Mathematics - III
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Lecture - 21
Solution of Homogeneous and Non Homogeneous Equations

Welcome to the lecture series and differential equations. Today's topic is Solution of Homogenous and Non Homogenous second order partial differential equation. We have learn in partial differential equations, the second order partial differential equations, the method of characteristics changing the normal form. And we have done one dimensional wave equation, all these things we had learnt for homogenous equation, today we will move to non homogenous equations as well.

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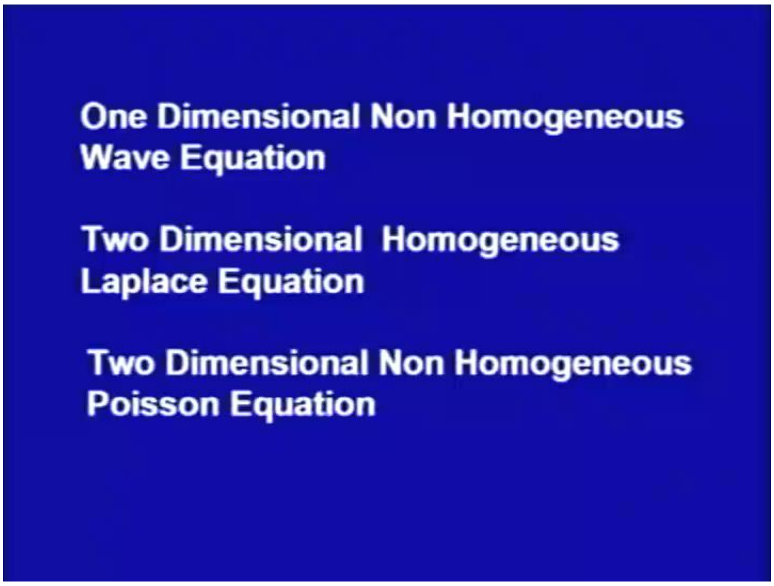

In today's lecture, we will cover the three things; one dimensional non homogenous wave equation, two dimensional homogenous Laplace equation and two dimensional non homogeneous Poisson equation, Poisson equation is actually non homogenous form of Laplace equation. So, let us start with the one dimensional wave equation with non homogeneous that is the right hand side is not 0.

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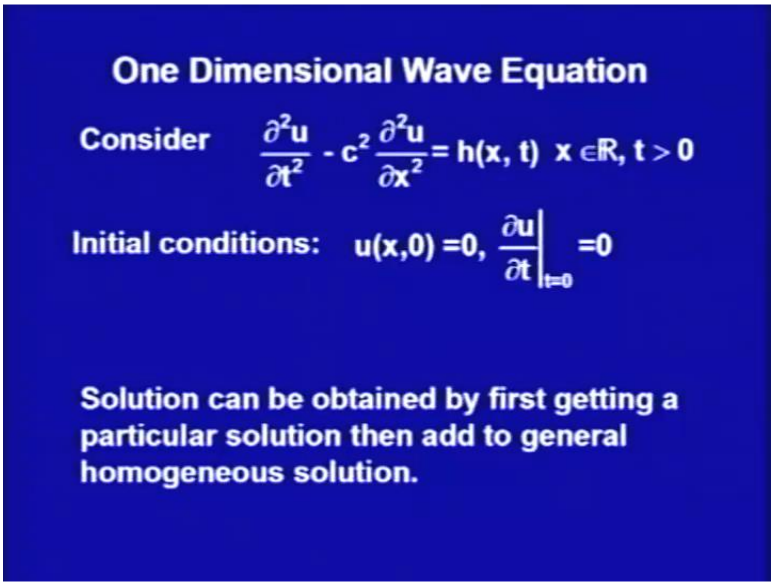

So, one dimensional wave equation we had learnt that it is del 2 u over del t 2 minus c square del 2 u over del x 2 is equal to 0 that was homogenous 1, now we are having on the right hand side h x t a function of x and t. In the whole region defined on the whole real line that is minus infinity to plus infinity and t positive, the initial conditions we do have that u x at 0 is 0 and del u over del t at t is equal to 0 is 0 that is initially at the time t is equal to 0, the unknown function u is 0 and the derivative of with respect to t at t is equal to 0 is also 0.

What we, if you do remember we have done the this homogeneous wave equation with initial conditions, where we had that initially the deflection was f x and initially the velocity was g x. Now, because we are talking about here this non homogenous equation we are using first this homogenous initial conditions, now the way how do we solve this equation, if we do remember in the ordinary differential equations, when we are talking about the non homogeneous equations, we first find out the general solution from the homogeneous equation.

And for the non homogeneous equation we used one particular solution and then using the super position principle we add up though both the solutions we get the solution of non homogenous equation, that is the method here also, but let us try one more method, which is based on the Duhamel's principle.

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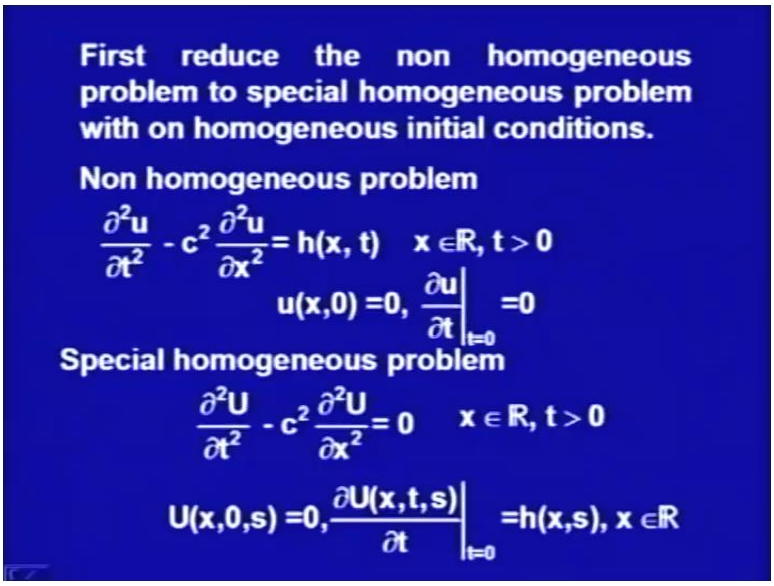

What is that, it actually does is that it reduce this non homogeneous problem to a special homogeneous problem with non homogeneous initial conditions. See, when the non homogeneous problem is del t 2 u over del t 2 minus c square del 2 u over del x 2 is equal to h x t in the region x belonging to R and t positive. And the initial conditions where homogenous conditions that is u x comma 0 is 0 and del u over del t at t is equal to 0 is 0.

Now, we change reduce it to a special homogeneous problem with non homogeneous initial condition, what is that special homogeneous problem we take del 2 u that is now I am taking another function capital U del 2 u over del t 2 minus c square del 2 U over del x 2 is equal to 0 that is the equation, we have said is that as equation in unknown function capital U this is homogeneous wave equation. In the whole region x in the minus infinity to plus infinity and t positive and the initial conditions we are having is the first U x comma 0 comma s is equal to 0 and derivative of u x comma t comma s with respect to t at t is equal to 0 is x x comma s for all x belonging to R.

And we say that if this is special homogenous problem initial value problem here, we do have del 2 u over del t 2 minus c square del 2 u over del x 2 is equal to 0, this is homogenous equation. Now, what we are taking this function capital U is in the three variables now, x t and s and if we can solve this and here the second initial condition about the velocity is h x comma s, where u you do see that is we do have is initially that in the non homogeneous problem this was the right hand side function of my non homogeneous wave equation, so which we are having is x comma s.

Now, if this problem can be solved, so now, this is actually homogeneous one with non homogeneous initial conditions of course, we do know. So, if this a special homogeneous problem does has the solution for each parameter s, now s is the parameter here we had meet, if it is can be solved for each parameter s. Then Duhamel's principle says, we can find out the solution of this, which can be related with the solution of this, now let us see what is the Duhamel's principle says.

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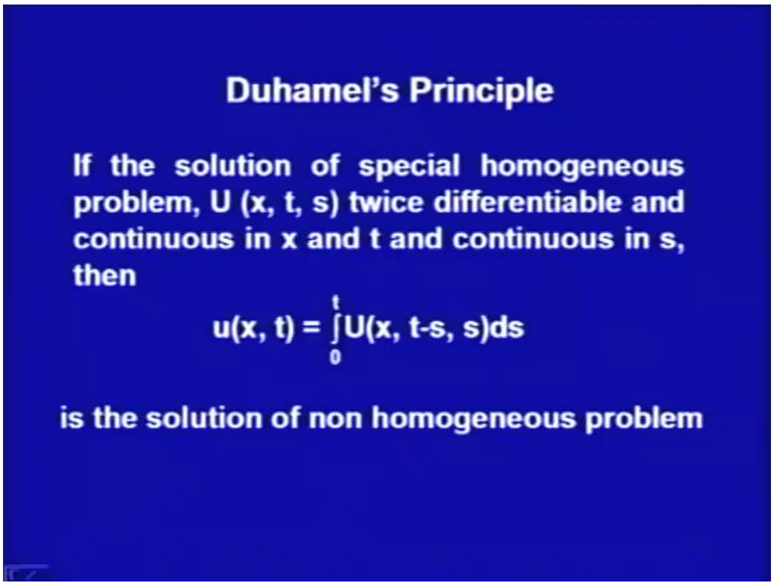

Duhamel's principle says, if the solution of this special homogeneous problem u x comma t comma s is twice differentiable and continuous in x and t, that is the function u x t comma s, which is in the three variable is continuous with respect to x and t and it is twice differentiable that is it has the second derivative with respect to x, as well as with respect to t and continuous in s. Then the solution of non homogeneous problem that is u x t can be given as integral 0 to t capital U x comma t minus s comma s d s, this is the solution of non homogeneous problem.

