Mathematics - III

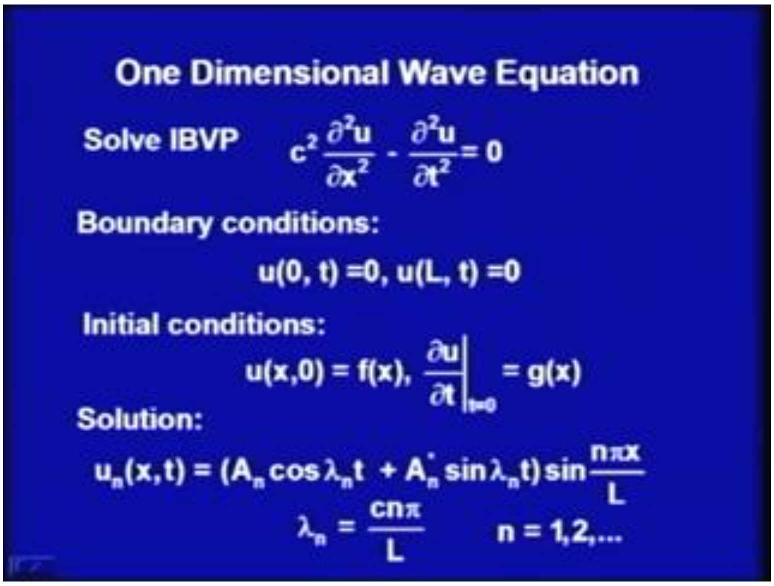
Prof. Tanuja Srivastava
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture - 20

Solution of One Dimensional Wave Equation

Welcome to the lecture series on differential equations for under graduate students. Today's lecture is on Solution of One Dimensional Wave Equation. In the last lecture, we had modeled one physical problem of vibrating string and we modeled it as a partial differential equations with boundary and initial conditions.

(Refer Slide Time: 00:58)

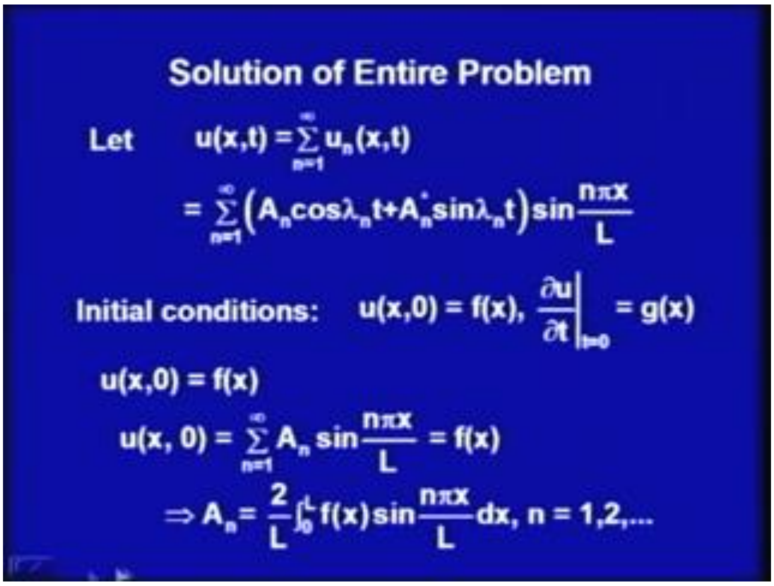


Actually we have got the partial differential equation as one dimensional wave equation c square del 2 u over del x 2 minus del 2 u over del t 2 is equal to 0. And we have actually find it out the boundary conditions for this as u 0 t is 0 and u L t is 0, because the string was fixed at both the ends. And the initial conditions we modeled as initially the deflection u x 0 is f x and initial velocity del u over del t at t is equal to 0 is g of x.

Actually in last lecture we had find out the solution of this boundary value problem, as u n x t is equal to A n cos lambda n t plus A n star sin lambda n t sin n pi x over L. Where, this A n A n star are the constants, which are to be established and this lambda n is nothing but, c n pi over L and is the solution was holding true for n is equal to 1, 2, 3 and so on even for some negative values of n also.

We have also checked in the last lecture, that when we try to find it out that is whether this solution satisfies the initial conditions. So, that we can establish this A n and A n star we obtain that for certain values of x we are not able to find out the values of A n and A n star that is in general this solution is not satisfying our initial conditions for all n. That says is now let us see, the solution of entire problem for that we will first see this that this given equation, this is linear with constant coefficient and homogeneous. So, using this fundamental result that is super position principle we could say is that if I take the some of this u n x t that will also be solution of this problem, so by that manner we had just getting the solution of entire problem.

(Refer Slide Time: 02:56)



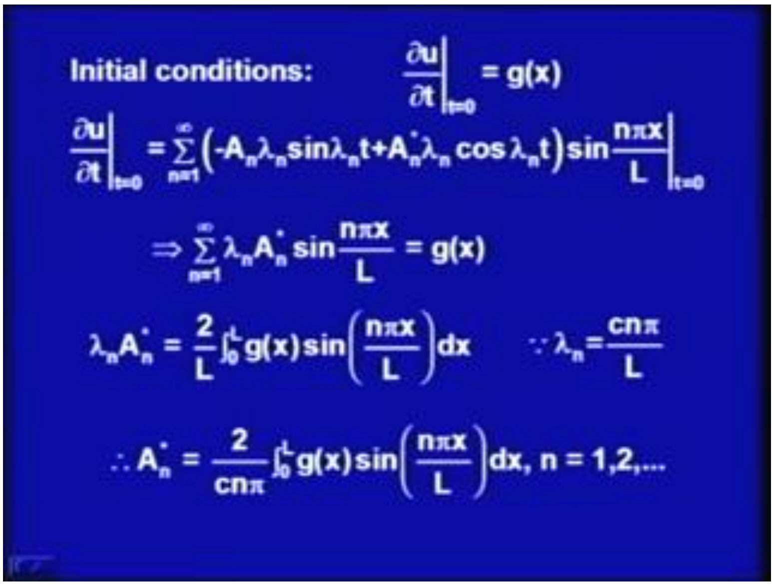
We let that let the solution u x t be summation n is running from 1 to infinity u n x comma t that is summation n is running from 1 to infinity A n cos lambda n t plus A n star sin lambda n t times sin n pi x over L. Now, we try to satisfy this initial conditions, this lambda n we know is that c n pi over L, try to see that is when this solution is satisfying initial conditions. The first initial condition we do have that u x comma 0 is f x and the second initial condition was del u over del t at t is equal to 0 is g x.

Let us go one by one u x comma 0 is f x; that means, if I put t is equal to 0 over here, what we would get in this solution if I put t is equal to 0 sin lambda n t that is it would be sin 0 which is 0 cos 0 is 1. So, what we would be getting is here is only in this coefficient term only A n and sin n pi x over L, so we are getting the u x comma 0 as summation n is running from 1 to infinity A n sin n pi x over L, condition says that it must be equal to f x.

So, it must be equal to f x, if I try to see is that is what it is for going for this solution of entire problem, we have to recall what we have learn about the Fourier series. So, here if we just see is this is giving is a series that f x this initial condition is saying is that f x is sum of certain functions, this sum of series what is this series, this series is cut containing sin functions. So, actually this is a Fourier sin series we could say, so what we just try to see is that we want this A n such that summation n is running from 1 to infinity A n sin n pi x over L is same as f x.

That says is if I just use that formula of the Fourier sin series, we do get is that A n should be of the form 2 by L 0 to L f x sin n pi x over L d x for all n 1, 2 and so on using that Euler's formula for the Fourier series. Now, you see the Fourier series we have defined for the periodic function, but if you do remember we have done the Fourier sin series where, we say it is that the function is need not to be periodic and if function we are just making this extension. So, odd extension and half range expansion and from there we had obtained is formula. So, we say is that this solution u x t will satisfy my first a initial condition, if my f x if this A n can be obtained as the coefficient of Fourier sin series of f x, now let us move to the second initial condition.

