Mathematics - III
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Lecture - 22
Fourier Integral & Transform Method for Heat Equation

Welcome to the lecture series on differential equation for under graduate students, we are doing partial differential equations. Today's topic is Fourier Integral and Transform Method for Heat Equations, we have learnt about the heat equation and it is solution using the Fourier series method. Today will learn the solution of heat equations using Fourier integral and the Fourier transforms.

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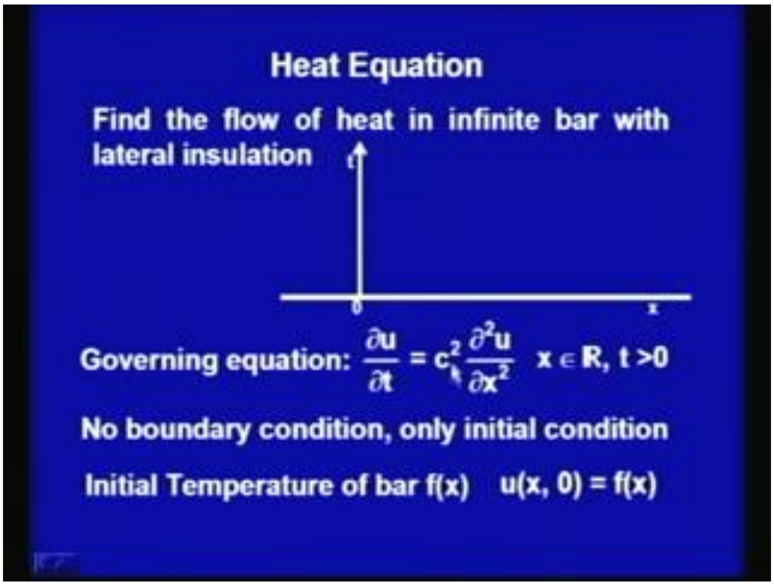

So, let us just try to see, what is our practical problem find the flow of heat in infinite bear with lateral insulation, that says is that now my bar is infinite one. It is ranging from minus infinity to plus infinity again it is laterally insulated that is says is that heat is not moving in the lateral direction it is move flowing only into the x direction.

Practically infinite bar what does it mean is, that is a very long bar a very long wire in this which heat is flowing. We have to now formulate this, so again the heat equation that the equation which is governing this flow of heat that would be same as that heat equation known to you del u upon del t is equal to c square times del 2 u over del x 2.

But now, my bar is infinite that is says is that in usual heat equation when we are solving we use to take the boundary conditions that is when the bar was the infinitive link, we use to take that temperature was the flow is 0 at the end points. Now, here because it is infinite one that is x is ranging from minus infinity to plus infinity, we do not have any boundary conditions.

But, we do have one initial condition that is initially the temperature of the bar is say f x that is what we are having is we are having u x at 0 is f x. So now, what we are, interested is if you see if you remember that is we have done this heat equation del u over del t is equal to c square del u x del 2 u over del x 2, for x belonging to 1 interval, so is 0 to l and then t is 0.

There we had the boundary condition at 0 as well as that l that is u 0 t is 0 on u l t is 0, for all t. That we have solved it using the Fourier series method, so we have got the solution in the terms of the Fourier series of the function where we had the function initial condition f x and that we had is that is if we say is that it is periodic with period 2 l that is we have taken the odd extension of that function f x which was defined on a finite length. Now, here what we are having is infinite bars, so there is no use of taking that kind of solution or it is natural that we think in the terms of Fourier integral, because we are having it infinite bar that is my function f x now cannot be periodic.

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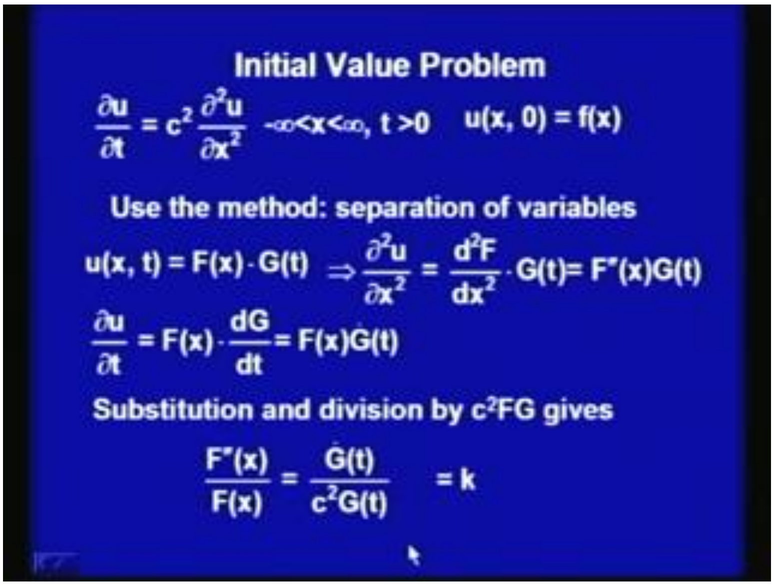

So, we do have one initial problem that del u over del t is equal to c square times del 2 u over del x 2 x is ranging from minus infinity to plus infinity t is positive with initial condition u x at 0 is f x. Now, how to solve it again we will go with the same method that is variable separable method, so separation of variables will use.

Let us, say the solution u x t is the function is product of two functions 1 of x only another is of t only F x into G t. So, if I differentiate partially this function with respect to x and with respect to t, we do get is that the second derivative with respect to x of u x t would govern only that is the second derivative of F with respect to t with respect to x only and G t would be the constant that is we have got it f double dash x G t. Similarly, we require here the first partial derivative the respective to t.

So, d u over d t would be f x dg by d t or we can write F x dot into g dot means is that derivative with respect to t. Now, substitute this del 2 u over del x 2 and del u over del t now at this given equation, and then divide it by c square F G what we do get if I do substitute it here I would get is F x times G dot t is equal to c square times F double dash x G t. If I divided by c square F G, so let us write it this first time that is c square times F double dash into G t divided by c square F x G t, what we would get it F double dash x divided by F x is equal to G dot t divided by c square G of t.

Now, this left hand side is having only one variable x the right hand side is having only one variable t and we are saying is that these ratios are equal the c square is certainly a constant. These ratios could be equal only if this ratio does not contain any variable that is this ratio must be equal to k.

So, let us say that it is some constant k what it says is now we have got two ordinary differential equation 1 from here that f double dash x is equal to k of F of x or F double dash minus k F and another is equal to 0 and another g dot t minus c is square k times G t.

