Mathematics- III
Prof. Tanuja Srivastava
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture - 19

Second Order Partial Differential Equations - II

Welcome to the lecture series on differential equations for undergraduate students. Today's lecture is on the solution of Second Order Partial Differential Equations. In last lecture, we had learnt about the second order partial differential equations, we have classified those equations and change them to the normal form. Now, we will go for the solution of second order partial differential equations, first we will stick to the linear second order partial differential equation; they are whom, which are homogeneous one. So, as in the case of second order ordinary differential equations, which are homogeneous, here also the superposition principle holds. That is the first result, which holds is that is fundamental theorem.

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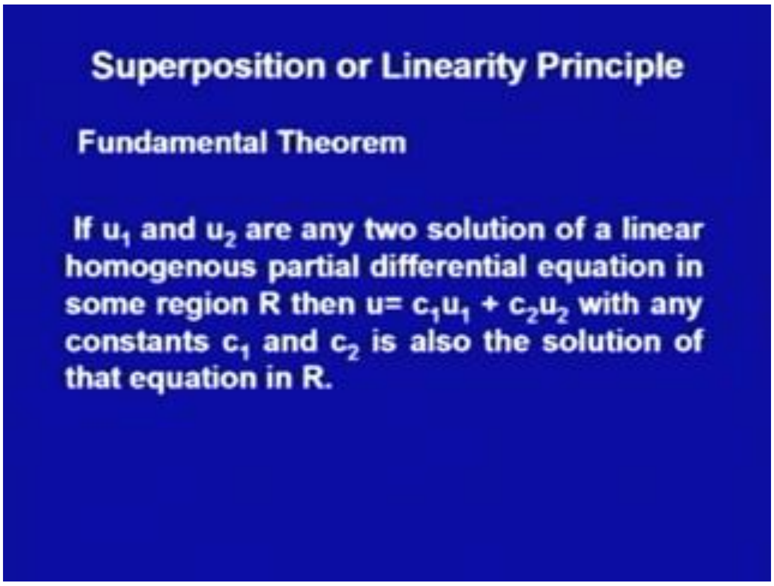

So, the result says is Superposition or Linearity Principle, the result is, if u 1 and u 2 are any two solutions of linear homogeneous partial differential equations in some reason r. Then, a linear combination u as c 1, u 1 plus c 2, u 2 with any constant c 1 and c 2 is also the solution of that equation in that reason r. For being a solution of any differential equation, we do know that the function must have the second order derivatives, which are coming in that equation and it should satisfy that equation. So, using this is the first result, which is holding true, let us just try to see certain second order partial differential equations, which can be solved as linear ordinary differential equations before that; let us come to the proof of this fundamental theorem.

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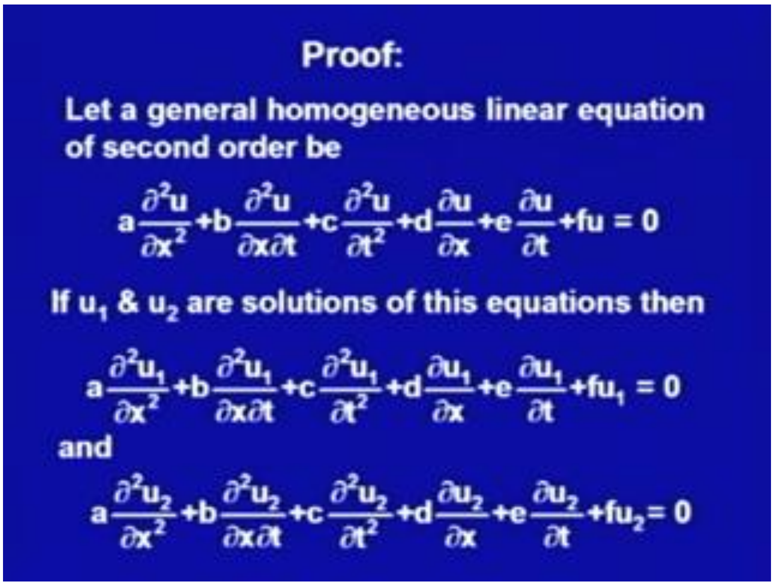

This is a very simple one, let a general homogeneous linear equation of second order be a times del 2 u over del x 2, plus b times del 2 u over del x del t, plus c times del 2 u over del t 2, plus d times del u over del, plus e times del u over del t, plus f u is equal to 0.

So, we have taken completely linear second order partial differential equation, if u 1 and u 2 are two solutions of this equation that says they must satisfy this equation. That says is, I should get a times del 2 u over del x, plus b times del 2 u del x over del t, plus c times del 2 u over del t 2, plus del 2 u, d times del u 1 over del x, plus e times del u over del t plus f u 1 is equal to 0. Similarly, for u 2 must also satisfy the equation, now we have to show its linear combination c times u 1 plus, c 2 times u 2 is also a solution.

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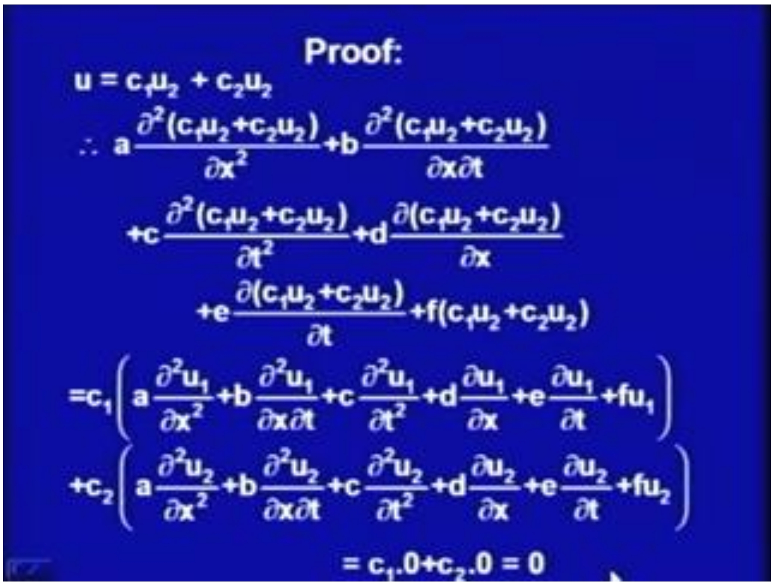

Now, I am taking u to be as c 1, u 1 plus c 2, u 2 that says is, if I put this function in our differential equation, we would get the second derivative with respect to the variable x of c 1, u 1 plus c 2, u 2. The second derivative of with respect to x and t as c 1, u 1 plus c 2, u 2 and so on, the second derivative with respect to the t c 1, u 1 of c 2, u 2 and the first derivative with respect to x of the function c 1, u 1 plus c 2, u 2.

And, the first derivative with respect to t of c 1, u 1 plus c 2, u 2 and the f times c 1, u 1 plus c 2, u 2 this must satisfy the equation. So, their left hand side of the equation is this. Now, this we can break as, because it is linear one, we do know that the derivatives property, we can write it as c 1 times, a of a times, del 2 u 1 over del x 2, plus b times del 2 u 1 over del x del t, plus c times del 2 u 1 over del t 2, plus d times del u 1 over del x, plus e times del u 1 over del t, plus f of u 1 and c 2 times similar terms for the u 2.

So, that is a times del 2 u, u 2 over del x 2, plus b times del 2 u 2 over del x del t, plus c times del 2 u 2 over del t 2, plus d times del 2 u 2 over del x plus, e times del 2 u 2 over del t, plus f of u 2. Since u 1 and u 2 both are solution we had seen that this is equal to 0 that is this satisfying the equation, similarly this is also equal to 0. So, what we are getting is c 1 times 0, plus c 2 times 0, that is 0. This u is also a solution of this differential equation, this is what we are saying is that, if we do get two le solution of a linear second order homogeneous equation. Then, their linear combination is again a solution to that a homogeneous equation.

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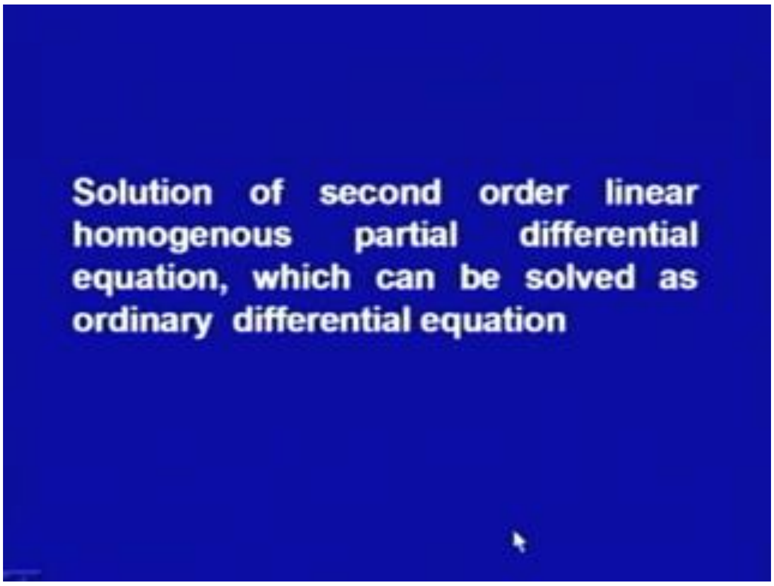

Now, let us come to the solution of second order linear homogeneous partial differential equation, which can be solved as ordinary differential equation. Let us, see what does it says, we will see it through one example.