So, what we are doing is we are changing our non homogeneous problem that is non homogeneous wave equation with homogeneous initial conditions, we are changing it to a special homogeneous equation to with non homogeneous initial conditions that we do know how to solve. That we are doing is with one parameter s as well and then with that parameter s we are integrating it and we are saying is that this is the solution of this, let us say that is whether it is solution of our non homogeneous problem, let us prove this principle.

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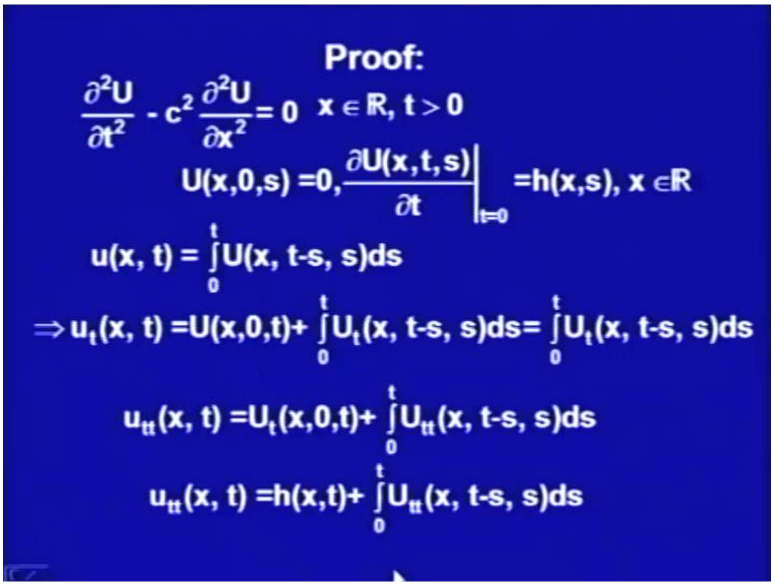

We are saying is that capital U is the solution of this problem U x t we are saying is that this should be this one. Now, let us try to see that can be satisfy this is satisfying the our non homogenous equation or not, so first find out it is derivative with respect to t, now when we are finding out the derivative of this is small u x comma t with respect to t, we see that u x t is actually integral of from 0 to t capital U x t minus s s d s.

That is we are having a function, which is depending on t as well as the limit is also depending on t we want the derivative of this integral. We have already done in first course this differentiation and the integral sign where the limit is itself a function of the variable. So, using that one we do have it is U x comma 0 comma t plus integral 0 to t, the partial derivative of this function with respect to t that is U t we are writing that is the partial derivative with respective t of this function x t minus s is integrated with respect to s.

Now, my initial condition says u x comma 0 comma s is 0 for all s, so when I am keeping is here s as t, this u x comma 0 comma t this would be actually 0. So, what I would get is that first partial derivative of u x t is integral 0 to t the partial derivative of capital U x t minus s comma s with integrated with respect to s. Similarly, I go again to find out the second derivative, that is again differentiate this function with respect to t again we will use the same formula. So, what we would get U t x comma 0 comma t plus integral 0 to t U t t.

Because, now I want the partial derivative of this function with respect to t again, that is it is second partial derivative with respect to t that is it is del 2 u over del t 2 with of course, that arguments if the function is of f x t minus s s integrated with respect to s. Now, this second condition is about the second initial condition is del U over del t of this function at t is equal to 0 is h x comma s, so that will substitute over here, so what we get the second partial derivative with respect to t of u x t is h x comma t plus 0 to t second derivative U t t of x comma t minus s comma s integrated with respect to s.

Similarly, we can find out for this the derivative with respect to x, now when we will find out the derivative with respect to x, then this integral does have the fix limit that is then the limits are not variable.

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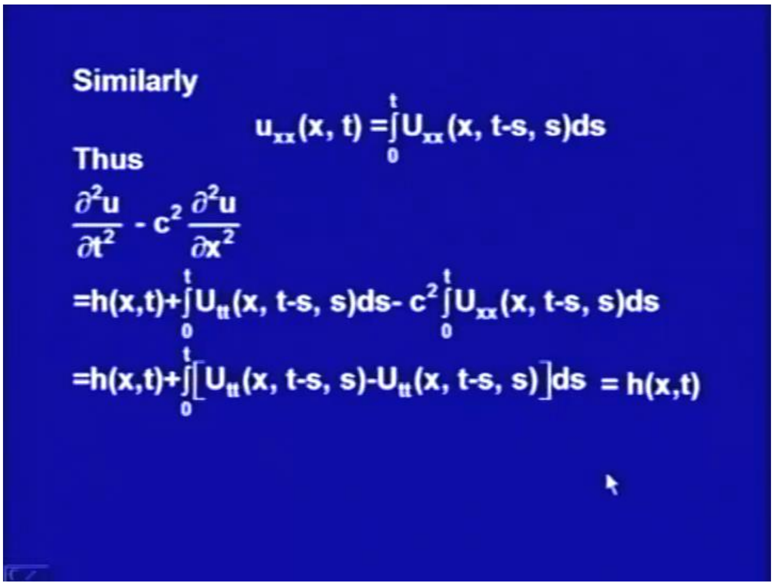

Again using that formula for the differentiation under the integral sign two times we would be using that is strides differentiable, we do get the second derivative of u x comma t is integral 0 to t partial derivative with the respect to x of capital U the function is of x comma t minus s comma s integrate it with respect to s in the whole range 0 to t. Now, substitute this u t t and u x x in our non homogeneous equation our equation is del 2 u over del t 2 minus c square del 2 u over del x 2.

So, if I substitute it here what I do get the second derivative with respect to t we have got h x comma t plus integral 0 to t U t t x comma t minus s s d s minus c square times second derivative with respect to x that is from here integral 0 to t U x x x comma t minus s comma s d s. Now, rearrange these terms what I do get h x t as such plus these two integrals we could write it as the U t t x t minus s comma s minus U t t x comma t minus s comma s d s.

Now, if you see capital U is the solution of our homogeneous problem or homogeneous equation, homogeneous equation was sorry here it has to be c square are also, the c square is also going to come over here. So, this c square U x x that would be now a special homogeneous problem which says is del 2 u over del t 2 minus c square del 2 u over del x 2 is 0, since capital U is satisfying that equation; that means, this equation must be 0.

Hence, what we do get is this is nothing but, h x t that is it is satisfying our non homogenous wave equation that says is u x comma t integral 0 to t capital U x comma t minus s comma s integrated with respect to s is a solution of non homogeneous wave equation, more over it is going to satisfy our initial conditions as well. Now, how to find out the complete solution, to find out the solution of that special homogeneous problem we will try to use the D' Alembert's solution.

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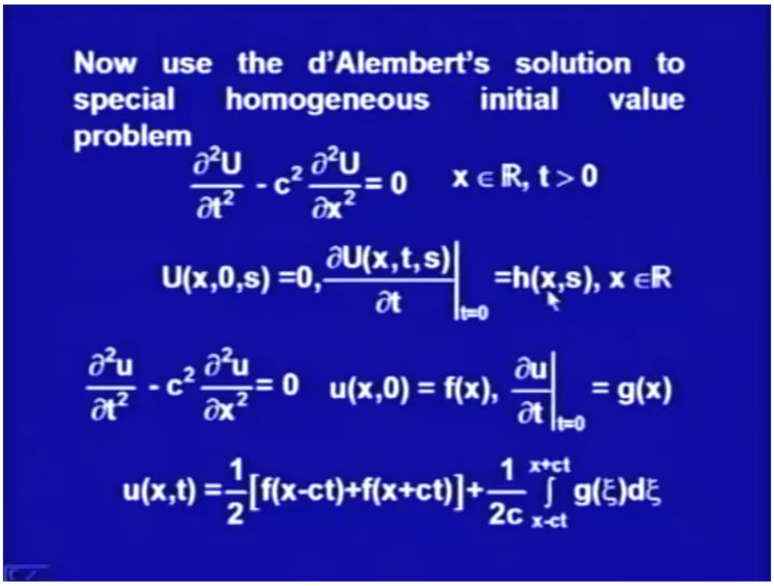

Because, that is only initial value problem and only initial value problem when we are talking about we do find out the D' Alembert's solution is more nice and more elegant way of to finding out the solution. So, we do have this problem homogeneous wave equation del 2 U over del t 2 minus c square del 2 U over del x 2 is equal to 0 and initial condition U x comma 0 comma s is 0 and U t x comma 0 comma s is h x is for all x belonging to R and this equation is on the whole real line with t positive.