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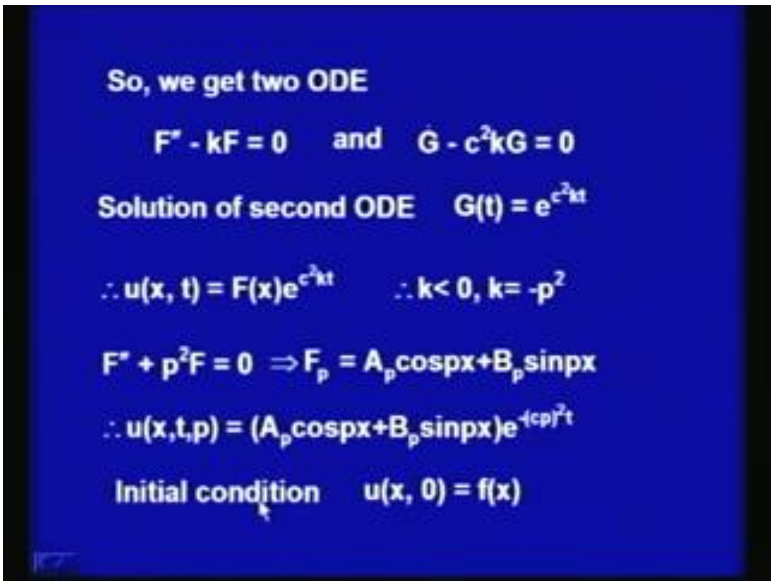

So, we have got the two ordinary differential equation F double dash minus k F is equal to 0 and G dot minus c square k g is equal to 0. Now, let us take usually we use to take this first equation to solve the first, but now let us just try with this second equation, because the second equation is first order equations much easier to solve.

So, let us try with this second equation first, if I am taking this second equation this is first order equation del G upon del t is equal to c square k G, we do know it is solution is nothing, but e to the power c square k t of course, there must be some constant that constant let us take it as 1.

Now, if this is the solution of second equation, let us say the, what kind of the solution of the first equation we do have. First is second order equation, we have already done a lot many times this kind of equation in this partial differential equation itself. It do have that is depends upon that what is the value of k is. If the value of k is 0 we would get the solution of the form a x plus b if the value of k is positive. We would get the solution in the form of a times e to the power x root k x and plus b times e to the power minus root k x, and if k x is negative we two get the solution in the form of a cos root k x plus b sin root k x kind of thing. So, finally, what we would get the solution of our partial differential equation that is we said is F x in to G t.

So, it would be a fixed that is the solution of the first equation into e to the power c square k t now, whatever the form of solution we do have for the different values of k.

That simply says is that function a x plus b are a e to the power root k x plus b root e to the power minus root k x and so on, what we do have is that, as t is increasing, because that would be a function of x only and t is in all the way here.

So, if t is increasing, this function would go on increasing, what it says is with respect to the time t, it will go on increasing while we do know that heat flows from the higher temperature to the lower temperature. So, it cannot go on increasing with the time t, so that is not of any practical importance if I am taking k to be positive or k to be 0, because at k to be 0 it would be only depending on x. And that is not with it is not depending on t again that would not be of any physical significance.

So, we have to take this k as negative only then it would have and practical significance this heat flow of the heat. So, we do take k to be negative, so let us say k is minus p square, k is minus p square means is my G t would be e to the power minus c square p square t.

Now, this equation first equation now we do, come the first equation will now become F double dash plus p square f is equal to 0. We do know it is the general solution is A times cos p x plus B times sin p x, since we are using this p we are writing it as the constants as A p that is depending upon p whatever the p would B.

To find out this A p and B p, we have to write the complete solution and then we have to use the initial condition. So, what we are getting is u x t, now I am using this third parameter p, because p we are using here k is equal to minus p square as A p times cos p x plus B p times sin p x e to the power minus c square p square into t.

Now, at t is equal to 0, this initial condition u x comma 0 is equal to f x, if I try to satisfy for every p, I would be getting different A p and B p and that what we would be having is that is we are not able to satisfy like that. So for doing it, we require hmm either we use as usual we use to do is that is we sum up all these p in the integer values we have taken the reason for direct to take it as integer value use to be. That we were having boundary condition that has given it certain h that is we have got this function was not in the exponential forms or something like that kind of things.

Now, here we do know this f x and there, what f x was defined in a finite interval, so we could take it odd extension or even extension depending upon where we are left with the cosine series or the sine series. But, here this f x is not periodic rather we cannot make it periodic in any manner, because it is already defined on the whole real line. So, it is better I used the idea of Fourier integral, that is I should use this p as a real number and we take for all p positive 0 to, because p has to be positive, so p positive 0 to infinity.

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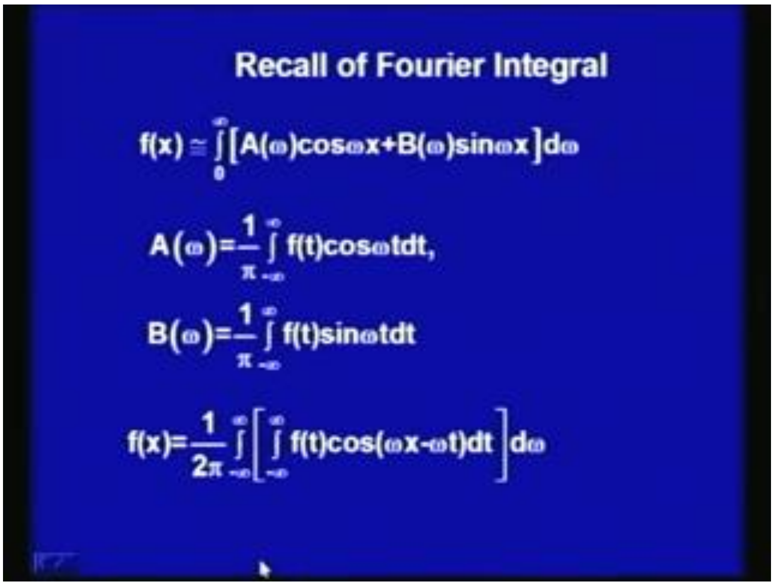

Let us, just see what is the Fourier integral which I am talking about I am just recalling it, because you have done long back. So, we do have that any function f x can be written as integral 0 to infinity A omega cos omega x plus B omega sin omega x d omega, where this A omega is 1 upon pi integral minus infinity to plus infinity f t cos omega t d t and B omega is one upon pi integral minus infinity to plus infinity f t sin omega t.

This we had learn in when we are doing is that Fourier, the function which are not periodic we had learn in Fourier integral. So, the same technique same method we would apply over now over here, so what we do get is that if I just write it at what we do get effects as 1 upon 2 pi integral minus infinity to plus infinity of integral of minus infinity to plus infinity f t cos omega x minus omega t d t d omega. That is what I have done is that is induce this A omega and B omega over here and simplified it would give me this one that is any function f x we could write as this in this form.