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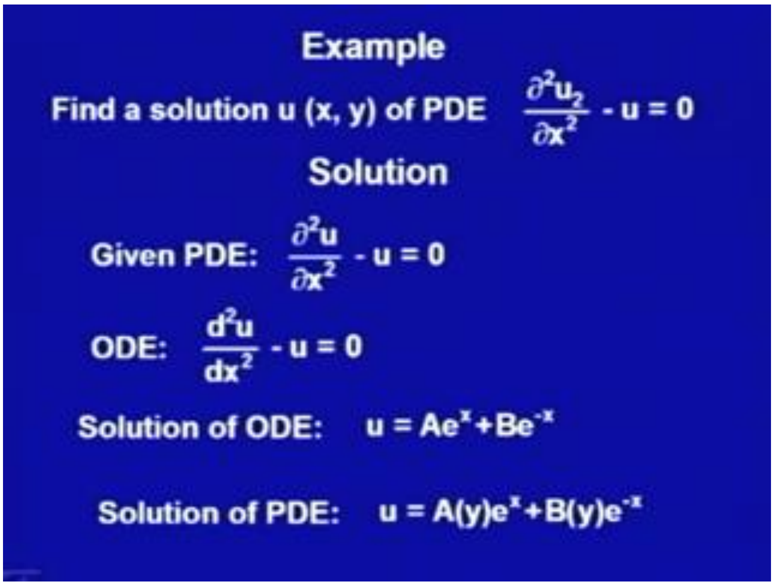

Find a solution u x y of partial differential equation del u over del x 2 minus u is equal to 0, this is not u 2, this is simply u solution. Here, this equation is involving the derivative with respect to one variable only x, we are having del 2 u over del x 2 minus u, so rather than treating it as a partial derivative, we can treat it as ordinary derivative, second order one that says is I can change it to the d 2 u over d x 2 minus u is equal to 0.

Now, this is second order linear differential equation, homogeneous one, we do know how to find out the solution using that characteristic. We are getting is latin roots as solution of this is a times, because it is lambda square minus 1 is equal to 0, that is lambda is equal to plus minus 1, we would get the general solution of the form a times e to the power x, plus b times e to the power minus x.

This is solution of this ordinary differential equation, here what we are assuming that u is function of x only. But, actually what we have been given we have been given a partial differential equation, which says it is a function of two variables x and y, that says I should get my u as a function of x and y. In ordinary differential equation, we have treated this a and b as general constants.

Now, what I could do it here is, because they are the partial derivatives, I could say is that is a and b are some functions of y, that is we could say the solution of this partial differential equation as a of y, e to the power x plus b y, e to the power minus x. That is rather than having this as a general constants, we are getting actually the solution in the form of two arbitrary functions, where these functions are function of y only, we are not involving any term of x over here. So, we have the solution of this partial differential equation general solution, where we are getting is this arbitrary functions are involved, let us do one more example.

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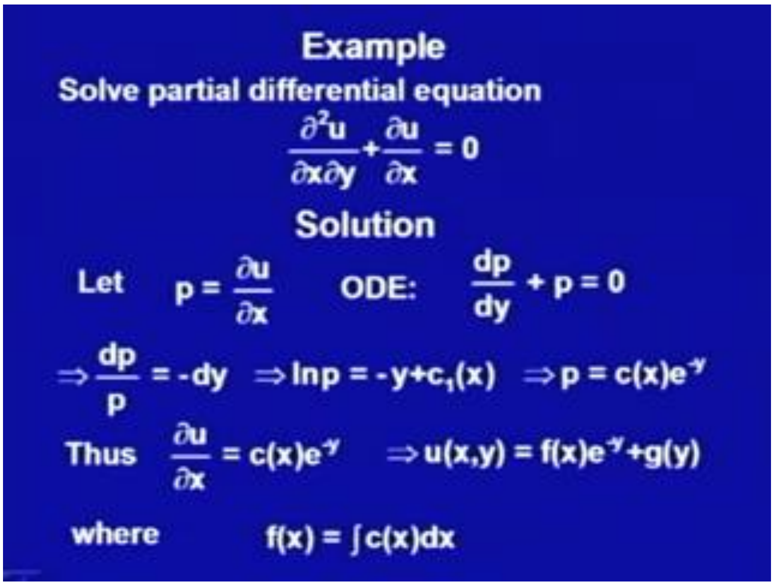

Solve partial differential equation del 2 u over del x del y, plus del u over del x is equal to 0. Now, let us see in this equation we are having here the derivative with respect to both the variables and here is a derivative with respect to the one variable. Now, if I treat this del u over del x as some function of x and y, then this del 2 u over del x del y, that I could write it as we are assuming is that interchange of the variable is possible in the derivatives, so I could write this as del by del y of del u over del x.

That is, let if I assume p as del u over del x, then this differential equation, I can change to the ordinary differential equation as the derivative of p with respect to y, plus p is equal to 0. Again, this is first order linear differential equation to solve it with using the method of variables separable, we could get that d p by p is equal to minus d y, integrate on both the sides, we get that log p is equal to minus y plus c 1.

Now, this constant as in the previous example, we have taken because here, what we are taking it this is the derivative with respect to y. So, whatever integrating we are getting is the constant, that now we have to assume that a function of x, so we are assuming it as c 1 of f. Now, take the both the sides the logarithmic of, so we get p is equal to c x times e to the power minus y, this is the solution of this differential equation.

Now, what is p, p is del u over del x, now since again it is involving the derivative with respect to one variable only, I can treat it as again as a single as ordinary differential equation with respect to one variable. The only thing is, when I would integrate it out, I will take the constant should be a function of the other variable y. So, this says is this is del u over del x is equal to c x, e to the power minus y, integrate it with respect to x on both the sides, what we get is u is equal to c x, e to the power minus y.

So, e to the power minus y, that would be treated as constant, because when we are taking the partial derivative the function volume y is only y would be constant. So, this is f x times e to the power minus y and the constant, we are treating as g of y, because we are integrating with respect to x, what is this f x, actually in this one, we are having is this c x. So, we have to integrate this c x, so f x is nothing but the integral of c x with respect to x.

So, what we have got the solution of this given partial differential equation as a f x times e to the power minus y plus g y, where this my f and g are again any arbitrary functions. These are the differential equations, which we have solved as ordinary differential equations, because the derivatives involved, we can break it into as the derivative with respect to one variable only. And, we can treat the equations as derivative involving the one variable only and so we have treated it in this manner.

Let us, take it as I said is that a normal form is also making simplified solution, so let us just treat one more example.

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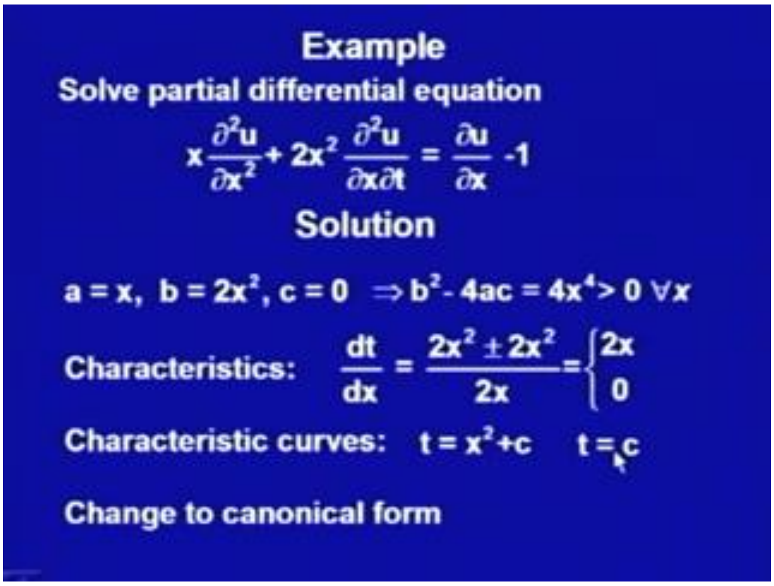

Solve the partial differential equation x times del 2 u over del x 2 plus 2 x square del 2 u over del x del t is equal to del u over del x minus 1. Now, you see, I am not having this coefficients as constant only, we are having is the coefficients or the function of x, so this is a principally linear equation actually. Now, for solving it, we will first find out the type of this equation now, we are going to solve it using the canonical form; that is we change it to the normal form.

And then, will try to solve with the method just now, we had learn that is changing the partial differential equation into ordinary differential equations. So, here the coefficients of del 2 u over del x 2 is x, so a is x, b is 2 x square and c is 0, so b square minus 4 a c would give me 4 times x to the power 4, because b square and 4 a c is 0, which would be positive for all x, because it is x to the power 4 even power. So, it will always be positive, that says this equation is hyperbolic for all x.

So, the characteristics could be obtained by the b plus square root of b square minus 4 a c, that is square root of 4 times x to the power 4, that is 2 x square and b is 2 x square. So, 2 x square plus minus 2 x square upon 2 a, that is 2 x, now when I will take this plus x, I would get 2 x plus 2 x square, that is 4 x square divided by 2 x, that is 2 x or I would get, when minus sign as 0.

So, we have got the two characteristics, one is given by d t by d x is equal to 2 x, another is given by d t by d x is equal to 0. So, we will get the two characteristic curves, one is integrating d t by d x is equal to 2 x gives t is equal to x square plus c, another d t by d x is equal to 0 integrating, it gives t is equal to c. So, now we will change it to the canonical form, for that we require the transformations based on this one.

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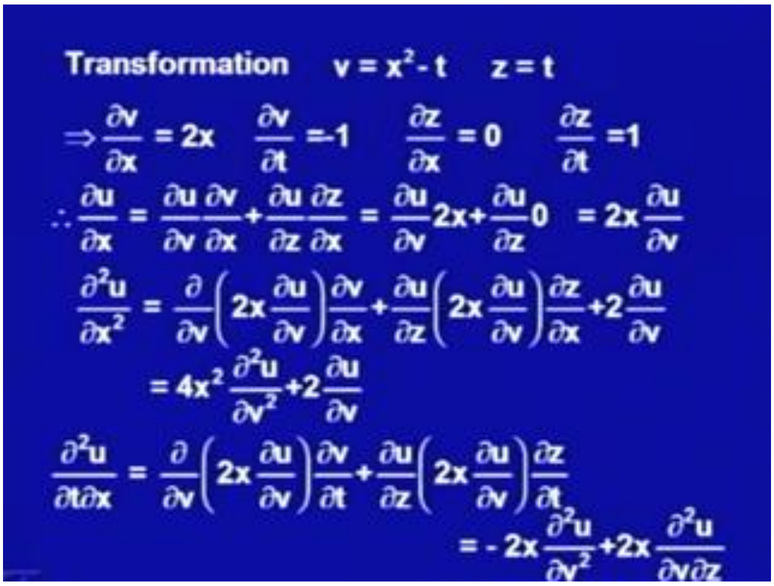

So, this transformation, we would introduce x square minus t and z is equal to t, using this transformation again will go that is del v over del x as 2 x, del v over del t as minus 1, del z by del x as 0 and del z by del t as 1. So, now find out del u over del x using the chain rule, del u over del v, del v over del x plus del u over del z, del z over del x. We would get it as del u over del v, because the derivative with respect to x of del v, this is 2 x and here it is 0, so we would get 2 x times del u over del v.

