Mathematics - III

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Module - 2
Lecture - 17

First Order Partial Differential Equation

Welcome to the lecture series on Differential Equations for under graduate students. Today's topic is First Order Partial Differential Equations. In last lecture, we had learnt about partial differential equations, there we had seen that partial differential equations do have solution in the form of arbitrary functions.

We have seen in some of the first order partial differential equations, that the form of, if that is the unknown function which is in the terms of arbitrary function, that could be decided using side conditions. In one example, we had seen using initial condition, we have done it. Then in some other examples, we had seen as we are changing our differential equation. The argument of that function arbitrary function is changing and we were pointing about or thinking about, that can be find out a method, which decides the argument of that function. So, today we are going to find out this method, which we are calling method of characteristics, the characteristic is that unknown argument that is in the arbitrary function, what is that argument?

This method of characteristic actually says is, that if I could change the differential equation in such a form that on x t plane. It is having one function or one curve on which it is ordinary differential equation; let us see it one by one.

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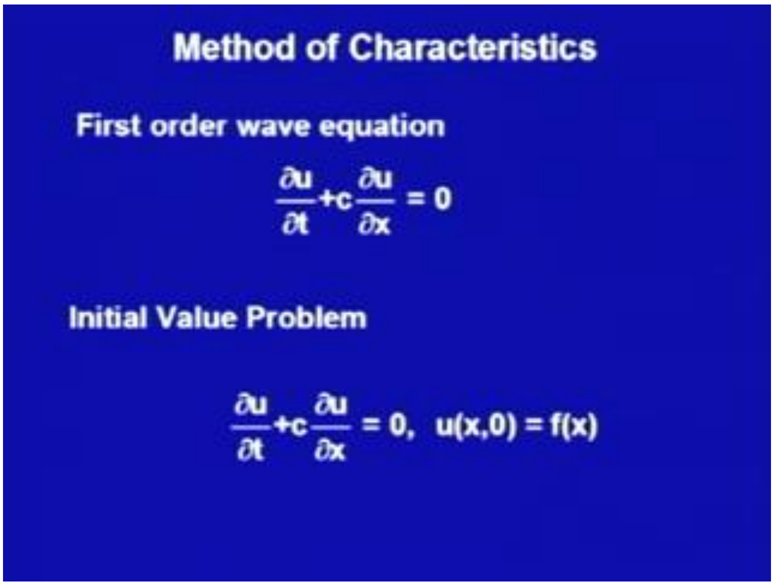

We would try to see it with the help of our first example, that was first order wave equation, del u upon del t plus c times del u upon del x is equal to 0. We actually, we would like to know the unique solution, so we would could move to the initial value problem, that is we can define the initial condition as well that at t is equal to 0 u x comma 0 is f x. Again here, the form of f, we have not take unknown, we just taking in general. Then, what we try to do is we assume that this differential equation, the function is u x t, that is it is surface on x t plane. We take a particular curve on that plane and try to see that with that curve, this surface is changing to it is curve or that is a function.

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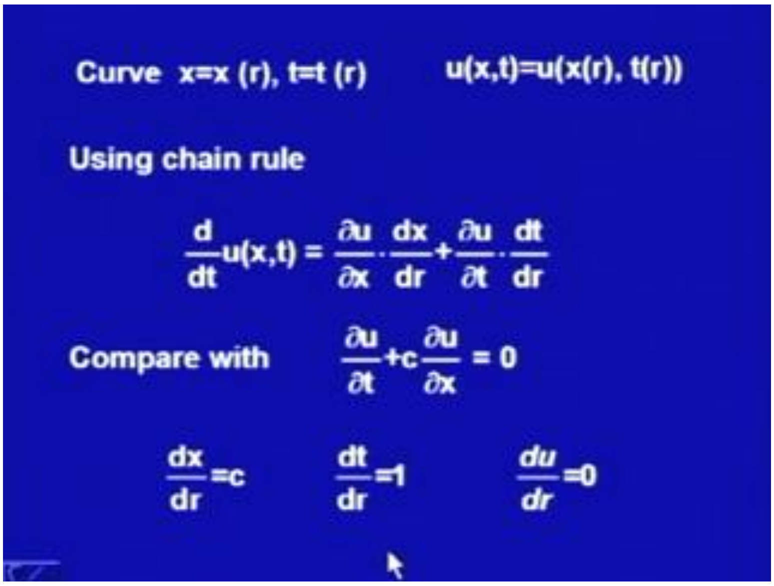

So, we just move in the x t plane, we take a curve and we assume that my function u x t is constant along that curve. So, in x t plane that curve we are denoting or we are using the parametric equation of that curve, that is what we are saying is x is a function of r and t is a function of this you do know that is this is called as parameterization of any equation of the function.

So, let us take x is equal to x r and t as a t r, were r is the parameter of that curve, then my u x t that would be actually that u is a function of x and t and x is a function of r and t is also function of r. So, you can right like this one, what we are trying to do is we are trying to find it out along this curve, this function u x t becomes a constant. So, that we do says that u x t. that is the surface is same as that particular curve.

Now, let us use the chain rule, so that we can get the derivative of u in the terms of r, so d by d r of u x t would be del u over del x times d x by d r and del u over del t times d t over d r. Now, compare it with this given equation del u over del t plus c times del u over del x is equal to 0, now the coefficient of del u over del x, here is c and here it is d x by dr.

So, I do get d x by d r is equal to c, similarly the coefficient of del u over del t here is constant and here the coefficient of del u over del t is d t by dr. So, d t by d r would be 1 and d by d r of u x t is 0, so we write it du by d r is equal to 0. Now, so what we have got, we have got actually, that we are changing this partial differential, we are getting is a relationship between x and r, t and r and u and r, that is with respect to that parameter. We are getting all our first order ordinary differential equations, we do not know, how to solve them.