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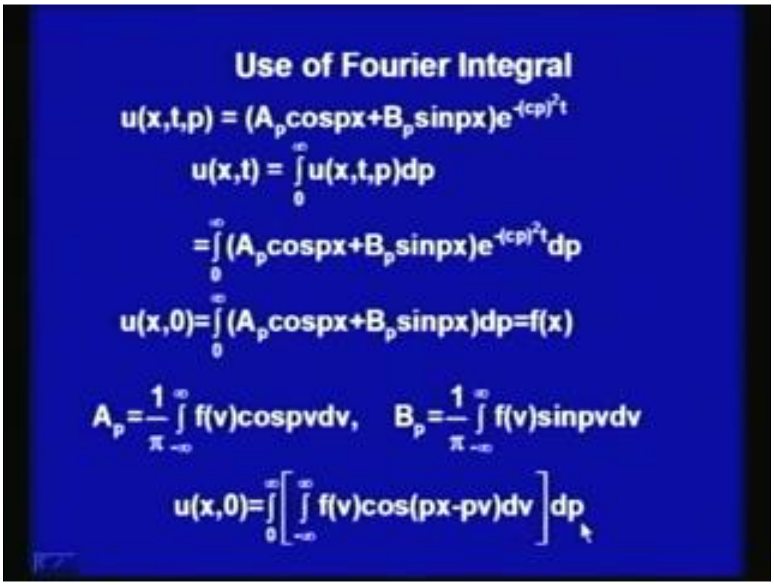

So let us see, we are going to use this Fourier integral in our solution, we have got the solution depending upon a parameter p as u x comma t comma p as A p cos p x plus B p sin p x into e to the power minus c square p square t. Now, let us define u x t as integral over this parameter p, so we are taking is u x t as 0 to infinity u x comma t comma p d p. Now if I write it out this one, what we are going to get it? We do get it at 0 to infinity A p cos p x plus B p sin p x, that is u x t comma p into e to the power minus c square p square t integrated with respect to p that is d p.

Now, the initial condition given is that u x at t is equal to 0 we are getting u x comma 0 as f x. So, first evaluate what will be u x comma 0, that is at t is equal to 0, I would get this term as 1, so I would get A p cos p x plus B p sin p x d p is equal to f x, that is f x, I should get diff i, I should get A p and B p in such a manner that I do get f x as this integral.

So, using the just now we had done the formulae we do know that is we could write A p as 1 upon pi integral minus infinity to plus infinity f v cos p v d v and B p as 1 upon pi minus infinity to plus infinity fv sin p v d v.

Now, substitute this in our u x t that is in place of A p, I write if I substitute this in place of B p, if I substitute this integral. And then write this whole integral, what I would get up, in first if I substitute over here I would be getting is after getting it that is cos p v and cos p x, sin p v and sin p x we could substitute it and we could write it as 0 to infinity minus infinity to plus infinity f v cos p x minus p v d v d p.

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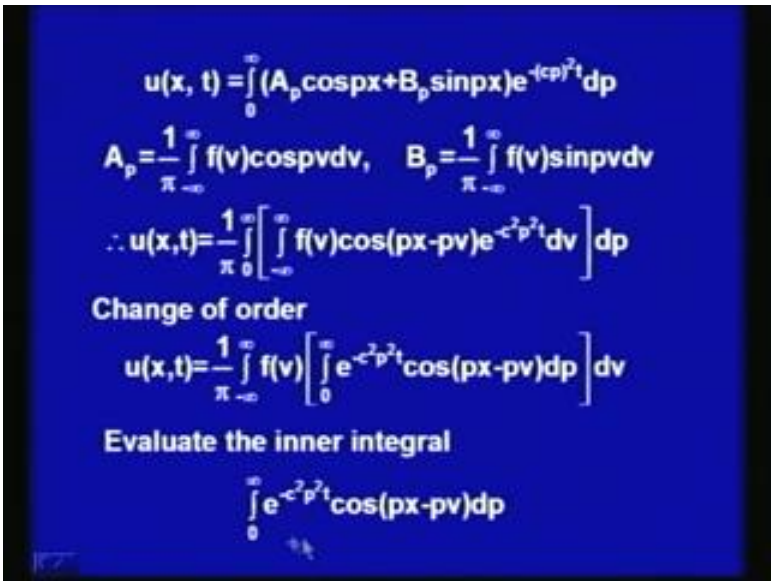

So, what we would get, u x comma t as this one, so substitute A p and B p this is what we are getting the solution of our heat equation or our initial value problem with A p and B p defined at is.

So, now let us simplify this solution I am writing it as substituting this A p and B p over here, I would get 1 upon pi integral this 1 upon pi is constant that 1, I am taking out integral 0 to infinity A p is the integral minus infinite into plus infinity f v, cos p v and then cos p x d v d x. Similarly the B p would be getting is f v sin p v and then sin p x and then e to the power minus c square p square t and this d p.

So, solve again simplify as usual, that is we do write cos p x cos p v plus sin p x sin p v is nothing but cos p x minus p v. So, we are getting is this integral as this function solution is u x comma t 1 upon pi 0 to infinity minus infinity to plus infinity f v cos p x minus p v into e to the power minus c square p square t d v, then integrated with respect to p that is d p.

Now, what this f is actually a initial condition, let us simplify it little bit more if I do allow that a change of order of integration is possible, that is the integral with respect to v and with respect to p, if I can interchange them and suppose that these integrals are existing then this inter change is possible. So, let us say that these integrals are existing, so the interchanges possible. I do get it first is integral minus infinity to plus infinity, since this integral is with respect to v, and the second integral is with respect to p and f v is not containing any term of p and taking this out of this integral with respect to p, so f v.

And then remaining functions with respect to p, that is integral 0 to infinity e to the power minus c square p square t cos p x minus p v d p. Now, this inside integral let us just try to find out this inner integral, inner integral is 0 to infinity e to the power minus c square p square t cos p times x minus v d p. Now, what we are having is, minus p square and then cos p something like this 1. So, we have to evaluate this integral.

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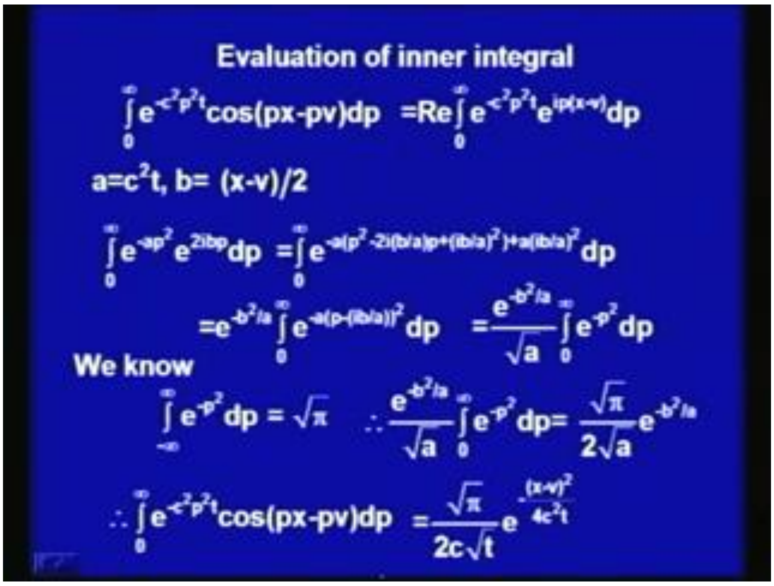

Lets come to the evaluation of this inner integral, if I see this cosine of theta, we can we do know that e to the power i theta, I can write as cosine theta plus i sin theta that l, I lets formula we do know very well. So, cosine theta plus i sin theta if i do take their cosine theta is nothing, but the real part of e to the power i theta.

So, we do know from the complex integration that real if I have to integrate real part of any function it is same as that integral of real part of the integral. So, we are getting it to this function if I do take, this function is actually the real part of this function. So, what the function we are having e to the power minus c square p square t can we do know that e to the power i theta I can write as cosine theta plus i sin theta, where l I lets formula we do know very well.

