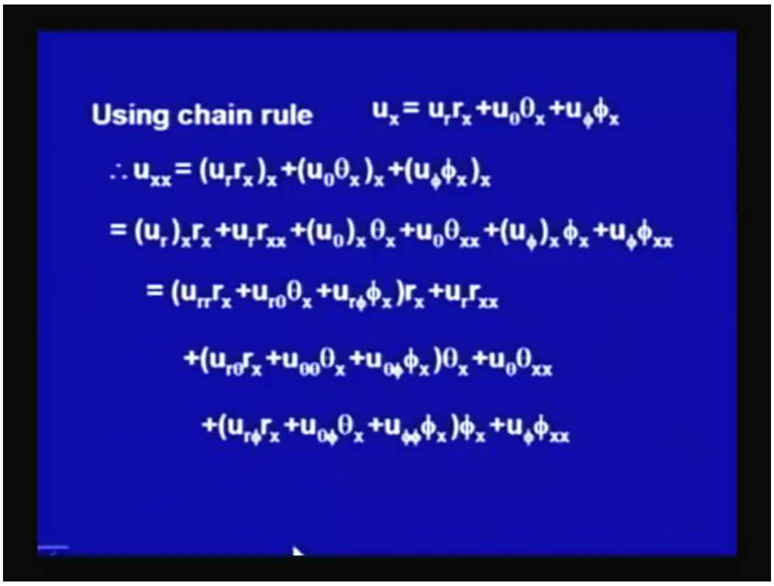

See using chain rule u x can be written as u r r x plus u theta theta x plus u phi phi x, now what are you seeing is that is I have extended that chain rule to the three variables now. The variables are that is u is the function of x y and z and we are changing this x to function of r theta and phi, so we have to apply these three various rule. Again, if I have to find out the second derivative with respect to x using similar manner we could write it out that is the derivative of u r r x with respect to x u theta theta x with respect to x and u phi phi x with respect to x.

Using the product rule, we could write it as the derivative of u r with respect to x into r x plus u r r x x plus u theta derivative with respect to x into theta x and then u theta the derivative of theta x with respect to x that is theta x x. Similarly, the derivative of u phi with respect to x multiplied with phi x and then u phi the derivative of phi x with respect to x that is phi x x.

Again, for finding out the derivative of u r with respect to x again we will use the chain rule and that says is we would be getting as the derivative of u r with respect to r that is u r r r x plus u r theta theta x plus u r phi phi x. Now, you see as again using the, that is the derivatives the order of the differentiation can be interchanged then this term u r r x has been as such.

Then, using the chain rule again for the derivative of u theta with respect to x, we get it u r theta r x plus u theta theta theta x plus u theta phi phi x times theta x and this u theta theta x x as such. Then using the chain rule, again for u phi the derivative of u phi with respect to x we get u r phi r x plus u theta phi theta x plus u phi phi phi x times phi x plus u phi phi x x.

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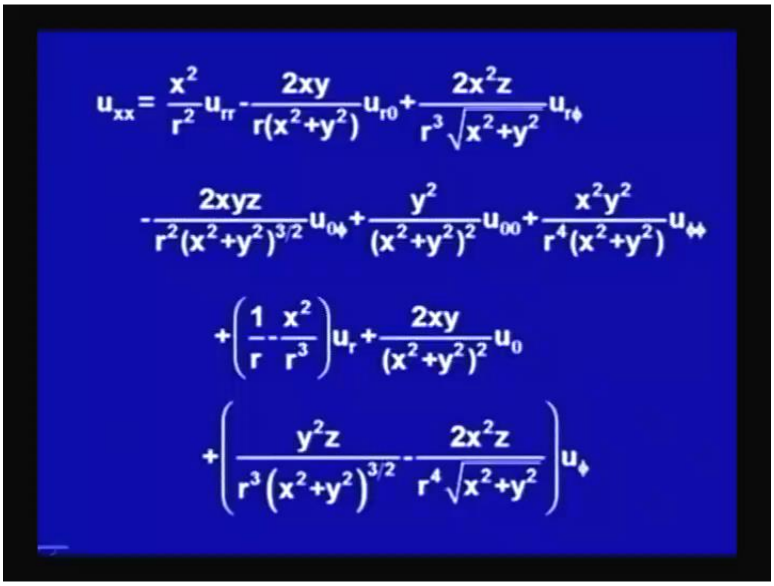