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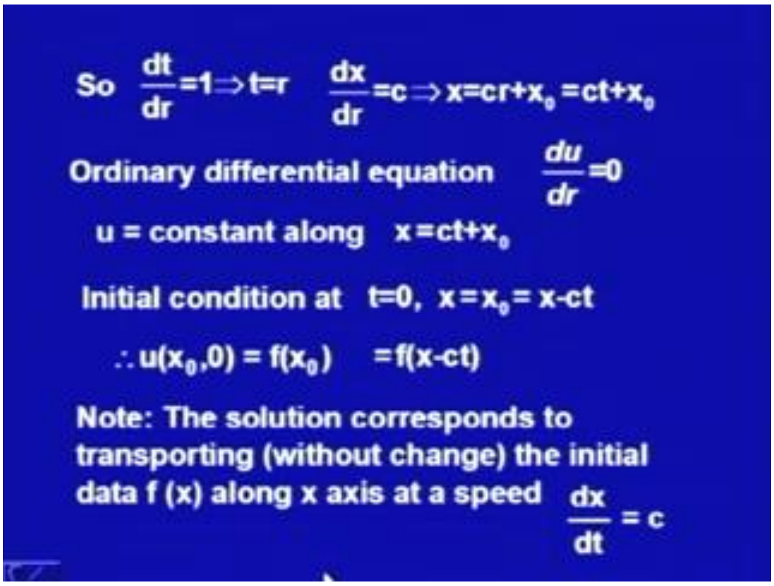

So, let us try to see one by one, let us first take d t by d r is equal to 1, what this implies this says t is r, so now our parameter we have fixed up as the second independent variable t. Now, d x by d r is equal to c this say is that x should be c r plus x naught, that is a constant and since r is t, the parameter is same as the second variable t.

So, I would take it as ct plus x naught, this x naught, why I am taking this constant as x naught, because if u see x is equal to ct plus x naught in x t plane, then this is a line. So, this constant is nothing but the x intercept on the ct plane on the x t plane. And, the ordinary differential equation, now what we are remaining is whether in the unknown function u, that is d u over d r is equal to 0, what it says is that, if I try to solve it, that u must be constant along this correct curve ct plus x naught.

This is what, we are getting is that, is when we are changing it to this one, we are getting this is constant along this curve x naught. So, now I use the initial condition at t is equal to 0, at t is equal to 0, x would be x naught, now this x naught, I would write from this equation is x minus c t. Then, now I am using the initial condition for my partial differential equation, that says is u at x comma 0 is f x.

Now; that means, we are using t is equal to 0, at t is equal to 0 x is x naught, because we are now not working x and t are not separate, we are talking about that both x and t and along certain curves. So, we are talking about along this line, so t is equal to 0, my x is x naught, this is given as f of x, so it should be f of x naught. Now, x naught along this line is nothing but x minus c t, so I would write it as f of x minus c t.

Now, you do remember this is, how we have got, that the unknown function, we have got the solution as f of x minus c t, the argument was x minus c t and that f the arbitrary function we have define by the data, that is what is the initial condition f. So, now what we are talking about is in this method, we are seeing is that our partial differential equation is we are changing to an ordinary differential equation.

In such a manner, that my unknown function u remains constant along that curve, that is we try to find out a curve, given this differential equation. We try to find out a curve, such that my unknown function becomes a constant along that curve and that curve is going to give me the argument of our function.

So, in this particular equation, which if you do remember we have called it a first order wave equation or it is transport equation. Let see that is what it is actually doing it is transporting the initial data effects along the x axis at a speed c d x by d t, that is the speed of x with respect to the other variable t is c.

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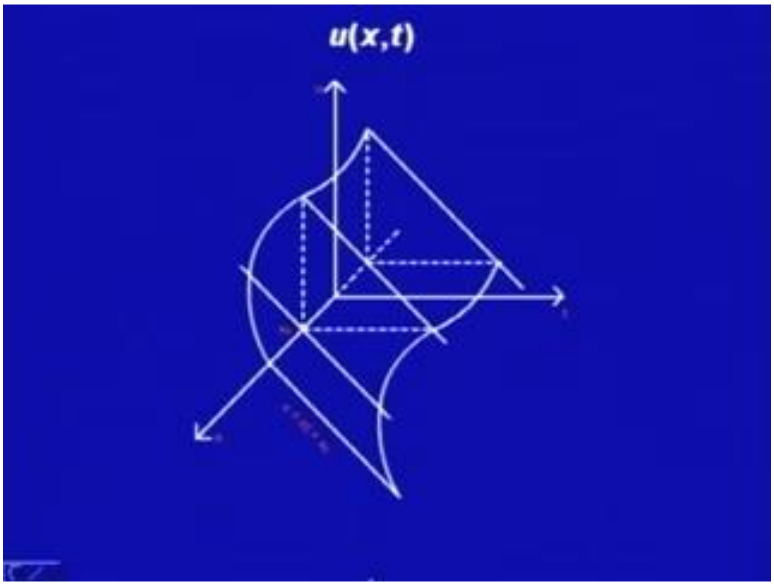

Now, you see that is, how it is coming up if this is x y plane, so we are getting is that from this points x naught, we do have this characteristic and so you again see it, that from these points the data, this is the surface let see, how we are moving.