So, cosine theta plus i sin theta if I do take their cosine theta is nothing but the real part of e to the power i theta. So, we do know from the complex integration that real if I have to integrate a real part of any function it is same as that integral of real part of the integral. So, we are getting it to this function if I do take this function is actually the real part of this function.

So, what the function we are having e to the power minus c square p square t cos p x minus p v and the other function is e to the power minus c square p square t e to the power i p x minus v its real part is this one, because e to the power i p x minus v we can write as cos p x minus c plus i sin p x minus v. So, using that one we do says this real part.

So, now, let us evaluate this integral complex integral, and then we will go for the real part of that integral and that would be equal to this one, so let us say this integral. For this integral we will simplify little bit, because this integral with respect to p and we are having c and t and x and v, so many thing.

So, let us assume a is equal to c square t, so what I would get e to the power minus a p square, and then assume this x minus v as 2 b, so that I would get e to the power i 2 b p. So, what we do get with this constant changing, now I am evaluating with this integral 0 to infinity e to the power minus a p square e to the power i 2 b p d p with respect to p.

So, take this exponent all the exponents in one side and then make it a whole square. So, what we are having is minus a p square plus 2 i b p, so minus I am taking out, so b square and it would be b upon a minus 2 i times b upon a p. So, what the term I do require. So, this is what p square minus 2 i b upon a times p.

So, what the function we do require the term we require to make it the whole square would be square of i times b upon a. So, square of i b upon a, that I have added over here. So, that I have to subtract over here from there if I am subtracting I have to get it out as plus sin. So, I do get a times i b upon a whole square.

Now, simplify it, since this term is e to the power a times i b upon a whole square this is not containing any term of b, we can take it outside the integral sin we do get it i square b square upon a square, so a would be cancelling out i square is minus 1. So, we do get e to the power minus b square upon a, and the term which is inside this integral that is e to the power minus a p minus i b upon a whole square, integrate it with respect to p.

Now, make the change of variable take square root of a times p minus i b upon a as some other variable say v. Now, because all the things we are using here, so I would not be making it v I would again change it to the p, with that one what we do get is that d p would be changed only to root a d p.

So, what we do get it actually this integral as e to the power minus b square upon a upon square root a, limit 0 to infinity since this quantity inside this square one that is with multiplied with square root a we are making the diff change. So, if p is going to 0 the whole quantity will go to 0, if p is approaching to infinity the whole quantity will approach to infinity. So, the limits would not change and it is e to the minus p square d p.

Now, if you do remember this is not our in nice function which we could say we could integrate this is not an elementary function which for which the integral is known to us. But, if you do remember when we are doing is this Fourier series, and for integrals we had evaluated this integral ones.

And, that integral was which we had evaluated that was minus infinity to plus infinity e to the power minus p square d p is equal to square root pi. We had evaluated a by changing it to the polo coordinates and making the multiplication of integrals as the multiple integral.

So, if this function I do take e to the power minus p square this function is even function with respect to p that says is this integral minus infinity to plus infinity e to the power minus p square d p I can write as 2 times integral 0 to infinity e to the power minus p square d p and that is given as u square root pi. So, I do get the integral 0 to infinity e to the power minus p square d p as square root pi by 2 na.

So, this integral is square root pi by 2, now substitute this to this integral what I would get it is square root pi by 2 square root a as is as such, and this e to the power minus b square upon a, this evaluation of this integral is square root pi by 2.

Now, if I see this integral, this value this is real value it is not having any complex term it is real value. So, the real part of this integral is this one which is would be same, so what we have actually got? We have got that integral 0 to infinity e to the power minus c square p square t cos p x minus p v d p. This, I have to substitute the value of a and b whatever we had assumed over here from there we do get square root pi by 2, a is c square t. So, square root a would be c times root t, e to the power minus b square upon a b square means I would get x minus v whole square upon 4 a is c square t. So, I would be getting minus x minus v whole square upon 4 sq c square t. now this is what we have got this inner integral.

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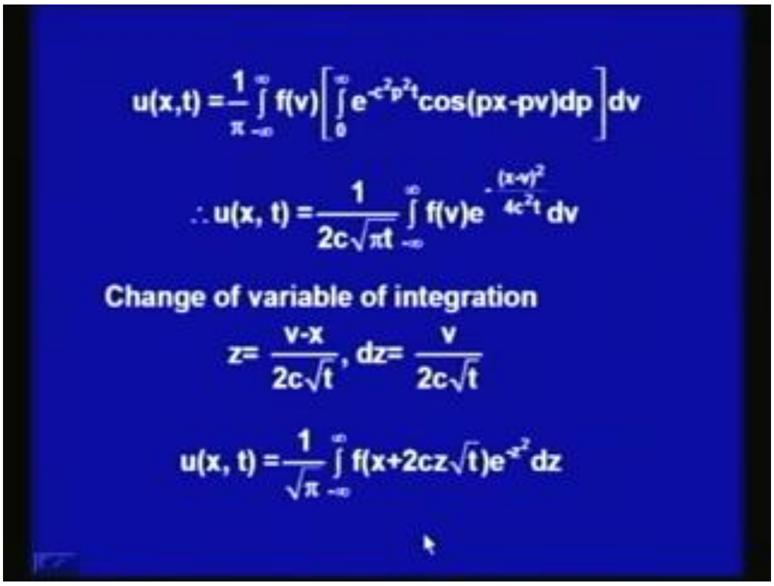

Now substitute this inner integral in our solution, the solution was u x t as 1 upon pi integral minus infinity to plus infinity f v, then integral of 0 to infinity e to the power of minus c square p square t cos p x minus p v d p. And whole thing is being integrated with respect to v. Now, this in our integral just now we had calculated now let us substitute, it we get 1 upon 2 c square root pi t. Because, 1 upon pi is there, and we have got square root pi upwards, so though a square root pi divided by pi would be 1 upon square root pi that is what they are getting the constant and since this is the constant. So, we can take it outside the integrals sign. Then if we and what is the we have got e to the power minus x minus v whole square upon 4 c square t d v, this integral is with respect to v.

Now, change the variable of integration, now rather than writing it with respect to v, let us use some new variable. The new variable if you see is we just want to simplify the term of this exponent, I am introducing new variable z as v minus x upon 2 c root t 2 c is that is what is the here, either I write x minus v whole square or v minus x whole square that does not matter.

The thing which will matter is since this integral is with respect to v, so this d z when I would be writing if I am writing it as v minus x, I would get the positively that is it should be d v, d z is equal to d v upon 2 c square root t. So, if I change this variable of integration as v is approaching to minus infinity the z will also approach to minus infinity because all others are constant.

Similarly, as v is approaching to the plus infinity z will also approach to plus infinity. So, what we would get another formula, 1 upon square root pi, because this 2 c root t that will go with the d v, and I would get as d z v e I would get from here as 2 c root t z plus x, so x plus 2 c z root t and e to the power minus z square dz.