So, this is what you have got u x x in a similar manner, we can use the chain rule. So, u x x we could now substitute the values of that r x and r x x and all those things, so we do get is now you can see it in your calculation yourself, you would be getting is x square upon r square u r r minus 2 x y upon r times x square y square u r theta plus 2 x square upon r cube square root of x square plus y square u r phi minus 2 x y z upon r square x square plus y square to the power 3 by 2 u theta phi plus y square upon x square plus y square whole square times u theta theta plus x square y square upon r to the power 4 times x square y square u phi phi.

Plus 1 upon r minus x square upon r cube times u r plus 2 x y upon x square plus y square whole square times u theta. And then we would be getting is that phi xx times u phi, so y square z upon r cube upon times x square plus y square to the power 3 by 2 minus 2 x square z upon r to the power 4 times square root of x square plus y square times u phi. All these we had already calculated, so we had substituted finally u x x we have this in our expression.

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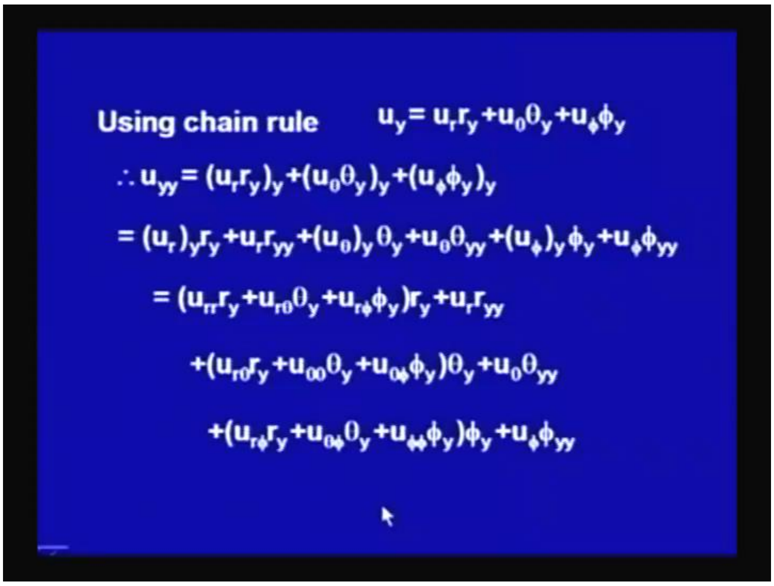

Similarly, we can get using the chain rule u y as u r r y u theta theta y u phi phi y, and then second derivative again using similarly that is first using the chain rule. Then using the multiplication rule u r y r y u r r y y and so on, we could get it in a similar manner. Again using the chain rule for u r the derivative of u r with respect to y derivative of u theta with respect to y and the derivative of u y with respect to phi with respect to y we would again use the chain rule and we do get final is as u r r r y plus u r theta theta y plus u r phi phi y times r y.

And then u r times r y y as such, similarly, for this u theta we could write u theta with respect to y u r theta r y plus u theta theta theta y plus u theta phi phi y times theta y plus u theta theta y y as such. Then again using the chain rule for the derivative of u phi with respect to y u r phi r y plus u theta phi theta phi u phi phi phi y into phi y and u phi phi y y as such again we have seen is we are using this as the order of differentiation is not making a difference in our calculations.

Now, again we had already calculated what is the co efficient of u r r from here we would get r x r y whole square. The co efficient of u r theta we would be getting from here theta y r y and from here again r y theta y, so we would get 2 times r y theta y, similarly for u r phi r y phi y and from here again r y phi y that is 2 times r y phi y.

Similarly, we can get phi u phi theta from here and u phi theta from here, theta phi from here as phi y theta y phi y theta y.

And, the co-efficient of u theta theta would be theta y times theta y that is theta y the whole square co-efficient of u phi phi would be phi y the whole square. And then we would be having this that co-efficient of u r as r y y co-efficient of u theta as theta y y and co-efficient of u phi as phi y y, so now substituting all these knowing all these, because you have already calculated r x r y r y y theta y theta y y and so on.