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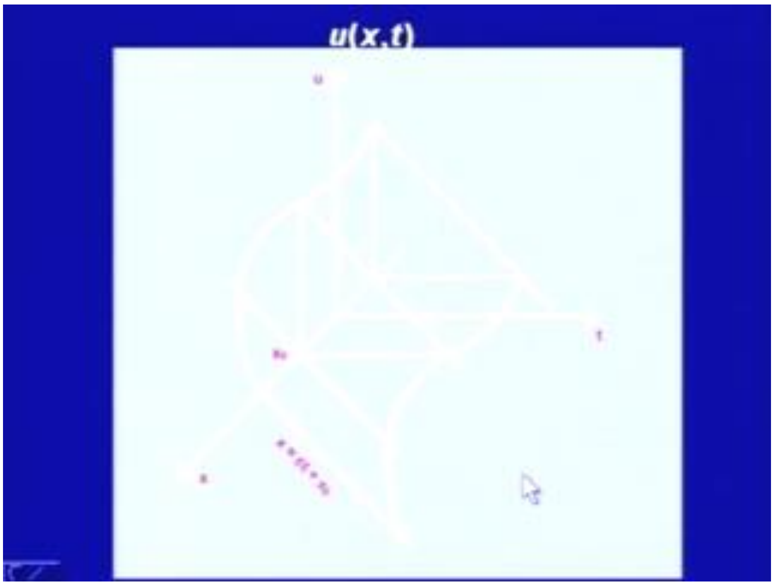

You see, this is my characteristic line x is equal to c t plus x naught, this is in x t plane. Now, as I move this line, so this is what is actually the function u is this surface, this surface which is the function u as I am moving some along this characteristic line in the on the x axis. So, from here to here this is distance is x naught, that is why the intercept is x naught.

So, now, if I am moving towards this point then my surface u, this is constant along this line, what we are getting is u would be at one particular point. If you see is this surface this is you just try to see it in the three dimensional one, so what we are getting is that is u dimension, you would be getting it here the constant along this whole line on this surface.

So, what we it is doing is, it is transporting my initial data f x, initial data along this x axis, so we are getting is my surface is you see that surface is changing. So, as we are changing from this point to this point, we are just changing my moving this line parallel to this One. Then, my surface is moving in this direction.

Then, when we are changing it one, the surface is moving in this direction, what we are getting is that is along this surface. This is changing or this is the transporting my initial data on this surface, this is what we are meaning, so this method is called the method of characteristic.

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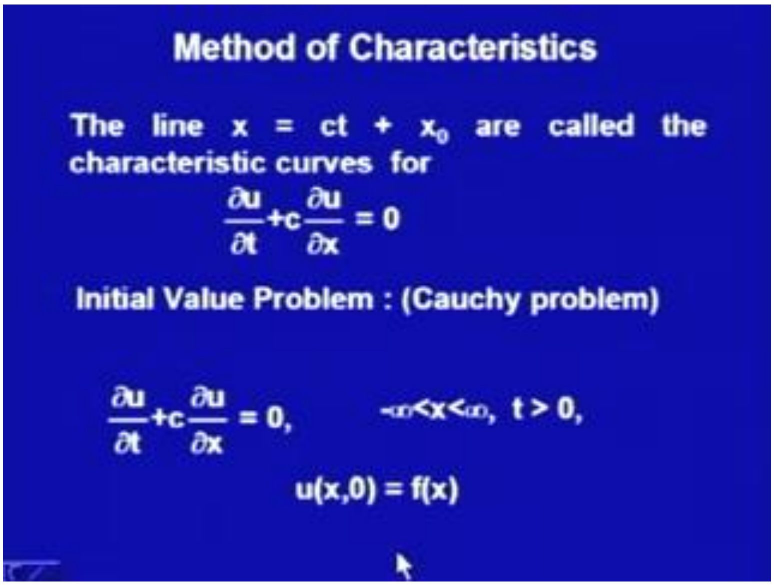

And, this line x is equal to ct plus x naught, this is called the characteristic curve for this first order wave equation del u by del t plus c times del u by del x is equal to 0. Now, this initial value problem, which we had started that del u by del t plus c times del u over del x is equal to 0, in the whole reason minus infinity, that is my x was on the whole real line. But, t was positive on it, t as I said is that x y z, we are taking as the dimensional spaces.

Cartesian coordinates and t as the time, time has to be positive, then if I had use the initial condition at t is equal to 0, that time t is equal to 0, if it is f x, what we had seen in this one. That is, if I could find out a characteristic curve in u x t space, such that that curve has to be in the x t plain, along which my surface u x t is constant, that is this f x remains constant at that of all x.

Then, this is called this is a having unique solution and it is called the Cauchy problem, why this says is that along this characteristic curve. So, let me call this characteristic curve, here it was coming as a line. So, in general let me just try to because, the whole idea of all these things is little bit above the level of here, so I will try to explain the things with the help of examples only.

So, let me just say is that general solution; that if along a curve gamma in x t plane or that is in the plane of independent variables. If, I could find out a curve along, which the unknown function is constant with respect to all those independent points. Then, it guarantees that, they would be that initial value problem, will have a unique solution, this is, what is called the Cauchy problem.

Now, let us see in the one example; that is how we are going to do if you do remember that is we have already done this one. We had find out that f of x minus this is the characteristic and from this characteristic, we had find out that is x naught is x minus c t that is the argument and would initial condition, we could find out that particular function.

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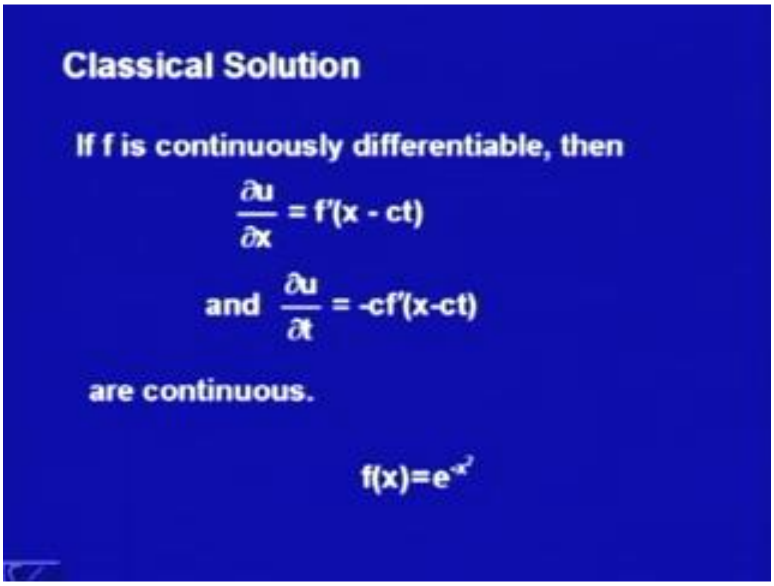

