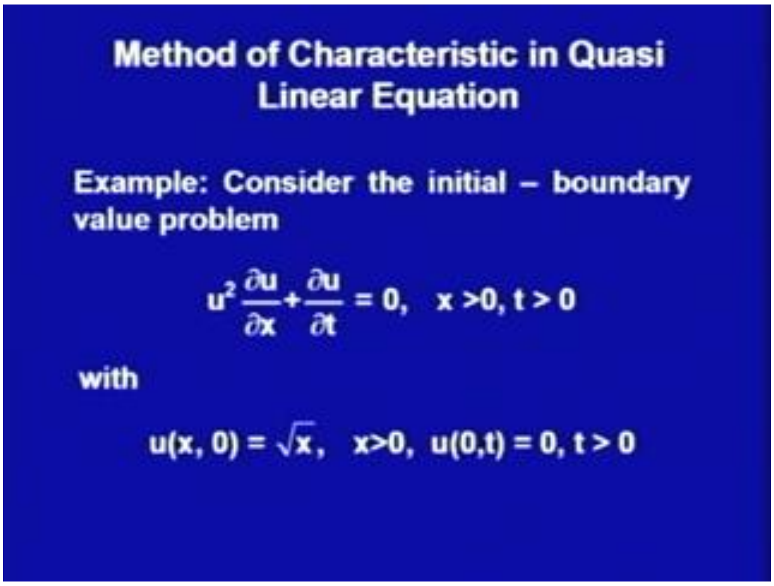
Mathematics - III
Prof. Tanuja Srivastava
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture - 18
Second Order Partial Differential Equations-I

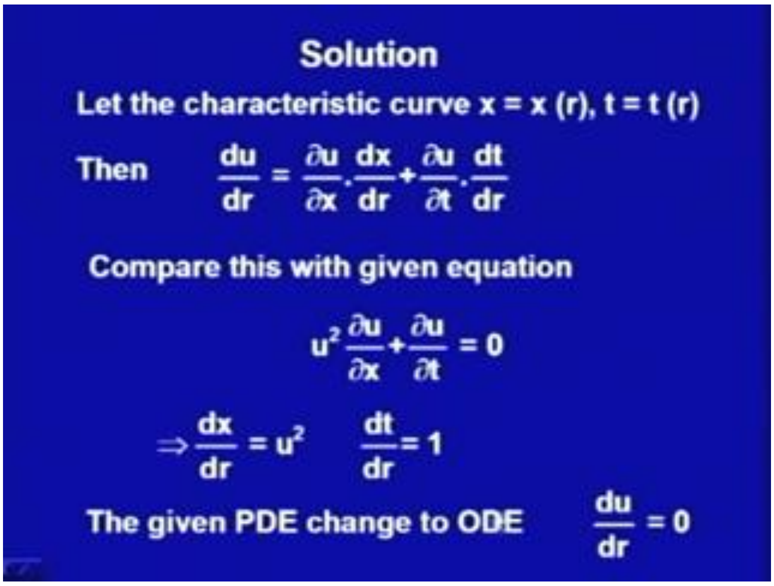
Welcome to the lecture series on differential equations for undergraduate students, today's lecture is on, Second Order Partial Differential Equations. Till now we had learnt about the, first order partial differential equations, we had learn, the method to solve the first order partial differential equations, using the method of characteristics, that we had learn in the linear, and semi linear equations. So, before starting the second order partial differential equation, let us just do one example, which is showing is, how to solve the first order partial differential equation, which is quasi linear.

(Refer Slide Time: 01:03)



So, method of characteristics in, quasi linear equation, let us see the example, consider this initial boundary value problem, u square del u over del plus del u over del t is equal to 0, for x positive and t positive, with, initial condition u x 0 is equal to root x and for x greater than 0 and the boundary condition u 0 t is 0 for t greater than 0, we are just going to use this method of characteristics.

(Refer Slide Time: 01:35)

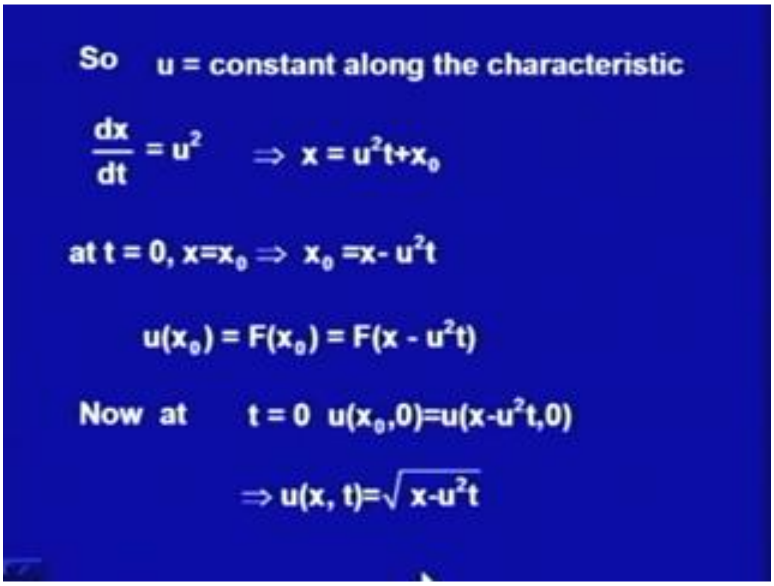


That says is that, we will change, the function to the, the partial differential equation to the, ordinary differential equations, using the characteristic curve, for that curve we assume that, it should be in the, x t plane. So, we assume the parametric form as, x as x r and t as t r, in that case, if we use the chain rule we do get, that du by d r, we can write as del u over del x times, d x over d r plus del u over del t times d t over d r.

So, we get and compare this with the given equation u square del u over del x plus del u over del t is equal to 0, because is a quasi linear. So, the coefficient of the derivatives may be, the involving the function of u and the lower derivative, so here the lower derivative would not be anything, so it is just u square. Now, if I compare with this, given equation, so we do get is, that is the d x by d r would be u square, and d t by d r would be 1.

So, d x by d r is u square, d t by d r is 1 and du by d r is 0, so what we are getting is that, this given partial differential equation changes to the ordinary differential equation in what manner, as du upon d r is equal to 0, now let us see is that, what it is saying is d t by d r.

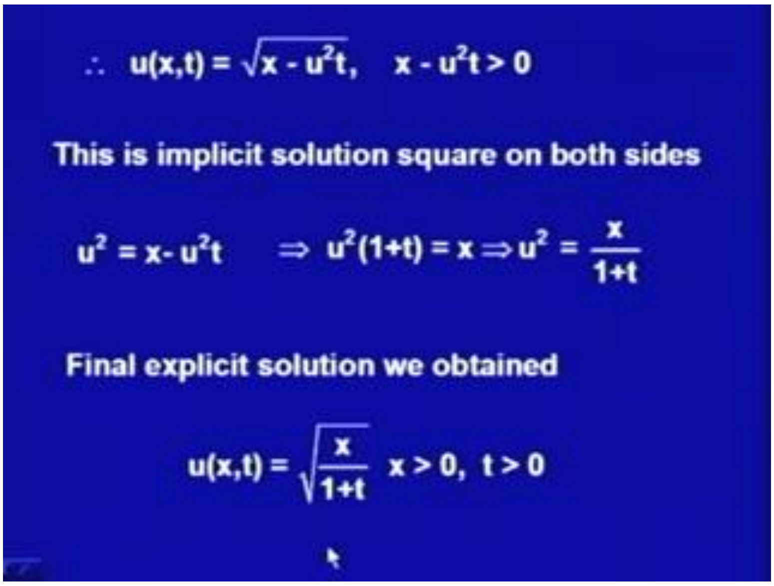
(Refer Slide Time: 02:56)



This is saying is, that u is constant along the characteristic, moreover d x by d t is u square, because d t by d r was, 1 that says is, t is same as r. So, I can write, d t d x by d r as u square as d x by d t is equal to u square, now u is constant, because we have changed the ordinary differential equation, we have got du by d r is equal to 0, that says u is constant, that means, u square would be constant here. So, treating it as a constant along this characteristic, if I try to solve this differential equation, we get that x is equal to u square t plus x naught that is constant, with respect to x.

Now, at t is equal to 0, now we are using this initial condition, that is that t is equal to 0, x must be equal to x naught from, from here, if I try to find out what is x naught, x naught would be nothing b u t x minus u square t. So, the condition, which has been given that, u at x naught, is F of x naught, that is I would get F as, F of x minus u square t. Now, at t is equal to 0, u x naught 0 that is u of x minus u square t at 0, so what we would get the solution, u x t, now substitute this would, we get the solution as F x t, what is F x t, F x t, if the initial condition was, given to was, was square root x. So, we are get, that is F of x is given as, square root x, so we have got, u x t as a square root of x minus u square t.