So, the second derivative del 2 u over del x 2, we have got del upon del v, 2 x of del u over del v, why this we have to integrate differentiate this with respect to x. So, we are differentiating it with respect to v and then d del v over del x, then we are differentiating it with respect to z and then del z over del x. Now, this is actually, we can treat it as a product of two functions; that is one function is 2 x another function is this one.

So, we are first taking the derivative of this function, so this function has been kept as such and then the derivative with this function with respect to x, that is 2 and this function is as such del u over del v. So, what we have got 4 x square del 2 u over del v 2 plus 2 times del u over del v, that is just this part is 0, from here we would be getting is 2 x, that is 4 x square del 2 u over del v 2 and this part is as such.

Then, we go for del 2 u over del t del x, so with respect to t, we get it differentiate it with respect to t. In the t, this 2 x is taken as constant, so will not be having that term. And when, we keep this del v over del t as minus 1 and del z over del t as 1, we do get is minus 2 x del 2 u over del v 2 plus 2 x times del 2 u over del v del z.

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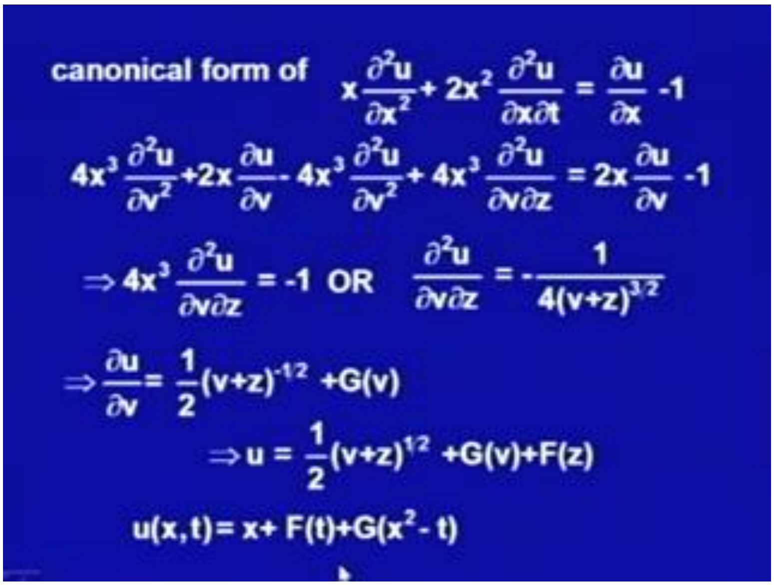

So, the canonical form of this equation, because this equation is involving the derivatives del 2 u over del x 2 and del 2 u del x del t and del u over del x. So, I am just putting all these values over here, del 2 u over del x 2 multiplied with x, so we are getting 4 x cube del 2 u over del v 2 plus 2 x over del u over del v minus 4 x cube del, then we get 2 x square times del 2 u over del x del t.

So, again minus 4 x cube times del 2 u over del v 2 plus 4 x cube times del 2 u over del v del z del u over del x, we have got 2 x times del u over del v and constant minus 1 as such. Now, simplify this one, we say is that is derivative with respect to v, the second derivative the coefficient here is 4 x cube, the coefficient here is minus 4 x cube, this is been cancelled it out.

The terms involving del u over del v, that is the first derivative with respect to v, we see here is plus 2 x, here is the right hand side is also plus 2 x. So, we get it, this is also cancelled it out, the term which is being is left is this one, the second derivative with respect to both the variables v and z. So, we have got 4 x cube del u over del 2 u over del v del z is equal to minus 1, now we are not talking in the terms of x and t, the transformation we have to use for v and z, so we have to write x in the terms of v and z.

So, if you do write it out we do get is del 2 u over del v, del z is equal to minus 1 upon 4 times v plus z to the power 3 by 2. Now, you see, I am getting this left hand side as the second derivative with respect to v and z and this right hand side is not involving any derivative, we are just getting the function, which is involving v and z,

Again, we will try to solve it, using this changing it to the ordinary differential equation, what we will do is, first we will try it say del u over del z as some function p and then integrate with respect to v and then go for with the z or other way manner. So, let us see first, I have taken this integrate it with respect to z, that says I would get del u del v as half of v plus z to the power minus half plus a function of v. Because, we are integrate it with respect to z. So, the constant has to be a function of v.

Now, again integrate it with respect to v, what I would get, u as integration of this with respect to v half of v plus z to the power half plus integration of this with respect to v. So, rather than calling it lets call this is small G v, then this would be capital G v, because this is not the same function, this would be integral of this function, what you are getting here, so let us make it as small g.

So, then this is capital G v plus a constant with respect to v, that is a function of z, so now, what we have got, we have got the solution in the form of v and z, my original equation was in the form of x and t. So, let us transform back, these variables v and z in the terms of x and t, what we would get, we do get u x t as v you have got as x square plus t something like that one.

So, we do get is that half of v plus z to the power half, that is x plus the function z is t, so f of t and v is x square minus t, so g of x square minus t. That is, I have got the solution u x t as x plus a function of t plus a function of x square minus t, if you do remember this is, what we have got our characteristic one characteristic. We have got as x square minus t at t is equal to x square plus c and another characteristic, we have got t is equal to c.

So, it is similar to that, what we are getting in the first order equations, that is in the terms of characteristic, we are getting is that argument of the functions and solutions, we are getting in the arbitrary functions. So, till now, we had done the three examples, first two examples, we have done simply that is some equations, which were of the nice form and we could directly integrate them with respect to one variable or the other variable.

Then, we have since a little bit general equation, which we said is we transform it to the canonical form, canonical form says is we have got our left hand side, very simpler one, which is involving only or which can be checked as a with a homogeneous one. That is with where we could treat it as the functions with v and z separately and then we integrate it out.

But, in all these examples, we had seen, that we are getting solutions in the terms of arbitrary functions. So, how to determine this arbitrary functions, in the ordinary differential equations, we have got the solution in the terms of arbitrary constants and those arbitrary constants, we have find it out using the side conditions, that is either initial conditions or the boundary conditions.

So, here also for getting this form of this f and g that is getting these arbitrary functions, we require the side conditions. So, we get the solution of this unknown functions f and the general function f and g, if we use the side conditions, in ordinary differential equations. Also, we had found it out that those side conditions, which we called either initial conditions or boundary conditions, they are coming from the physical problems.

So, here also there should come from the physical problems, so before getting that what kind of this side conditions we should have let us try to formulate a physical problem in the form of solving a partial differential equation.

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So, we would be modeling some physical problem.

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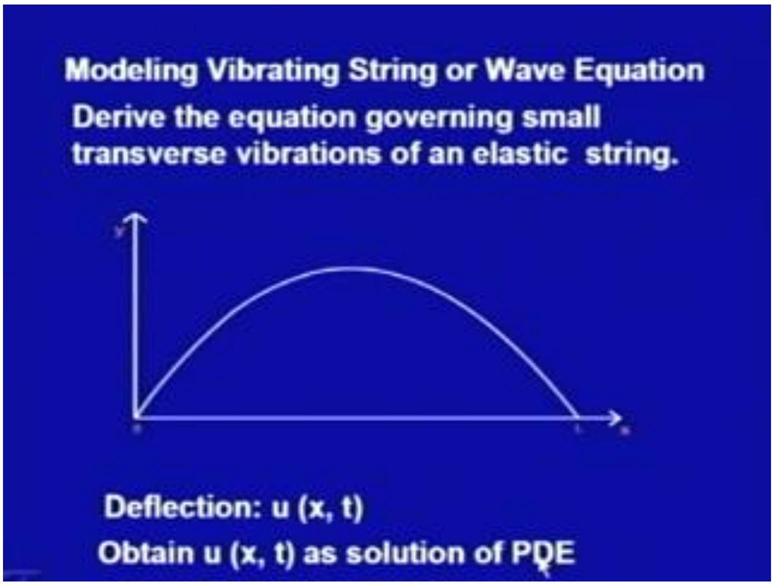

Here in this example, I am taking the modeling as vibrating string or we are here, what we get is modeling of a wave equation. So, what we want derive the equation governing a small transverse vibrations of an elastic string, what does it mean, it simply says is that suppose, I do have an elastic string, which is being fixed at the two ends. That is, I do have one fixed is that is we are taking as the initial point and then another end is been fixed at l.

Now, this is the perfectly elastic string, that says is, if I deform it, it will come back to the original formation with certain vibrations. So, we take this elastic string and we want to find it out, that is, if I deflect it out or pull it out some time, then it will give the vibrations and we want at a particular time t. So, what we are saying is this is the my variable x, that is the vibration would be actually in this form and the deflection would be this form.

So, we want the solution x t that is at a particular time t at a particular point x, what will be the deflection, that is what u x t we would be interested to find it out, if we had deflected it of deformed it, so we are interested to find out the deflection in this elastic string. Now, to model it as we know that we require little bit more assumptions, so that we could model first a simple equation, which we could solve and will go for the further ones.

So, let us for simplifying it or for modeling it, we require certain assumptions, what are those assumptions, let us see the physical assumptions, which we do require.