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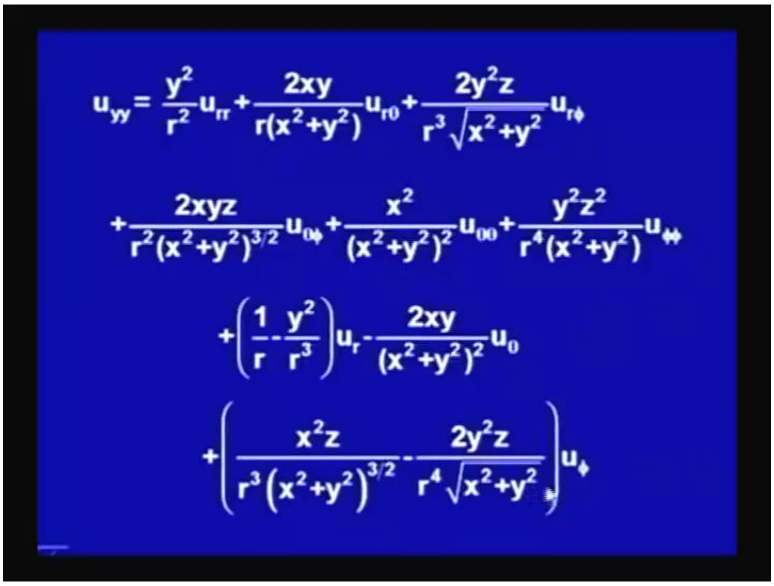

We do get it finally, as y square upon r square u r r plus 2 x y upon r times x square plus y square u r theta plus 2 y square z upon r cube times the square root of x square plus y square u r phi. Plus 2 xyz upon r square times x square plus y square to the power 3 by 2 u theta phi plus x square upon x square plus y square whole square u theta theta plus y square z square upon r to the power 4 times r to the power 4 times x square plus y square into u phi phi.

And then the comes u r u theta and u phi 1 upon r minus y square upon r cube times u r minus 2 x y upon x square plus y square the whole square times u theta and x square upon z upon x square into z upon r cube times x square plus y square to the power 3 by 2 minus 2 y square z upon r to the power 4 square root of x square plus y square times u phi.

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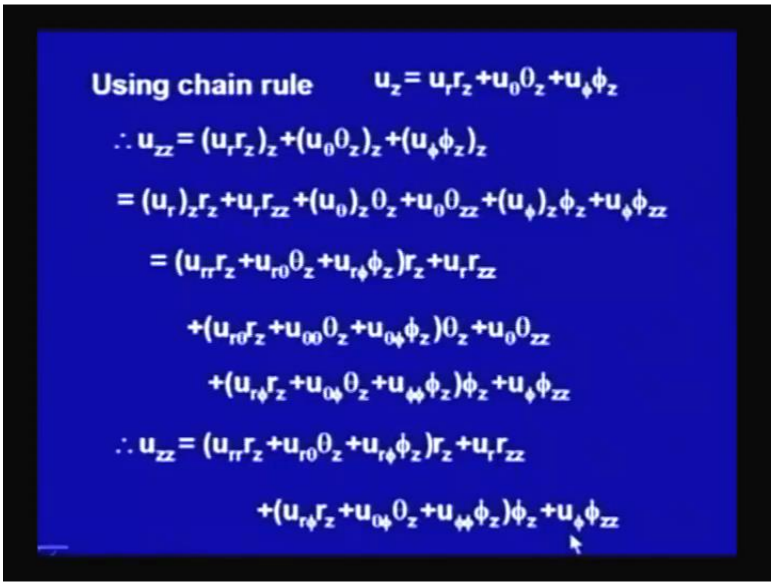

Similarly, for z u z again we use the same kind of chain rule u r r z u theta theta z u phi phi z, the second derivative again first using this chain rule that is differentiation of the sum same as the sum of the differentiations. And then using the product rule for the differentiation as that is the derivative of first function into the second function plus the first function into the derivative of the second function like that. Again using in this, the derivative of u r with respect to z and u theta with respect to z and u phi with respect to z, again using the chain rule for this we would get it finally, as u r r r z plus u r theta theta z plus u r phi phi z times r z plus u r r z z.