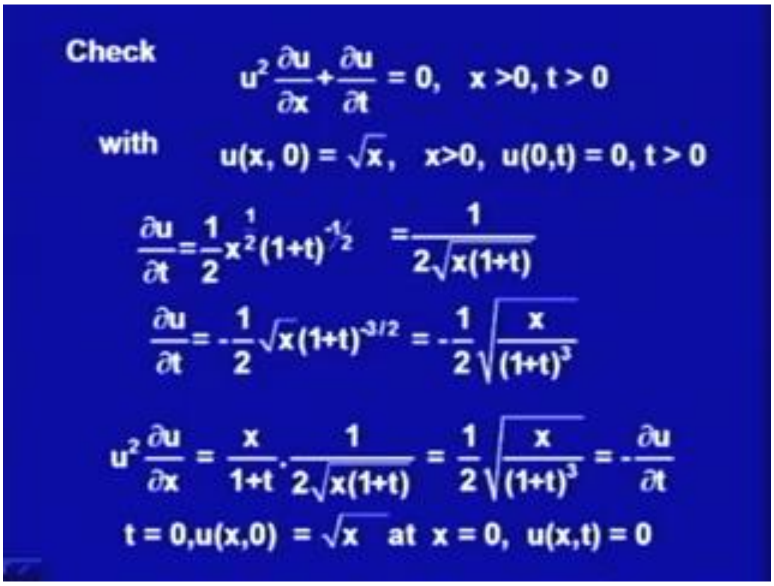
(Refer Slide Time: 04:36)



So, we have got, the solution as u x t as the square root of x minus u square t, and the solution is valid, and the case is mean, this x minus u square t is, positive, now this is implicit solution, because we had involving u, here as well as in the solution. So, to find out explicit solution, that is u in the terms of x and t only, let us simplify it, so what we do is, square on both the sides, we get u square is equal to x minus u square t, which says is that, u square times 1 plus t is equal to x or u square is x upon 1 plus t.

So, what we would get, u as square root of x upon 1 plus t, and this is square root, we would we want is, that is real solution, so x and t both has to be, positive. So, for all x positive, and for all t positive, with the given partial differential equation, has got the solution as, square root of x upon 1 plus t, this is explicit solution, explicit solution, because u we are getting in the terms of x and t, let us check that is, whether it is satisfying our given differential equation, we can check that this is the solution.

(Refer Slide Time: 05:52)

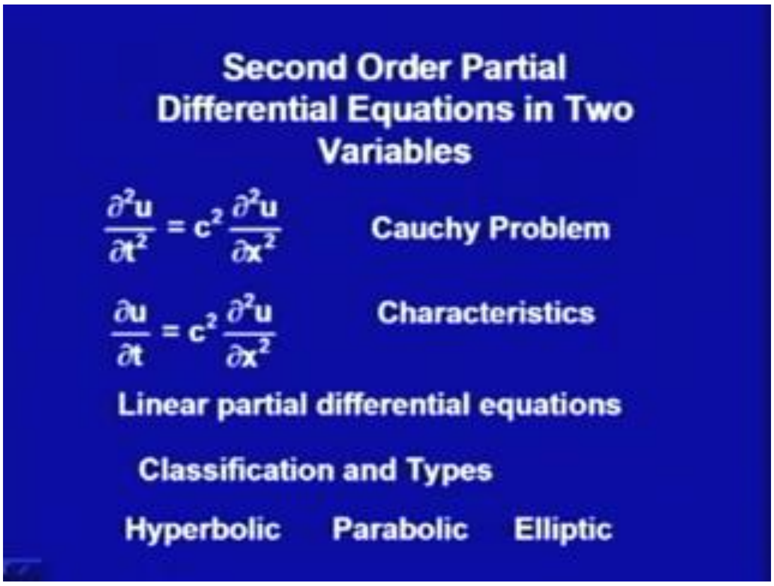


The given differential equation is u square del u over del x plus del u over del t is equal to 0, and the solution we had obtained, with a initial condition, that u x comma 0 is square root x and u 0 comma t as 0. So, we have got the solution, that u x t is square root of x upon 1 plus t, so first find out, del u over del t, that is half times, x to the power half, into 1 plus t to the power minus half, and if I differentiate that, that we can write it as, 1 upon 2 times square root of x into 1 plus t.

Now, if I differentiate u x t, that is square root of x upon 1 plus t, as with respect to x, we do get it as, minus 1 by 2 square root x 1 plus x to the power minus 3 by 2, so we are getting is minus 1 by 2 x upon 1 plus t cube. Now, p u t it over here, u we have got a square root of x upon, 1 plus t, so u square del u over del x is x upon, 1 plus t into del u over del x is 1 upon, this is del u over del x, this is 1 upon, 2 times square root of x into 1 plus t, which is same as, 1 upon 2 times square root of x plus 1 plus t whole cube, which is same as actually minus of, del u over del t, that means, this would satisfy this equation.

Moreover, if I p u t, t is equal to 0 in the other solution, I would get an only a square root x, and if I p u t x is equal to 0, we would the whole solution would be 0, for all t, so this is what, we are getting it. So, this solution, which we had obtained, that is satisfying this one, so this is the method of characteristic, in semi, in quasi linear equation in the first order partial differential equation, now let us come to the second order partial differential equations.

(Refer Slide Time: 07:58)



First we will treat, again the equations in two variables, only that is one is space variable and one the time variable, we will just take it, this kind of equation that del 2 u over del t 2 is equal to c square, times del 2 u over del x 2, where I do have only two independent variables, x and t normally I will take, the in two independent variables as x and t. Of course, we can take x and y as well also, that does not matter, just only thing is that two independent variables, and the dependent variable or the unknown function is depending only on the two variables.

This is also, second order partial differential equation with the, two variables del u over del t is equal to c square del 2 u over del x 2, what we will try to see, as in the case of, first order partial differential equation, what we tried, how to find out the solution, we said that the solution is a surface, because it is in the two variables. So, we try to find out in the, plane containing this with the, variables x and t, a curve which is, which says is that, my function is constant along that curve, that is what we call the characteristic curve.

And, if this character, we are getting is that, this is happening then we said is that, Cauchy problem said is, that is we would always get, a solution and with boundary value problem or initial value problem, we can get the unique solution. So, here, just like as the first order case, here also we will try to, find out the Cauchy problem, that is we will try to find out, the characteristics that is, can we get, some curve in x t plane such that, u is constant along that, and that is guarantying us a solution, to the with the side conditions.

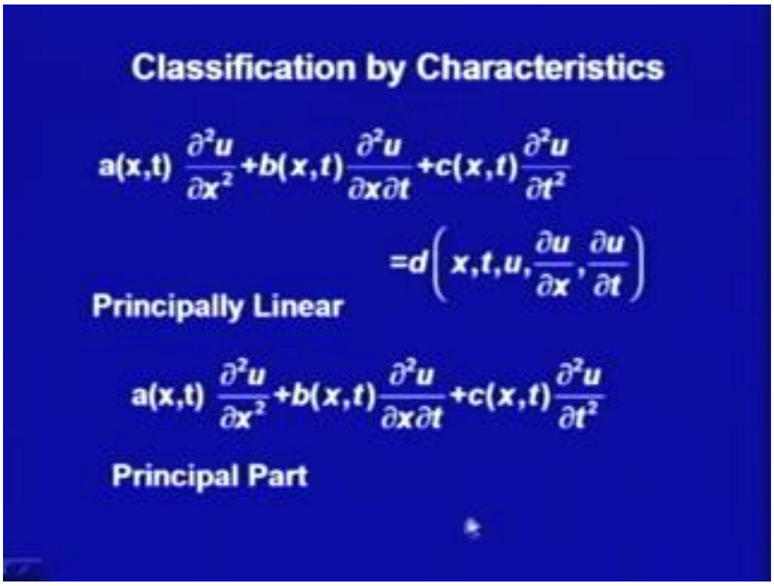
So, for that again we would, look for the characteristics, because it is a second order equation. So, rather than getting, one characteristic we may get more than, one characteristics and according to this number of characteristics actually, we are characterizing or you could say is we are, making the types of the second order partial differential equations. So, first we would be considering only, linear partial differential equations, linear partial differential equations means that is, my derivatives they are occurring linearly.

Moreover, because we are treating second order partial differential equation, or rather than taking, strictly linear partial differential equations, we will treat that is, the second order derivatives, they are occurring linearly. That, is their coefficients, are not depending on u, it may depend on the independent variables x and t, that is you could say is linear and semi linear kind of things, we are talking about.

In this case, this characteristics, that is the number of characteristics, when we try to find out, the curves in x and t, such that u is constant along them, then we do find it out, that these number of characteristics, are going to actually decide about the, kind or the type of the equations. And, this is, what is the, we are calling the classification and types, basically we do get three types of a classification says is, either the equation is Hyperbolic or it is Parabolic or it is Elliptic.

These three types are very important in the theory of, partial differential equations, because they give how to solve these particular problems. Because, a unlike in the ordinary differential equations, we are not having a general solution for all the cases or some kind of things, rather we would be getting is, that is first deciding the type of the equation. And then we will try to, change it to the certain forms, and then we will try to find out the solutions, those things we will learn later on.