So, if we I compare it by usual that is what we have already done, there we had this our wave equation del 2 u over del t 2 minus c square del 2 over del x 2 is equal to 0. And the initial conditions where u x comma 0 is equal to f x and del u over del t t comma 0 is equal to g x. Now, if I compare my wave equation is all right the same similar kind of things, the only thing is that here we are having capital U, here we are having a small u if I compare the initial condition my f x is coming as 0 and this second initial condition g x is h x comma s.

So, what we will do get the solution for this the solution was if you do remember we have done that u x comma t the solution of this initial problem was initial value problem was half times f of x minus c t plus half of plus f of x plus c t plus 1 by 2 c integral minus x minus c t to x plus c t g x i d x i, where g x was the second initial condition initial velocity, now if I compare it my f is 0 here and g x here is h x s, so what we do get.

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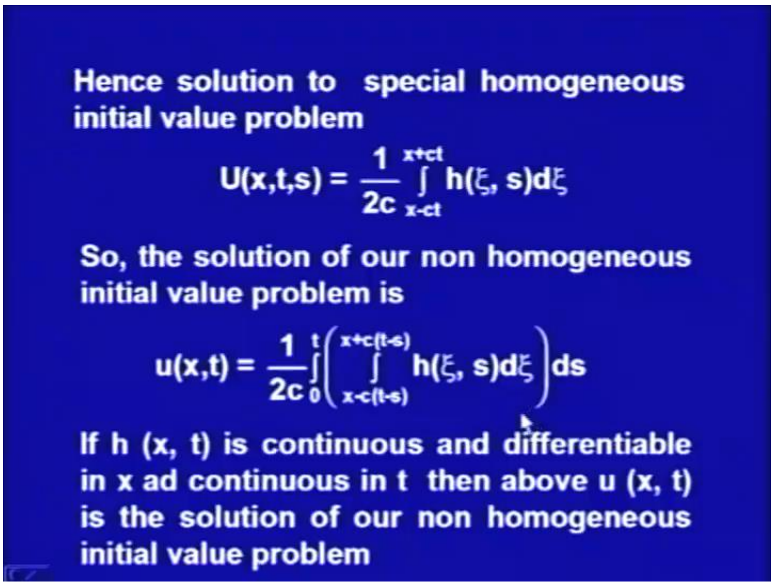

The solution capital U x t comma s as 1 upon 2 c integral x minus c t to x plus c t ((Refer Time: 15:39)) comma s d xi. So, this is the solution of my special homogeneous problem, so what will be the non homogeneous problem solution, initial value problem solution that we can find it out using our Duhamel's principle, which says is that it should be integrated around. So, we do get is integrate this whole thing from 0 to t with respect to s, so this 1 upon 2 c is constant that I have taken out integral 0 to t integral of...

Now, the function has to be u x t minus if you do remember, so here t minus s if I am writing here that is at the place of t I have to replace it by t minus is this. So, what I am getting is well limits as x minus c times t minus s to x plus c times t minus s and the integrant is h xi s d xi and this whole thing whatever this integral we are getting that we have to integrate with respect to s. So, this is what is this solution of our non homogeneous wave equation, with homogeneous initial conditions.

Now, if I do have a problem in which I do have non homogeneous initial conditions, what we do you is. Since, we had find out the solution with respect to the homogeneous initial conditions, we do know that in the similar manner we can find out the solution for the non homogeneous initial conditions how we could do is, before that let us see. We says that is this solution would be existing if my this right hand side function h x t is continuous and differentiable in x and continuous in t, then this u x t is solution of our non homogeneous initial value problem.

That is what the condition we do require is only that the integrals must exists for integrals to be existing I do require that this integral that is this function must be continuous. And since, we are using double integral and partial derivative with respect to t we do require that is it should be x and t that is it just requires the with respect x it should be differentiable and with respect to t it should be only continuous.

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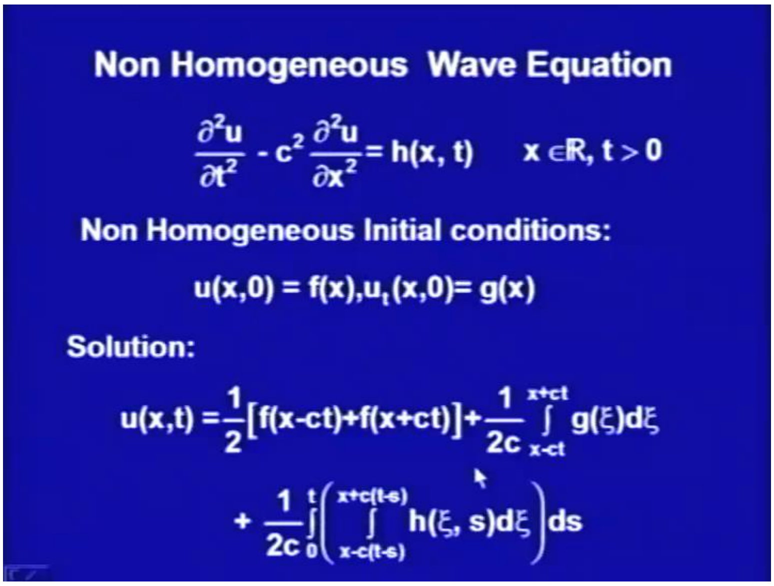

Now, if I do have this non homogenous wave equation del 2 u and del t 2 minus c square del 2 u over del x 2 is equal to h x comma t. And non homogeneous initial conditions as well that u x comma 0 is equal to f x and u t x comma 0 is equal to g x, then we can find out the solution, first what we do is we take the homogeneous initial conditions and then take this, then we can change it to the special homogenous problem with non homogeneous initial conditions.

In that non homogeneous initial conditions we can fix up this non homogeneous initial conditions of here from here itself. And then we can find out the solution of homogeneous wave equation with initial conditions as non homogeneous, we can get it using D' Alembert's formula. And then we can use this Duhamel's principles for finding out the solution of this non homogenous equation and that we can add it up that is what it is simply says is...

Whenever we do have non homogeneous problem, the basic method says is we convert it to the homogeneous that is we just simply put right hand side 0 and solve it that gives me solution of homogenous one. And for non homogenous one we find out a particular solution and that we add and that we can do using the super position principle. It says is super position principle is as self for the homogeneous ones.

But that actually makes is that if it is non homogenous and we are adding the homogeneous solution of homogenous equation and particular solution of non homogenous equation, we are getting the solution of non homogeneous equation. That principle can be extended with the initial conditions as well, that is if the initial conditions or homogeneous or non homogeneous, we can still change it that is we can use it, we can change this initial conditions homogenous that is we write the right hand side is 0.

And then, if we can find out the solution of this non homogenous and we can add it up and that will give me the solution with this initial conditions, non homogenous initial condition. So, that principle we would be using here what it says is for this we can use the solution as because, for non homogeneous conditions the D' Alembert's solutions for the homogenous wave equation is half of f x minus c t plus f of x plus c t plus 1 upon 2 c integral x minus c t to x plus c t d g xi d xi, where my f is this initial condition and g is the function difference giving the second initial condition.

Then, we just add up the solution of this non homogenous problem, which we just have find out using the D' Alembert's principle, which says is it should be 1 upon 2 c integral 0 to t of integral x minus c minus t minus s to x plus c plus t minus s h xi s d xi and whole integrate with respect to s. Now, let us do again the solution would exist if I do have that my the function f g and h are continuous and differentiable that is my function u x t should be continuous and to as differentiable, so that it is it can satisfy these equations.

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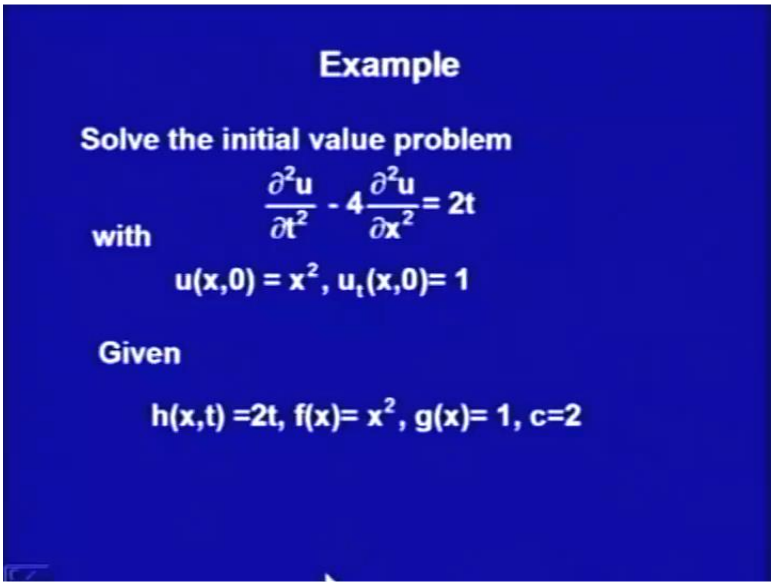

Let us do one example, solve the initial value problem del 2 u over del t 2 minus 4 times del 2 u over del x 2 is equal to 2 t with non homogeneous initial conditions u x comma 0 is equal to x square and u t x comma 0 is 1. Now, what we do is first we will solve the homogenous equation with these non homogenous initial conditions using the D' Alembert's principle. And then using that this Duhamel's principle, we would write the solution of this non homogeneous equation with homogeneous initial conditions and that would give me the solution of the whole problem by adding up.