(Refer Slide Time: 06:06)

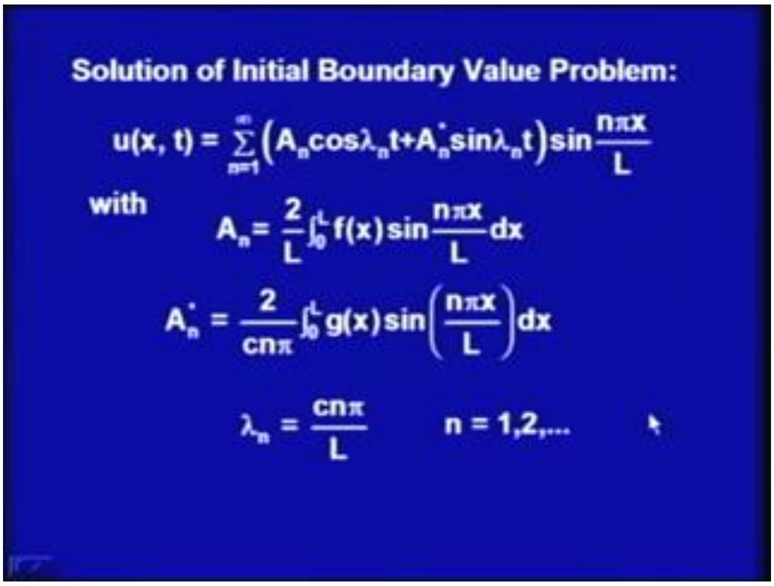


Second initial condition is del u over del t at t is equal to 0 is g x, so first let us find out del u over del t, del u over del t that is it should be thus differentiating with respect to t we get it summation n is running from 1 to infinity minus A n lambda n times sin lambda n t plus A n star lambda n cos lambda n t sin n pi x over L. Now, evaluate it at t is equal to 0 t is equal to 0 says is that sin lambda n t that sin 0, so it should be 0 and cos lambda n t that cos 0 that it should be 1. So, what we would be getting is actually summation n is running from 1 to infinity lambda n A n star sin n pi x over L now this has to be according to this condition this has to be equal to g x.

Now, again what we are getting is the second initial condition is saying is that we should get this A n stars in such a manner that g x this lambda n A n star is the coefficient of Fourier sin series of g x that is again we are talking about the half range expansion with the odd extension of g x. So, using that again Euler formula lambda n A n star we want to be equal to 2 by L 0 to l g x sin n pi x over L d x, so what we have obtained now again lambda n is actually c n pi over L.

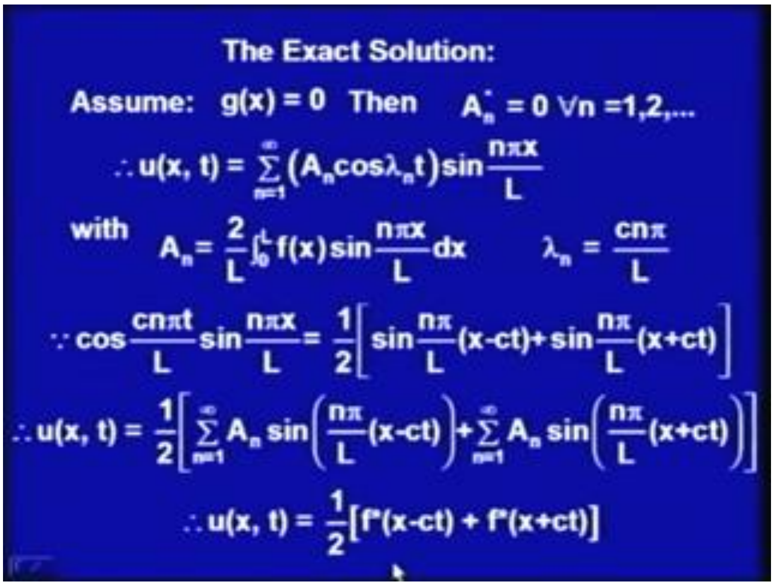
So, we have got A n star should be of the form because, lambda n is this c n pi over L, so we just take it this side 2 up on c n pi 0 to L g x sin n pi x over L for n 1, 2, 3 and so on. So, we have got that if we use this Fourier series and we use this superposition principle of linear homogeneous equations, then we can get the solution which is satisfying the boundary condition as well as the initial conditions.

(Refer Slide Time: 08:14)



So, what we have got finally, the solution we have got, solution as u x t summation n is running from 1 to infinity A n cos lambda n t plus A n star sin lambda n t sin n pi x over L with A n as 2 up on integral 0 to L f x sin n pi x over L with respect to x and A n star as 2 up on c n pi times integral 0 to L g x sin n pi x over L d x lambda n is c n pi over L and this is true for all n 1, 2, 3 and so on. So, we are getting this solution of our initial boundary value problem, let us try to validate it that is does it really satisfies our equations, so let us just try to see what this interpretation of this solution.

(Refer Slide Time: 09:07)



For that let us find out the exact solutions, so first we are for simplicity we are assuming that the second initial condition, which is giving us initial velocity g x that we are assuming as 0. Then, we are trying to see can we find out the exact solution of our initial boundary value problem, it says is if g x is 0 then A n star which is actually the coefficient of Fourier sin series of g x because g x itself is 0. So, each one of that is coefficients would be 0 or we could see is that because, that is integral involving g x that would be 0.

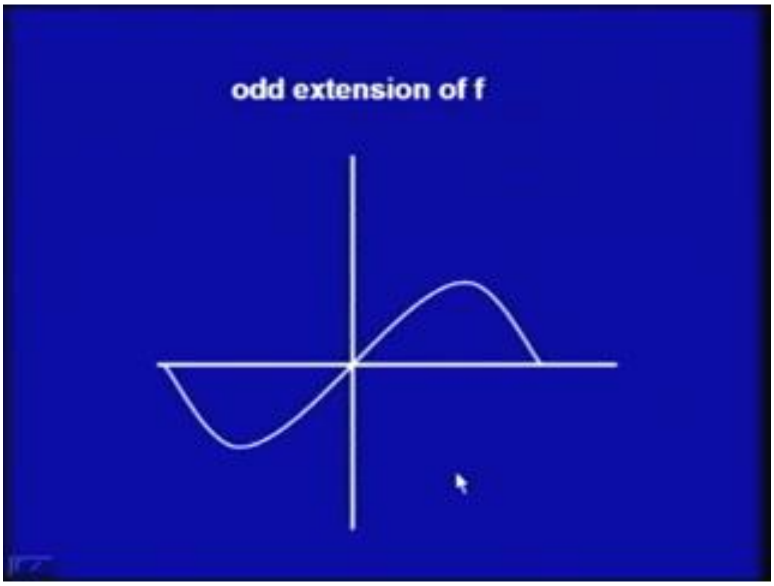
And my solution would be summation n is running from 1 to infinity A n cos lambda n t sin n pi x over L. Let us try to see what this we are getting with A n as 2 by L 0 to L f x sin n pi x over L d x, now we are having here is cos lambda n t, lambda n is c n pi over L that says is we are having here is cos c n pi t over L and this is the term sin n pi x over L this A n is as constant. So now, we do not require this bracket actually, so break because, this is a single function.

So, we are getting let us just try to see what this function I could write cos c n pi t over L and sin n pi x over L, using the simple trigonometric formulas we could write it out that is it is half of sin n pi over L x minus c t plus sin n pi over L x plus c t. Now, if I substitute this over here what we get, we get u x t as substituting instead of this one this complete thing and breaking this series into two series, we get is 1 by 2 times first series summation n is running from 1 to infinity A n sin n pi over L x minus c t plus another series second series summation running from n is equal to 1 to infinity A n sin n pi over L times x plus c t.