(Refer Slide Time: 12:10)

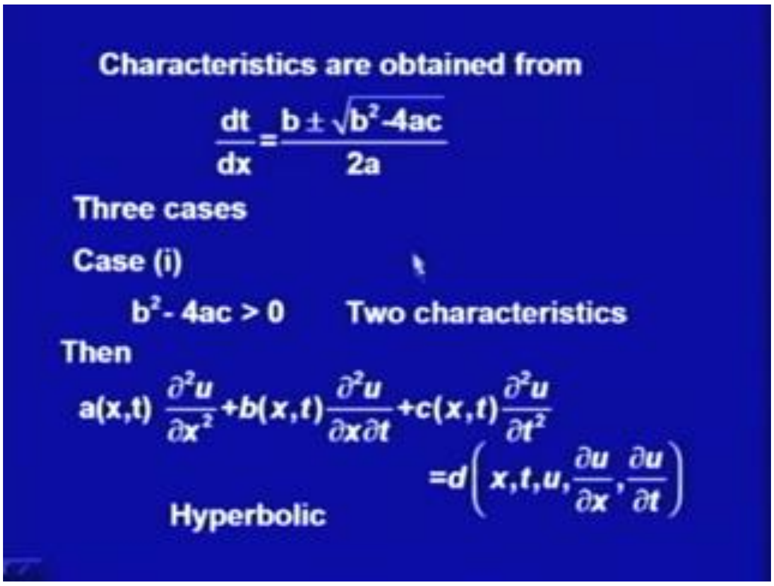


Let us, first learn this, classification by the method of characteristics, so classification by characteristics, let us take, this general linear differential equation, which is linear. In the second order derivatives, that is I am taking del 2 u over del x 2 with coefficient a, which may depend on x and t, b u t not on u. So, let us say in general I am writing, a linear second order partial differential equation as, a x t del 2 u over del x 2 plus b x t del 2 u over del x del t plus c x t del 2 u over del t 2 is equal to a function. So, here this d we are denoting is as a function, d of x t u del u over del x del u over del t.

Why I am not writing here anything, why I am calling it a function, here what we are saying is, that I am taking this, second order differential equation, which is linear in the second order derivative, with the first order derivative, it may contain the coefficients of, containing u also, or u may contain the coefficients as, any other thing. So, we are just treating, this part as the, general part, we are talking about, the linearity only in the second order derivatives.

This, kind of equation we are calling, principally linear, and this left hand side, of this equation, that is a x t del 2 over del x 2 plus b x t times del 2 u over del x del t plus c x t times del 2 u over del t 2, that is all the second order derivatives, and their coefficients, this part is been called the, principal part of this, principally linear differential equation. Now, with this differential equation with this principal part, only this principal part is going to play, a role to find out the characteristic, and how we are finding it out.

(Refer Slide Time: 14:07)



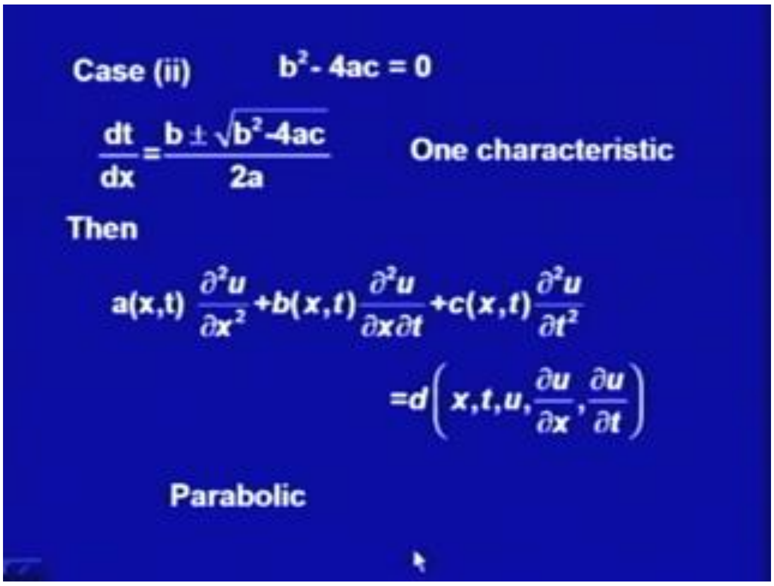
The characteristics are obtained from, that principal part you see is, that is you have got the coefficients a b and c. Where a was the coefficient of the, second derivative partial derivative, with respect to the one variable x, and b is the coefficient of the second partial derivative is a join, that is derivative with respect to x and t both, and the c was the coefficient with the, derivative with respect to the second variable.

So, we get the characteristics, we are obtaining by this linear, this ordinary differential equation d t by d x is equal to b plus minus square root of b square minus 4 a c upon 2 a, now here b and a b c all are the function of x and t. Now, because this, square root of b squares minus 4 a c, this will give me three cases, the first case, when b square minus 4 a c is positive what I would get, I would get here that is it is positive.

So, some square root some real number, so b plus minus that is I would get two characteristic that is I would get the two equations, which will give me the characteristics. That is one is, with b plus square root of b square minus 4 a c upon 2 a, another is b minus square root of, b square minus 4 a c upon 2 a, both would be real. So we will get, two characteristics, and in this case, this given equation a x t del 2 u over del x 2 plus b x t del 2 u over del x del t plus c x t times del 2 u over del t 2 is equal to, d times x d of x t u del u over del x del u over del t, this is called hyperbolic.

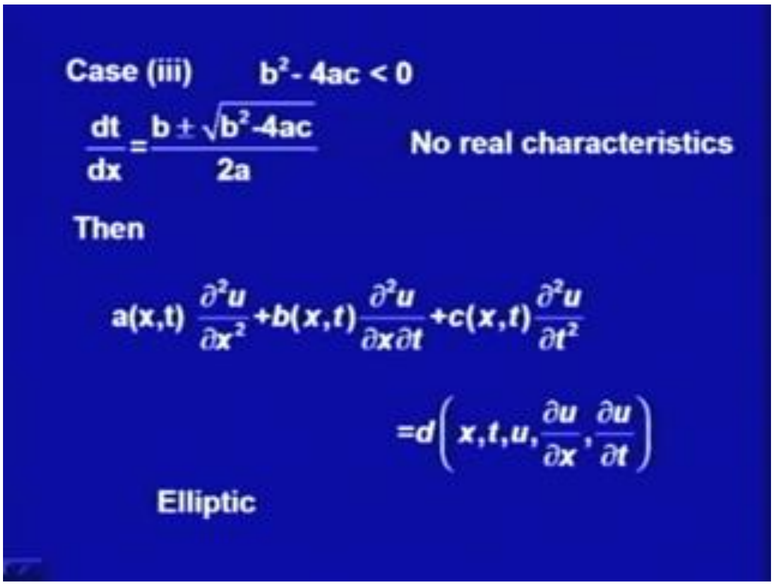
So, when we are getting is, in this equation, these coefficients we are getting such that, b square minus 4 times a c, this is coming as the positive, we call this equation to be hyperbolic.

(Refer Slide Time: 16:07)



Second case would be, that is, this b square minus 4 a c is 0, in that case, what I would get, this equation d t by d x is equal to b plus minus square root b square minus 4 a c upon 2 a, because b square minus 4 a c is now 0. So, I would get only single, equation d t by d x as b by 2 a, that says is, only one characteristic, and in this case, my equation a x t del 2 u over del x 2 plus b x t del 2 u over del x del t plus c x t times del 2 u over del t 2 is equal to, d of x t u del u x over del u t, this is called parabolic.

(Refer Slide Time: 16:53)

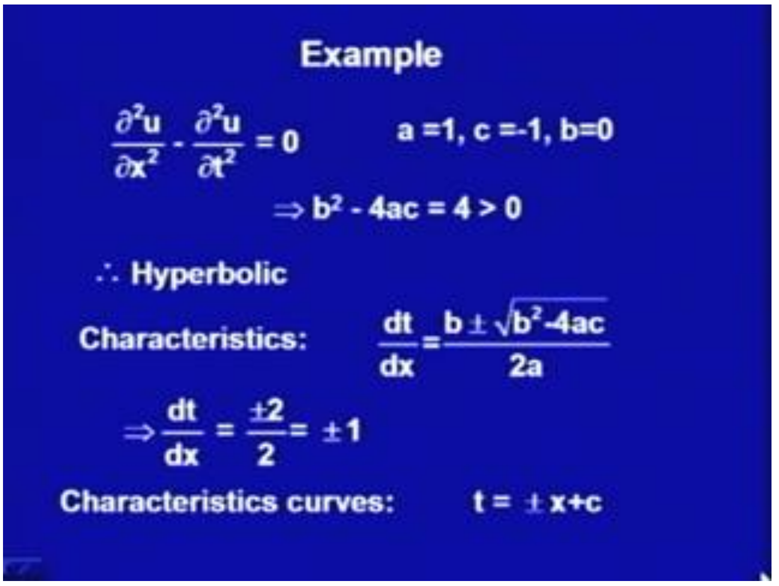


Then, the third case, when this b square minus 4 a c is negative, what it says is in this characteristic, what I would be getting, I would not be getting a real root, what I would be getting is that, imaginary roots. And, if are the complex roots that are we are not getting, any real characteristics, so in this case, this equation is called elliptic, so we are finding out the, for this equation a x t del 2 u over del x 2 plus b x t del 2 u over del x del t plus c x t times del 2 u over del t 2 is equal to a function of x t u del u over del x and del u over del t, this equation we would call hyperbolic, parabolic or elliptic according to my b square minus 4 a c is positive, equal to 0 or negative.

And in that cases, the characteristics we are finding out, using this equation d t by d x is equal to b plus minus square root of b square minus 4 a c upon 2 a and depending upon these cases, we are getting, that is either one or two or none real characteristics. All these things when I am talking about, I am talking about, at a particular point x and t, let us, at a particular point t is equal to t naught and x is equal to x naught, we are talking about these things, these ah types of the equations.

So, if these a b and c are constant, then of course, the equations would be, of these types hyperbolic parabolic or elliptic for all x and t, if they are depending on x and t, then for each x these types would change, let us just try to do certain some examples to understand this concept.