So, we are being given here right hand side that we have denoted h x comma t that is 2 t and initial condition we had use that u x comma 0 should be f x. So, f x here is x square and u t x comma 0 is 1, so g x is 1 moreover in this initial condition in the wave equation here we to use the have c square. So, c we could says 2 because, in solutions we are having is x plus c t and x minus c t kind of thing, so c is 2 here.

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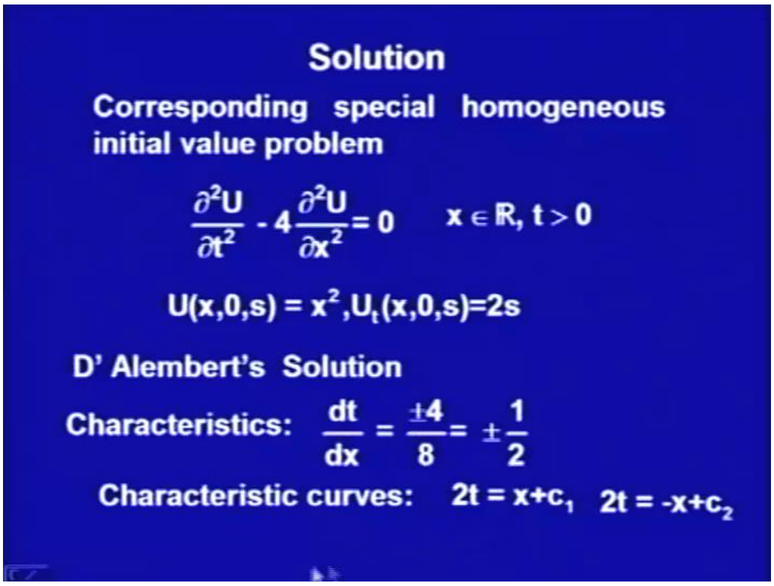

Now, let us first get the D' Alembert's solutions for the homogeneous problem with the, so corresponding special homogeneous problem del 2 U over del t 2 minus 4 del 2 U over del x 2 is equal to 0 with U x comma 0 comma s is equal to x square and U t x comma 0 comma s is equal to 2 s. So, now what we have done is we have changed our non homogeneous problem to the homogeneous problem with non homogeneous condition, after the initial condition it is for non homogeneous. So, we have done it the only thing what we have changed is that is rather than using that g x I am using here is 2 s.

So, using it D’ Alembert's solution if you do go ahead what we will do is, we will first find out the characteristic of this equation and then change the transform it and do it. So, characteristics for this here b square is b is 0, a is minus 4 and c is 1, so we would be getting is electic 1 d t by d x we would be getting is b square minus 4 a c is plus minus 4 by 8 that is 1 by 2. So, characteristic curve we would get 2 t is equal to x plus c 1 and 2 t is equal to minus x plus c 2.

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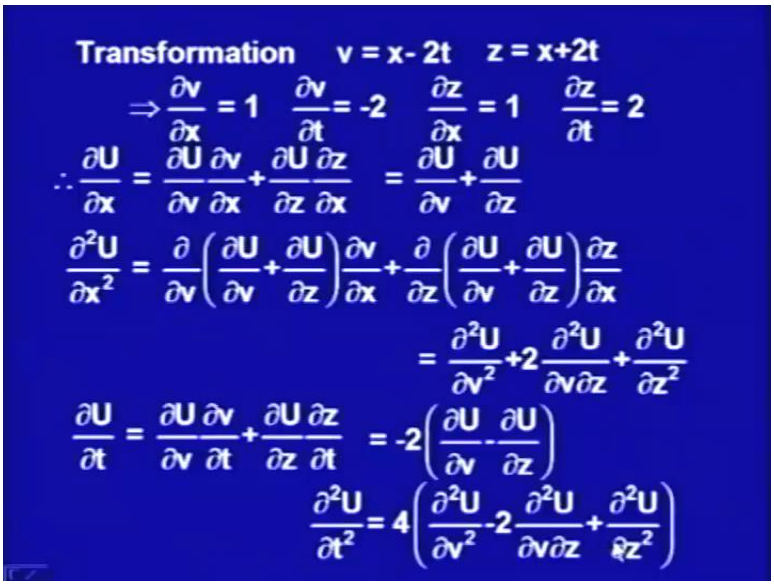

From here we can transform it using the transformation v as x minus 2 t and z as x plus 2 t, we are changing it to the canonical from. So, for this we do know that del v over del x would be 1 del v over del t is equal to minus 2 del z over del x would be 1 del z by del t would be 2 using this chain rule del u over del x, we could write del u over del v times del v over del x plus del u over del z times del z over del x just putting del v over del x as 1 and del z over del x is equal to 1 we get it is del u over del v plus del u over del z.

Again the second derivative simply says is that differentiate this with respect to x again, again you would use the chain rule that is del upon del v of this whole function del u upon del v plus del u upon del z times del v over del x plus derivative with respect to z of this whole function del u upon del v plus del u upon del z times del z over del x. Now, again del v over del x and del z over del x we are nothing but, the 1, so we would get it as del 2 u del v 2 plus 2 times del 2 u over del v del z plus del 2 u del z 2.

Since, we had assume that our x function u is continuous in x and t that guarantees that we can get that del 2 u over del v over del z or del 2 u over del z over del v that would be same. Similarly, we can go ahead with the derivative with respect to t again using the chain rule in this similar manner first differentiating with respect to v then derivative of with respect to t and then differentiating with respect to z and derivative of z with respect to t. So, d v over d t here is minus 2 and del v over del z over del t is plus 2 we do get is minus 2 times del u over del v minus del u over del z. Again use the second derivative that is again differentiate it with respect to t we do get it should be 4 times del 2 u del v 2 minus times del 2 u over del v del z plus del 2 u over del z 2.

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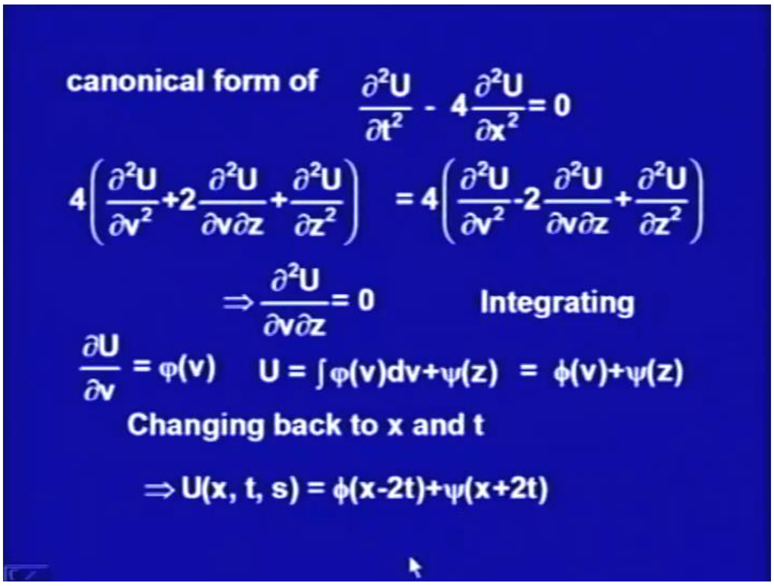

Now, substitute this in our given equation, so we do get the canonical form of del 2 U over del t 2 minus 4 times del 2 U over del x 2 is equal to 0 in this we are going to substitute, now the derivative with respect to v and z. So, we are getting is 4 times del 2 U over del v 2 plus 2 times del 2 U over del v del z plus del 2 U del z 2 that is actually the derivative of del 2 U over del x 2. So, this I am keeping with this, this should be equal to 4 time del 2 U over del v 2 minus 2 times del 2 U over del v del z plus del 2 U over del z 2.

Now, if we just simplify it we do get is that del 2 U over del v del z is equal to 0 that is it has been changed this has changed to the canonical form, we do get very easily that is the terms involving the second derivative with respect to v and second derivative with respect to z, they are equal on both the sides they would be cancelling it out. Now, this equation we can solve by first differentiating with respect to z and then differentiating with respect to v.

So, integrating with respect to z first we do get del U over del v as phi of v why because, this is 0. So, we will get the constant with respect to z, but that may contain a function f v because, it is the partial derivative with respect to z, again integrating it with respect to v we do get U as integral of phi v d v plus a constant, constant may involve the variable z. So, let us write that is a function of xi z, this we have already done this is again just for you could say is your practice that is how we are using this D' Alembert's and this canonical form to solve the equations, since this integral would be only the function f v let be write it out another phi, so phi of f v plus xi of z.