So, what this first series let us see first series we said is that is A n is 2 up on l 0 to L f x sin n pi x over L d x. We do know that if this is the A n, then the series A n sin n pi x over L that is A n into sin n pi x over L this series is sum is from 1 to infinity is actually Fourier sin series of the function f x or what we are calling it that a half range expansion of f x with odd extension. Now, in this series if I see we are getting is A n is as such, but instead of x we are having is here x minus c t, so let us just write it as f star x minus c t, what is f star, f star is odd extension of f x, so odd extension of f at x minus c t.

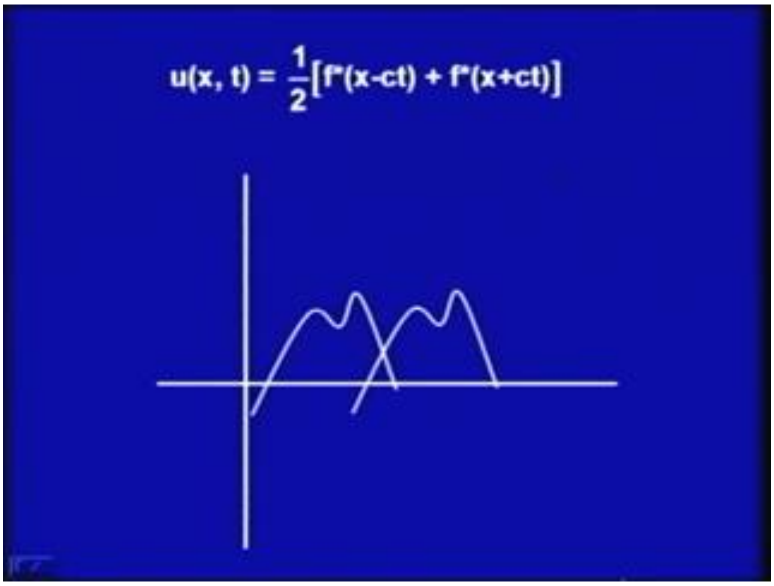
Similarly, the other series also can be written as f star x plus c t, what this we are meaning by let us just try to interpret. So, what we have actually got if I simplify the second initial condition with initial velocity as 0 and initial deflection at t is equal to 0 as f x we are getting the solution of our initial boundary value problem as 1 by 2 of f star x minus c t plus f star x plus c t, where f star is nothing but the odd extension of the given a initial condition function f x.

(Refer Slide Time: 13:28)



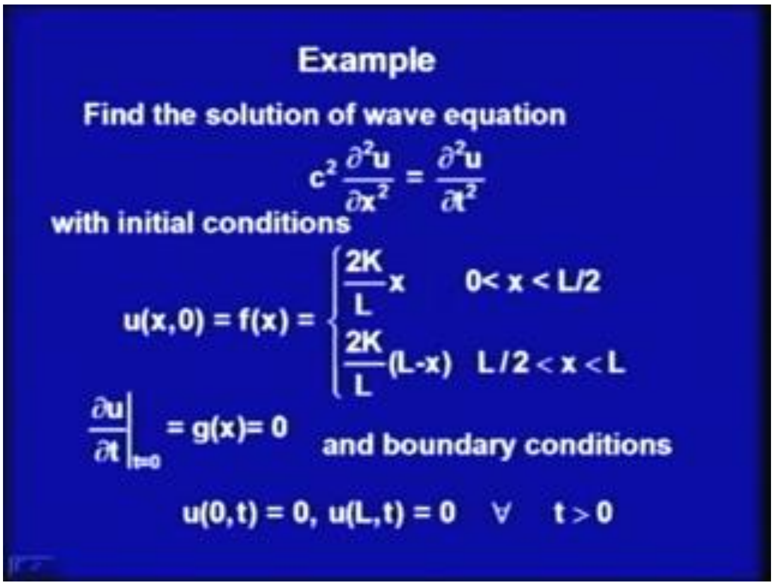
Let us see what it is interpreting, if you do remember we have started one with one string that we had modeled. So, let us see that is other way it that I do have our function initial function suppose initially the string is at this moment that is at this position. So, I do have this the function f it is odd extension simply says is that f of minus x should be minus of f of x, that is whatever we are having over here we just extend in the reverse order in the second series. So, this is what we are knowing is the odd extension of f and what we mean by f of f star x minus c t.

(Refer Slide Time: 14:12)



So, suppose another function I am taking another example, suppose I do have f x function is of this kind. Then if my x is a starting from here, then f x minus c t would be starting from here, if my f x is starting from here, my f x plus c t would be starting from here. So, we are saying is that we have shifted it, so we are calling it one dimensional wave equation, why we are getting the solution if you do remember that u and x t we have got in the last lecture, the solution as a sin wave kind of thing for n is equal to 2. And if we are changing this time t with the time t it is only shifting towards, so that is why it is called one dimensional wave equation.

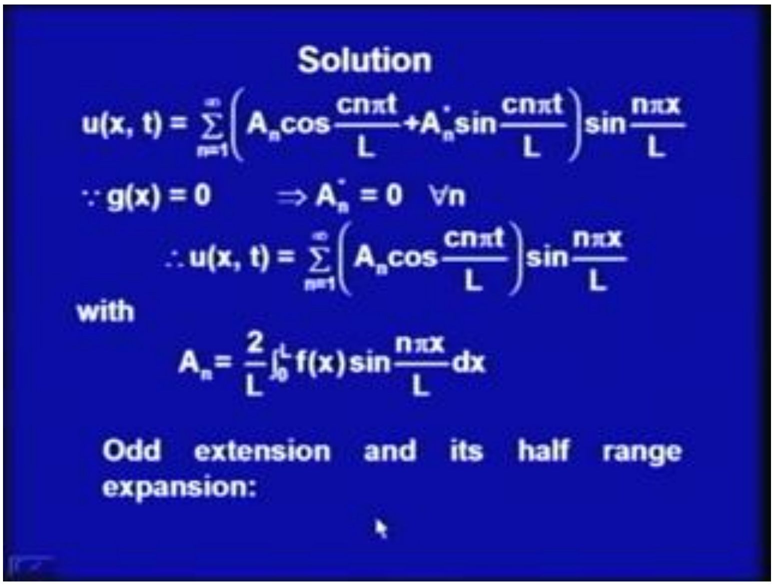
(Refer Slide Time: 15:01)



Let us try to do one example, where we are treating this some particular function f x, so find the solution of wave equation c square del 2 u over del x 2 is equal to del 2 u over del t 2 with initial condition u x 0 as f x is equal to 2 K by l x and for in the interval 0 to L by 2 and interval L by 2 to L 2 K by L L minus x. So, we do remember that we could just observe it that this is not a triangular wave and the other initial condition that initial velocity is 0.

We are not using here, the boundary conditions are same because, it is being fixed over here. So, we are just trying to find out the solution of this wave equation, already we had find it out that, so this boundary condition same as that it is fixed at both the ends, so u 0 t is 0 and u L t is 0.