(Refer Slide Time: 18:52)

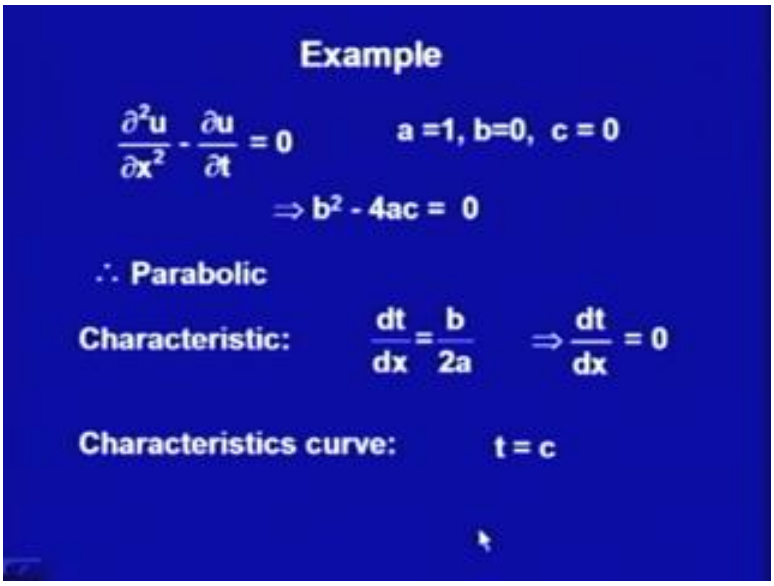


Let us take the first example, del 2 u over del x 2 minus del 2 over del t 2 is equal to 0, this is one equation, second order partial differential equation it is, coefficient of the second order derivatives are constant. So, it is principally linear, in this one if I treat, the coefficient of del 2 u over del x 2 that is 1, so a is 1, del 2 u over del t 2, this coefficient of second with the, the second partial derivative, with respect to second variable is minus 1, and there is no term containing the derivative with respect to both x and t variable, so b is 0.

What we are getting b square minus 4 a c from here, we would be getting is 0 minus 1 into minus 1 that is plus 4, which is positive, so this equation is hyperbolic. And, the characteristics would be given by, b plus minus square root of, b square minus 4 a c upon 2 a. So, we get here, since b is 0 and b square minus 4 a c is 4, we would be getting plus minus 2 by 2, that is plus minus 1.

So, we are getting the two equations d t by d x is equal to 1 and d t by d x is equal to minus 1, and that would give me the two characteristics, those characteristic curves we would be calling, t is equal to x plus c 1 and t is equal to minus x plus c 2. So, here I have writing the both the curves, in single equation, that plus minus x plus c, so here the c is the constant, with respect to x. So, we can treat it, in 1 equation as c 1 another equation is c 2, so the constant is, may be different, may be same depends upon, that is what is side conditions, we do get.

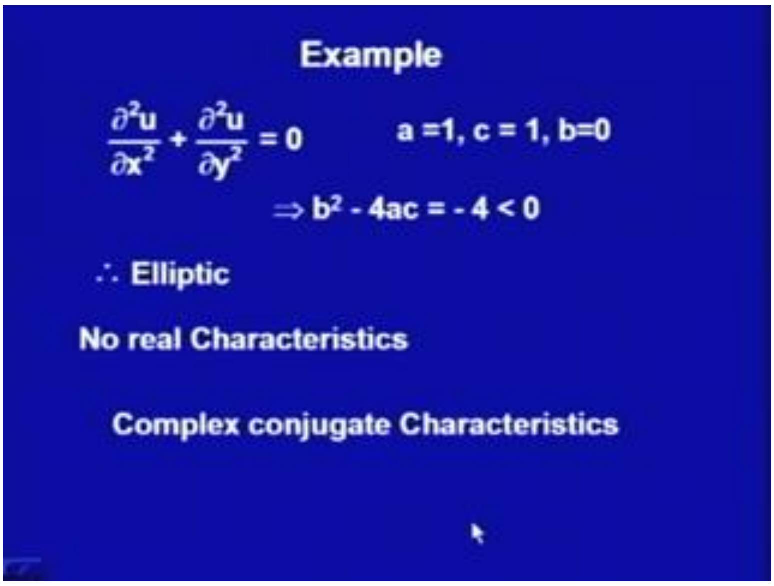
(Refer Slide Time: 20:45)



Let us do one more example, del 2 u over del x 2 minus del u over del t is equal to 0, now you see, the coefficient that, term containing the second order derivative is only single, one then there is one term, which is containing the first order derivative. So, it is a constant, again it is, of the form of, the principally linear equation, in this one a, is 1 b is 0 and c is 0, so what I would get, b square minus 4 a c, b is 0, c is 0, so b square minus 4 a c would be 0 that says is this equation is Parabolic.

Now, what will be the characteristic, characteristic we are getting is by b upon 2 a, because b square minus 4 a c is 0, so b is 0 actually here. So, what I would get is, the d t by d x is equal to 0, so what will be the characteristic curve, characteristic curve would be that, t is a constant, so this is only single characteristic, we are getting this is a Parabolic equation.

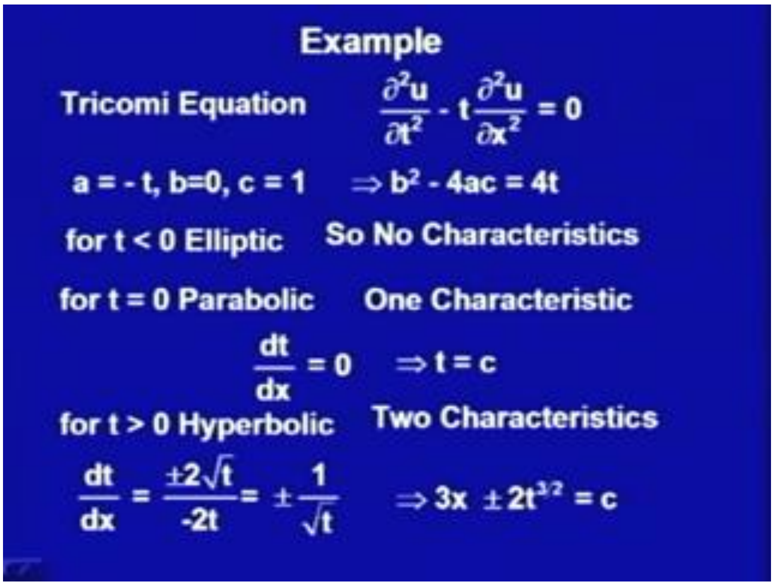
(Refer Slide Time: 21:48)



Let us do one more example, del 2 u over del x 2 plus del 2 u over del y 2 is equal to 0, here you see is, I have taken, the 2 independent variables as x and y, rather than x and t. So, we will treat y as similar as, the t, now again this is, second order differential equation, the coefficient of second order derivatives they are constants, and we are not having any other terms, so again we can term it as a, principally linear actually this is linear one, here what is a the coefficient of this one is 1, b is 0, because I do not have any, second order derivative, with respect to both the variables and we do have, c is equal to 1.

So, what will be b square minus 4 a c, b is 0, so minus 4 a c would be minus 4, which is negative, that says is I will not get any, this is simply negative, says is it is elliptic and I will not get, any real characteristic. Now, from these characteristics, and these three typical kinds of equations, when all these examples, we have got, that my coefficients were constants, now let us do one example, where the coefficients are not constant

(Refer Slide Time: 23:13)



So, let us take this Tricomi equation, which is del 2 u over del t 2, minus t times del 2 u over del x 2 is equal to 0. Now, again it is a second order differential equation, where the coefficients of second order derivatives, of with respect to t, this is one and with respect to x it is minus t. So, it is not taking the term u, it is depending on x and t, so again it can be treated as, principally linear equation, now what is been given, a is the coefficient of, with respect to, derivative with respect to x, the coefficient is minus t, b is 0 and c is 1, this says is b square minus 4 a c, b is 0 minus 4 a c would be 4 t.

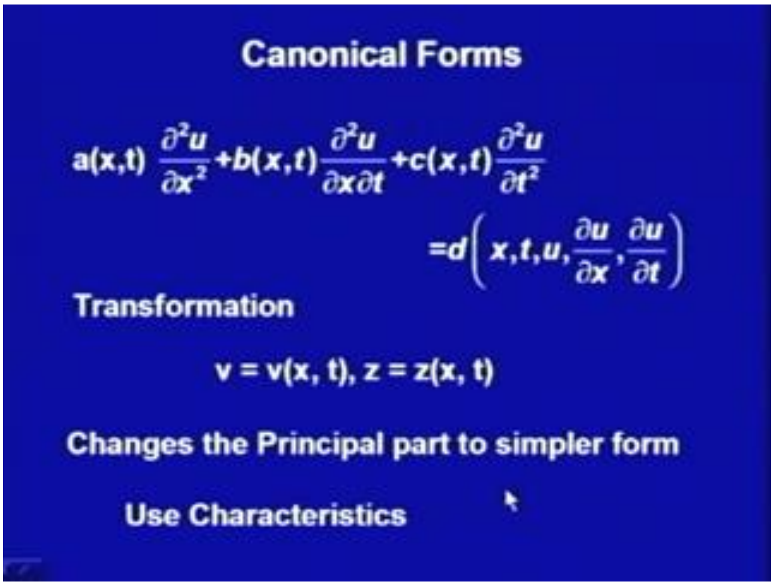
Now, this is depending upon the value of the t, so now, if t is positive negative or 0 accordingly we will get the, type of equation. So, let us take, if t is less than 0, then this equation is elliptic, and we will get no real characteristic, if t is equal to 0, then this equation would be, Parabolic and we will get, single characteristic, and if t is positive, then we would get, and in the case of this t is equal to 0, the one characteristic we would get d t by d x is equal to 0 says is, t is equal to c.