Now, change back it to in x and t, so what we have got u x comma t comma s as phi of x minus 2 t plus xi of x plus 2 t. Now, what would be this phi and xi for that we require to determine this phi and xi, which are unknown functions general functions we use the initial conditions.

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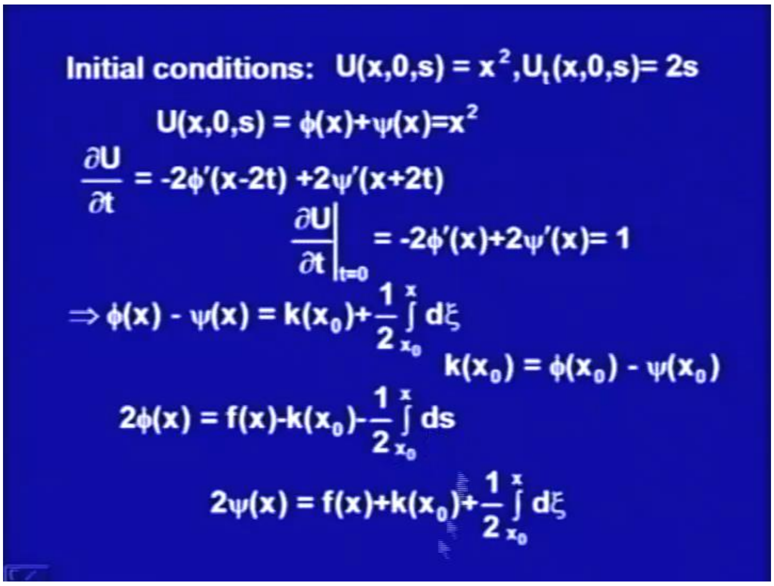

Our initial conditions are been given as U as x comma 0 comma s as x square and U t x comma 0 comma s as 2 s. So, U x comma 0 comma s as phi x plus psi x is equal to x square and the derivative of this U x t is that would be minus 2 phi dash x minus 2 t plus 2 times psi dash x plus 2 t where this dash means the derivative with respect to this evaluate it at t is equal to 0 this is given as 1 g x was given as 1, so here it should be one.

When we are keeping it over here we do get it as phi x minus psi x is equal to k x naught plus half integral x naught 2 x 1 with respect to xi that is d xi these xi d xi. So, now we have got one equation over here involving phi and psi and another involving phi and psi from these two ones we do get the solution and k x naught is of course, a constant when we are integrating it with respect to x, we are getting is 2 phi x is equal to f x minus k x naught minus half x naught 2 x d s 2 psi x is equal to f x plus k x naught plus half x naught 2 x d xi. So, this s we are just taking out rather than we are using xi because, this s may should not confuse with our parameter s.

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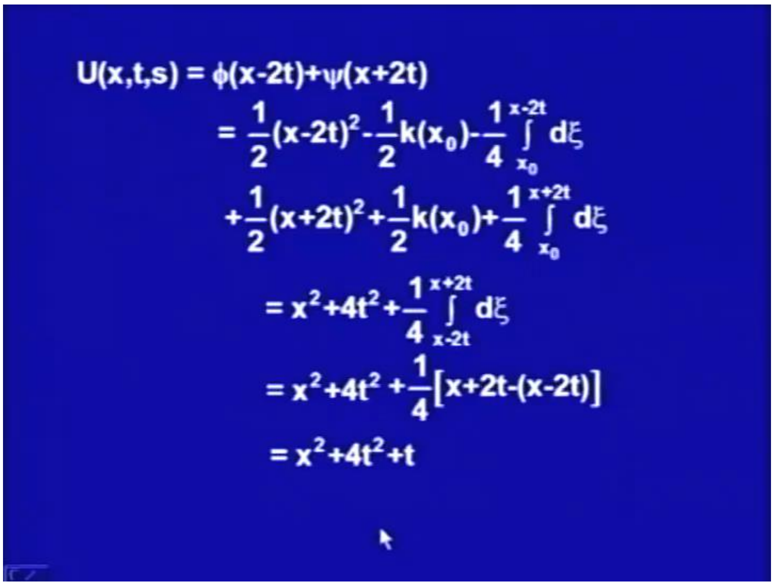

So, what we get the solution as U x comma t comma s as phi of x minus 2 t plus psi of x plus 2 t, now replace phi and psi in the terms of f and g we do get half of x minus 2 t square minus half of k x naught minus 1 by 4 x naught to x minus 2 t d xi and psi x plus 2 t is half of x plus 2 t square plus half of k x naught plus 1 by 4 times x not to x minus 2 t d xi. Now, add it up what we do get it x square plus 4 t square it should be 2 t square x minus 2 t whole square and x plus 2 t whole square, so it should be x square plus 4 t square and this k this constant what is we cancelling out because, the same constant and both this one.

And this integral that is from x naught x minus 2 t and minus sign I could write it as x minus 2 t to x naught. So, we do get finally, the integral 1 by 4 times integral x minus 2 t to x plus 2 t d xi, now evaluate this integral, this integral d xi that we would get xi, xi is evaluate it from x minus 2 t to x plus 2 t, what we would get 1 by 4 x plus 2 t minus x minus 2 t we add it up we just get it only t. So, we have got the solution u x comma t comma s as x square plus 4 t square plus t.

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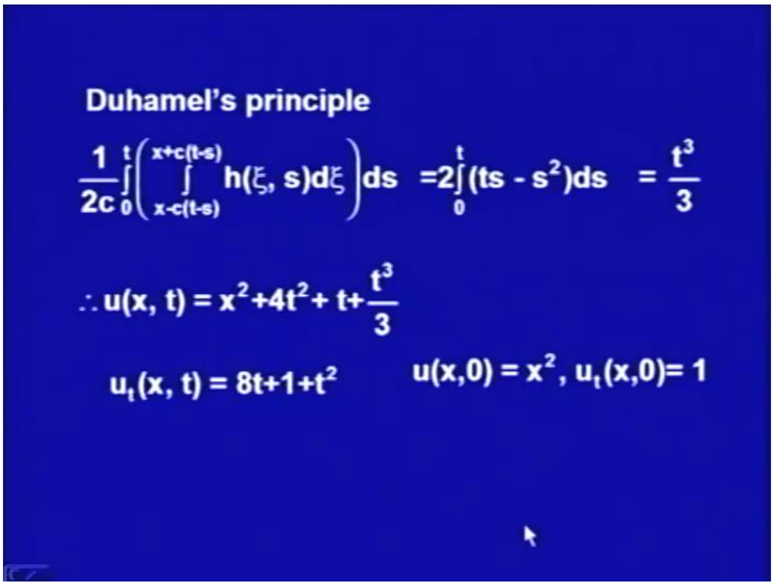

Now, we just go with the solution of non homogeneous problem using the Duhamel's principle, which says is that should be 1 upon 2 c 0 to t integral x minus c times t minus s to x plus c times t minus s h xi s d xi d s, now x xi s nothing but, 2 s. So, we get it as 2 s integral from x minus c times 2 t minus s with respect to xi, so that is we are not having any function of xi over there.

So, I would get simply 2 s and then x plus c t minus c s minus x minus c t plus c s and that would give me only c times t minus s twice c times t minus s twice c that would be getting cancel it out, what we would be left is 2 times 2 s is there, so 2 s multiplied with t minus s. So, integral 2 have taken outside integral 0 to t s times t minus s that is t s minus s square integrate it with respect to s, now integrate it here we have to integrate it with respect to s.

So, this will give me s square by 2 t times s square by 2 evaluate it from 0 to t I would get t times t square by 2 and this 2 is getting cancel it out, so I would get t cube. Similarly, this 1 minus 2 s square you could say s square does has integral would be s cube by 3 evaluate it from 0 to t I would get t cube by 3. So, what we have got t cube minus 2 t cube by 3 that gives me t cube by 3, so this is what we have got the solution of this special homogenous is this non homogeneous problem with homogenous initial condition. And previously we had to find out the solution of homogeneous problem with non homogeneous initial conditions.

So, we just add up these two solutions, we do get the final solution of our problem initial value problem as x square plus 4 t square plus t plus t cube by 3. Now, let us see that is whether it is a solution of our given equation or not, you can just find out it is derivatives with respect to t and with respect to x second derivatives, try to put if I am finding out the second derivative with respect to x, so here I would get it simply two.

And if I find out the second derivative with respect to t, you can just get it out that is here I would be getting it 8 and this would be 0 and this would be first derivative would be 3 t square that is t square, then we would get 2 t. So, we do get is that is when we are keeping it in our equation, we can get that it is satisfying our equation we would be getting it equal to 2 t. And when does it satisfy our initial conditions, let us see is that is this the derivative of this with respect to t is 8 t plus 1 plus t square.

So, from here actually you can see is that is second derivative would be 8 plus 2 t, so you can just get is that is 8 plus 2 t that is you would get the solution as 2 t. So, now initial conditions are at t is equal to 0, the function should be x square, so here if I put t is equal to 0 all these terms would be 0 I would get only x square. And u t at t is equal to 0 is given as 1, so if I keep t is equal to 0 put t is equal to 0 over here I would get only one that is satisfying our equation as well it is satisfying the initial conditions, so this is the solution.