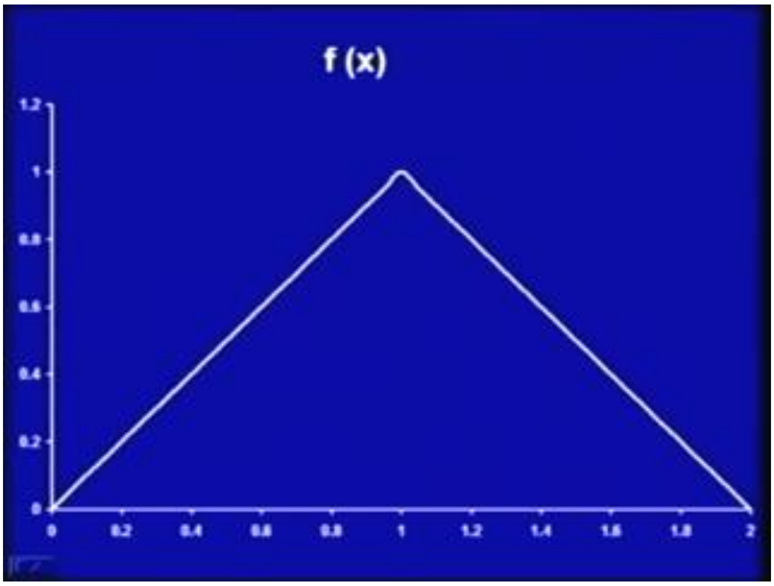
(Refer Slide Time: 16:02)



We had established that the solution of wave equation is summation n is running from 1 to infinity A n cos c n pi x t over L plus A n star sin c n pi t over L times sin n pi x over L. Where, this A n and A n star we are finding out as the Fourier coefficients for the Fourier sin series of g x and f x, so let us first see that is because, our initial condition gives my initial velocity to be 0 g x is 0, so this A n star would be 0 for all n and this A n.

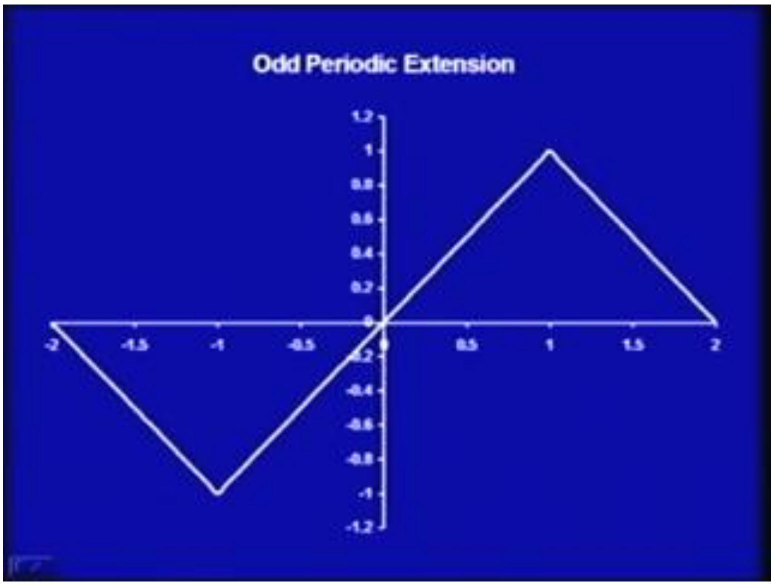
So, actually we will get the solution as containing only unknown function, unknown terms A n’s this A n's can be determined as the Fourier sin series for the function f x, what is our f x is this is also we had established. So now, let us try to see this function f x which is given to us that is a triangular function.

(Refer Slide Time: 17:13)



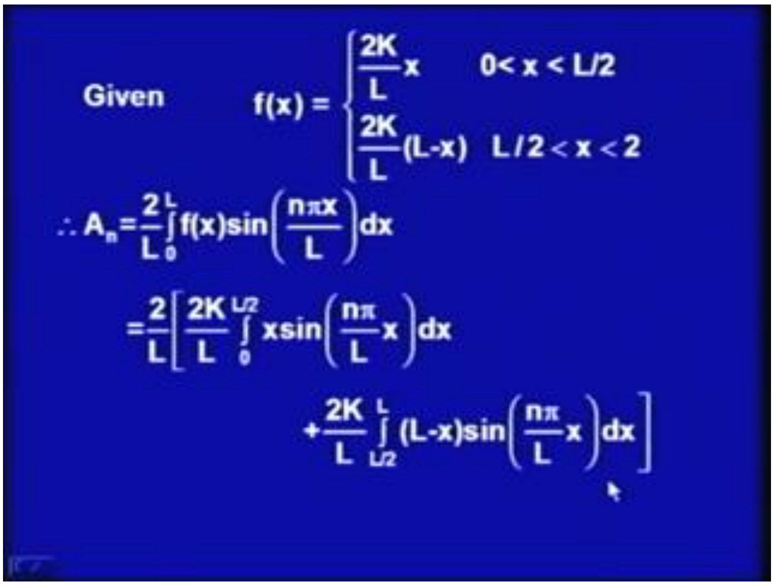
Let us see that is what it shape and it is this is what the function is been given to us that is it is 0 to L at L by 2 it is increasing and then after L by 2 it is decreasing.

(Refer Slide Time: 17:22)



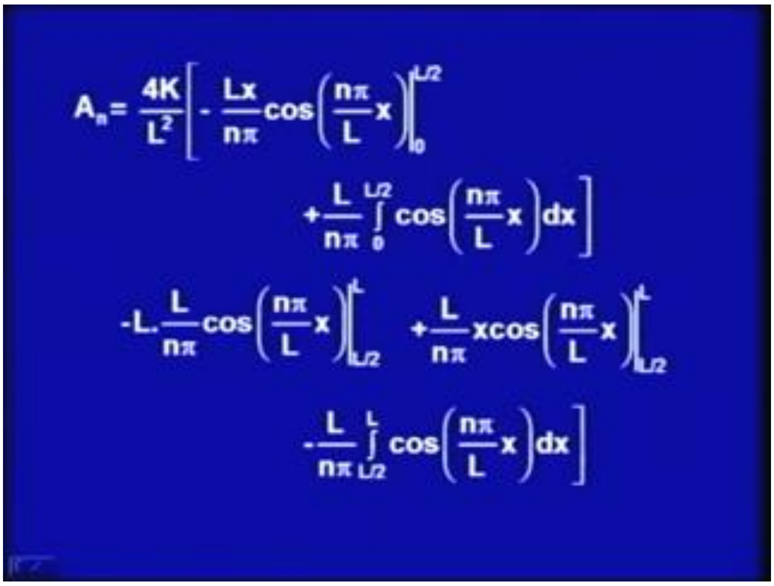
It is odd periodic extension means is if I go first this 0 to L. So, then minus L to 0 this function should be just in the reverse order, this is odd extension, now we say is that this is periodic one we had taking it up.

(Refer Slide Time: 17:36)



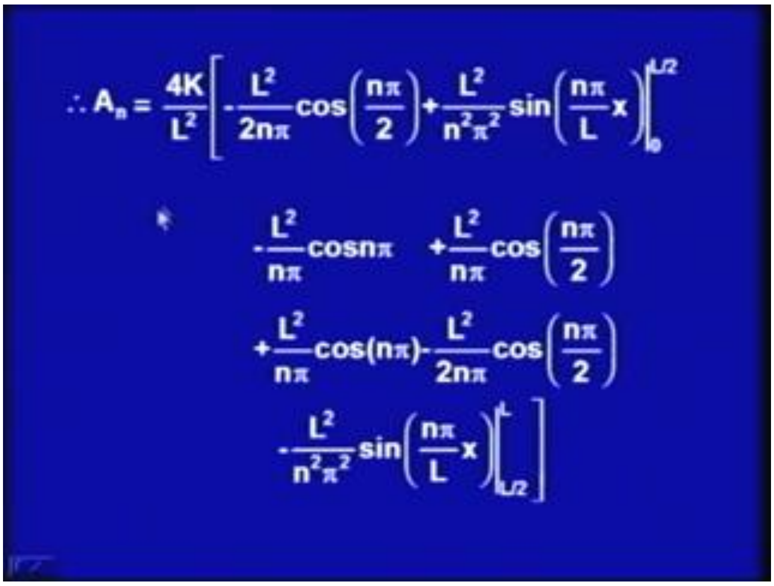
Now, there is the functional form, so A n we are going to calculate A n should be 2 by L 0 to L integral f x sin n pi x over L d x, now substitute the values of f x in the interval. So, we have this the we have to break up this integral into two integral parts, one is from 0 to L by 2 another is L by 2 to L this should be L. So, 2 by L as such the function here is 2 K by L is that is constant 0 to L by 2 x sin n pi over L x d x plus 2 K by L L by 2 to L L minus x sin n pi x over d this one.