So, now if f is continuously differentiable, that is the initial condition function f, this is continuously differentiable, what does it mean, it says is that it is partial derivatives del u over del x and del u over del t they are also continuous. Then, we call it to be a classical solution, that is then u x t is equal to f x minus ct, that would be called a classical solution.

Say for example, if I if you do remember in the last lecture, we had used One function as f x, as e to the power minus x square, this is differentiable function, it is continuous, it is derivative. So, will also remain continuous, how many times we differentiate it, so this is a classical solution, moreover if you do remember, we had use one more function f.

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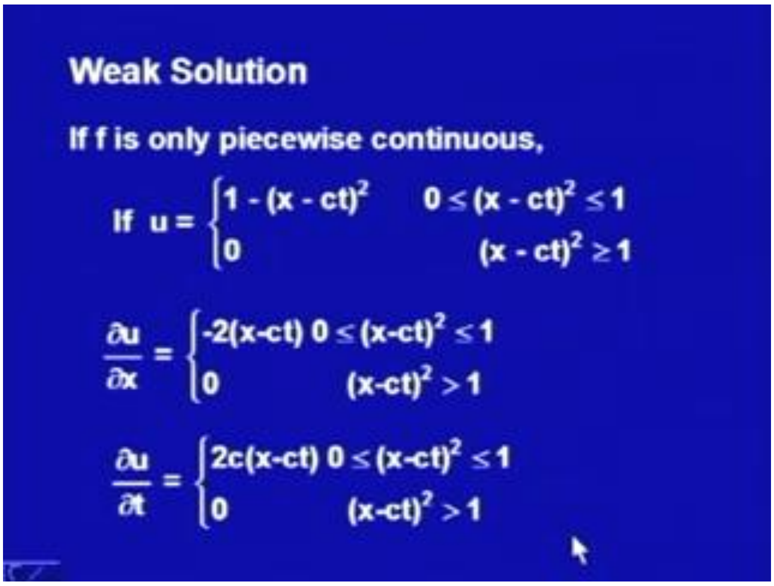

So, let say what we are seeing is, if f is only piecewise continuous, that is it is partial derivatives may not be continuous. So, if you do remember we have taken one example of this form, this function is continuous, piecewise continuous, differentiable, but it is derivative. If you do remember del u over del x as minus 2 times x minus c t in the recent x minus ct, whole square line between 0 and 1 or that is x minus c t is from minus 1 to plus 1. And if, it is outside we are getting it, that this is both of these derivatives, they are not continuous. Then, we call this solution as a weak solution, of course, it is satisfying the equation, but its derivatives are not continuous, then it is called the weak solution.

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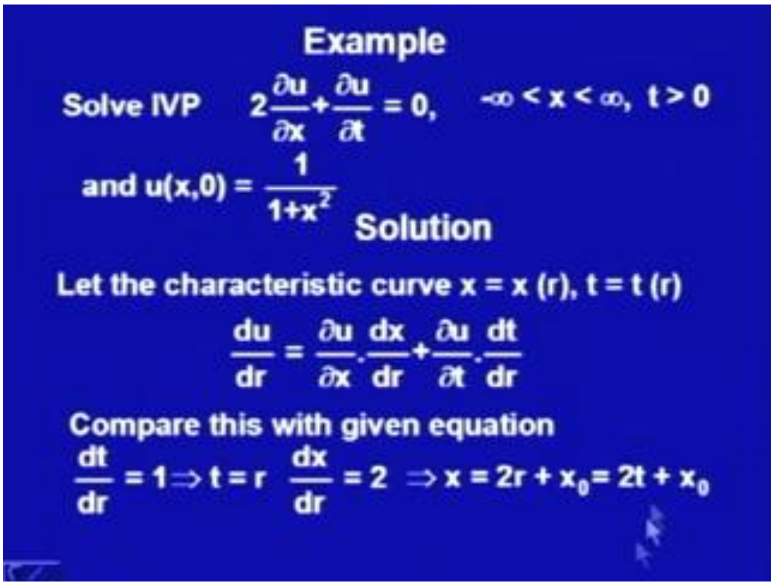

Let us, do one example solve the initial value problem, 2 times du over d x plus du over d t is equal to 0 with x on the whole real line t positive and initial condition at t is equal to 0 is given as u x comma 0 is 1 upon 1 plus x square. Now, we see this is again my first order wave equation, let us just try to find out this one, the characteristic curve, we are assuming in x t plane as x r and t r with the parametric forms, then we would be having u x t as u x t, x r and t r.

So, using the chain rule du over d r would be del u over del x into d x by d r plus del u over del t plus d t by dr. Compare this with this given differential equation del u over del x I would get as so let us try from here d t over d r, we are getting as one this says is t is equal to r. So, my parametric parameterisation r, we are changing it to the second variable t, that is I can say is my function u x comma t can be given as u x a function of t and t in the single variable we are getting. And del u over del x is coefficient d x by d r is 2 which give me that x as 2 r plus x naught at 2 t plus x naught because r is same as t, from the first equation.