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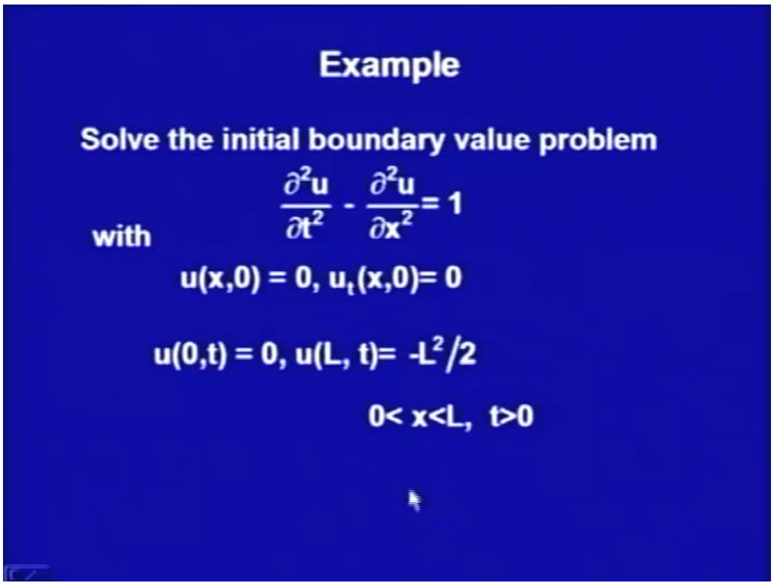

Now, let us come to another problem, solve the initial boundary value problem del 2 u del t 2 minus del 2 u del x 2 is equal to 1 with initial conditions u x comma 0 is 0 and u t x comma 0 is 0 and boundary conditions u 0 comma t is 0 u L comma t is minus L square by 2 where this, equation is for x 0 to L and t is positive. So, you do remember that is if we are having is that our bound is been fixed into 0 to L that we are getting is this boundary condition and this initial condition.

So, now we are having actually non homogenous equation, homogeneous initial conditions. But, with boundary condition, how to solve this one, we can solve this kind of equations using just take this non homogenous equation, we just take the homogenous part use this boundary conditions and use the Fourier series method. So, what we do is now we will go with the method which says is that find out a particular solution and find the general solution for the homogenous one.

So, what we do is that is we take one particular solution of this non homogenous problem and the boundary value problem then will treat with this homogenous equation with these boundary condition of boundary value problem, find out the solution using the Fourier series method . And then, put that our particular problem over there and find out the using this initial condition find out the solution, so let us do that is how we are going to do it.

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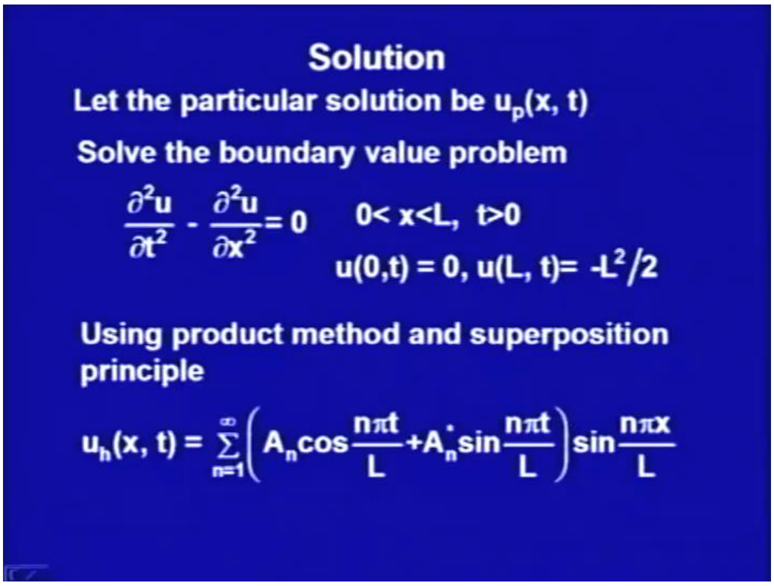

So, first this particular solution let I am taking it here in general let us say that particular solution be up x comma t first. Then solve the boundary value problem del 2 u over del t 2 minus del 2 u over del x 2 is equal to 0 with boundary conditions use 0 comma t is equal to 0 and u L comma t is equal to minus L square by 2. In last lectures we have done there is how to solve this one dimensional wave equations with boundary conditions using the Fourier method, using that variables separable method the this one.

So, we have got this using this product method and super position principle, if you do remember we had got that, now I will call this solution of this as homogenous that u h x comma t that we have got as summation n is running from 1 to infinity A n cos n pi t over L plus A n star sin n pi t over L sin n pi x over L. Now, if I just use the our this solution u p x comma t as the particular solution of our non homogeneous equation.

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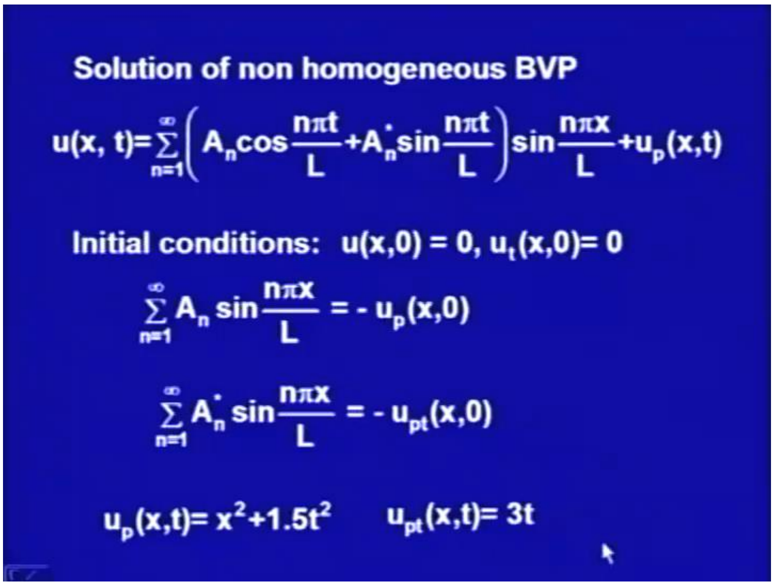

Then the solution of non homogeneous boundary value problem would be u x comma t summation n is running from 1 to infinity A n cos n pi t over L plus A n star sin n pi t over L into sin n pi x over L up x comma t do you remember that there we are having c also here sc is 1, so this c would be 1. Now, we try to satisfy the initial conditions over here, initial conditions says is at t is equal to 0 this function should be 0 and the derivative of this function at t is equal to 0 with respect to t the derivative with respect to t at t is equal to 0 should also be 0.

So, let us try to satisfy these initial conditions, there says that is one when I am putting t is equal to 0 this cos 0 is 1 and sin 0 is 0. So, I would be getting only from here the only term A n and sin n pi x over L, so what we would be getting is summation n is running from n 2 1 to infinity A n times sin n pi x over L plus up x comma t that should be 0 that an putting on the right hand side.

So, it will satisfy initial conditions that is if I find out this A n such that this is happening now what does is says is that I should take my whatever be that particular solution, that particular solution at t is equal to 0 whatever we are getting my A n should be actually the coefficients of Fourier sin series of that function. And similarly, when I take the derivative with respect to t here I would be getting is minus and pi over L A n times sin n pi t over L and plus n pi over L A n star sin n pi t over L that is says is that n pi over L would also come over here, that A n star times n pi over L sin n pi x over L should be equal to the derivative of this partial this particular solution with respect to t at t is equal to 0.

So, what it says is that I should get the solution as using the Fourier series I could get that is whatever this particular solution I do have, that evaluated at t is equal to 0 what function I do get that should be a function of x, we do get it in the terms of the Fourier series. And similarly, we should get over here in the A n stars we could get from the other initial condition, now let us do one example suppose in this particular one I do take my particular solution as x square plus 3 by 2 t square.

Now, this would satisfy our equation because, our equation says is del 2 u over del x 2 minus del 2 u over or del 2 u over del t 2 minus del 2 u over del x 2 is equal to 1. So, the second derivative with respect to t will give me here only 3 by 2 is into t is that is 3 and second derivative with respect to x will give me 2 only, so 3 minus 2 would be 1. So, this is actually a solution of our non homogeneous differential partial differential equation satisfying that one, so this is a particular solution.

Now, in this particular solution if I put t is equal to 0 what I do say get is that A n I should get as the coefficient of Fourier sin series of minus x square. If I take that is derivative with respect to t I would get it as 3 t and at t is equal to 0 that would be 0, that says that A n stars I would be getting it actually n pi over L times A n star would be 0 or A n star would be 0. So, what would be getting the solution, we would be getting the solution as summation n is running from 1 to infinity A n cos n pi t over L times sin n pi x over L plus x square plus 3 by 2 t square, where, A n I have been given as the Fourier coefficients of the Fourier sin series of minus x square, that is A n should be integral 0 to L minus x square sin n pi x over L d x, so that is how we are solving this boundary value problem also as one example.