(Refer Slide Time: 18:15)



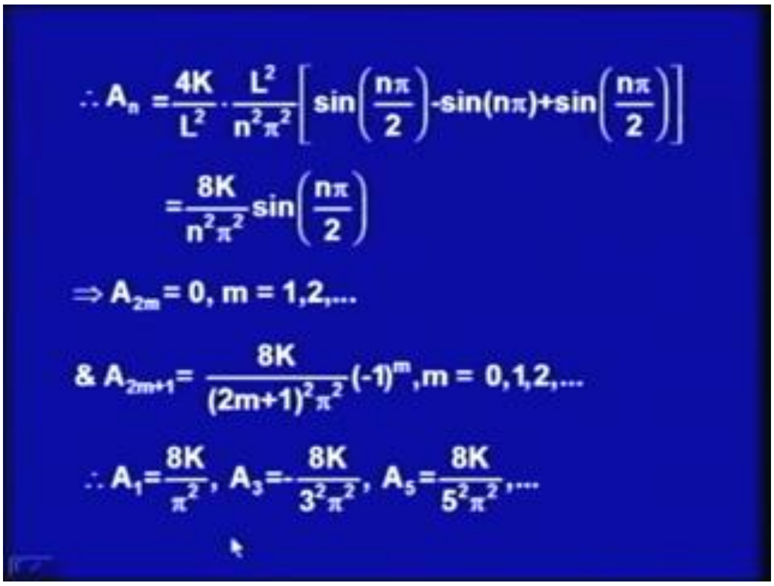
So, A n would be integrating by with the by partition, first integrant minus L x over n pi cos n pi over L x evaluate it from 0 to L by 2 plus L up on n pi 0 to L by 2 cos n pi over L x d x. Then the second integral L minus x, so with the L 1 we have integrated it out L is constant over here, the integral is L up on n pi cos n pi over L x evaluated from L by 2 to L, then minus x to minus x sin n pi x over L from L by 2 to L again by with the partition L up on n pi x sin n pi x over L evaluated from L by 2 to L and the integral would be minus L up on n pi L by 2 to L integral cos n pi over L x d x, again evaluate these values and solve those integrals what we do get.

(Refer Slide Time: 19:19)



Four K by L square as common that constant minus L square up on 2 n pi cos n pi by 2 because, cos 0 is 1 plus L square up on n square pi square sin this integral n pi x over L evaluated from 0 to L by 2. This evaluation is L square up on n pi cos n pi, this integral evaluation is L square up on n pi cos n pi by 2, this evaluation is L square up on n pi cos n pi minus L square up on n pi cos n pi by 2 and the last integral is minus L square up on n square pi square sin n pi over L x evaluate from L by 2 to L again this two evaluation we just go on and simplify the terms.

(Refer Slide Time: 20:13)

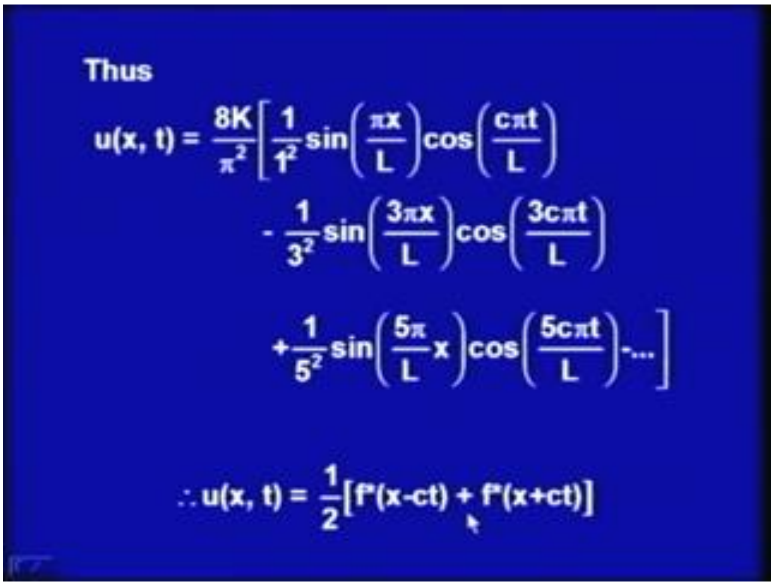


We do get 4 K by L square minus L square up on n square pi square times sin n pi by 2 minus sin n pi plus sin n pi by 2. We do know that sin n pi is 0 and sin n pi by 2 plus sin n pi 2 by 2, so we get it 8 K upon n square pi square L square L square is cancel it out sin n pi by 2, sin n pi by 2 we do know that whenever n would be even this sin it would be sin n pi and that would be 0. So, for even values that is A 2 m would be 0 for m is equal to 1 2 and so on.

And for odd ones A 2 m plus 1 it would be 8 K up on n we are changing as 2 m plus 1 whole square pi square and sin pi by 2 we are getting is minus 1 to the power sin n pi by 2 is actually minus 1 to the power m with the odd ones. So finally, we have got if I just try to find it out m is equal to 0 that is A 1, A 1 would be 8 K up on pi square m is 0, so minus 1 to the power 0 is 1, A 2 from here it would be 0 A 3; that means, m is equal to 1 m is equal to 1 that is minus 1 to the power 1 that will give me minus sign and here we would get 8 K upon 3 square into pi square.

Similarly, A 4 would be 0 A 5 again we are going, so m is equal to 2, so minus 1 to the power 2 that is plus sign. So, we are just getting is that is actually the series we are getting is 8 K up on you could say is m square pi square for A 2 m plus 1 kind of thing with plus and minus signs.

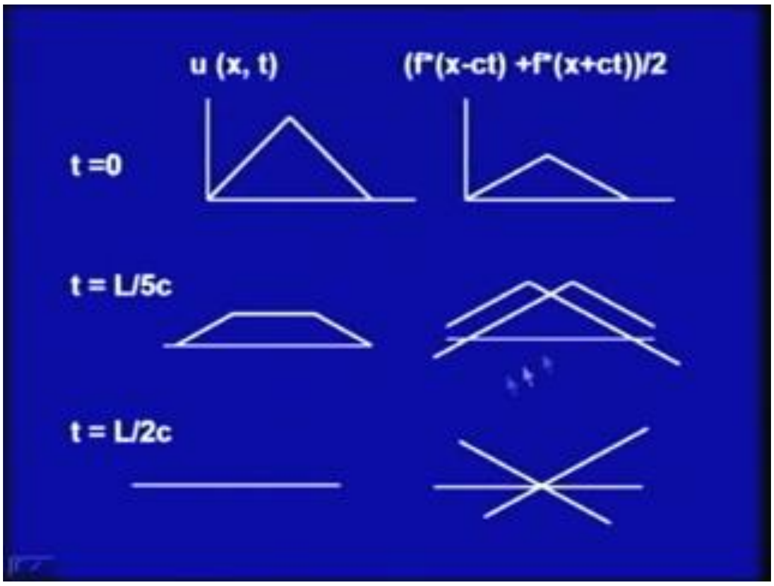
(Refer Slide Time: 22:14)



So, let us put it in our series and get the solution, so we are getting the solution u x t as 8 K up on pi square 1 up on 1 square sin pi x over L then cos c pi t over L n is L here minus 1 by 3 square sin 3 pi x over L cos 3 c pi t over L and so on, we just go. Again if I just write this sin you could say sin a cos b with the using the formula half of sin a plus b plus sin of a minus b.