And then the chain rule over here would give me u r theta rz plus u theta theta theta z plus u theta phi phi z times theta z into plus u theta theta z z and the third one the phi u r phi r z plus u theta phi phi theta z plus u phi phi phi z into phi z plus u phi phi z z. Then, you see we have got that is our theta was the function of x and y only, it was not having any it is not a function of z, so theta z derivative of theta with respect to z or the second derivative of theta with respect to z they are 0, that is this middle term goes out this is simply 0. And here also, this term we would get is this term would goes to 0, so finally, we would be getting as u r r r z again this term will also be 0. So, finally, we would be getting is u r r r z plus u r phi phi z r z plus u r r z z and then this third term u r phi r z and u phi phi phi z times phi z and u phi phi z z this term will also go out.

So, now the co-efficient of u r r is nothing but r z whole square co-efficient of u r theta from here is, this theta z is not the point this is 0 the co-efficient of u r phi from here is phi z r z and the co-efficient of u r phi is here some here is also r z phi z. And then the co-efficient of u phi phi is phi z square and u r co-efficient is r z z and u phi co-efficient is phi z z.

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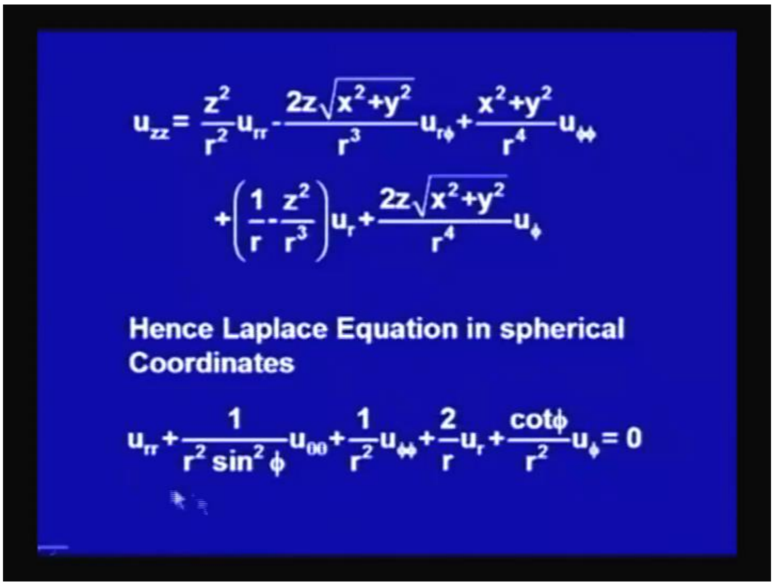

Substitute it we would finally, get z square upon r square times u r r minus 2 z x square plus square root of x square plus y square upon r cube times u r phi plus x square plus y square upon r to the power 4 times u phi phi and the second order derivative. And then u r and u phi the co-efficient of that is r y there phi y y, so r r z z and phi z z, so that is 1 upon r minus z square upon r cube u r plus 2 z square root of x square plus y square upon r to the power 4 times phi y.

So, now we have got u xx u y y and u z z in all these terms you see is that is this the co-efficient of u r r in u xx was x square upon r square in u y y y was y square upon r square and in u z z it is z square upon r square. So, if I add up u xx plus u y y plus u z z i would be getting is x square plus y square plus z square divided by r square, that is same as 1.

Similarly, we just see is that all those terms we can spread it on one sheet, and then we can see that some of them are canceling it out some of them are being simplified. And finally, what we would get the Laplace equation in spherical co-ordinate as u r r plus 1 upon r square sine square phi u theta theta plus 1 upon r square u phi phi plus 2 upon r u r plus cot phi upon r square u phi.