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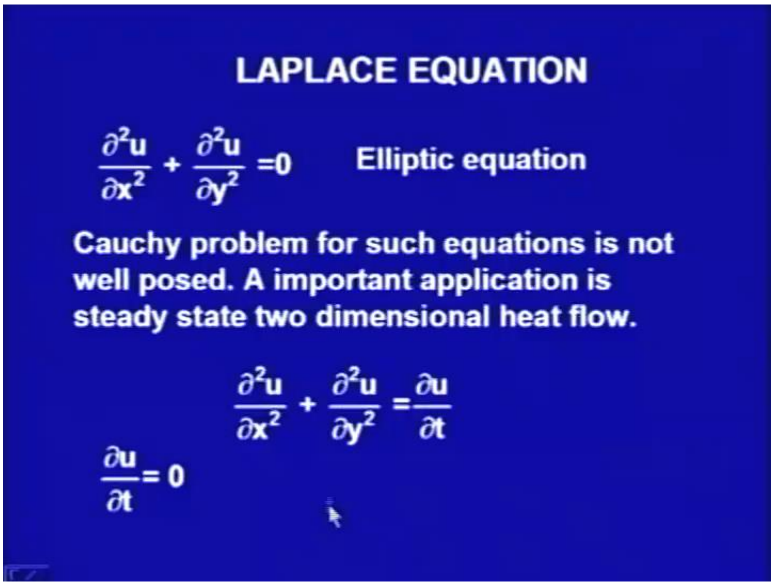

Let us, move to one more practical problem which says is that in heat conditions we have already done this heat equations, this equation which we call the Laplace equation del 2 u over del x 2 plus del 2 u over del y 2 is equal to 0, this we have a turnaround in the first lecture and second lecture, we had already find it out that this is in elliptic equation, it is not having any characteristic.

Moreover, we are knowing is that this is not well posed Cauchy problem is not well posed, that is says is if I use any initial conditions or boundary conditions. Then on any characteristic curves is not necessary that that give me a unique solution that is it is not the Cauchy problem for such equations is not well posed. An important application of this is coming from the steady state two dimensional heat flow that is heat equation this is two dimensional heat equation del 2 u over del x 2 plus del 2 u over del y 2 is equal to del u over del t.

If we say is that from here heat flow is a steady; that means, it is independent of time, if it is independent of time that says del u over del t would be 0. And then this two dimensional heat equation would be change into our Laplace equation, this is one of our particular important. So, let us see this is what is this elliptic equation because, elliptic equation just Laplace equation is elliptic and from here the Cauchy problem is not well posed that is we may not get unique solution, so how do how to find out the solution of this.

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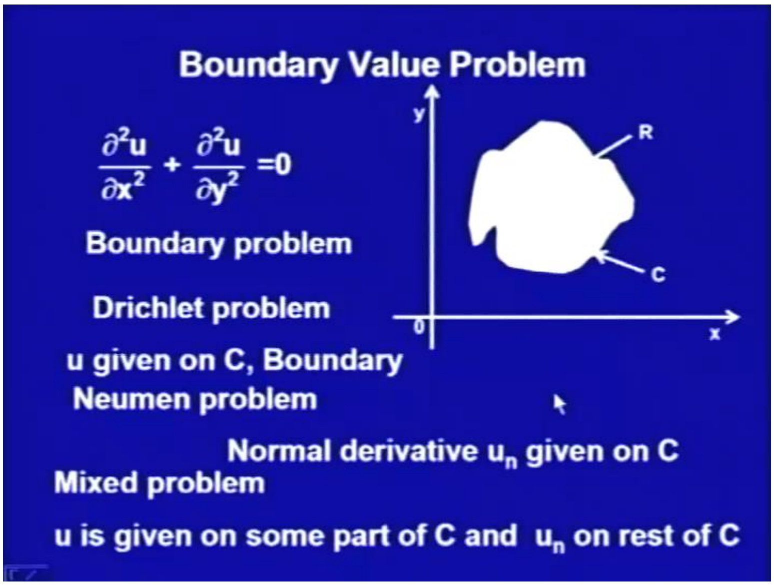

Let us, talk about this two dimensional Laplace equation, if it is on some region that is it is defined on this some region on the two dimensional x, y and the some region. Since, the Cauchy problem is not well posed what we say is that, boundary value problems that is what the boundary what the kind of sin it conditions we can have, actually you posing the different side condition if it is on the some region or if it is working. Then, we can have different kind of pose, different kind of boundary conditions.

And from those from boundary conditions, we do have we named them different problems, one is that we could put one boundary condition as my u is given on the boundary of this region, that is C this is then it is called the drichlet problem. If it is given that is because, this Cauchy problem is not well posed what it says is that, it is normal derivative with respect to t may not be having the unique solution, that says the another problem which will give me the solution that is called the neumen problem.

In which we do have that normal derivative u n is given on this boundary c or we may have mixed kind of problems, in which on some part of this c, the u is given and on the rest of the part of this C this u n is given. So, let us just talk about this drichlet problem, that is we are talking about the boundary value problem with Laplace equation, where boundary conditions are given for the u on the boundary of the region.

So, for simplicity let us take the first one, that is directly from the one dimensional heat equation if I am just changing it to two dimensional heat equation with initial with del u over del t as 0 the steady state one. So, let us say the rectangular reason, so I am just considering the drichlet problem and rectangular region.

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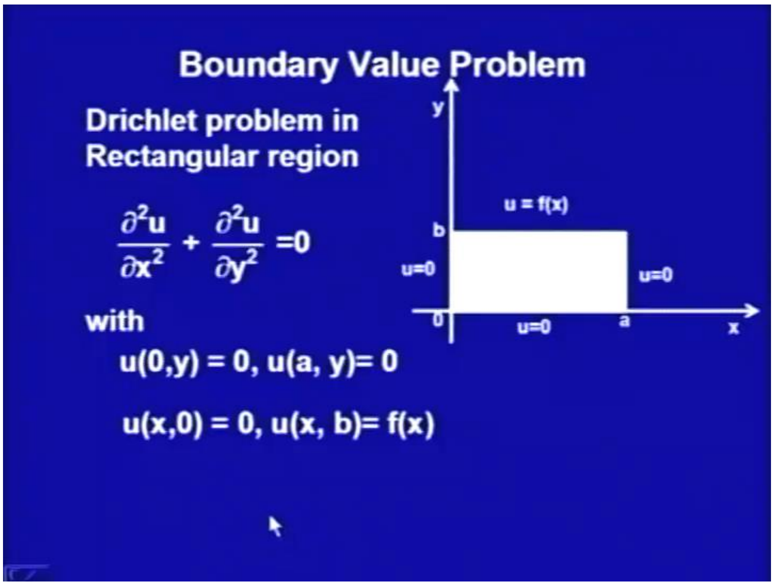

So, this is the region in which we are talking about this region and the boundary conditions we are having is that is on the 3 sides of the boundary, we are putting that is u is 0 and on the upper side of the boundary we do have that it is f x. So, let us pose the problem we do have del 2 u over del x 2 plus del 2 u over del y 2 is equal to 0 with boundary conditions that u 0 is y is 0 u 0 y is 0, u a y is 0 and u x comma 0 is 0 and u x comma b is f x.

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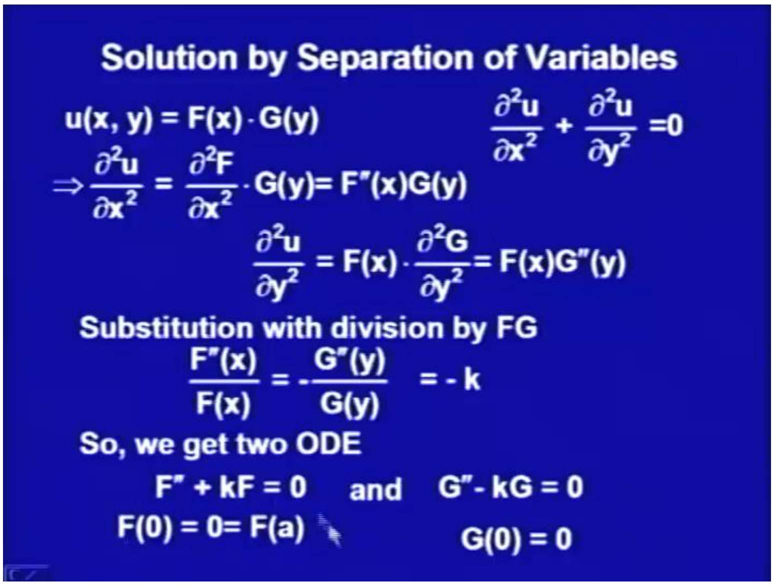

Let us see how to solve it we just go by the separation of variable method that is using the product function F x and G y we do get del 2 u over del x 2 as F double dash x G y del 2 u over del y 2 as F x times G double dash y. Now, substituting this in the given F equation del 2 u over del x 2 plus del 2 u over del y 2 is equal to 0 and dividing it by F into G what we do get F double dash x upon F x is equal to minus G double dash y upon G y.