We would get it in the same manner it is half extension of f star x minus c t plus f star x plus c t, where my f is given triangular function and f star we are talking about just now I which I had shown that your odd extension and we take it that it is periodic with the interval minus L to plus L, so we had establish this solution this is satisfying our initial condition.

(Refer Slide Time: 23:19)

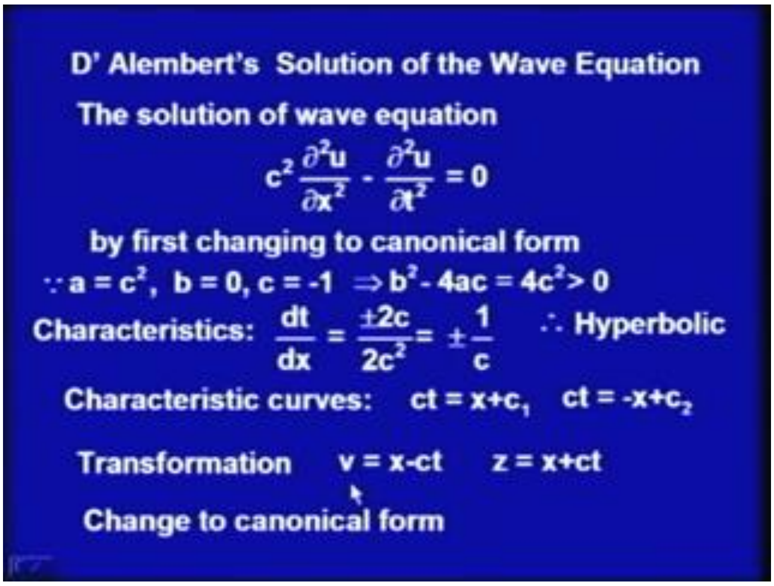


Let us interpret this result, if t is equal to 0 u x t you see that is here ((Refer Time: 23:30)) again I am coming up u x t is this one, you want to establish it. So, let us try to see, if I take f star x minus c t plus f star x plus c t and 1 up on 2 over here, at t is equal to 0 we are been giving this is the function from 0 to l like this one. Now, when I put this odd extension and add it up we would be getting it the similar form, when t is equal to L up on 5 c we have taken, it is going in this manner this is u x t actually what if you see is in this manner, this is the function f star x minus c t, so at t is equal to 0 it would be f star x plus f star x that is it is simply f star x.

If I am shifting it with t is equal to L by 5 see that is we are shifting it towards L by 5 we would be getting it that is it would be shifted towards. So, shifting means is now the function is starting from this is plus 1 and this is the minus 1 that is I have to shift my origin from 0 to c t and 0 to minus c t. So, when we are shifting 0 to minus c t we have to come over here, when we are shifting it 0 to plus c t it would start at 0 the function would start like this one, in this manner if I see my u x t would be of this form.

And if we are just going as L by 2 we are shifting it at the L by 2 you see is that is we would be shifting the function plus c t; that means, this manner minus c t means this manner this line is coming over here and this line is coming over here and in that case my u x t would be simply this line. So, this is how we are establishing that is what kind of solution we would be getting for this one dimensional wave equation, we just find it out that it is shifting once.

(Refer Slide Time: 25:40)



In this one we had seen that the solution of one dimensional wave equation, we had find it out we try to use the product method, in which we had assume that the two functions of x and t are separate. And then we find out the solution changing them to the ordinary differential equations, then satisfying the boundary conditions we find it out and we find out that many functions are satisfying, then none of those functions in general words satisfying the initial condition,. We use this fundamental theorem and said is that it is sum of all those functions because, the functions are infinite many we had summed it as infinite many.

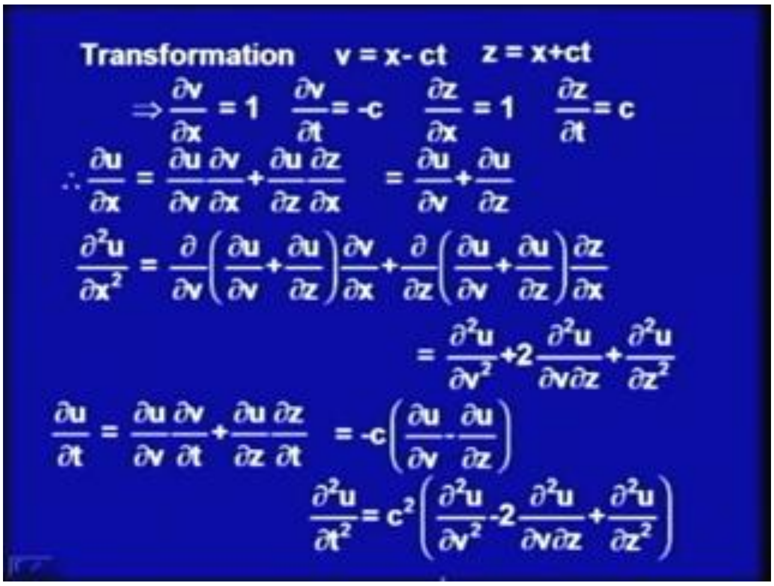
And use this Fourier series tool and find it out that we could say is it is in the term of odd extension of the initial condition function, in which we are getting this wave equation solution. Let us see, another method which we have done as using the characteristics finding the type of the equation and then finding out the solution in the terms of general of arbitrary functions and then try to find out that arbitrary function using this side conditions.

So, let us first use this wave equation and solution, the name this D' Alembert's solution of wave equation that I will explain little later. The solution of wave equation we want c square del 2 u over del x 2 minus del 2 u over del t 2, now just try to find out the characteristics and type of this equation, we have already come across this kind of equation we do know that this is hyperbolic, let us do it again to changing this canonical form.

So, a is c square b is 0 c is minus 1 thus say is b square minus 4 a c is 4 c square, which is positive because, c square is positive 4 is positive. So, it is a hyperbolic equation, it must have two characteristics, the characteristic should be given by b plus minus square root of b square minus 4 a c up on 2 a, b is 0. So, b square minus 4 a c is 4 c square, so it is square root is plus minus 2 c and a is c square, so up on 2 c square, so we have got a the characteristic is been given by the equation d t by d x is equal to plus minus 1 up on c that is we have got two equations d t by d x as 1 up on c and d t by d x is minus 1 up on c.

If I integrate we will get the two characteristic curve with plus sign when d t by d x is equal to 1 by c is by integrating we get c t is equal to x plus c 1, the other one we would get c t is equal to minus x plus c 2. now, we want to change it to the canonical form, so we will use the transformation using this characteristic curves, so the first transformation we would use is x minus c t, another transformation we will use is x plus ct. So, we are transforming the new variables v and z, rather than x and t, v is x minus ct and z is x plus c t, now change it to the canonical form this given equation. So, we require del 2 u over del x 2 in the form of del partial derivative with respect to v and z. And similarly the partial derivative with respect to t in the form of partial derivative with respect to v and z.