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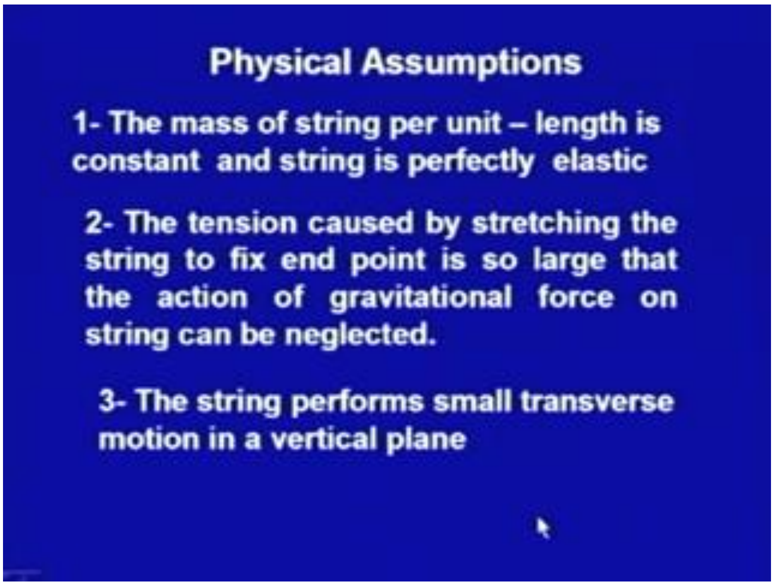

First is that the mass of string per unit length is constant and the string is perfectly elastic, what we are assuming is that the string is homogeneous and it is perfectly elastic means is, it is coming back to the original position. The second is the tension caused by the stretching the string to fix end point is so large, that the action of the gravitational force on a string can be neglected.

That is what we are saying is, that is the tension which we have made it; that is because we had fix it up to the other end. So, the tension which is been created that is, such a large that is the gravitational force, which is pulling it down, that is negligible; that means, this is totally governed by the tension only.

The third assumption, we are making is the string performs small transverse motion in a vertical plane only, that is the when we are deflecting it and then we are leaving it, the motion which is been occurred, that is only in the vertical plane. That is, this where the string is moving, it is none of the time, this horizontal motion is been done in the vibrating string.

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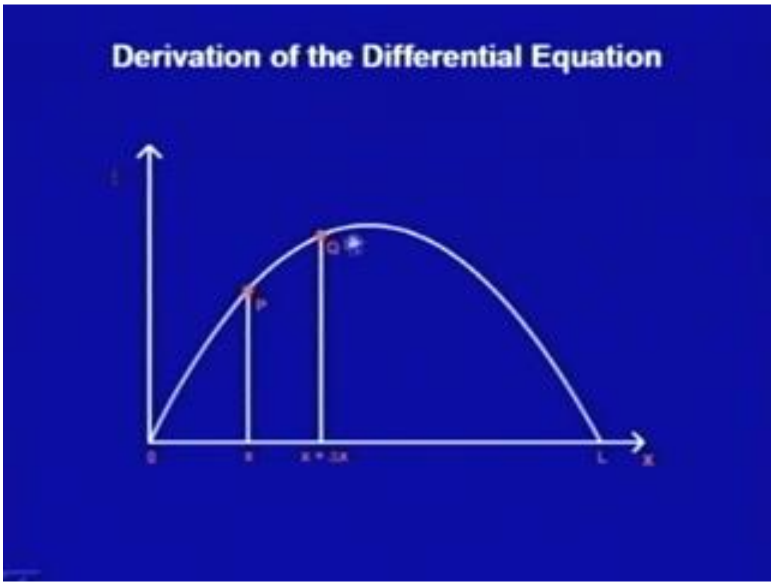

Let us, see that is how to derive this differential equation for this, so let us come back again to our model, this is the string which we have done, let us take a small portion on this string this p q. So, correspondingly the points which we are having on the x axis is your x and x plus delta x, at this point, what we are having is that is because, it is governed by the tension only.

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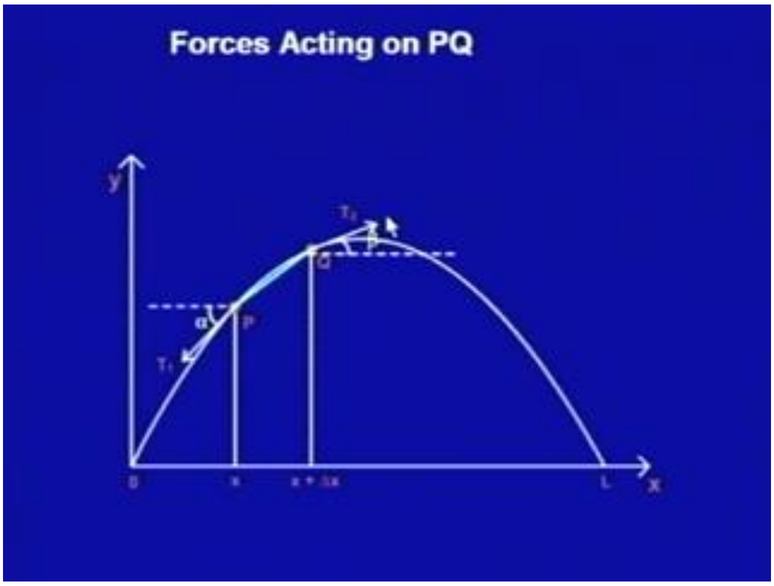

So, we first find out what are the forces, which are acting on this p and q, the forces which would be acting on this p and q would be the tensions.

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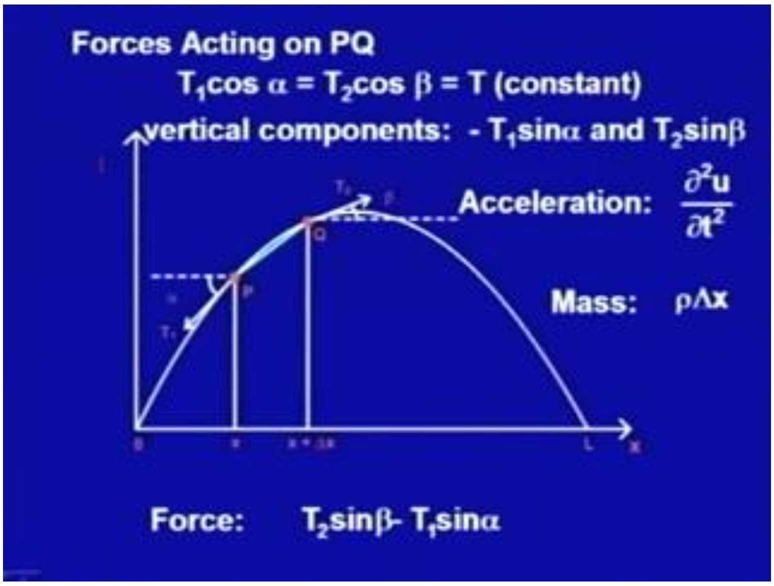

They, would be in this form and what we do have is, that is the tension would be totally because the motion is vertical, the tension would be on they would be tangential to the curve of this string. That is at this p this tension would be t 1 and at q the tension would be t 2 and the now we just see is that is what will be its factor. So, we get is in the horizontal one, that is what we are assuming is with the horizontal line, this tension t 1 that is this tangent is making the angle beta, at the q and angle alpha at p.

What we are assuming that horizontal, you part of this tension, this would be creating you this should be the constant, because we are seeing is that, there is no horizontal motion. So, it should be constant, what it says is that this side this motion is this force is t cos alpha here, this force is t cos beta that should be a constant, so let us say t.

Moreover, the vertical component of this t 1 and t 2 for t 1 vertical component would be minus t 1 sin alpha, why minus sign, because you see is that, this tension it towards the downward side. And, for this the vertical one would be t 2 sin beta, these are the forces which are acting at these two points p and q on this string, what would be the acceleration at which it is moving is that the acceleration. We do it, should be because, we are having only with the vertical planes; that means, my velocity and acceleration both are with respect to t only.

So, we do have acceleration that is the second derivative of u with respect to t that is del 2 u over del t 2, moreover we had assume that the mass is homogeneous. So, let us say if that what is the string here we do have is let us just call it delta x times, so the mass is, if constant mass rho, so mass would be rho times delta x. So, now total force is which are acting on this string would be t 2 sin beta minus t 1 cos alpha, that is the vertical ones t 2 sin beta minus t 1 sin alpha.

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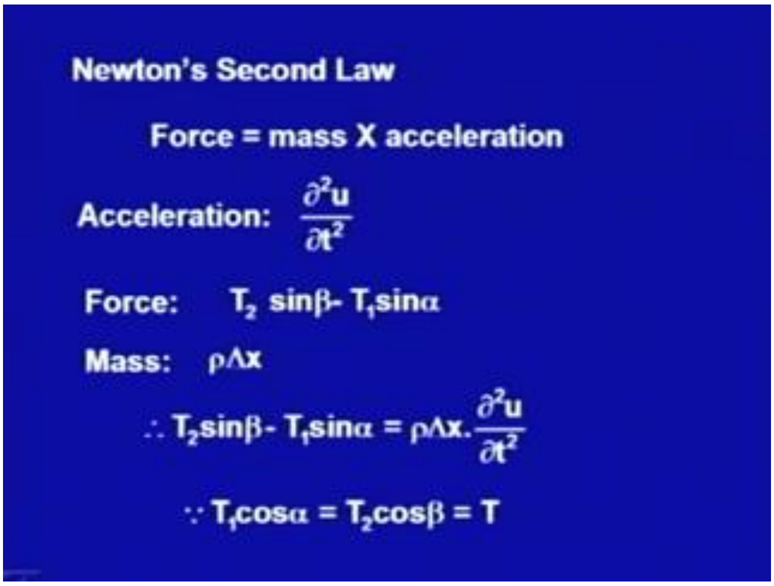

Now, what we would use is, we just model it using the Newton second law, which says is that force is mass into acceleration. Just now, what we had find it out, the acceleration we had find it out, that it should be del 2 u over del t 2, force we had find it out, that is it is t 2 sin beta minus t 1 sin alpha acceleration and the mass, we had find it out as rho del x.