And for the case, t greater than 0, the equation would be hyperbolic, and the characteristic would be given as, two characteristics here, what is my b is 0, and b square minus 4 a c is 4 t. So I would be getting it, plus minus 2 times square root 2 t divided by 2 a, a is minus t, so 2 minus 2 times t that is what we are getting is, plus minus 1 upon square root t.

So, what will be those characteristics by solving these equations, we would get it, simply 3 x plus minus 2 t to power 3 by 2 is equal to c, actually we are getting two equations. One is 3 x plus 2 t to the power 3 by 2 is equal to c 1, another equation 3 x minus 2 t to the power 3 by 2 is equal to c 2. So, depending upon the value of t, here we are getting these, 3 types in the same equation, that is same equation can be, elliptic, parabolic or hyperbolic.

So, this is one example, now let us see, how this characteristics or this finding out the characteristics, help us in finding out, solution of partial differential equation. But before going to the solution of this second order partial differential equation, let us see one more, step in between, which says is, we can give a normal form of, a general second order principally linear differential equation, that is called canonical forms, they are also termed as normal forms.

(Refer Slide Time: 26:20)

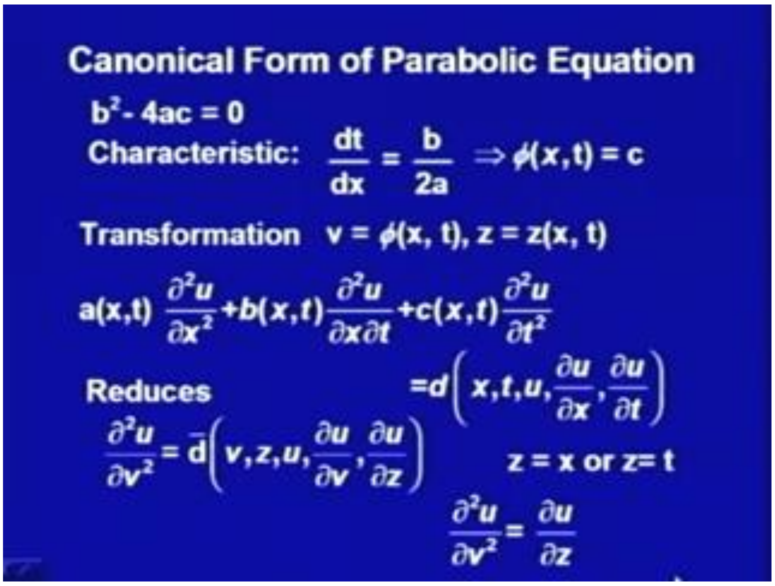


What is this canonical form, it is saying is that is, because I we were having the, equation in two independent variables x and t, if I change them, to another variables, say v and z. So, let us see, if I do have this, general principally linear, second order differential equation a x t del 2 u over del x 2 plus b x t del 2 u over del x del t plus c x t times del 2 u over del t 2 is equal to, a function of x t u over del u over del x and del u over del t, if I introduce some new variables, instead of this variables independent variables x and t.

If I introduce, some other new variables v and z, again they has to be function of x and t, so if I introduce two another independent variables v and z, other function of x and t. How this transformation, this transformation should be such a manner, so that I get this, principal part, in a simpler form, plus this right hand side part is also, something very simple form, what do we mean by simpler form, you see we have done three examples, first three examples, where we have got, the equations were hyperbolic parabolic or elliptic.

The principle parts of those equations are, actually coming up as a normal form, let us see, one by one these kinds of cases, that is, how to find out these transformation, and why do we call them as the normal form. What they are helping us in finding out the solution, so let us try to see here, that is if this changes, principal part to the simpler form, we call it the canonical form, how we are using this transformation, as I said is, let us see, we are going to use the characteristics.

(Refer Slide Time: 28:25)



So, let us take the first case, as canonical form of parabolic equation, parabolic equation means, b square minus 4 a c is 0, that says is I would get only single characteristic, given by the equation, d t by d x is equal to b upon 2 a. If I integrate this one, I would get, integral as phi x t is equal to c, so we are just writing, that is may be, whatever it may be depending upon whether b and a are the function of x or t or something like that, we do get it is integral as, phi x t is equal to c, this is single equation.

So, what we will, introduce the new variables, it is transformation, we would introduce, v as phi of, that is whatever this integral we have, got from this characteristic equation, that is integral this gives me the characteristic curve. So, we are taking, that characteristic of the left, part of that characteristic curve, as that phi of x t, that is the one variable, another variable z, because this is giving the parabolic equation, and this is giving me, only single characteristics.

So, the other variable, we are taking, as any variable, any function of x and t, only thing we keep in mind, that this function, has to be differentiable, because all these variables should be, in such a manner, that they should, posses the second order derivatives. So, that I could, transform m them into the, equation, and moreover, I should not get, them to be, they should that, this z has to be independent of v, linearly independent of v.

How we do get this z linearly independent of v, that we will understand, in the example actually, this transformation if I introduce in this given equation, then this would change my this equation to as, del 2 u over del v 2 is equal to d bar, now this is another function. v z u del u over del v del u over del z, that is the principal part, is been transformed, that is here, in the original equation, in the principal part we were having, we might have, all the three kind of, derivatives.

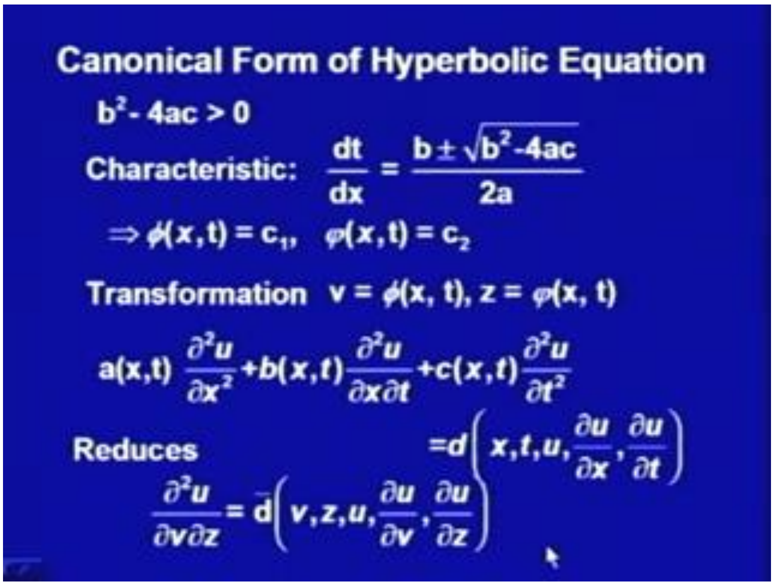
While, when I introduce this parabolic equation, I introduce this transformation, in the principal part, I may get only, the second derivative, with respect to one variable only. And of course, the right hand side, can contain the function of v z u and the derivatives, with respect to v and z first are, first order one. So, this you see is that principal part has been, transformed now, how do we choose this z, as I said is z has to be chosen independent of this, a normal practice is, that is we choose z as, the first variable the or the other variable.

So, we do, use z is equal to z or, z is equal to t, whether we should choose x or t, that depends upon that, the choice has to be such that, this right hand side is, very much simplified. In certain cases, some special cases, we may get it only, a derivative with respect to the second variable, that is my equation would reduce to del 2 u over del v 2 is equal to del u over del z directly.

Now, you see, if this we compare with our examples, we have got in the second example, where we had del 2 u over del x 2 minus del u over del t is equal to 0, that was the parabolic equation. So, in that case, this could be called as the normal form of the parabolic equation, so that is, why we are calling those, examples as a, a typical kind or the normal form of the, or canonical form of the equation.

But, it is not necessary, that we may always get, this kind of thing, but we try to, use the transformation in such a manner, because here we do have a choice, that we have only one equation, that is one variable we are transforming, the other is with our choice. So, that choice we, treat it as, or it should be, linearly independent of v, and it should simplify my right hand side, to maximum extend.