(Refer Slide Time: 29:09)



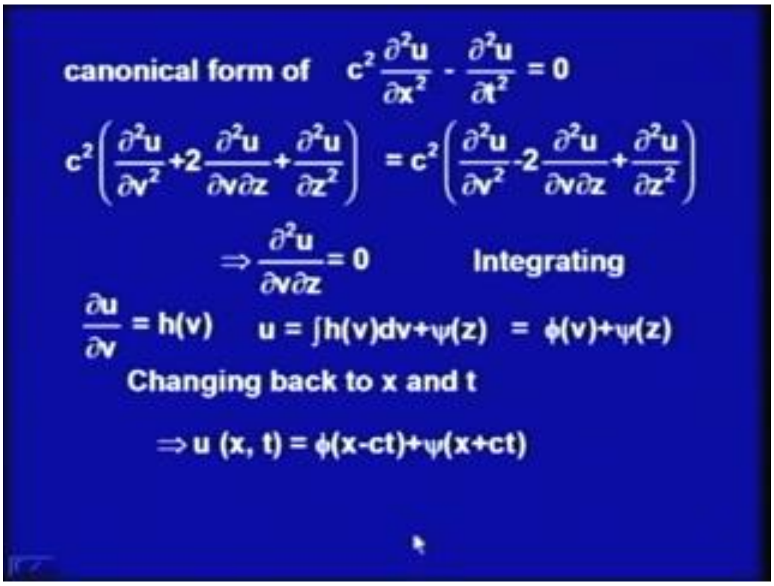
So, we want to change it to the canonical form using this transformation v as x minus c t and z as x plus c t. So, first the partial derivatives del v over del x as 1 del v over del t as minus c del z over del x as 1 and del z over del t as plus c, so using the chain rule we could get del u over del x as del u over del v times del v over del x plus del u over del z times del z over del x. Now, substitute this del v over del x and del z over del x we get it is del u over del v plus del u over del z.

So, the second derivative would be that is del 2 u over del x 2 again differentiate del u over del x with respect to x. So, again using the chain rule it is del up on del v of del u over del v plus del u over del z because, that is what del u over del x into del v over del x plus del over del z of del u over del v plus del u over del z times del z over del x, again substitute this del v over del x and del z over del x from here as one, we do get it as del 2 u over del v 2 plus 2 times del 2 u over del v del z plus del 2 u over del z 2.

We have assume that the continuity of this partial derivatives that is we had assume that del 2 u over del v del z and del 2 u over del z del v they are same. So, we are getting this one, similarly it does get del u over del t using the chain rule del u over del v plus times del v over del t plus del u over del z times del z over del t. Now, del v over del t and del z over del t we are substituting from here we get it minus c and plus c, so we get it minus c times del u over del v minus del u over del z.

So, again using this chain rule again on this one with respect to the t again because, we require the second partial derivative with respect to t, that we would get. Because, here if you see is in this case also, what we have to do is that is rather than using it x we have to put it t. So, that is giving as again c, so c square del 2 u over del v 2 minus 2 times del 2 u over del v del z plus del 2 u over del z 2, so we have got the both the derivatives with this one.

(Refer Slide Time: 31:40)



So, now our given equation c square del 2 u over del x 2 minus del 2 u over del t 2 is equal to 0, this will change to now we substitute this whatever we have got in the terms of this 1 c square del 2 u over del v 2 plus 2 times del 2 u over del v del z plus del 2 u over del z 2 minus del 2 u over del t 2. So, let us put it in the right hand side, so c square times, this is what del 2 over del t 2 del 2 u over del v 2 minus 2 times del 2 u over del v del z plus del 2 u over del z 2.

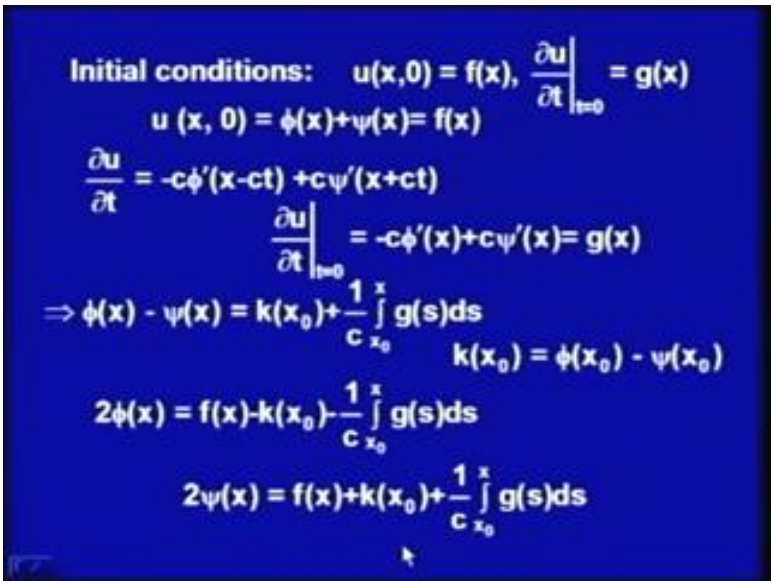
So, we get on both the sides the terms involving del 2 u over del v 2, here is also, here is also the coefficient for both c square, so this will be cancelling out. Similarly, del 2 u over del z 2 the coefficient of this c square here, the coefficient of del 2 u over del z 2 here is also plus c square. So, both these terms will also be cancelling it out, what we would be getting is here the coefficient is plus 2 c square, here the coefficient is minus 2 c square; that means, I would be adding it up.

So, what we are getting is del 2 u over del v del z is equal to 0, actually we would be getting 4 times c square 4 is non 0 constant, c square is non 0 constant. So, the final canonical form this differential equation we are getting is del 2 u over del v del z is equal to 0. Now, this partial differential equation we can solve using the integration 1 by 1, so let us go ahead with the integration first with respect to z that says I would get del up on del v as h of v because, this is a partial derivatives.

So, when we are integrating the constant integration of 0 is the constant, but because we have integrate it with respect to z. So, the constant may involve the function containing v, so we are getting is it h of v, now again integrate this function with respect to v that says is I would get integral of h v with respect to v plus constant that constant has to be can involve z because, we are integrating it this with respect to v. So, this may involve z, so we say is that plus psi of z, now this is a function of z only and here integral h v d v this is a function of v only because, this we are integrating with respect to v only, so we will get a function of v only.

Let us, write this function as phi of v, then I could write it as my solution u as phi of v plus psi of z. Now, let us come back to the original variables x and t, the transformation we had used is v as x minus c t and z as x plus c t, so changing back I would get the solution u x t as phi of x minus c t plus psi of x plus c t. This is the general solution of our one dimensional wave equation, now to find out the particular functions that is what these functions phi and psi are because this is a general arbitrary functions we will use the initial conditions.

(Refer Slide Time: 35:00)



The initial conditions given are that u x comma 0 is f x and del u over del t at t is equal to 0 is g x. So, first use this u x comma 0 that is phi of x minus c t since t is 0, so it would be phi x and psi of x plus c t since t is 0 it should become psi x, so we get phi x plus psi x should be f x. Then, we come to the second initial condition our solution u x t was phi of x minus c t plus psi of x plus c t, so if I differentiate it with respect to t first derivative of phi x minus c t.

So, here I am denoting it phi dash x minus c t that what we are doing is, we are using again the chain rule. This is the derivative with respect to x minus c t, the derivative of function phi with x minus c t and the derivative of x minus c t with respect to t is minus c. So, we are getting is minus phi dash x minus c t, similarly for psi of x plus c t we are writing this dash means the derivative of psi with respect to x plus c t and then the derivative of x plus c t with respect to t is c. So, dash whenever we are introducing it says is that the derivative of the function with that argument.

So, del u over del t we have got as this one, now we have to evaluate it at t is equal to 0, so if I am evaluating at t is equal to 0 what I would get is here from here minus c phi dash x plus c psi dash x this is given as g x. Now, we have got two equation, one is phi x plus psi x is equal to f x, another is minus c phi dash x plus c psi dash x is equal to g x from here this equation let us because, we are having is that this now derivatives, which we are talking about we are saying is with respect to that arguments.