Now, if I keep it over here, what we are getting is t 2 sin beta minus t 1 sin alpha as rho times del x times del 2 u over del t 2. Now, divide this equation by a constant, so the constant which I am going to take that is the horizontal component of the tangent t 1 and t 2, that we have taken as that to be the constant t. So, we just divide it by the t, this is what we have taken, so divide it by this t, what we do get.

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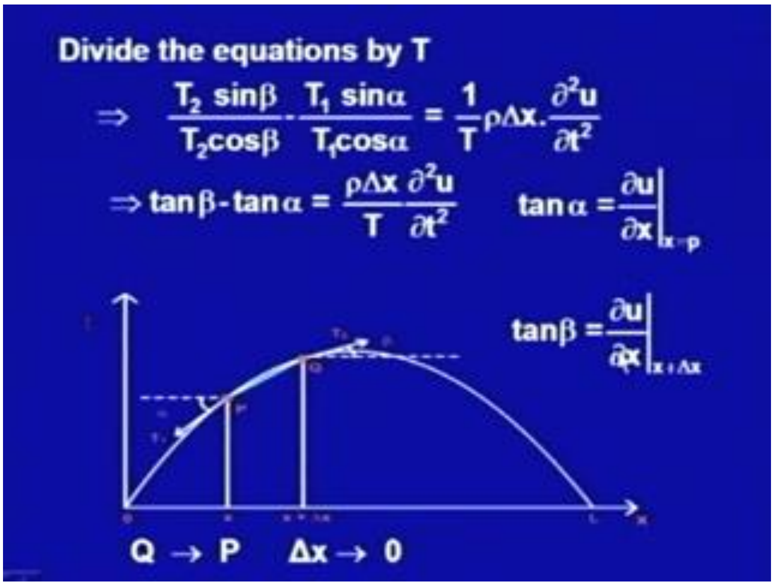

T 2 sin beta over t minus t 1 sin beta over t is equal to 1 by t times rho delta x del 2 u over del t 2. Since, c 1 cos alpha and t 2 sin beta, they are same as equal to the t, so t here I am replacing by t 2 cos beta, at this place t, I am replacing by t 1 cos alpha, what it says is, this would give me tan beta, this will give me tan alpha. And what, we are getting is tan beta minus tan alpha is equal to rho de del x upon t times del 2 u over del t 2.

Now, what is this tan alpha and tan beta; let us again come to our figure this alpha is the angle, which this t 1 is making with horizontal 1. So, what would be tan alpha, tan alpha would be actually and this tangent is actually tan tangent at the point p, what it says is that tan alpha would be nothing but the slope of this string. At the point p, similarly tan beta is nothing but the slope of this string at the point q.

Now, slope you do remember that is we are finding it out, because this string we are seeing is with respect to the variable t only, that is we would be getting is the slope is with respect to x. So, del u over del x, if it is tan alpha is del u over del x at x is equal to p and tan beta is del u over del x at x plus delta x or at x is equal to at point q. That says is, now it will transform this equation as del u over del x at p and del u over del x at this is del u over del x at x plus delta x this is del u over del x at x.

Now, we will take here in this one itself, that as q is approaching to the p that is, if I take this q is approaching to p this delta x will approach to 0. Now, let us see that, this is an important part, which the next slide, I would be using, so what we have got this equation now, if I write these in this form.

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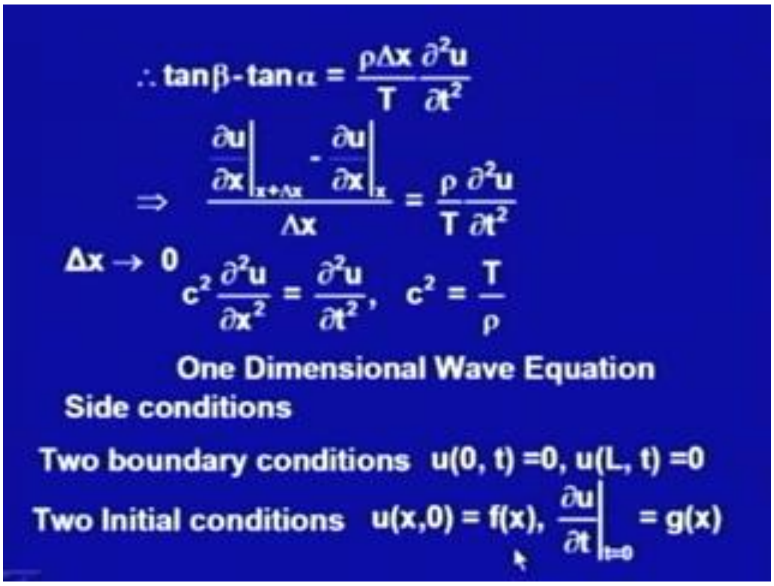

That is tan al beta minus tan alpha is equal to rho del x over t times del 2 u over del t 2, this can be written as del u over del x at x plus delta x minus del u over del x at x divided by delta x, this delta x, I have taken this side, here it is rho by t times del 2 u over del t 2. Now, as delta x is approaching to the 0, what we do get this is nothing but the second derivative of or this is the derivative of del u over del x with respect to x again, that is the second partial derivative with respect to x.

So, what we would be transforming del 2 u over del x 2 is equal to rho by t times del 2 u over del t 2. Now, this rho is the mass homogeneous mass and t is that constant tangent along the horizontal one; that also has to be positive, actually this both rho and t has to be positive. So, if I include this rho by t on this side that says is t by rho, this has to be this constant has to be positive, that make it sure that this is positive.

We are introducing here the constant c square, because this is homogeneous mass and this is your constant tangent, so we are getting it a constant, but that constant has to be positive. So, make it sure that it is, always positive we are introducing this constant as c square, so now what we have got the differential equation which is governing the deflection at a time t with the point x.

At that point x is been governed by this differential equation, which is partial differential equation c square del 2 u over del x 2 is equal to del 2 u over del t 2, this equation we will recognize this is nothing but our one dimensional wave equation. Now, we will learn it, how to solve this equation, but before solving, we do know that till now, whatever solutions, we have got.

We have got all them in the term of arbitrary function and to determine a unique solution, we require side conditions and those side conditions has to come from this physical problem only. So, let us first find out what are those side conditions, which we could get in this physical problem.

First thing is that, we have took this string is been attached to the two end points, let 0 and l, that says is we must get two boundary condition, that is whatever be this function u x t, that should be here, because it is been fixed up at these two points. So, it should be 0 over there that says is two boundary conditions we are getting, that u at x is equal to 0. For any time t, u 0 t must be 0 and u L t must be 0, that is the deflection or those two points must be always 0.

Then, since the deflection is going to depend upon, what is the initial velocity of the one and since my time t is starting from any 0 point that is, we could say is initial conditions. So, we will get actually two initial conditions also for this problem, what are those initial conditions, one is that, what is the originally at time t is equal to 0, what is the form, what is the shape of the string.

Because, my shape of the initial time, that is what is the shape of the string the deflection will depend upon that, so initially what is the shape of this string. Let us, say this is the form of f x and how it would accelerate depending upon the initial velocity. Velocity means is the derivative with respect to t, at initial point t is equal to 0, let us say this is g x. So, what we have got? We want to find out the deflection u x t in the vibrating string, at any time t, at any point x, which is been governed by this.

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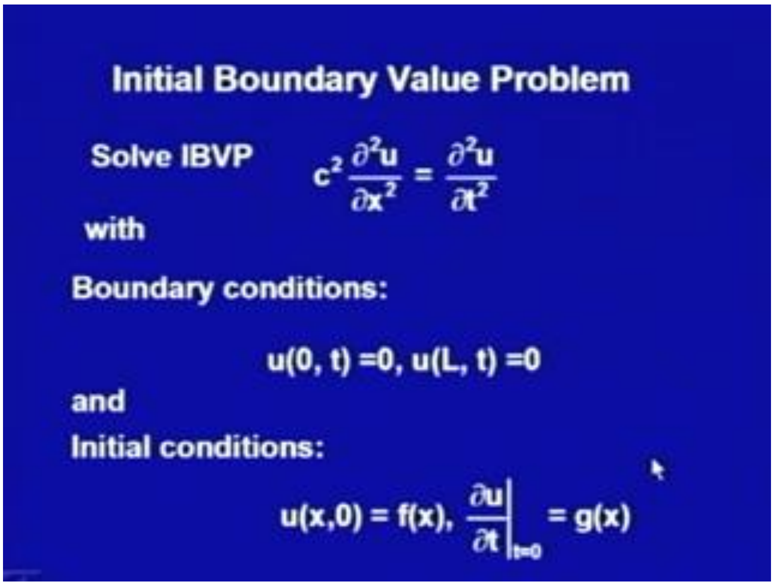

Initial boundary value problem, that is we have got, now to solve this initial value problem, which is involving the second order partial differential equations c square del 2 u over del x 2 is equal to del 2 u over del t 2 with boundary conditions u 0 t is equal to 0, u l at t is equal to 0. And 2 initial conditions, that u x at 0 is f x and del u over del t at t is equal to 0 is g x. Let us see that is to find out the solution of a physical problem, we have to solve this initial boundary value problem, let us see how to solve this.

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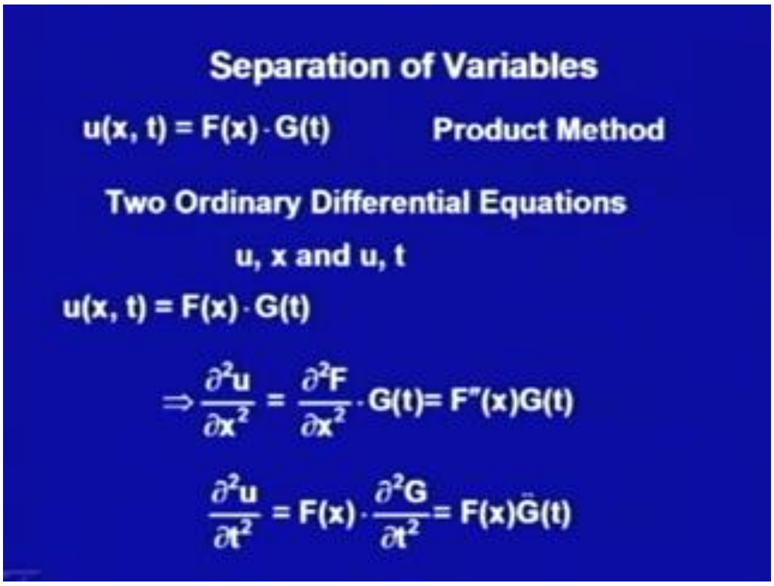

First we would use one method, which is we are calling separation of variables, here what we are meaning is, we are saying is that the function u, which is of the two variables x and t, that can be separated as a function of x and a function of t. So, we are also calling this is a product method, because what we are assuming is that u x t is a product of two functions, one function involving only x, another function involving only t.