(Refer Slide Time: 32:47)

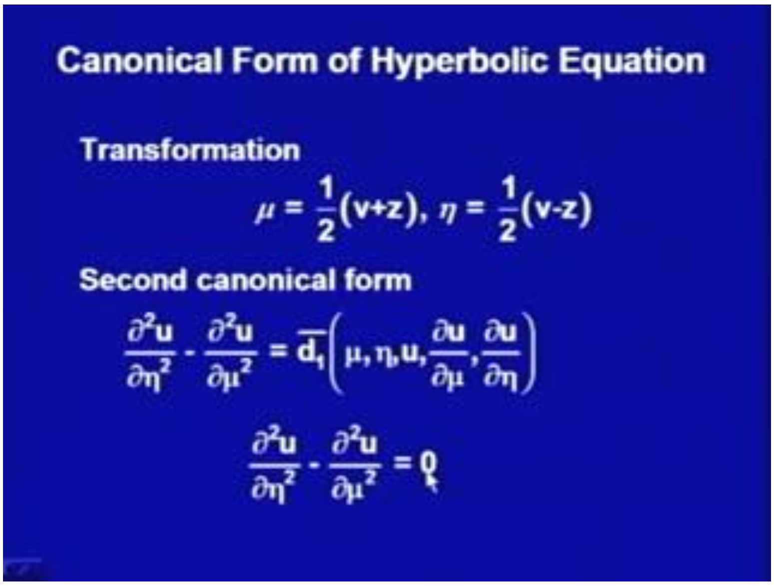


Let us take, the Case of canonical form of hyperbolic equation, hyperbolic equation means, b square minus 4 a c has to be positive, that says my characteristics would be given by, d t by d x is equal to b plus minus square root of b square minus 4 a c upon 2 a. This will, because b square minus 4 a c is positive, so we will get the two equations, so let say that, two equations, so two characteristics, I would get let say, phi of x comma t as c 1 and psi of x comma t as c 2.

Because, we would be getting this, two differential equations their solutions, we are writing is, a actually the solution would be t is equal to x, and this explicit 1, implicit 1 we are writing, phi of x t and psi of x t is equal to c 1 and c 2. So, we introduce the transformation, v as phi of x t and z as psi of x t then this given equation will transform, so now what we would do is we find out, the del 2 u over del x 2, in the form of new variables v and z, the explicitly how to do all these things, we learn with through the examples.

So, that this equation is reducing to, this kind of equation, that is the derivative of, u with respect to v and z, the second derivative with respect to first and second variable both. And, this should be some function of v z u first derivative with respect to v, and second first derivative with respect to z, again this is called the canonical form of Hyperbolic equation.

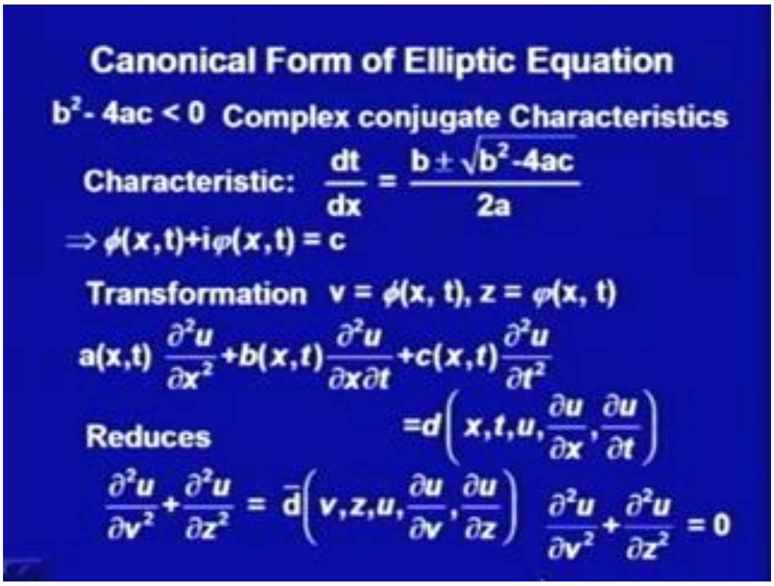
(Refer Slide Time: 34:37)



Now, this is not matching with our examples, what we do here, is that if we use, another transformation after is, what is that transformation, mu as v plus z, and eta as v minus z, with half of this 1 or you could say is we are using is, v as mu plus eta and z as mu minus eta. So, this mu and eta the new variables here introducing, if I introduce this new variables, then our, that canonical form, would get change to del 2 u over del eta minus del 2 u over del u 2 is equal to, some function of mu eta u, and the first derivative with respect to mu, and the first derivative with respect to eta.

This is also known as the, canonical form of the hyperbolic equation, now if I compare this with, with our first example, there we have got, del 2 u over del x 2 minus del 2 u over del t 2 is equal to g. So, this left hand part side, this is matching with left hand side of our first example, so that left hand side is, actually the normal form of the, hyperbolic equation. Again, when you are introducing this thing, in the special cases, the right hand side exactly, we could get it as 0, that is matching with that, equation and we do know that, this equation is nothing but the one dimensional wave equation that is a special case.

(Refer Slide Time: 36:01)



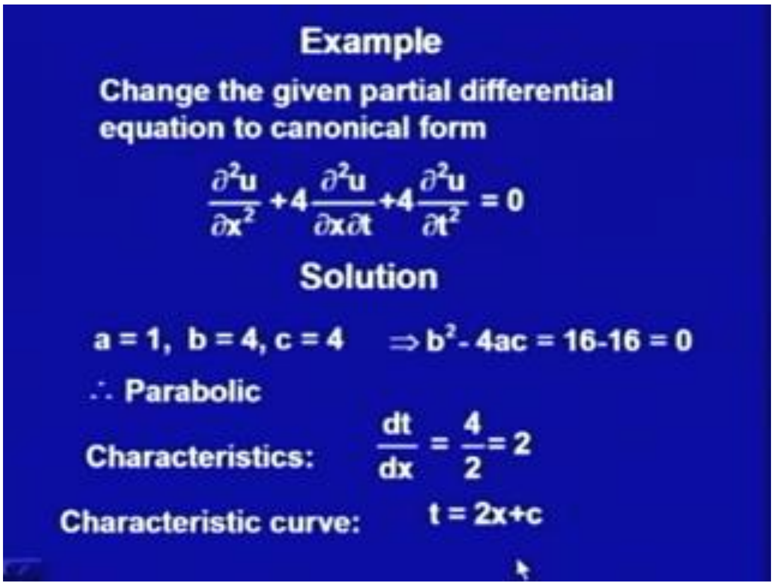
Now, come to the canonical form of, elliptic equation, that means, b square minus 4 a c should be negative, in this case, what we will get complex conjugate characteristics, that is characteristics, would be given as, b plus minus square root of b square minus 4 a c upon 2 a. Since, this is b square minus 4 a c is negative, so whatever this square root we could write it out as, b plus minus iota times, some real number, that iota means, square root of minus 1 I.

If I try to solve this equation in that form, let us say the solution would be, the complex conjugate, that is phi x t plus i of psi x t is equal to c. Then, the new transformation, which we would introduce, they would be of the form, that is v is equal to phi x t and z is equal to psi x t. So, what we are getting is actually we are getting the complex conjugate equations, and from there the solution also we will get the complex conjugate, so we just try to see is, that is we are treating this, as one equation and from there, we are getting this complex conjugate solution.

So, both this phi and psi they are real parts, so we are, introducing the transform, using these real parts. If I introduce these transforms, we get our given equation, to be transformed to the form of, del 2 u over del v 2 plus del 2 u over del z 2 is equal to a function of v z u, again the first derivative with respect to v and the first derivative with respect to z. In certain special form cases this right hand side, may reduce to 0.

And then we do get, that our example three, even if here, the left hand side is, same as that the, left hand side of the our example three. So, the first, second and third examples, which we have done, where we have tried to find it out the, type of the equations, they are basically the canonical forms of those, hyperbolic, parabolic and elliptic equations. This is if all, also we do know that this is nothing but the 2 dimensional Laplace equations, it is see is, that is how we are really transforming it.

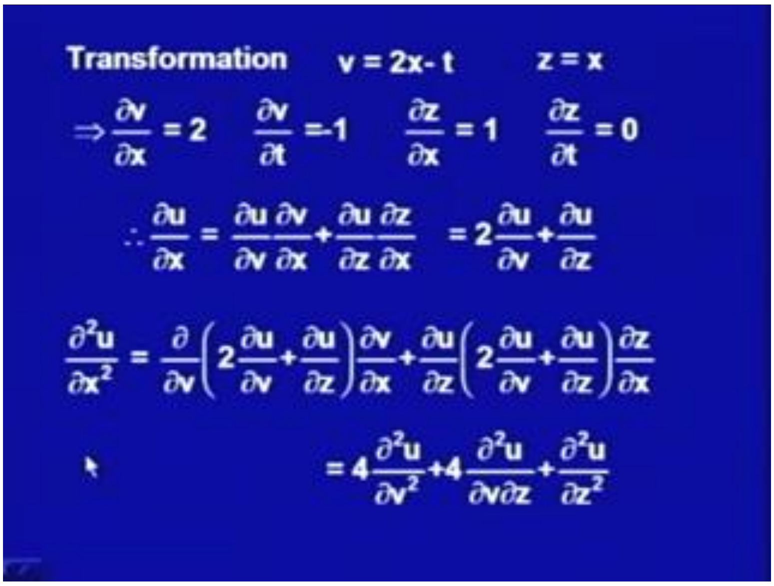
(Refer Slide Time: 38:23)



So, let us do some examples, change the given partial differential equation to the canonical form the equation is, del 2 u over del x 2 plus 4 times del 2 u over del x del t plus 4 times del 2 u over del t, t is equal to 0. Now, you see that, is our equation is not very difficult in that since, our coefficients are with are constants, so let us first, check it that is what is the kind of, what is the type of this equation. So, we do have is, a is equal to 1, b is equal to 4 and c is equal to 4, so b square minus 4 a c would be, 16 minus 16, that is 0, So, this is a parabolic equation.

Parabolic equation means, that is my, characteristic would be given by, b upon 2 a b is 4 a is 1, so b upon 2 a, that is 4 upon 2, that is 2. So, what we would be getting is the characteristic curve, would be t is equal to 2 x plus c, so this is the, one characteristic only we are getting, now change it to the canonical form.