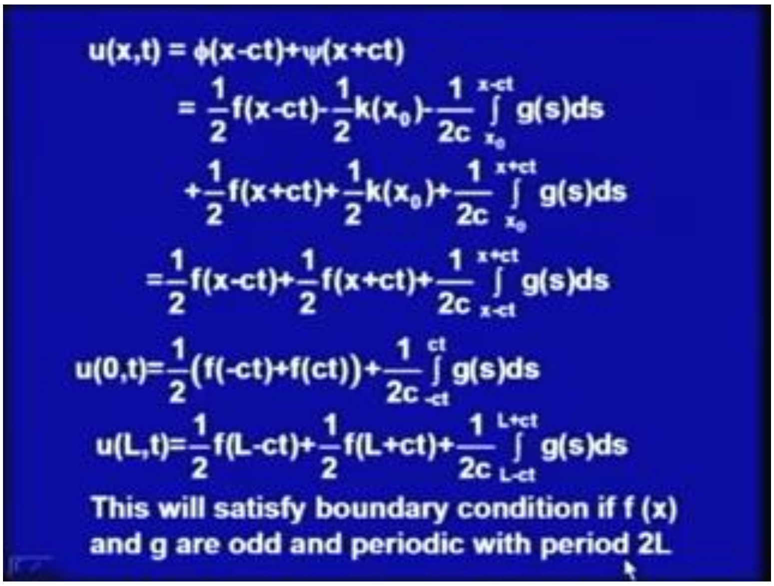
Since, t has been taken 0 this is the derivative with respect to x, this is also derivative with respect to x we are getting this a differential equation x only we can just integrate it back what we would get minus c times phi x plus c times psi x is equal to integral g x from x naught to x, why we are talking about x naught to x because, we have taken t is equal to 0. So, what we are assuming let that at t is equal to 0 x is x naught, so now just change this minus c I am taking common.

So, what we are getting is phi x minus psi x is equal to k x naught plus 1 up on c integral x naught to x g s d s, just to differentiate it that is with this x and this x I am just using that variable of integration as s. What is this k x naught, we do know whenever we would be integrating we require certain constants, and that constant is being decided with the initial condition, initial condition we are having is t is equal to 0. So, whenever t is equal to 0 I would get initially that x is equal to x naught, so in that case what we are getting is k x naught would be actually phi of x naught minus psi of x naught.

Now, we have got one equation over here, another equation from here, if I try to solve these two what we get, if we just add it up we get 2 of phi x is equal to f x minus k x naught minus 1 by c x naught to x g s d s. And if I just subtract the two ones I would get 2 of psi x is equal to f x plus k x naught plus 1 by c x naught to x g s d s. Now, that says is these arbitrary functions phi x and psi x, we have got in the terms of the initial condition, which is given as u x comma 0 as f x and initial velocity as g x.

So, the initial velocity term we are using here is in the integrant, it is been coming as integrant and integral of that function. So, now we see is in this method which we had used here to change it to the canonical form and then solve the differential equation, we had find out that the solution which we had obtained there. So, now if I just substitute this phi x and psi x in our solution let us see that is what the solution we are getting.

(Refer Slide Time: 39:48)



Our solution is u x comma t is equal to phi of x minus c t plus psi of x plus c t, so now, substitute this phi and psi, phi is half of f of x minus c t minus half of k x naught minus 1 by 2 c x naught to x minus c t g s d s, we are putting phi x in the argument x minus c t. So, everywhere this x has been change to x minus c t except that x naught because, x naught is the constant, so here this integral is also from x naught to x minus c t, then with the psi x plus half f of x plus c t plus half k of x naught plus 1 up on 2 c x naught to x plus ct g s d s.

Now, just simplify it what we would be getting is that these terms half of x minus c t half of f of x plus c t. Now, what we are saying is x naught is initial point that is we have started at t is equal to 0 x is x naught c is positive t is positive, so minus c t would be negative what it says is that this integral is actually the x naught 2 x minus c t; that means, it is in the negative side.

So, we can write using the properties of definite integral, we can write it as plus integral from x minus c t to x naught and this integral is from x naught to x plus c t. So, both these integrals being added up, again using the properties of definite integral we get x minus c t to x plus c t this integral of g s d s. Now, you see this solution is similar to that whatever we have got the solution of the entire problem in the previous section, where we had use the Fourier series, we had find out the solution using the product method, it was a lengthy process here we have got very neat and clean solution.

But, the thing is that this solution or this method of solution would be applicable when the function or when the equation is of such a form. Where, I could change it to the very simple canonical form and we can integrate it out and we are getting is the exact solution, this solution is called the D' Alembert's solution of the wave equation. Does it satisfy the initial conditions, the boundary conditions initial conditions from the initial conditions we have got does it satisfy the boundary condition.

Let us see u 0 t from here that is x is equal to 0 what I would be getting is f of minus c t and this would be f of plus c t and this integral would be minus c t to plus c t g s d s. It should satisfy the boundary condition that says is I should want it to be equal to 0, this is equal to 0. Similarly, u L t half of f of L minus c t plus half of f of L plus c t plus 1 up on 2 c L minus c t to L plus c t g s d s boundary condition says that u 0 t and u L t both should be 0.

Now, let see how this could be 0, what should be this my function f this could be 0 if f of minus c t is same as minus of f of c t that says is f is odd function. Moreover, this integral would be 0 if my g is an odd function, then we do know that integral minus a to plus a g x d x is 0 if g x is odd. So, the first boundary condition gives me that is I should have my f and g both as odd function, then let us come to this one, this would be 0, so again if I just try to use the a similar explanation I want that L minus c t f of L minus c t should be same as minus of f of L plus c t.

And again this integral should be 0 or if I combine this explanation of the first one with the second one what we are saying is, we want that this function whatever we have got that is odd function from a minus L to plus L. And then it should repeat because, after L minus c t that is 0 to L whatever we had, we want that it should be odd and then we want that it should repeat that is L minus c t whatever we are getting I should it negative of that in the l plus c t.

So, what we want is that my function should be odd and periodic with period 2 L both the functions f and g. So, what we are saying is this D' Alembert's solution as such it is very easy to find it out, but it is applicable in special conditions, where I got this Fourier series solution, which we had find it out that was applicable everywhere, whatever be the function we could find it out while as here we require that f must be an odd periodic function of that the period 2 L.

So, that is what we are saying is this will satisfy the boundary conditions, if f x and g x are odd and periodic with period 2 L. This is what we have got the solution of second order one dimensional wave equation we had learn, so now I just summarize that is what we had learn in the partial differential equation, we had learn that is how to solve the first order partial differential equation, how to characterize them using the method of characteristics, we had find out that is the solution.

In all the partial differential equations we had find out that the solution is coming in the form of arbitrary functions, rather than arbitrary constants. And those arbitrary functions must be decided by the side conditions, those side conditions could be initial conditions or the boundary conditions as we had learn in the ordinary differential equations, here we had learn that it may be a combination of both initial and boundary conditions.

So, we have got one more kind of problem that we have called initial boundary value problem in partial differential equations. The second order differential equations we had again learn the type of equation, using those characteristics we are changing them to the canonical form, we had learn that they are three types which are very important in theory of partial differential equations, they were hyperbolic, parabolic and elliptic.

We have seen certain examples, we have done one simple examples of parabolic and hyperbolic equations. We have try to model one physical problem, which is turned out to be that governing equation is a one dimensional wave equation, which is hyperbolic we try to find out the solution of the complete problem that is initial boundary value problem. We try to find out with using the one method, which is called the product method and then we say is find it out that it was not satisfying the initial conditions and we had gone to the method of Fourier series.

And there we had find out the complete solution, then we had done for the one dimensional wave equation another solution that we called D' Alembert's solution that was actually changing the equation to the canonical form and finding out the solution, there we had learn that it has little bit limitations than the Fourier series method, so that is all in today's lecture.

Thank you.