So, we have got the solution of our heat equation or rather you could say we have got this as the heat flow in an infinite bar as either by this formula where 1 upon 2 c square root pi t integral minus infinity to plus infinity f v e to the power minus x minus v whole square upon 4 c square t d v, where your f is nothing but the function which is governing the initial temperature are which is giving the initial temperature of the bar.

Or we may write in this another formulation where we are writing it as 1 upon square root to root pi. So, you are finding it out that all these constant and all these terms have been simplified the term only argument of f is been there and that is also in the terms of x. So, we are getting is x plus 2 c z root t dz. So, this is what we are getting is this solution of our heat equation. Now let us see is that is if I do give any particular value that is do an example, how do we do get this solution.

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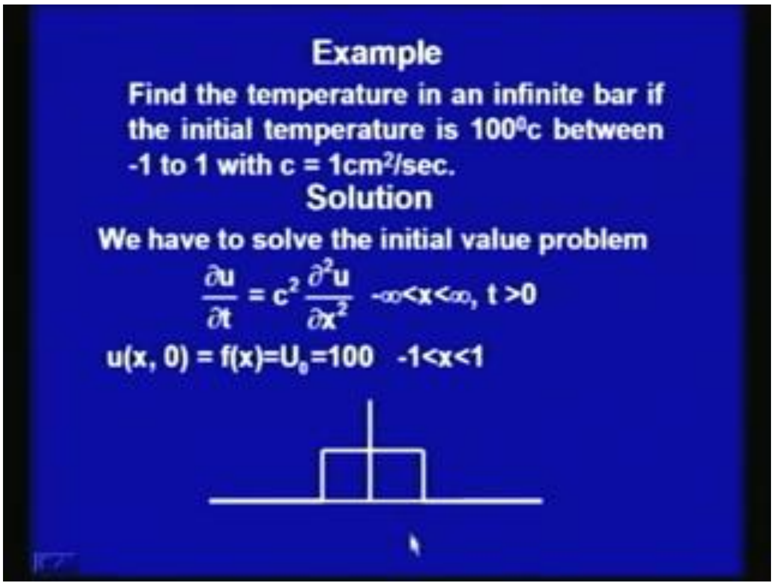

So, let us do one example, find the temperature in an infinite bar if the initial temperature is 100 degree centigrade between minus 1 to plus 1 and the bar, that is and the bar we do have on the x side only the temperature is from minus 1 to plus 1 and c we are having is 1 centimetre square per second.

So, let us see what we have to solve this initial value problem, del u over del t is equal to c square del 2 u over del x 2, because it is infinite bar. So, my x has to be from minus infinity to plus infinity t is positive, initial condition we have been given that u x 0 is f x which is actually a constant, constant is been given as 100 degree centigrade that is it is 100 in the range minus 1 to plus 1 and it is 0 is there.

If I just go ahead for solving this kind of equation, what we had been actually given, that if this is mine infinite bar what we are been given, that the initial condition is saying is that the temperature between this is 100 degree centigrade and all other points it is 0. So, this is my f x, this you see this is my f x initially which is been given. Let us just move to the solution, so I am not going to solve this equation again, just now we had solid using the Fourier integral.

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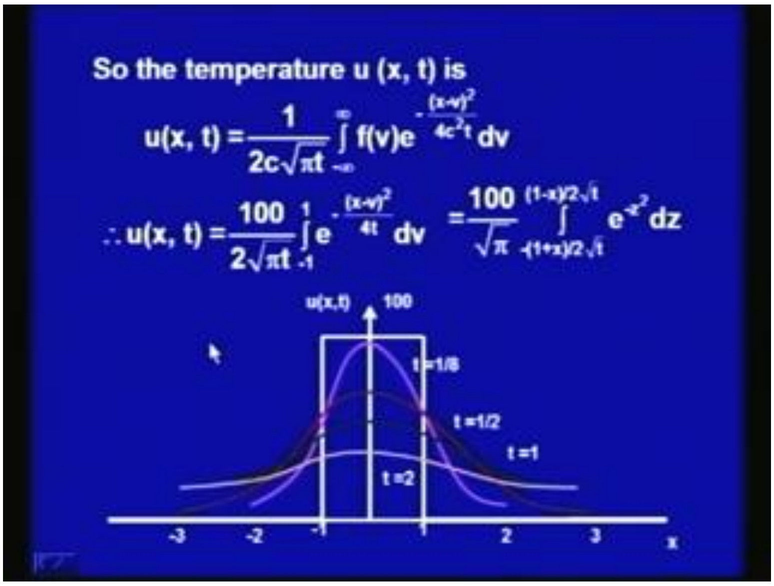

We have got the solution as 1 upon 2 c square root pi into t minus infinite to plus infinity f v e to the power minus x minus v whole square upon 4 c square t d v, that is the first formulation we are using. Now here I will substitute the value of f, f is 100 between minus 1 to plus 1 and 0 is what we are assuming is always integrals that is whenever we do have a limit from minus infinity to plus infinity. The function is define in a finite range and other range if it is 0 even in then finite range we take the integral to be 0.

So, accordingly that we will substitute c is equal to 1, f v as 100 between minus 1 to plus 1 and 0 is where, what we do get, 100 that is the constant I could take outside the integral sign. So, 2 times c is 1 is square root by t and since it is 100only in the range minus 1 to plus 1, so the integral reduce from minus 1 to plus 1 e to the power minus x minus v square upon 4 c square t d v.

Let us, try to solve this integral we do know, this integral as I said is not very nice function elementary function. So, it is it cannot be evaluated as such, what we could do is, we could find out the values of this integral which have been available in the tables. This is in engineering it is being called as error function and those who are knowing little bit probability in distribution, they might be understanding that this is something is density function of an normal distribution and this we could say is that the probability under the normal probability curve, and they are they can use this tables of those probability curves to find it out the solution the values.

Let us see is for different values of t, how does this look alike, if I change it to the that usual v minus x upon 4 c square t as a z, we do get is in this 1 and in the limit would be certainly changing it to this 1. So, let us see is that, how this function would look like for different values of t, when t is equal to 0, my function u x 0 is given this 1, so this is this one, when if we just put t is equal to 0 here we will get actually this one if I put t is equal to 1 by 8 you see, this is the line which is this little bit pink colour kind of line, this is coming as this curve is giving me the heat at t is equal to 1 by 8 as i move t 2 1 by 2.

This red colour line, this is what we are getting is that heat equation this solution u x t heat flow if i take t is equal to 1 we are moving to this 1 and if t is equal to 2 we are having this orange colour of flying. So, what we are having is, that is as my t is increasing from 0 to this 1, initially we had this square kind of this dark was having only the heat.

Now, at t is equal to 1 by 8 time the heat has been float that is heat is now till here, and it is having this kind of function. When t is equal to 1 by t, the heat has been float till this point and we had the flow of the heat in this are this heat the temperature is this 1, and so on the temperature is moving that is with the time t whole bar is getting heated and the temperature is flowing. And in this manner, we are going is that and the heat is been equally being distributed you could says that this is how we are getting it like, so this is what is the flow of heat.