(Refer Slide Time: 39:33)



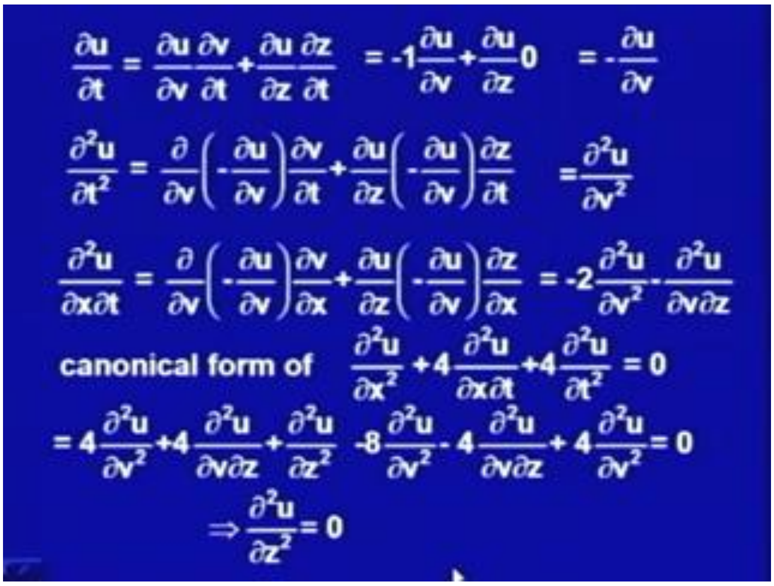
So, transformation we would introduce, v as 2 x minus t, what we have got the curve is, that x is equal to, t is equal to 2 x minus c, that we are writing as 2 x minus t. And, the other form z is equal to, x you can just check that, this is now going to transform it, so now what we would be using, we would be using the chain rule, z I have chosen as x, we do know that is, here z the choice of z is, just ours that is, we have chosen z such that, it should be linearly independent.

Linearly independent means is, if I find out all the derivative with respect to x and t for both the vary variables, del v over del x is 2 del v over del t is minus 1 del z over del x is 1 and del z over del t is equal to 0. So, if I take the, jacobian 2 minus 1, 1 0 I would get it 1, so this is not 0, so that says is, it is linearly independent, now using the chain rule, we get, del u over del x, we can write, del u over del v into del v over del x plus del u over del z into del v del z over del x.

Now substitute this del v over del x and all those del u del z over del x, so here, what we get is 2 times del u over del v plus del u over del z. Now, find out the second derivative, that is derivative of, del u over del x, so we are again, differentiating with respect to v and writing the derivative with respect to x. So, del of del v of del u over del x that is, 2 times del u over del v plus del u over del z times del v over del x plus, the derivative with respect to sorry, it should not be u should, it is del upon del z of, u times del u over del v plus del u over del z and multiplied with del z over del x.

Again, del v over del x is 2 and del z over del x is 1, what we do get is, that is, when I operate this, del upon del v over here, I would get 2 times, del 2 u over del v 2 plus del 2 u over del v del z. This place we would be getting is, this u is not there, so del upon del z, when I am operating over here, I would get, 2 times del 2 u del z del v, now what we are assuming this, that the continuity is being contain, so whether we write, del 2 u over del v del z or del 2 u del z del v both we are treating as same. Thus, we would get is 4 times del 2 u over del v 2 plus 4 times del 2 u over del v del z plus del 2 u over del z 2.

(Refer Slide Time: 42:11)



Similarly, we will find it out, del 2 u over del, del u over del t, again using the same kind of chain rule del u over del v del v over del t plus del u over del z and del z over del t, now del v over del t, we have got minus 1 del z over del t is 0. So, we do get it as minus del u over del v, now the second derivative with respect to this, again we go with the same rule. So, del upon del v minus del u over del v del v over del t minus del over del z of minus del u over del v del z over del t.

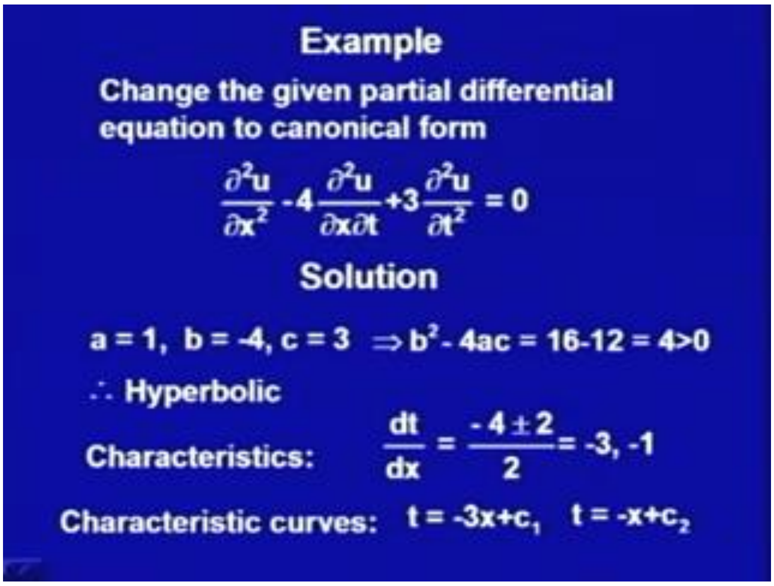
Again, put it that z del over del t is 0, so what I would be getting this minus 1 minus, minus will get plus 1, and I would be getting is only, simply del 2, u over del v 2 from here, we will try to find out, del 2 u over del z del t del u del t. So, del x del t, we are getting, so that is, we are differentiating with respect to x, so del upon del v of, minus del u over del v into del v over del x, plus del upon del z of minus del u over del v of del z over del x, which is again, we are getting del u del v over del x was 2 and del z over del x was 1. So, i would be getting, minus del 2 u over del v 2 minus del 2 u over del v del z.

Now, substitute these, in the given equations, so the canonical form of, this equation, this is the given equation, now I will substitute, del u over del x 2 del 2 u over del x del t and del 2 u over del t 2. Whatever we have obtained, that form of, our derivatives with respect to v and z, so the new variables, the first one if you do remember we have got, 4 of del 2 u over del v 2 plus 4 times del 2 u over del v del z plus del 2 u over del z 2.

Then plus 4 times del 2 u over del x del t del 2 u over del x del t is this one, this has to be multiplied with 4 times. So, I would get, minus 8 times del 2 u over del v 2 minus 4 times del 2 u over del v and del z, and plus 4 times del 2 u over del v 2 is equal to 0. Now we see is this, del 2 u over del v 2 with 4 coefficient it is with, minus 8 and this is with plus 4, I will get this cancel it out.

Moreover, this del 2 u over del v del z, that is with the 4 coefficient, and here is the coefficient is minus 4 again this is being cancelled it out, what is being left is, del 2 u over del z 2 is equal to 0, this is what is we are getting the Parabolic equation. So, now we have got, we have chosen, our second variable z as x and we have got the simplest form, where the right hand side has turned out, to be 0. This will actually help in solving the differential equations that will learn latterly.

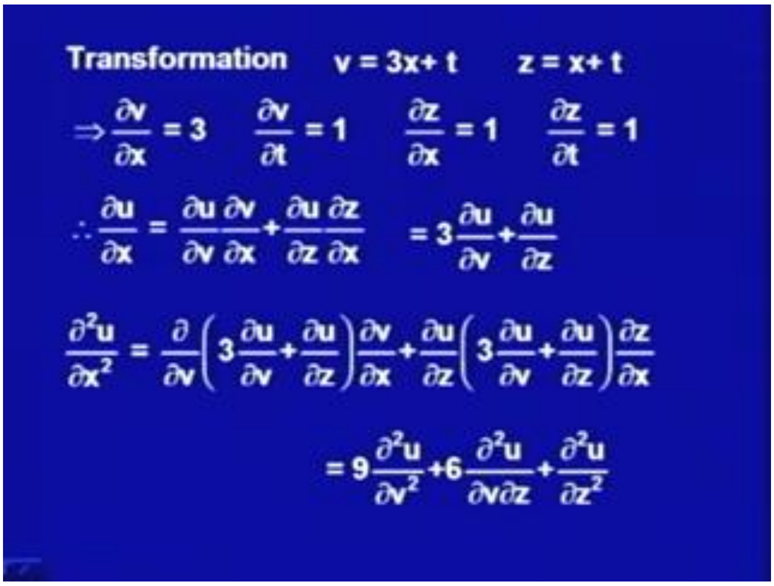
(Refer Slide Time: 44:55)



Let us do one more example, so now we do have this equation, del 2 u over del x 2 minus 4 times del 2 u over del x del t plus 3 times del 2 u over del t 2 is equal to 0. We want to change this differential equation to, it is normal form, what we do is, a again first find out, the type of the equation and then use the characteristic to change it to the normal form. The coefficient of del 2 u over del x 2 is 1, the coefficient of del 2 u over del x t del t is minus 4, so b is minus 4, similarly c is 3, that says is b square minus 4 a c would be, 16 minus 12 we are getting it, 4 that is positive.

So, this is Hyperbolic equation, the characteristic is now, d t by d x is equal to minus 4 that is b plus minus square root of b square minus 4 c that is, 4 square root of 4 is 2, so plus minus 2 by 2. That is, what we would be getting two equations, d t by d x is equal to minus 3 and d t by d x is equal to minus 1, thus we would get, two characteristic curves, one is t is equal to minus 3 x plus c 1, another is t is equal to minus x plus c 2.