Now, if I am assuming like this one, then in this case when what we will get actually our partial differential equation will change to two ordinary differential equations. And then, we would be able to solve those two equations separately and put our side conditions over there, to get the unique solution of that problem, how we are going to do it, let see.

The two equations we will get, one is independent variable x, another is independent variable t, so u x t is f x and f x into g t. So, if I differentiate it with respect to x, we are having in our equation del 2 u over del x 2 and del 2 u over del t 2, so if I am differentiating it twice with respect to x only. Partial derivative with respect to x; that means, the differentiation would affect only the part, which is involving x and this part which is involving only t, that will remain as a constant.

So, what I would get is del 2 u over del x 2 as the derivative of second derivative of this f with respect to x. So, rather than writing it as a partial derivative this could be treated as ordinary derivative with respect to x and it is so let us denote it by f double dash x, that is derivative with respect to x second derivative with respect to x of the function f into g t. Similarly, if I take partial derivative with respect to t of this function, that will affect only this function, but not this function, this function will remain constant.

So, I would get, if affects into the second derivative of g with respect to t, so either you write it partial derivative or we simply say is the derivative with respect to t. So, let us denote it the derivative with respect to t of g as g double dot, what we are just using the different notations, just to say that this dash. When, we are using it means derivative with respect to x, when this dots, we are using it is the derivative with respect to t.

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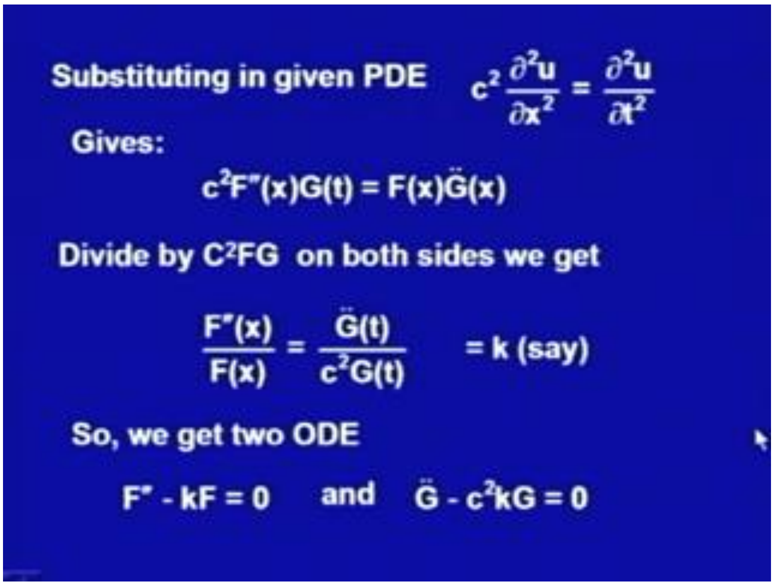

Now, if I substitute this in our given equation, c square del 2 u over del x 2 plus del 2 u over del t 2, this what we would get c square f double dash x G t is equal to f x G double dash t, it should be t. Now, since I am taking that u x t as f x into g t, that should be a solution of this equation; that means, this function f x into G t. This should satisfy this equation; that means, it must be equal, it must be equal says is that left hand side is involving this x and this is involving x and t.

Now, let us divide it by the whole equation by C square into F into G, what we are getting it here F double dash x upon x, because what we would be getting c square is cancelled it out with the c square, G t is been cancelled out with the G t. Similarly, here we would get g double dot t over c square G t, now if this is true then this must be true, this side is involving only function of x, this side is involving only function of t.

That is whatever this function be; this has to involve only x terms this has to involve only t terms, if it is satisfying this equation they must be equal, how they could be equal it simply says is that both these terms has to be a constant. So, let us say that constant to be k, now if I treat this, if take this as k, this has to be a constant k and then what I am getting is this equation is actually giving me two differential equations.

Ordinary differential equation, one equation from this first one f double dash x is equal upon F x is equal to k, which gives me f double dash minus k F is equal to 0. And, the second from here, which gives me G double dot minus c square k G is equal to 0, you see both these equations are linear. Second order linear differential equation with constant coefficients and homogenous right hand side is 0; we do know how to solve it, so we will solve them one by one.

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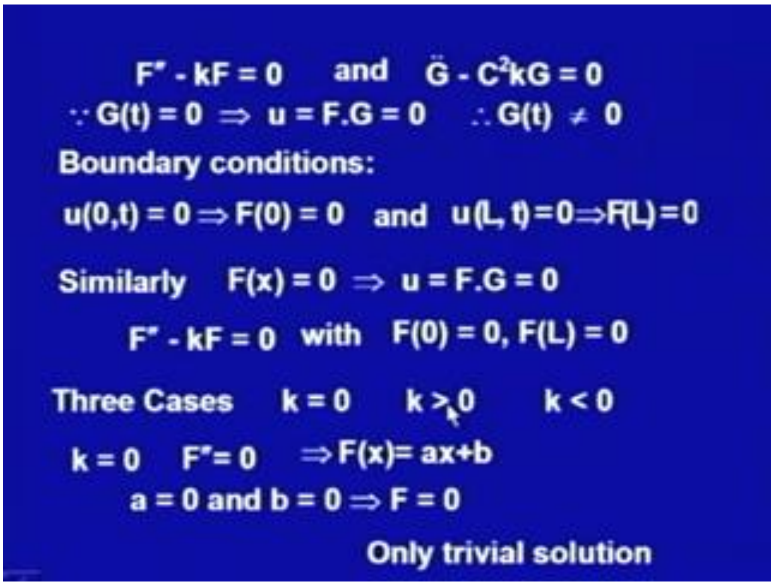

Let us take the first this equation f double dash minus k F is equal to 0, this is second order linear differential equation with constant coefficients. We do know that, it is characteristic would be lambda square minus k is equal to characteristic equation would be lambda square minus k is equal to 0. That says is before solving it, let us just move it to the side conditions that is our boundary condition, let us see.

If, I do get G t is equal to 0, then my equation u x comma t is F x into G t, that will give me F into G as 0, we do know that we require nine trivial solution. So, 0 is a solution is not required, because this 0 is solution to both these equations. So, G t should not be 0, similarly the boundary condition when I do keep u at 0 t is 0, what it says is since G t cannot be 0.

So, F x into G t is equal to 0, simply says is F at 0 must be 0, because G t cannot be 0 for any of the values, so f at 0 has to be 0. Similarly, F at L has to be 0, so the two boundary conditions, which we are having for our partial differential equation, in this method when I have taken this u x t as the product of two functions F x and G t. Those boundary conditions has transformed to the boundary conditions for the first equation, F double dash minus k F is equal to 0 with the boundary condition F at 0 is 0 and F at L is equal to 0.

So, let us now see this first problem that is boundary value problem for this equation, similarly if this boundary value problem or if I am getting any solution which says is F has to be 0, that is also not interested solution or not that is also a trivial solution, so we are not interested. Now, let us come to this one this boundary value problem, so this is second order linear differential equation, it is characteristic equation is lambda square minus k is equal to 0.

So, what we would be getting is first with this constant k, either this constant k is 0 positive or negative. Let us treat them one by one, if this constant k is 0, what will happen, I would get my equation F double dash is equal to 0. That says is, I would integrate it twice, so I would get first F dash as constant, then they b, a and then we get F as a x plus b.

Now, if I try to satisfy these boundary condition, that is at x is equal to 0, if I put F 0 is 0, that says is I would get b is equal to 0 and at L. I would get a L which is 0, that says is my a is 0, because L is not 0, that says is I will get this both the constants a and b as 0, which says is that F has to be identically 0 for all x. But this says is that, I am getting u x t as 0, which is a trivial solution and we are not interested. So, this when k is equal to 0, this boundary value problem gives only trivial solution, we do remember that we require non trivial solution.

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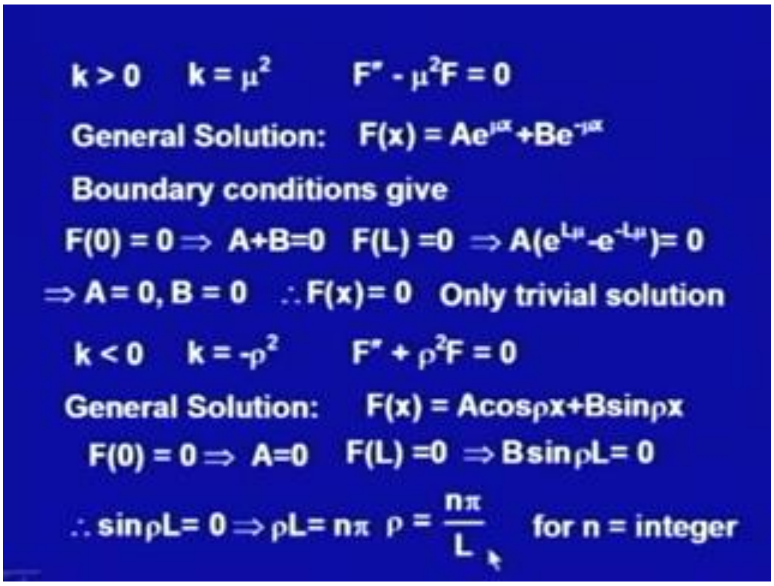

Now, let us come to the second case, when k is positive, k is positive means, we are having let say k as mu square. We are having the equation F double dash minus mu square F is equal to 0, so the general solution in this form of equation is a times e to the power mu x, plus b times e to the power minus mu x. Now, try to satisfy the boundary conditions, when at x is equal to 0, we get a plus b is equal to 0.