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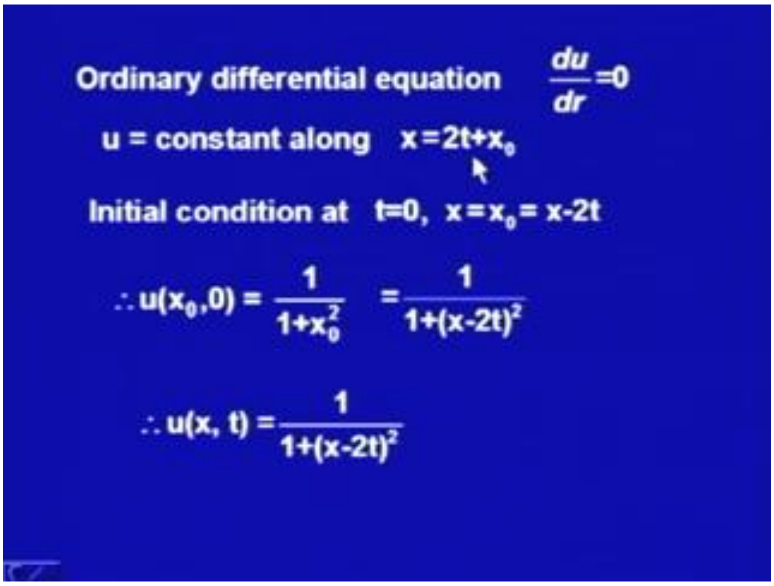

Now, the ordinary differential equation which my partial differential equation is changed that is change ordinary differential equation is d u by d r is equal to 0. It is solution, gives me that u is constant along my characteristics, what the characteristics we had find out x is equal to 2 t plus x naught. So, from here, if I use the initial condition at t is equal to 0, x would be x naught, so my characteristic x naught would be actually x minus 2 t.

So, what will be my solution, my solution would be u x naught at coma 0, should be f of x, if you do remember, we were having our initial condition is 1 upon, 1 plus x square. So, at x is at t is equal to 0 x is x naught in along this curve, so I have to replace it by x naught square that says is this has to be replaced as 1 upon, 1 plus x minus 2 t square because we want the general solution in the form of 2 t.

So, in the form of x and t, so we have got our general solution, we have got the general solution, u x t as 1 upon 1 plus x minus 2 t square, this is my unique solution in of the initial value problem. Since, we can find out a curve along which the unknown function u is constant, that says is that unique solution does exist and that solution we can find out in the form of its characteristic, the characteristic we are defining by this, because that constant value.

So, we are using the initial condition at the characteristic level gives me the characteristic function or the characteristic of this one. And then again using it we get the functional form, so we get the solution one upon one plus x minus 2 t square.

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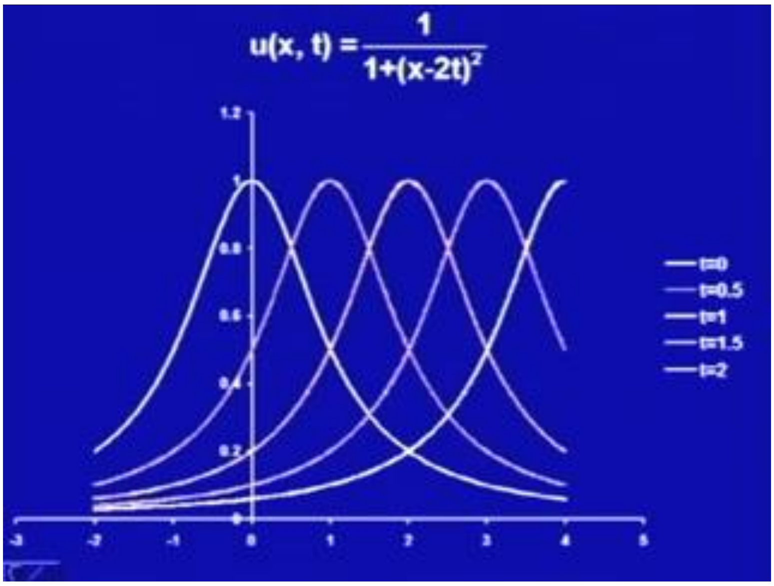

Now, let see what a graph of this function, so if I use t different values for t, at time t is equal to 0, it should be 1 upon 1 plus x square, this is the curve, at t is equal to 1. We are getting is the curve is this 1, at t is equal to 2, this third curve, t is equal to 1, t is equal to 0, t is equal to half, t is equal to 1, t is equal to 1.5 and t is equal to 2.

These curves, I have taken that is at different t, I have taken these curves, as it is a wave equation. This is the transporting equation, what it is doing is, so now you see is, if I try to make a surface out of this a would be getting this surface like this one, which is containing this kind of function, you can just imagine it in the three dimensional, one that is how this surface would be looking like. This is transporting this initial function, towards it is right, because my c was plus 2 over here. So, this speed at which it is transporting is along this x axis; that is it is at the speed of 2, so we are getting it that, this function we are transporting towards the right.

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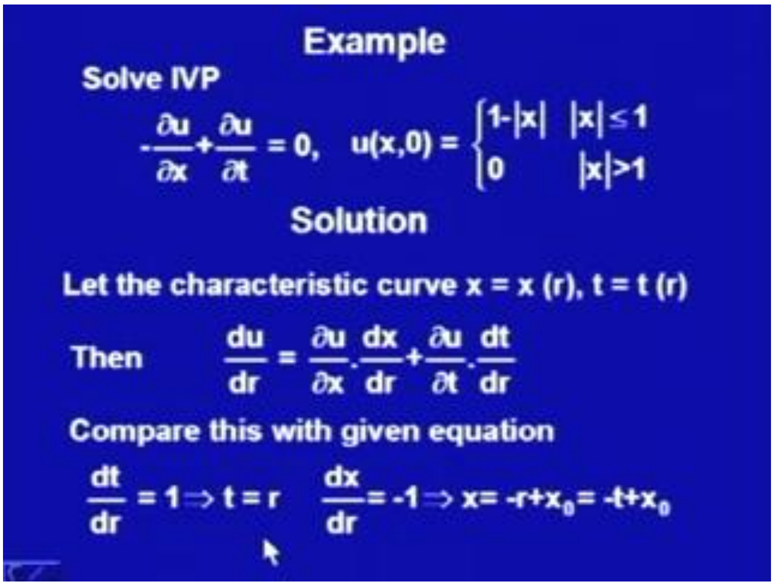

Let us see one more example, now my differential equation is minus del u over del x plus del u over del t is equal to 0 and the initial condition is u x comma 0 is equal 1 minus mod x, for mod x line is less than 1 and 0 for mod x greater than 1. You see here, again this is first order wave equation, c is here minus 1.