Since, left hand side is x and y right hand side is y if they has to be equal they must be constant. So, let us say that is constant is minus k, what we are getting is 2 ODE or ordinary differential equation one is F double dash plus k F is equal to 0 and another is G double dash minus k G is equal to 0. And here we do have the boundary conditions if we change we do get that F 0 is 0 and F a is 0 here the boundary condition we would be having is G 0 is 0. So, now we do have two boundary value problems, which is on the ordinary differential equations.

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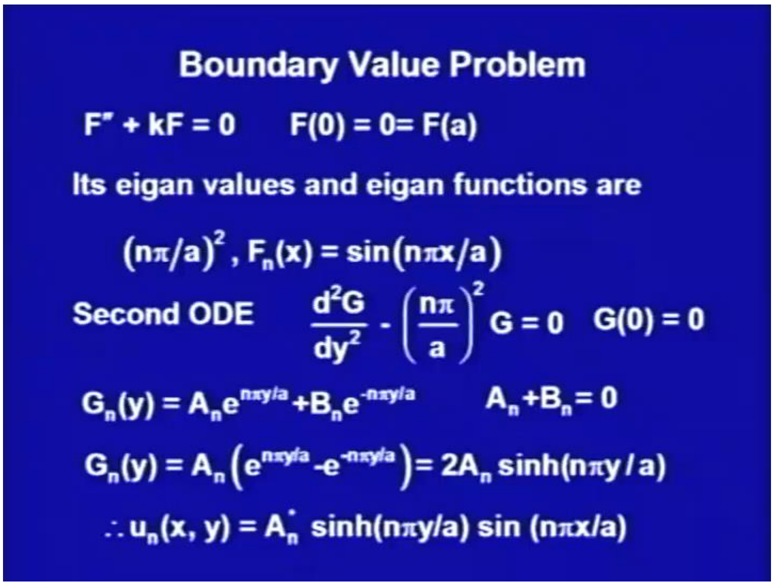

For this boundary value problem we do know that the Eigen values and Eigen functions are that is Eigen values are n pi over a whole square and Eigen functions let say the solutions F n x is sin pi x over a. Now, second ordinary differential equation is this one now if this is the Eigen values, then this k would be writing it as n pi over a whole square times G. So, this is one for this the solution is actually A n e to the power n pi y upon a plus B n e to the power minus n pi y upon a.

Now, if I put G is equal to 0 G at 0 is 0 this give me A n plus B n is equal to 0 or B n is equal to minus A n. So, we do get the solution as A n times e to the power n pi by upon a minus e to the power minus n pi y upon a that says is in the hyperbolic function 2 A n sin hyperbolic n pi y upon a. So, what we have got the solution u n x comma y is A n stars sin hyperbolic n pi y upon a into sin n pi x over a, this two A n I am writing as A n star.

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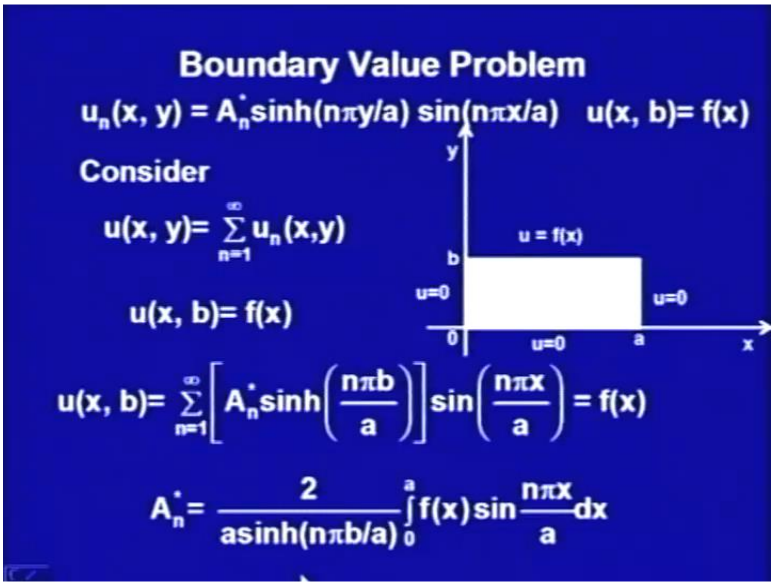

So, we do have seen that with this solution is satisfying the boundary conditions on the three sides, the solution we have got A n star sin hyperbolic n pi y upon a sin n pi x over a this is satisfying the boundary condition on the three sides. Now, let us see that is whether it would satisfy the boundary condition on the fourth side or not, certainly saying the solution you cannot find out this one, so we use this super position principle.

And let this solution u x y s summation n is running from 1 to infinity u n x comma y, where u n x comma y is this one. So, that says is now we want to satisfy this boundary condition, the last one that is u at x comma b is f x, so u x comma b if I am just using this u n x y over here, if I put our y is equal to b what I would be getting is A n star sin hyperbolic n pi y up n pi b upon a into sin n pi x over a is equal to f x.

Now, from here if you do see what this says is that, this coefficient A n star sin hyperbolic n pi b over a, this must be actually the coefficient of Fourier sin series of f x. Now, f x is a function which is defined from 0 to a at this point, so now, we have to have odd expansion of this and Fourier expansion for the odd extension of f x from 0 to a to minus a to plus a.

So, now we do have is A n star using that just gives 2 upon that is actually A n star times sin hyperbolic n pi b upon a should be 2 upon a. So, that whole thing I have taken this 2 upon a and sin hyperbolic and pi b upon a integral 0 to a f x sin n pi x over a d x using the Euler's formula for the Fourier coefficients of odd expansion of Fourier expansion for this f x. So, we get the solution of this our boundary value problem, where this boundary conditions we do have this one in the rectangular region as this one. So, now up this is what we have got Laplace equation with boundary conditions, we have got. Now, if I take non homogeneous Laplace equation, we do know that this is called Poisson equation.

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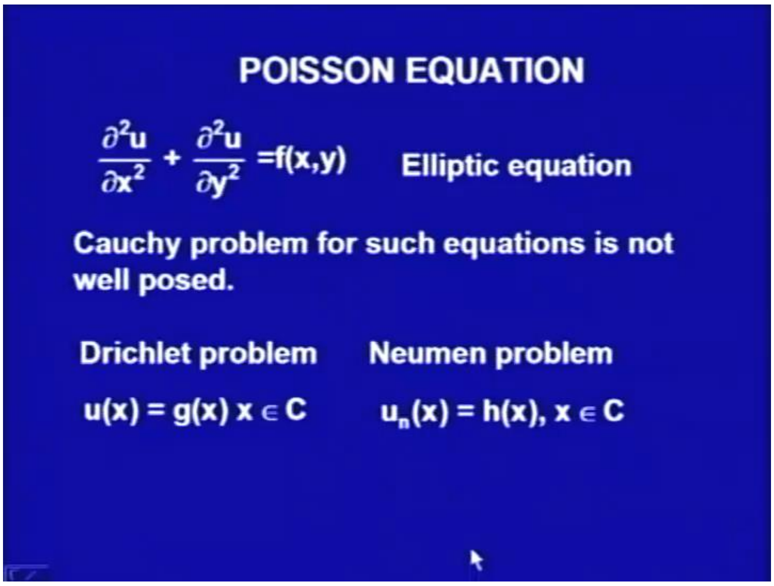

Del 2 u over del x 2 plus del 2 u over del y 2 is equal to f x y, how to go for the solution, normally we are having is this again we do know that this is elliptic equation, again we do know because this is elliptic equation. The Cauchy problem for such equation is not well posed, that says is we have to choose the initial conditions or the side conditions in such a manner. So, that we do get some solutions, unique solutions normal condition is that is whenever we are using non homogenous equations we use homogenous initial conditions or with homogeneous equations we use non homogenous initial conditions.

We do have is that many times homogeneous equations we do have that the solution is existing. Just now, we have done this Laplace equation also that homogenous equation and this boundary conditions, we could say that we could treat as initial conditions as well also we could find out the solution. So, if the problem is non homogenous we can change it to the homogenous one and take the particular solution of this non homogenous one and add it up and find out the using the initial conditions, we could find out the solution or we are having is homogeneous equation with non homogeneous initial conditions that already we have done.

So, if we do have non homogenous equation with non homogenous initial conditions, we just go ahead with the homogenous equation with non homogenous initial conditions and then add up the solution of particular solution of non homogenous equation. So, in this Poisson equation also we can define the drichlet problem as well as the neumen problem, in the drichlet problem it says is that this u should be initial condition should be g x for the boundary and the neumen problem says is that we do have the normal derivative, normal derivative is defined on this one.

So, either we take this non homogenous Poisson equation with g x and h x as 0 and or we do have this here as 0 and then g x and h x are as such, on all the cases as we have done in the Laplace equation similar manner and in the this wave equations we can do it in the Poisson equation as such. So, today we had learnt the one dimensional wave equation in the non homogenous one using homogenous initial conditions, then using non homogenous initial conditions, then to find out a boundary value problem using homogenous initial conditions and so on. Then we have gone to the Laplace equation using the boundary conditions and we had learnt about the Poisson equation as well, so that is all in today's lecture.

Thank you.