And f at L is equal to 0 gives me because a plus b is equal to 0, so we get b as minus a, so we would get a times e to the power L mu minus e to the power minus L mu is 0. Now, this function would be 0, if e to the power L mu is same as e to the power minus L mu; that would be same only when this both L mu is 0, L mu would be 0. If either L is 0 or mu is 0, but mu is not 0, because we have taken k is to be positive, that is k is not 0, so mu is not 0.

Similarly L is not 0, L is the other point, where we had fixed it up, that says is I would get only trivial solution, again that is a is 0, so we have got a is 0 and b is equal to 0, again we have got the trivial solution. So, this is also not of interest, let us come to the third case, when k is negative, let us assume that k is equal to minus rho square.

So, what my differential equation has transformed F double dash plus rho square F is equal to 0, we do know the solution general solution of this form of equation is a times cos rho x plus b times sin rho x. Now, we want to find out this a and b with the help this boundary condition. At 0, if I keep, I get a is equal to 0, because this is 0 at x is equal to 0, this cos 0 is 1, so is should get a is equal to 0.

Now, if I put the other condition that is F at L, I should get b sin rho L is equal to 0, now to get non trivial solution, I must get this b to be non 0. That says, I must get sin rho L is equal to 0; that means, I should get rho l as n times pi, where n is any integer positive or negative or we take that rho to be of the form of n pi over L. So, these are the only values for which I do have the solution of this boundary value problem involving the first equation F.

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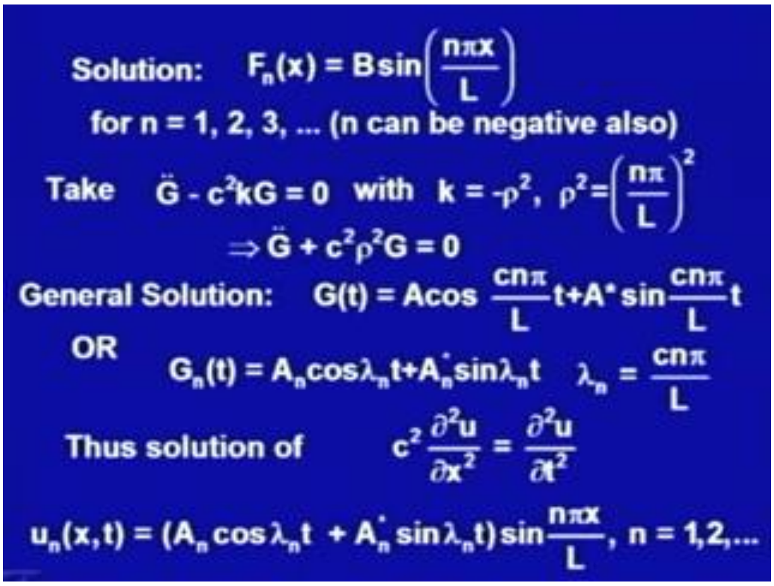

So, what we have got the solution, now this solution I am writing F n x B times sin n pi x over L, for n 1, 2, 3 and on. So, on this n can be negative the only thing is with this negative I could take this B to be negative, so that we are not distinguishing it over here, now take the other equation. Other equation says is, that the second derivative of G with respect to t minus c square k G is equal to 0.

Since, we have got that for k 0 and k positive, my f is coming at to be 0 only, so those are not the case in which I am interested or we are interested to get the solution. So, we will take the case when k is equal to negative or k is equal to minus rho square and rho is n pi over L or rho square is n pi over a L square, that is the only case in which I want this solution of this second equation.

If I keep this like this one, what it says is my equation would be G dot plus C square rho square G is equal to 0, again this is linear equation second order linear homogeneous equation. So, the solution of this general solution of this is of the form A cos c n pi over L t plus a star times sin c n pi L over t, because rho is n pi over L and c is the constant over here, so we are getting this as the solution.

Now, we require this A and A star are here arbitrary constants, let us write it again because, we are involving this n, so we are writing G and t, A n cos lambda n t plus A n sin lambda n t. So, what we have got with lambda n, I am writing as c n pi over L, so the solution, what we have got of our partial differential equation C square del 2 u over del x

2 is equal to del 2 u over del t 2 as u n x t as A n cos lambda n t plus A n star sin lambda n t times sin n pi x over L.

This was B, I have taken here as one some constant one let say, now this is one solution of our equation and this actually is varying for every n 1, 2, 3 and so on. It may be even then negative negative sign; again we have to take the B as the minus 1, so we are getting the different solutions of the partial differential equation in the very first lecture. We had seen that is partial differential equation may not have single solution; we may have more than one function as the solution of partial differential equation.

So, here we are getting is a lot many functions as the solution, where this they are varying with the n, what let us see these solutions, how they are looking like.

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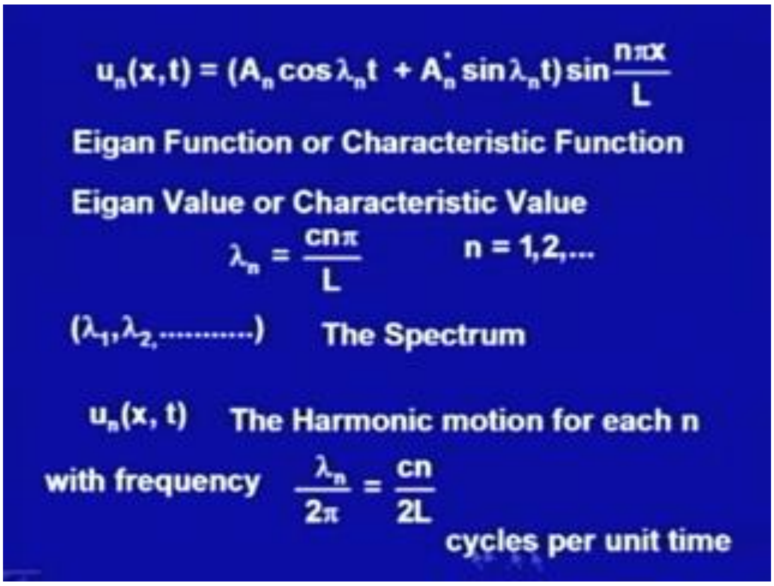

This solution this function u n x comma t, this we are calling Eigan function or characteristic function for this initial boundary value problem. Actually, we have not solved these initial values, we have not satisfied the initial conditions, just this boundary value problems. And this, lambda n this is known as the characteristic value or the Eigan value of this boundary value problem.

And this lambda 1, lambda 2, this complete set is known as the spectrum, this u n x t is called the harmonic motion for each n, let us see for fix t, that is how with it is moving. So, the harmonic motion with the frequency lambda upon 2 pi as c n upon 2 L, because lambda n is nothing but c n pi over L, so if I just keep it out, it would be coming out n times c upon 2 L.

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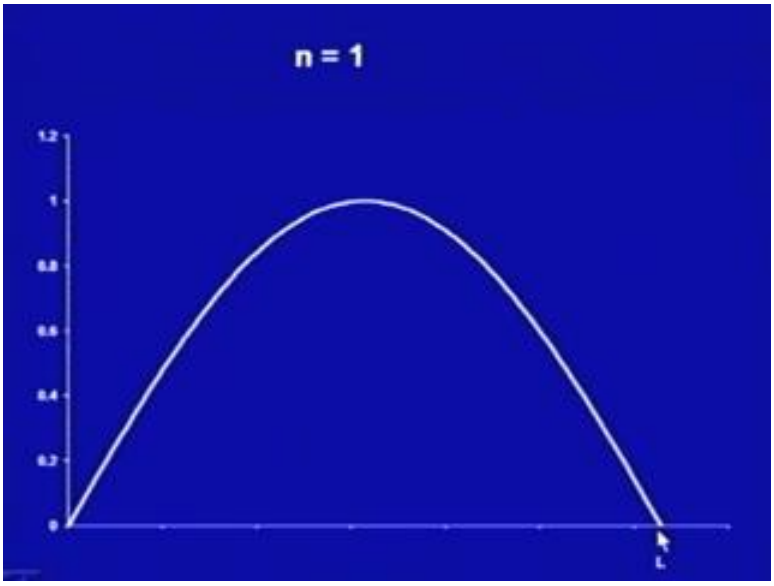

So, let us see that is how it is looking, if n is equal to 1, this is the solution u x t.

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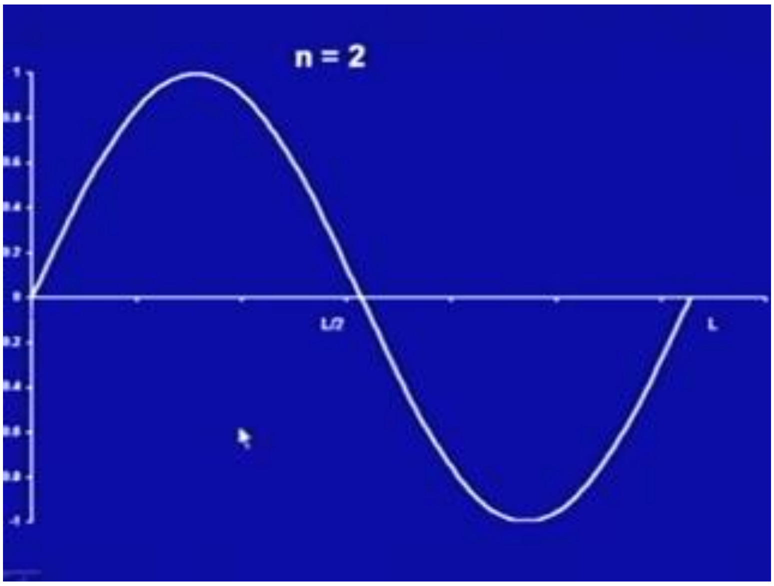

If n is equal to 2, the solution is this 1.