Let see, what happens to the solution, again we will go with this method of characteristic. So, we will assume the characteristic curve to be of the form, x r, t r and then we get this first order equation, ordinary differential equation d u or d r is equal to del u over del x times d x over d r plus del u over del t times d t over d r. Now, compare this ordinary differential equation with the given partial differential equation, we get the characteristic, what it gives d t over d r as 1, which says t is equal to r. So, the parameter is we are fixing at the second variable t, derivative of coefficient of del u over del x d x by d r is minus 1 gives x as minus r plus x naught r minus t plus x naught, so this is my characteristic curve.

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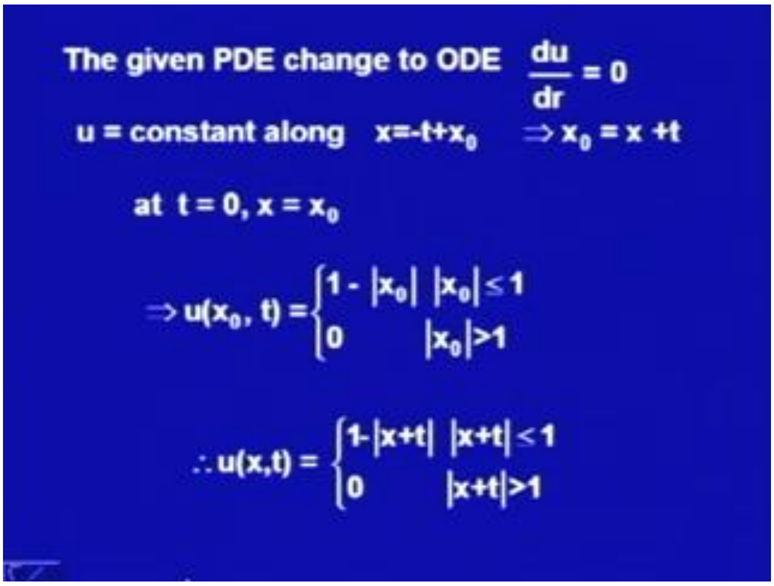

Now, the comparison of du over d r is equal to 0, that says is my partial differential equation has been change to the ordinary differential equation du over d r is equal to 0, r has to be t. So, we get that u should be constant along this characteristic curve x is equal to minus t plus x naught, that says is x naught must be equal to x plus t, what we are getting at t is equal to 0, x is equal to x naught. And the characteristic, we would define as x naught is equal to x plus t or this is x plus t is our characteristic.

So, what will be our solution, so at t is equal to 0, x is x naught, so u x naught t would be 1 minus x naught, for x naught less than 1 and for 0 for x naught greater than 1. Now, that says is now change this to the characteristic x naught 2, x plus t, we would get the solution as u x t as 1 minus absolute value of x plus t.

For x plus t, lying between minus 1 to plus 1 and 0 for x plus t lying between outside the reason minus 1 to plus 1 are you could say is here that x is lying between t minus 1 to t plus 1. And, x is lying outside t minus 1 to t plus 1, let see that is how this function is looking like.

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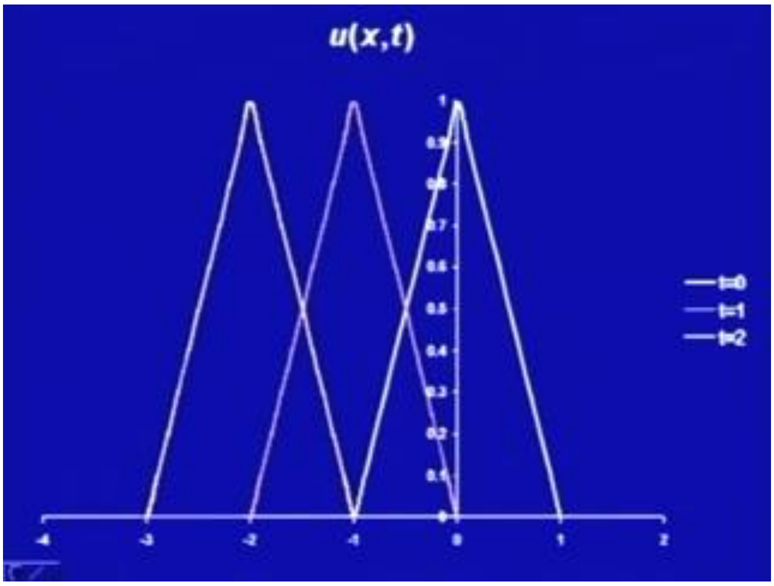

We are having this is at t is equal to 0, that is initially function is minus 1 to plus 1, because t is equal to 0, what we would be having is that is function would be 1 minus absolute value of x for minus 1 to plus 1. When, t is equal to 1, we would have that the function would be 1 minus x plus 1, absolute value of x plus 1 in the reason, x plus 1 is lying between 0 and 1 or x is lying from minus 2 to 0.