Now, we had use in this one because it was infinite where we had used this Fourier integral. If you do remember in the Fourier transforms, Fourier transforms we have got using the Fourier integral, we let not been nice if I do use Fourier integral in this solving this problem. Let us try to see can, we use this Fourier transform and we have actually done the not only the Fourier transform from the Fourier integral, we have got Fourier sin transform Fourier cosine transform. And then the Fourier transform, so all of them can be used over here. Now, let us see that is whether we can do those things or not before going to that let us recall those things. So, that we do remember that how do we can use it.

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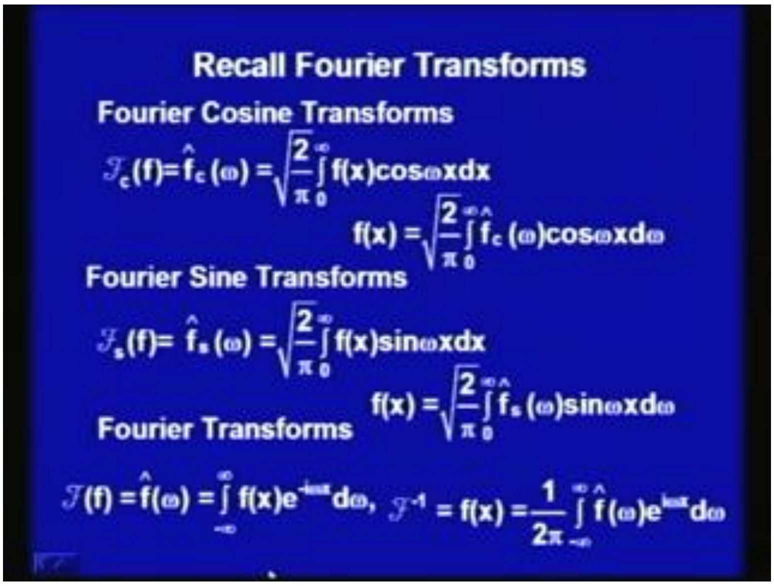

So, Fourier cosine transform should remember we had defined as that square 2 to upon pi 0 to infinity as x cos omega x d x and its inverse Fourier cosine transform was f x square root 2 upon pi 0 to infinity f c had omega cos omega x d x we had use this two kind of notations either cap this f c of f that is the Fourier cosine transform of f or we had used another term that f c had omega that is at omega we are getting it. Or Fourier sine transform.

Similarly we have defined as square root 2 upon pi integral 0 to infinity f x sine omega x d x if we do remember we have also used that is this constants may be differing, that is sometimes we are using the constants at four transform, as well as in the inverse transform the equal constant are sometimes in the transform. We are not using any constant and in the inverse transform you would be using 2 upon pi. So here also, inverse Fourier sin transform is square root 2 upon pi f is had omega sin omega x t x.

And, the Fourier transform we were having as Fourier transform of f we have define integral minus infinity to plus infinity f x e to the power minus i omega x d omega, and inverse Fourier transform as 1 upon 2 pi integral minus infinity to plus infinity f had omega e to the power i omega x d omega. Here, also the constant may differ that is either we use in the inverse the constant 1 upon 2 pi and here no constant or we use 1 upon square root 2 pi constant had both the sides, where you see in the Fourier transform we are having this integral from minus infinite to plus infinity.

While as, in the sine and cosine transform we are having this integral from 0 to infinity that what it says is my function f x is and the range 0 infinite, that is x is on the 0 to infinity and here x is on the minus infinity to plus infinity. Now, since we are using here our in heat equation the bar was infinite that is from minus infinity to plus infinity, so we would like to use this Fourier transform for having the solution. So, let us use this before using lets use some properties as well, because if without those properties we cannot use. The Fourier transforms of derivatives because we would be using them in the differential equation that is we would be using it with respect to derivatives.

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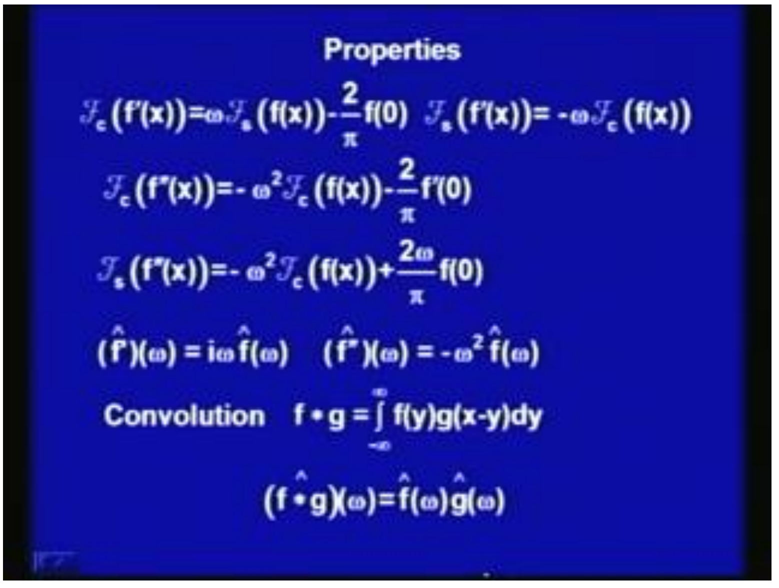

So, this is one important properties which we have to recall, so Fourier cosine transform of the first derivative is nothing but omega times. Fourier sin transform of the f minus 2 upon pi f 0 and the Fourier sin transform of the derivative of f is minus omega times Fourier cosine transform of Fourier cosine transform of second derivative of I just go ahead using it here again with this one we do get the formula minus omega square Fourier cosine transform of f x minus 2 upon pi f dash 0.

And, second derivative Fourier sine transform as this 1 minus omega square, it should be Fourier sine transform of f x plus 2 omega upon f 0. And the Fourier transform of the derivative is i omega times Fourier transform of f and second derivative we just get it minus omega square Fourier transform of f.

One more important property was convolution that is if I define convolution of two functions f and g as minus infinity to plus infinity f y x minus y d y. Then Fourier transform of the convolution is nothing but the multiplication of the Fourier transforms.

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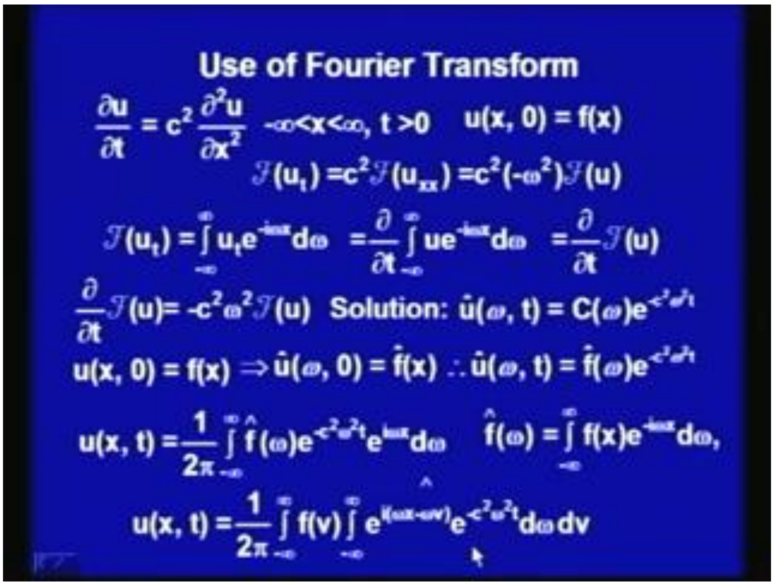

So, let us use this Fourier transform in this equation, now assume this solution of this is actually u x t. Now what we would be assuming this that is first we would be taking the Fourier transform of u with respect to x only, so this is our initial value problem. Now, Fourier transform of if I take u as a function of x and the u x function of x and t we are taking the Fourier transform with respect to x variable. Then if I take the Fourier transform in this whole 1 this is u t of course, it will involve x its Fourier transform with respect to the variable x is c square Fourier transform of u x x, this is the derivative with respect to x since I am taking the Fourier transform with respect to x the derivative with respect to x i can use the property of the derivative which is nothing, but minus omega square the Fourier transform of u and c square is as such.