You see, we are having here certain terms in the terms of x y and z, what we would do finally, is that is square root of x square plus y square that is x square plus y square that is nothing but r square sine square phi, so like that we would be getting. And similarly, tan phi is square root of x square plus y square upon z, so like that we would be substituting this those terms we would get in this simplified form. Sometimes, it is easy to write this differential equation in another manner also that is much easy for solving the equation what is that alternative for…

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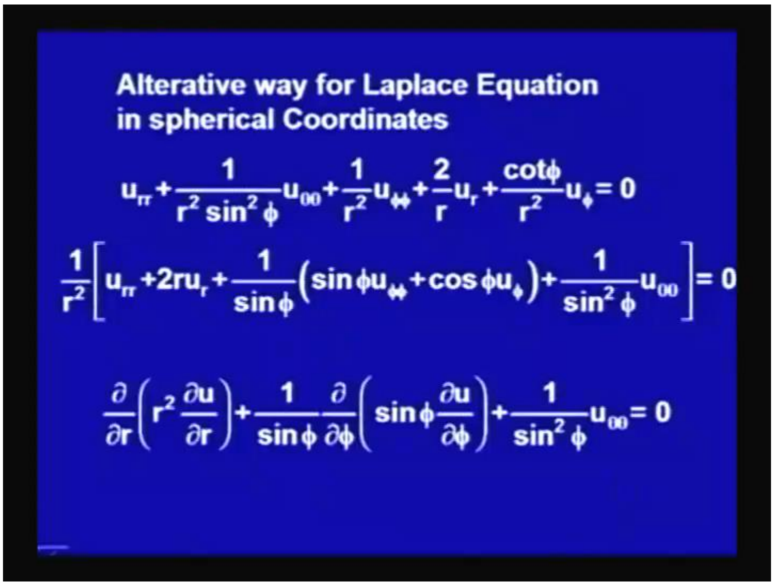

This is my equation Laplace equation in spherical co-ordinates we have got u r r plus 1 upon r square sine square phi u theta theta plus 1 upon r square u phi phi plus 2 upon r u r plus cot phi upon r square u phi is equal to 0. Now, you see here I am having 1 term as u r r and here I am having is 1 term of 2 upon r u r and we are having is 1 upon r square all the times in the deniers except at this point and this point.

So, first we will take 1 upon r square outside, if I am taking 1 upon r square outside, here I would get r square times u r r and then here I would get 2 times r times u r. So, we could simply write it out in that manner u r r plus 2 r u r that is this term have it in first. Then from here 1 upon r square is common it would be 1 upon sine square phi and 1 upon r square has gone from here also cot phi means we could write it as cos phi upon sine phi. So, again 1 upon sine phi from these two terms I would take it out.

If I am taking out this it should be 1 upon sine square phi why so we would be getting it as sine phi u phi phi plus cos phi u phi plus 1 upon sine square phi u theta theta that is this term as such. So, from here what we are getting is 1 upon r square is outside 1 upon sine phi we have taken outside u phi phi co-efficient I have to multiply sine phi and then I would get cos phi u phi, and this 1 upon r square sine square phi times u theta theta that is as such. This is equal to 0 this would be my equation.

So, now, because it is equated to 0 and r square this is the distance, this cannot be we are taking the points, so this is we are not taking it as 0. Some sphere of some radius r, so the boundary surface we are not taking it any point on the surface that would never give me the distance 0. So, we can take it that side and we would get it simply the equation like this 1 what is in the bracket 1 that is equated to 0.

So, this one let us take the first two terms we could say is that is if I take the function r square times del u over del r, u r means del u over del r if I differentiate it with respect to r using the product rule I would get it r square times. So, here actually I think forgotten r square, so r square has to be here. So, r square times u r r and then the derivative of r square is 2 r and then u r r. So, this is what you are getting is r square del u over del r.

Then, 1 upon sine phi now here if I see, the function if I take sine phi u phi its derivative with respect to phi would be sine phi times u phi phi. And then the derivative of sine phi is cos phi and phi as such, so it is del upon del phi sine phi del u over del phi and then the third term is as such. So, this alternative form is many times much easier to solve, so we do also use this alternative form rather in our example we would be using this alternative form.

So, this is what we have changed the, our Laplace equation into either cylindrical co-ordinates or spherical co-ordinates. Let us, try to see is that is, how to find out the solution, so let us first try the surface of the boundary as sphere or spherical symmetry. So, we would use this spherical Laplace equation in spherical co-ordinates.