So, what we are having is at t is equal to 0, it is this 1, at t is equal to 1 the function has shifted towards the left, at t is equal to 2 the function is shifted again more left to this 1. Now, we have seen the first order wave equation in which we have seen that c to be positive, when I have taken c is as 2, whatever be the initial condition, initial condition is just giving me the way shape of this wave that is all.

And, it is wave is shifting, that is what you are saying is that is if the wave is like this one, then it is going out, if it is triangular 1, it is just going like this one. So, what we have got, when c was positive, in all of our examples, where I have not given any particular value of c, all the places I have taken c to be positive or more explicitly, I have taken in all the examples in the last lecture also as c as 1.

We had got all the time, that from t is equal to 0, as we are moving towards the positive values of t, I am getting that my equation was my wave moving towards the right. Here, I have that taken the first example, where my c is negative and what we are finding out that it is shifting towards the left.

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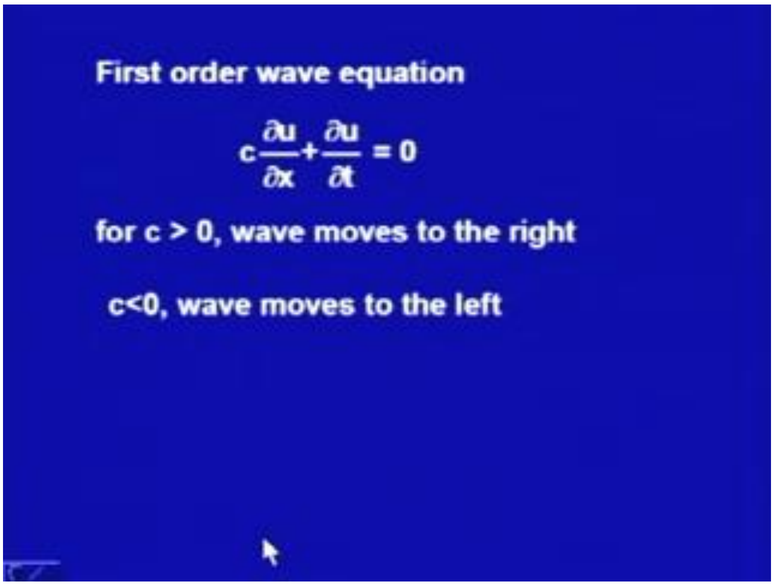

So, let us summarise from these examples, what we have got that in the first order wave equation first c to be positive wave moves to the right and for c indicative wave moves to the left. So, the sin of c defines the movement of waves, so that is why this first order wave condition, wave equation is actually called is that is transport equation, whether it is transport towards the right are transporting towards the left.

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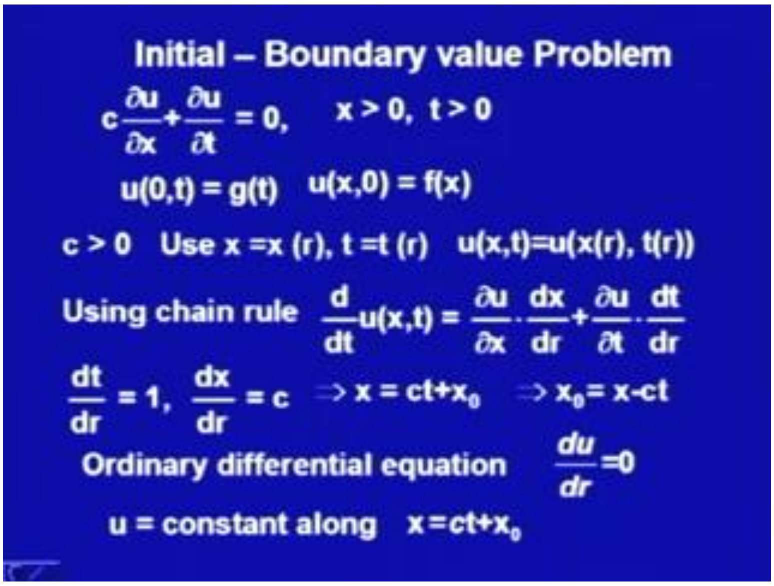

Now, let us come to another problem, where this it is not only initial value, but we are having is initial and boundary value problem. What the difference, in the first kind of equations, what we have seen that is again I would be going with this first order wave equation, we had that my x was 1 whole real line. Now, suppose I do come to the semi finite plane; that is let say that is semi finite plane, let say that is positive plane.

So, again I am using my first order waver equation, but now my x is also positive and t is positive, then I would put one boundary condition on x also. So, what will be the initial condition is, this boundary condition would be that u at x is equal to 0, t is of function g t and initial condition is that u x comma 0 is f x. Let us see that, what a solution would be does the start I give me solution.

First, I will consider the case when c is positive, we are seen in the last example that is c positive and negative was giving me the wave is moving towards right or left, that was in the initial value problem. There, I was not having this condition of this boundary condition; moreover the x was on the complete plane; that is from minus infinity to plus infinity.

Again, I go to this method of characteristic, so will use the characteristic curve as the parametric form x is x r and t is t r. So, u x t would be u x r, t r using the chain rule again we would get d over d r of u x t is del u over del x d x by d r del u over del t d t by d r, which will give me. Now, compare it d t by d r as 1 and d x by d r as c, so both these equations would give this give t is equal to r and from here we would get x is equal to c r plus x naught or r is same as t.

So, this will give me x as ct plus x naught, again this x naught is the x intercept, so the characteristic, I am getting is x naught as x minus ct. You see is that is when we are changing, since my equation was not changing, whatever be the other conditions my characteristic will always be same. So, the characteristic is actually, because of the equation, the ordinary differential equation would be d u by d r is equal to 0, that says is u should be constant along this characteristic line at this characteristic curve.