Now, what is this f Fourier transform of u t, if I do allow the diff interchange of differentiation sign and integration sign that is, let us write what would be Fourier transform of u t this is nothing but by definition minus infinity to plus infinity u t e to the power minus i omega x d omega u t is nothing but del u by del t.

Now, if I say is that I can interchange the integration and differentiation sign, because this is function is involving only x, so there is no t terms, so I could treat it as the derivative of u into e to the power i omega x. I can write it as del by del t integral minus infinite to plus infinity u e to the power minus i omega x d omega, what it is, nothing but this is the Fourier transform of u.

So, you could write it as del upon del t f, f of Fourier transform of u, what it says is, I am getting Fourier transform of del u del u over del t as del by del t of Fourier transform of u. Now, if I write in this manner this equation if I do substitute, what I do get, I do get del by del t f Fourier transform of u minus c square omega square Fourier transform of u.

Now, I am taking this new function rather than as we use to have a f x or something now what we are having is this Fourier transform of u this is the function of omega, so this is a function of omega I am having. So, this is now my what my partial differential equation if I do see this has been changed to the ordinary differential equation, where I am having is the, where it has to change with the respect to del t we are having and this function is the omega and t. So, if I do solve it this differential equation with respect to t this is the first order equation we do know its solution would be or.

Now, I am using another notation that is for Fourier transform of u with respect to x at and t is as such we are writing it you had omega t is c omega e to the power minus c square omega square t. Now, why I have used this u omega, this would be actually constant, since we are using the partial derivative with respect to t. So, this constant may depend upon the other variable that is omega.

Now, this constant c omega that can be determined using my initial condition, now what is initial condition if I take the Fourier transform of this initial condition also, because initial condition is also a function of x. Then what I would get you had omega 0 is if had x that is the Fourier transform of u at omega 0 is same as Fourier transform of x.

Now, substitute this one what we do get that use this initial condition in the solution, what I would get, my solution would be flourier transform of u with respect to x that is you had omega t is f had omega e to the power minus c square omega square t. So, now to get u x t, what I have to do is, I have to find out the inverse flourier transform of this right hand side function.

So, let us write it out u x comma t as 1 upon 2 pi integral minus into plus infinity if at omega into e to the power minus is minus c square omega square t inverse Fourier transform, so I have to multiply it with the e to the power i omega x integrate it with respect to omega. So, now, if I write what is f had omega f had to omega f had to omega if I use the definition it is minus infinity plus infinity f x e to the power minus omega x d omega, so d x it should be d x.

Now, because x is having both the places the solutions also I want to x, so I would use here we and then if I substitute this in this equation, what I do get, u x t as 1 upon 2 pi integral minus into plus infinity, if we integral minus into plus infinity e to the power i omega x minus omega v e to the power minus c square omega square t d omega d v, what I have done is substituted this one, and then change the order of the integration.

Now, why I have changed the order of integration, one is with respect to omega, so this is with respect to omega and this is with respect to v, again we see is that this is the similar form. We have got when we are solving it using the Fourier integral there we had solve this one, so we are getting it the solution is similar manner.

But, you see is that is here I do not have to go for that separation of variables solving diff many diff too differential equations. And then making it idea that is how to integrate or how to get the solution of the entire problem using the integration, what we have done is directly use the Fourier transform and we had come up with the solution.

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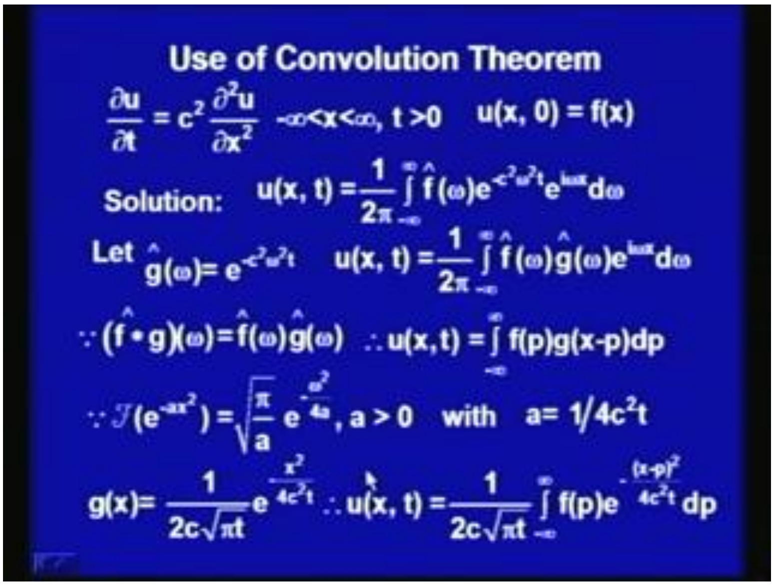

Now, you can simplify it again and you will get this in solution, from here such we are getting it this e to the power i omega x and this f v and this e to the power c square omega square. If you see is u h, what we could get one more idea, cant we use this convolution theorem, you see is that is what we are saying we are solving this problem initial value problem, del u upon del t is equal to c square del 2 u upon del x 2 x is ranging and the whole real line t is positive initial conditions u x comma 0 is f x.

And, when we use the Fourier transform we got the solution as 1 upon 2 upon by integral minus infinity to plus infinity if had omega e to the power minus c square omega square t into e to the power i omega x d omega, what we are saying is, this we are having Fourier transform of f and into another function e to the power minus c square omega square t this is also a function of omega, function in omega.

And, multiplication of these two, I want inverse Fourier transform this what it says is that if I treat this e to the power minus c square omega square t if I say this is also a Fourier transform of some function g that is what I am saying is, let g had omega that is the Fourier transform of g x is e to the power minus c square omega square t. Then, what we are getting I am writing it is f had omega into omega into g had to omega and it is in of Fourier transform, what it says is, if I do have my convolution theorem, convolution theorem says is if convolution 0 Fourier transform of f convolution g is f had to make into g had to omega.

So, what I would get f convolution g as inverse Fourier transform of f had to omega into g had to mega and this is what we are getting is f had omega into g had to omega. So, the solution should be the convolution of f and g, what is this g, that is we have to find it out. So, this is what I was explaining that this says is u x t should be 1 upon 2 pi integral minus infinity plus infinity if had to omega g had to omega e to the power i omega x, that is inverse Fourier transform of f had to mega into g had to omega.