(Refer Slide Time: 46:16)

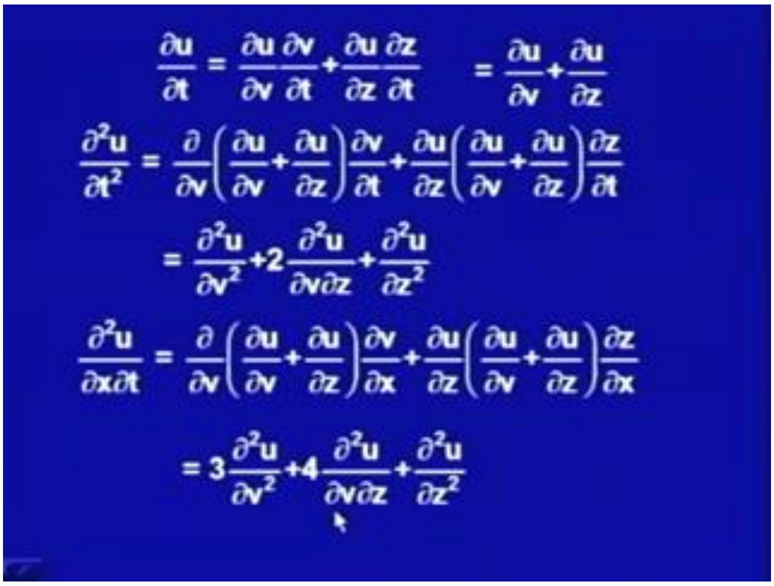


So, the new transformation what we will introduce, v as 3 x plus t and z as x plus t, this will give me del v over del x as 3 del v over del t as 1, del z over del x as 1 and del z over del t as 1. So, again using the chain rule will transform, will get del u over del x and del 2 u over del x like that one, so del u over del x, by this again using the chain rule, we would get, del v over del x is 3, and del z over del x is 1, so 3 times del u over del v plus del u over del z.

Now, go to the second derivative, differentiate this once more with respect to x, so again using the chain rule, we are differentiating it with respect to v, and then getting the v with respect to x, then with respect to z, and with respect to z with respect to x. So, we would be getting is that, del upon del v of 3 times, del u over del v plus del u over del z, times del v over del x plus del of del z of, 3 upon del u over del v plus del u over del z, times del z over del x.

Again, we will keep, del v over del x as 3, and operate this upon, similarly del z over del x 1 and operate del upon del z on this one. We will finally, get 9 times del 2 u over del v 2 plus 6 times del 2 u over del v del z plus del 2 u over del z 2, you see here, when we are getting is 3, so 3 into3 this we will be getting, del 2 u over del v 2. We are again using, that del 2 u over del v del z and del 2 u del z del v they are equal, so here the coefficient is 3, because of this 3, and here is also the coefficient is 3, because of this 3 would get 6 and so on.

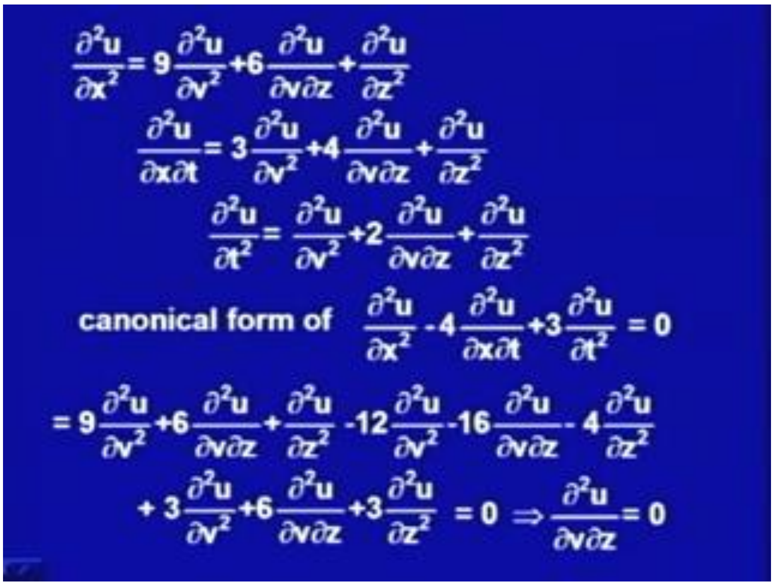
(Refer Slide Time: 48:10)



Now, let us move to the del u over del t, again using the chain rule, del v over del t is 1 and del z over del t is also 1, so we get it, as del u over del v plus del u over del z. So, the second derivative with respect to, t will differentiate again, this with respect to, this thing with respect to t. So, again using the chain rule, this would be del upon del v of del u over del v plus del u over del z del v over del t plus del of del z of del u over del v plus del u over del z del z or del t, which again when we apply this is one and this is 1.

So, from here, I would be getting del 2 u over del v 2, this term and this term is going to give us, plus 2 times del 2 u over del v del z and plus del 2 u over del z 2. From here itself, we will find out the, del 2 u over del x del t, that is what we will do is that, this del u over del t, this function will differentiate once more with respect to x, using the chain rule again we would get here, del v over del x and here, del z over del x and this is derivative with respect to v, and this is the derivative with respect to z, del upon del v, del v upon del x, we have got 3 and del z upon del x as one. So, what we are getting is 3 times del 2 u over del v 2, from here we would get, 3 times del u over del, del 2 u over del v del z and, from here one times del 2 u over del z del v, we would get 4 times del 2 u over del v del z plus del 2 u over del z 2.

(Refer Slide Time: 49:50)



So, now we have got all the three that is, del 2 u over del x 2 as, 9 times del 2 over del v 2 plus 6 times del 2 u over del v del z plus del 2 u over del z 2, del 2 u over del x del t as 3 times del 2 u over del v 2 plus 4 times del 2 u del v del z plus del 2 u del z 2 and del 2 u over del t 2 as del 2 u over del v 2 plus 2 times del 2 u over del v del z plus del 2 u over del z 2.

Now, substitute this, in our given equation to find out the canonical form, so the given equation is del 2 u over del x 2 minus 4 times del 2 u over del x del t plus 3 times del 2 u over del t 2 is equal to 0. So, substitute it here, what I would get, instead of this del 2 u over del x 2 we would write, 9 times this del 2 u over del v 2 plus 6 times del 2 u over del v del z plus del 2 u del z 2 minus 4 times this will multiply this with, so I would get the coefficient of del 2 u minus del v 2 as, minus 12 the coefficient of this second derivative with respect to both the variables as minus 16, and the derivative of this is minus 4 coefficient of this 1 plus 3 times del 2 u over del t 2.

That is multiply this with 3, and adds it over here, so 3 times del 2 u over del t v 2 plus 6 times del 2 u over del v 2 plus 3 times del 2 u over del z 2, this has to be equal to 0. Now, let us see in this equation, first take the terms involving, second derivative with respect to the variable v, del 2 u over del v 2 coefficient is 9, del 2 u over del v 2 coefficient is minus 12, del 2 u over del v 2 coefficient is plus 3, so 9 plus 3 is plus 12 and minus 12 this will cancel it out.

Coefficient of, del 2 u over del z 2, this is 1 plus 1 here, here it is minus 4, and here it is plus3, again will get 3 plus 1 is plus 4 and minus 4. So, again this term, involving that terms of del 2 u over del z 2 they are been cancelled it out, then what we have got the terms, involving del 2 u over del v del z, the coefficient here is 6, the coefficient here is minus 16 and the coefficient here is plus 6. So, 6 plus 6 12 and minus 16, what we are getting is, minus 4 times del 2 u over del v del z.

Since, minus also I can divide this equation by minus 4, what we would be getting is the actually, del 2 u over del v del z is equal to 0. So, this is what you are getting is, that for hyperbolic equation, this is the canonical form, which we are getting, these canonical forms are helping us in getting the solution. What you are seeing is that is, the equations which we are having, of course, in both the example I have, I have taken that little bit simpler one that is where the right hand side was the 0.

So, the only principal part we have change to the canonical forms, and the right hand side has become 0. But, this says is that, when we are using this kind of forms, it help us in solving the, differential equation, how to solve the differential equation of the second partial differential equation, of second order that we would see, the next. And, how these forms are actually helping us, in getting the solution of the differential equations that is also very important part.

So, we would, learn all these things in the next one, so today we had learn, that is second order differential, partial differential equations, which we have talked about the, principally linear second order partial differential equations. Where the principal part we have, kept in more eye and we said is that, this principal part, taking upon the coefficients of the, second order derivatives, we have derived that is three kind of or three types of equations, we have defined, those things we have called that is characteristics.

We had find out using those coefficients, and with the help of those characteristics, we had find it out that is, what is that new transformation, and with those transformation, we have changed those principal parts to, much simpler forms. The simpler form we have learned about the, two kind of equations, parabolic and hyperbolic, in parabolic, we got only one characteristic so for the transformation the other variable, we had used with our simple inlet that, the other variable has to be linearly independent of the first variable, and it should simplify the right hand side. We have got the simpler forms, that is involving only one variable or the derivatives with respect to the both the variable in the case of, your this hyperbolic equations. Let us see that is, how they are going to help us, in solution of the differential equations, so that is all for today.

Thank you.