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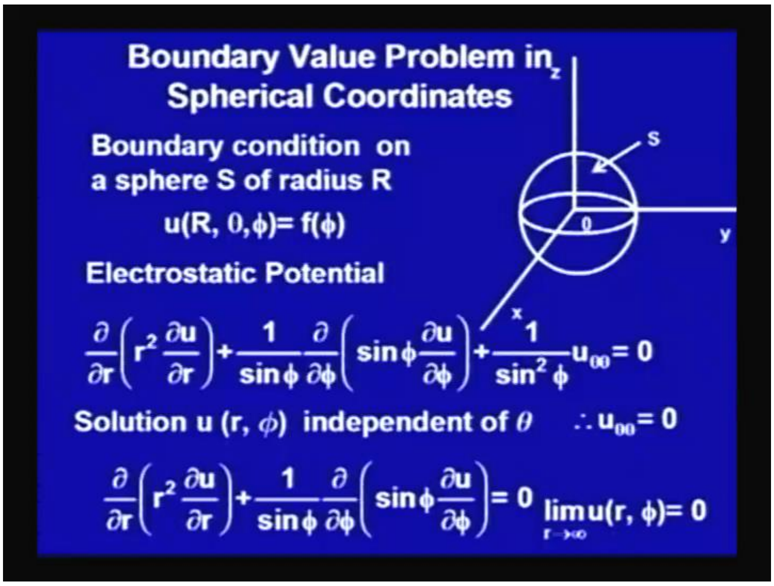

For this, let us see the boundary values problem here, how we could define, suppose my region of interest is this ball that is sphere and the boundary is the surface of this sphere, what we have been given if the function u is independent of theta on this surface that the boundary condition is given independent of this theta on this surface.

So, we want boundary condition on a sphere s of the radius R now this radius also we have fixed. We say is u R theta phi is f of phi that is independent of theta that says is if I make this kind of situation is arising in electrostatic potential, where R and the temperature where this we are talking about that we are taking a ball kind of things and we are talking about the potential is at this or outside this ball.

If I make it independent of theta, what it says is, that I would get my differential equation this laplacian equation would be this is govern by this laplacian equation. So, I have written this laplacian equation that what we have written in second form as here, what it would give me that u theta and u theta theta would be 0, that is this term will go out.

And my solution, if it is independent of theta u theta theta would be 0 that says is i would get only two terms del by del r of r square del u over del r plus 1 upon sine phi del upon del phi sine phi del u over del phi is equal to 0. So, now what we have defined, we have got that is if I am interested to determine the electrostatic potential on a spherical region. I could use it, on that what we have seen is it is independent of the third co-ordinate theta, the solution is independent of third co-ordinate theta. Then my solution would be u r phi and the governing equation then would be del upon del r r square del u upon del r plus 1 upon sine phi del upon del phi sine phi del u upon del phi is equal to 0.

One more thing we would like to say is that is this potential has to be go to 0 as r approaches to infinity that is we say is that on this all those potential all the inside this, because the boundary condition is on the surface. So inside the sphere, and outside the sphere, so inside the sphere the boundary condition will come over here outside this sphere, we do say is that is limit as r approaches to infinity u r phi is 01.

So, now, what we are having is we have been defining our boundary condition on the surface that is on the boundary S the function is been described on the boundary S. So, this is Drichlet kind of problem that is the first kind of boundary value problem, that is called the Drichlet problem. So, let us learn how to solve the Drichlet boundary value problems. where my differential equation is this and the boundary conditions boundary condition is this one using this one more condition under the limit case. So, we would learn, how to solve this boundary value problem the Drichlet problem that we would learn in the next lecture.

So, today what we had learnt, we have learnt that is if we have a three dimensional Laplace equation this is some important application partial differential equation which is arising in basically electro statistics or in fluid flow and heat flow kind of things, what we had learnt is that is when we are talking about heat flow and fluid flow it is two dimensional Laplace equation which we had learnt little earlier.

Now, we are talking about the electrostatic that is potential and we had learnt that is solution which we are discussing that is how to find out the solution this theory we are calling that is why potential theory. So, today we had made one kind of change if my surface is spherically symmetry or if I am changing my co-ordinates Cartesian co-ordinates to spherical co-ordinates. Then, I would go ahead with the, I am changing the Laplace equation into spherical co-ordinates defining the boundary value problems that is Drichlet problems the first kind of boundary value problems. So, next class we would our next lecture we would learn how to solve this Drichlet problem.

Thank you.