The solution has to be your convolution of f and g, the convolution of f and g if you do remember we have defined as minus infinite to plus infinity f p g x minus p d p. So, now, the position comes is that is what is my g, now let us see this g. in the Fourier transform, we have done this as an example there.

I have taken this example, just because we would use it somewhere you can just go ahead and find it out. It is we had find it out that Fourier transform of e to the power minus a x square this is a square root pi upon a e to the power minus omega square upon 4 a for any a positive.

Now here, what I am having is c square and t, t is positive c square is already positive, so this a positive is being satisfied see. If I choose my a as 1 upon 4 c square t I would get that g x would be 1 upon 2 c square root pi t e to the power minus x square upon 4 c square t x square, so a is actually my 1 upon 4 c square t that is what you are getting an. So, here is square root pi upon a that is so again you would be getting it in the form of that one, so we are getting this is my g x.

Now, if I substitute this g x in my u x t I would get u x t as so this constant is with respect to t, so this is complete constant. So, I could take it outside integral minus infinity to plus infinity f p into e to the power minus x minus p square upon 4 c square t d p. Now you see, this is also the solution which we have got in the first problem, so what we have seen is that is either I used this method of separation of variable.

Or I just come to this use the Fourier transform, in the Fourier transform also rather than finding out inverse Fourier transform directly if I do use this convolution result by method becomes much cleaner or you could say much nicer and elegant and we get the solution. So, all these techniques that is why this Fourier transform is called as technique, we are using this techniques we are simplify our problems.

Now, in all these things we have taken this infinite bar, now if I do have to find out the heat flow in a semi final bar, that is its not minus infinity to plus infinity, but says from 0 to infinity. In that case, my function would be defined from 0 to infinity and in that case we can use Fourier sin cosine transform.

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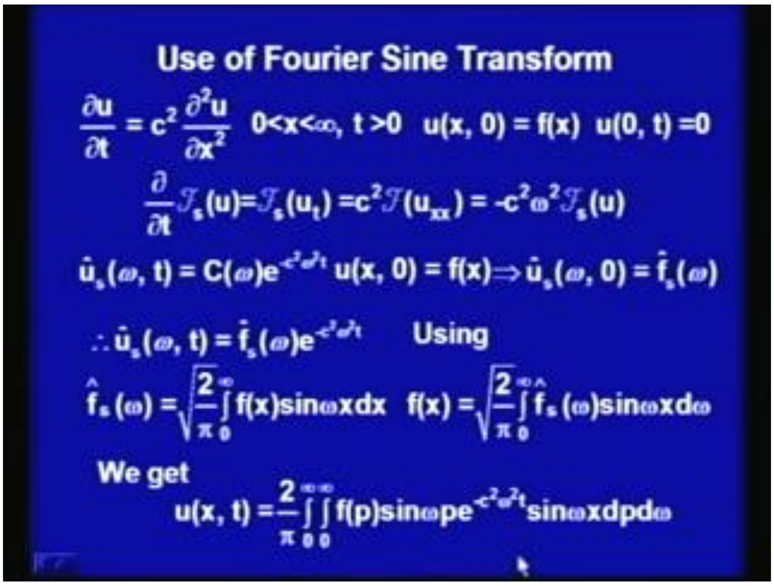

So, let us see is that is with the, we can use it, so first see is that is use of Fourier sin transform. So, now, we are having our problem the del u over del t is equal to c square del 2 u over del x 2, where x is from 0 to infinity and t is positive, so again we are having semi finite bar with lateral insulation.

And now, we can introduce one boundary condition as well, so with the initial condition u x at 0 is at t is equal to 0 is f x and we can now impose one boundary condition that u 0 t is 0. Now, we have to solve this one for this now we are not going to use again this separation of variable method.

Now, will directly use the Fourier sin transform, so what it says is I require the Fourier sin transform of del u by del t and del 2 u over by del x 2, again I will take the trans Fourier sin transform also with the respective x only because what we are having is that is here initial condition is with respect to x and with t we are getting is that is boundary condition is 0. So, what we do get is f at 0 would be u 0 0 that is so we would be getting it 0 itself.

So, we just use the function the Fourier transform of the Fourier sin transform sin transform with respect to x, again we use the Fourier transform of the Fourier sin transform of the derivatives that says is and as we have done in the Fourier transform this again we can write the Fourier transform of u t as del upon del t f Fourier sin transform of u which is same as Fourier sin transform of u t. The sin transform we are taking with respect to x this should be c square it should be equal to c square times Fourier transform of u x x that is according to this equation.

And, what will be the Fourier transform of second derivative with respect to the sine transform that is minus c square omega square f s u. The Fourier transform Fourier sin transform of u, and then what we do have the constant, that is at f at 0 from here if i u at 0 0 we would get. Because, we want u 0 t, u 0 t is given as 0 initial boundary condition that says is that constant which we would be getting here in the Fourier sin transform that would give me 0, so we would be reducing this one.

So, now, what again we have got the ordinary differential equation that is, this is now my function Fourier transform Fourier sin transform of u with respect to x and we are taking this equation derivative with respect to t only. So, we do get is ordinary first order differential equation which is easy and we do know its solution is c omega e to the power minus c square omega square t and the function what we would be getting is the Fourier sin transform of u at omega and t.

Now, this c omega we can determine using my initial conditions, what this initial condition gives me, it gives me that is if I take the Fourier sine transform both the sides it gives me that Fourier sin transform of u omega 0 is Fourier sin transform of omega for x it should be omega.

So, if I just substitute it, what I do get u omega had s omega t is f has s omega t f has at omega e to the power minus c square omega square t. So, we just find out u omega u x t as the inverse Fourier's sin transform of this function, so we can just use the inverse Fourier sin transform using the inverse Fourier sin transform this one. And for f has omega this is, what is for this sin transform for, so for f is omega we do use this one and for inverse Fourier transform we use this formula what I do get is, so we do get u x t is equal to 2 upon pi integral 0 to infinity integral 0 to infinity f p sin omega p e to the power minus c square omega square t sin omega x d p d omega.

So, what we are getting is again if you do see is that is because it is in the semi finite bar. You are getting the solution nice neat solution in the terms of these function now we do know that we can simplify we do get it sin omega p here and sin omega x. So, we can write it as cosine of omega p minus omega x mi plus omega x minus cosine of omega p omega x like that we do get it. This would give me neat one and then we can integrate it out, so this is what we are using this Fourier sin transform we can get the solution.

So, today we had learnt it, that the heat equation if I take the heat flow in infinite bar, infinite bar practically means very long wire very long bar that is all. We can solve it using rather than using this separation of variable method which gives was very lengthy one, then we had use this Fourier integrals we had also learnt that is we can use the Fourier transforms.

And, in the Fourier transform if we do use this one more is that this called convolution result we had find it out that the solution was coming at very nicely and neatly. Then if we are talking about semi finite bars then we could use the Fourier sign transform that is one method we had seen or we could use depending upon the other situations, what is the boundary conditions have been given or initial conditions have been given we can use other transform. So, today we had learnt the heat equation using the Fourier transforms and Fourier sin transforms.

Thank you.