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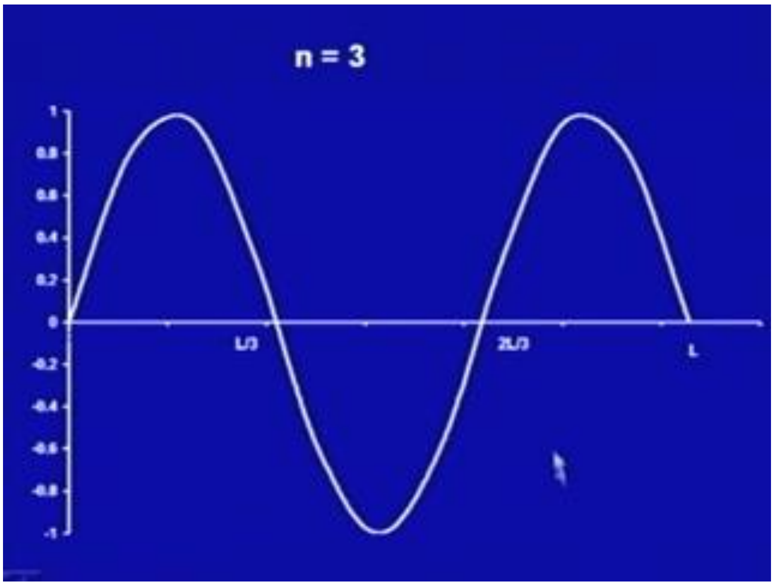

If n is equal to 3, you see that is in the first one we had only the strings a same kind of function, here the solution is like this, we are having the two points, it is going down.

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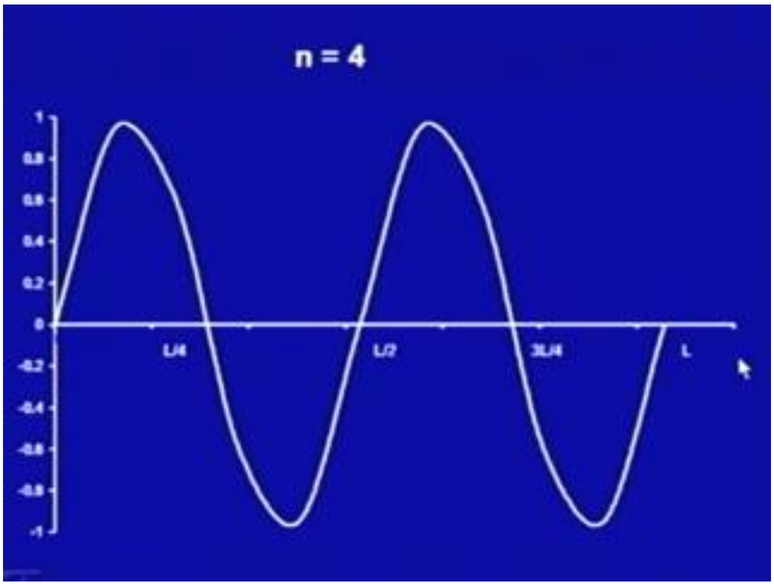

Here, it is at the 4 points, it is going down and so on.

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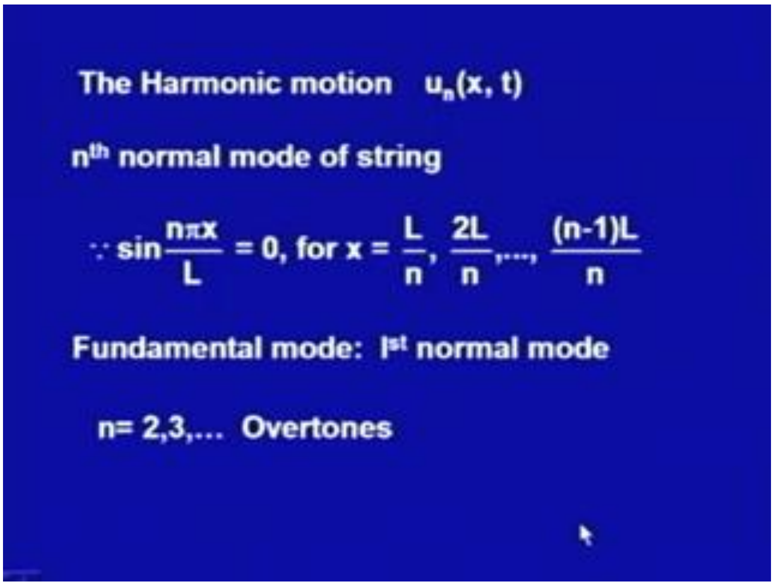

So, what we are getting this harmonic motion u n x t, we are saying is for different n values we call it nth normal mode of a string, for n is equal to 1, we do call it fundamental mode. Since, sin n pi x over L would be 0 for all x as L by n 2 L by n and so on, so we do get that nth normal mode of the string will have n minus 1 nodes, that is what we had observed that is when n is equal to 2. We have got that is, we are getting the u x t 0 at 1 point, then when it is 4, we are getting at 3 points.

So, the first normal mode is known as the normal mode, fundamental mode and other modes are been called overtones, that is how many times they are going up and down. This is vibrating string, this is been seen in the all though instruments, which are been played by string with the help of a strings and that is the term is coming as overtones and this ones. So, you could see is that this is a physical problem of that kind, but what we see is that is, if I try to see this solution u n x t.

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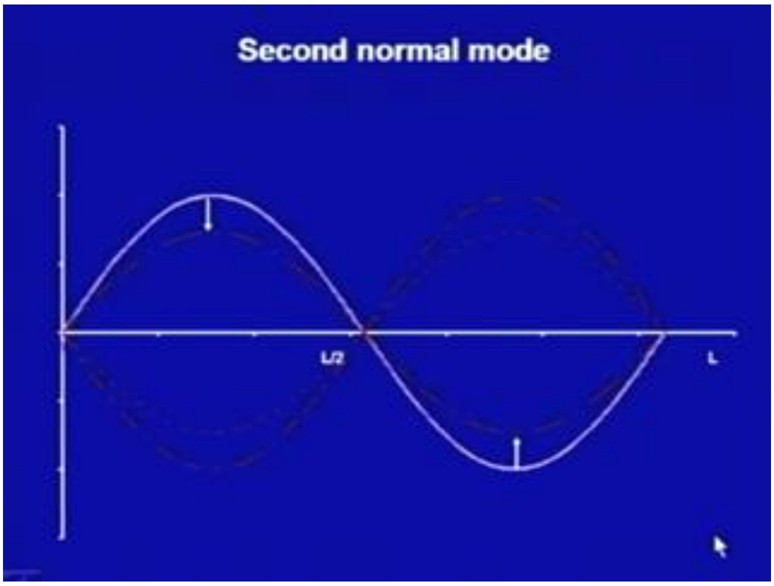

So, here is that is for different values of t, if I take the second normal mode, where I should get 1 point that is one node. We are getting is, that is as t is increasing to this, decreasing to this side, my function is moving downwards. While from this side, it would be that is after the node, it would be moving upwards.

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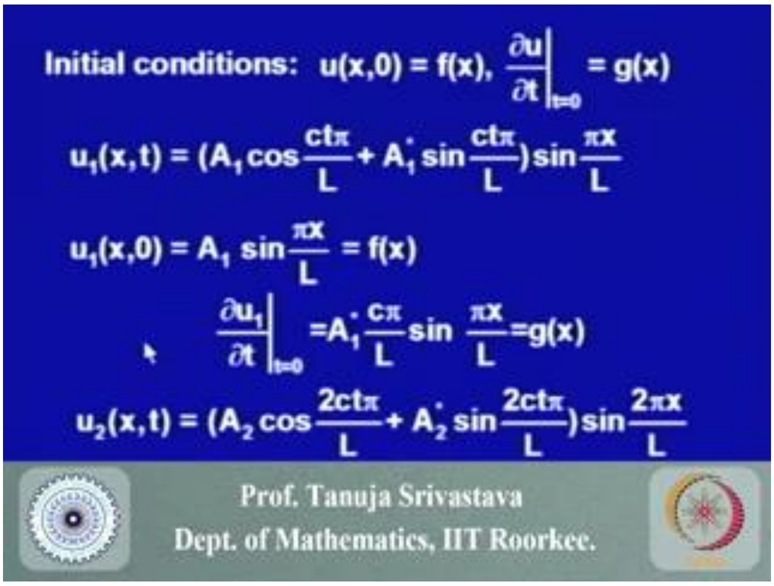

Let us, see this initial conditions, at t is equal to 0, u x 0 is f x and initial velocity is g x, so if I take the first one u 1, x t, A 1 cos ct pi over L, A 1 star sin ct pi over L, sin pi x over L, at t is equal to 0, if I put, I would get a 1 sin pi x over L as f x. And del 2 u del u L over del t, I would get A 1 star c pi L over sin times sin pi x over L is equal to g x.

Now, from here we do know that, for x is equal to L upon pi, I would get it as 0, so A 1 is, I am getting as f x is equal to 0 or A 1, we are not getting the function A 1, we could not be able to define.

Similarly, here A 1 star, I will not be able to define, when my x is l or you could say is L, 2 L, 3 L and so on, similarly if I go for the second one, u 2, x 2. Again, if I put t is equal to 0 we would get a 2 sin 2 pi x over L is equal to f x and the second initial condition would give me A 2 c star c pi L over L times sin 2 pi x over L of is equal to g x. Again, we see these constants, we would not be able to determine, when x is equal to your L by 2 L, 3 L by 2 or so on.

Neither, this A 2 nor this A 2 star, so what we have got we have got the solution of boundary value problems, which are many solutions we have got as u n x t as A n cos lambda n t plus A one star sin lambda n t times sin pi x over L.

So, today we had learned how to model 1, physical problem into the partial differential equation, second order partial differential equation. Basically, we had modeled our first one dimensional wave equation and we have modeled the side conditions; that is initial boundary value problems; that we try to solve using the method of separation of variable. That is with the product method and we have got many functions as a solution, where we are not able to satisfy the initial conditions. At that next, we will see that is, how to get the solution of complete initial boundary value problem, so today it is over here.

Thank you.