So, at using this initial condition and now I do not have only the initial condition, I do have this boundary condition as well, what it is actually doing this, let us just try to see this characteristic equation this is a line. Now, my plane is only the positive side of x and positive side of t, let see this characteristic, that is x naught, this is a constant x minus c t.

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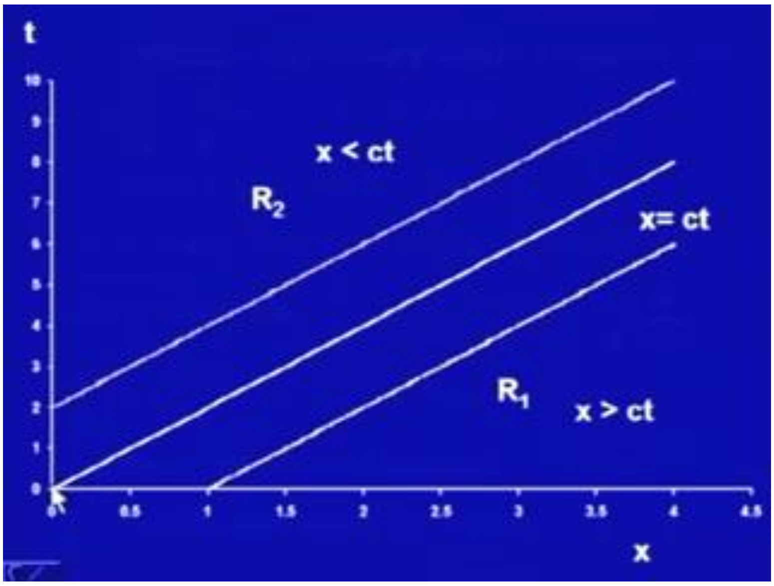

It is actually dividing my entire x t plane into so this characteristic is actually dividing my whole x t plane, you see we do have this line x is equal to c t, that is when the constant x naught is 0. This is dividing the whole plane into two parts, one is reason r 1, another, I am calling the reason r 2. In this reason r 1, x is greater than c t or you could say is that my x minus c t would be positive, what it says this characteristic line over here, this would be originating from the x axis.

And in this reason r 2, x minus c t would be negative or the characteristic line in this is originating from the t axis; that means, in the reason r 1. This characteristic line because we, if you do remember for initial value problem, we had use that the characteristic, that from the initial condition. Now, initial condition, where we have to take in the reason r 1, it is originating from x axis, that is I could take x is equal to 0; that means, I have to use in the reason r 1, the boundary condition.

And here, it is originating from the t axis, so that says is t is equal to 0, this would decide from where my characteristic is originating or what should be my x naught. So, I would get here this line would be deciding with using my initial condition. So, thus what we are getting is that my solution would be decided by both boundary conditions as well as by the initial condition. Boundary condition, we would be using in the reason; where of r 1 that is when x minus c t is positive and we are using the initial condition in the reason, where x minus ct is negative.

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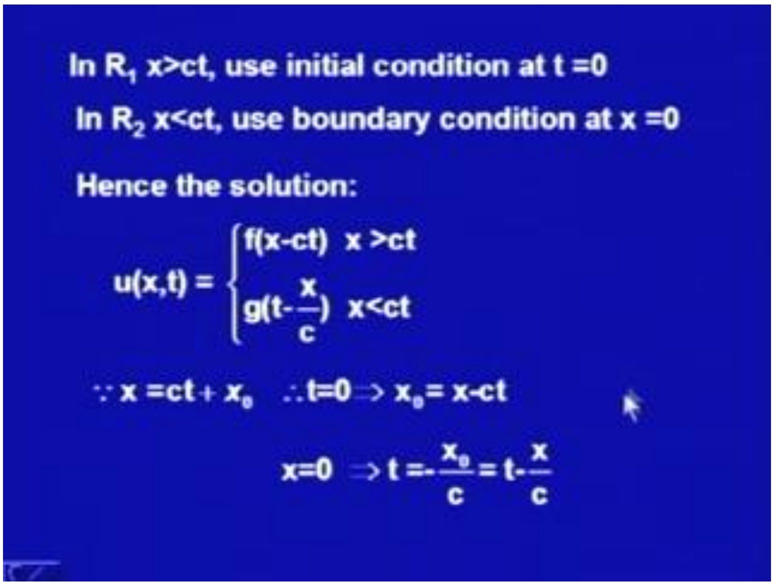

So, what we are getting is in the reason r 1, x is greater than c t and my characteristic was originating from the x axis. So, we do use t is equal to 0, we get and in r 2, this x is less than c t and we would be going to use this boundary condition. So, the solution would be now u x t as f of x minus c t, that was my now you see, you will find out that is, how I had find out this x minus c t, as I said is in the reason r 1, that is, when x is greater than c t, I have to use initial condition, initial condition at says that at t is equal to 0.

If I see my characteristic x is equal to x plus x naught plus c t, x would x naught and that x naught we would be saying is x minus c t. So, this is actually we had already done in our initial value problem, so my solution would be f of x minus c t. Now, let us come to this reason r 2, x is less than c t, this is what we are writing here, I have to use the boundary, because the characteristic is originating from the t axis.

So, I have to take x is equal to 0, at x is equal to 0, what is the value of t, so value of t is, how do we find it out, at x is equal to 0, let say value of t is t naught, how do that find out this t naught. So, let see at x is equal to c t plus x naught, so at t is equal to 0, x naught is x minus c t and at x is equal to 0, what I would get x is equal to 0. I would get c t plus x naught is equal to 0; that is t is equal to minus x naught by c.

So, t naught would be actually now x naught, what we are getting is x naught, if I am going to use, because this is has to be the same characteristics. So, x naught has to be decided by this x naught is nothing but x minus c t. So, whatever is this x and t, that x minus c t has to be same, so x naught, I am using x minus c t, so if it, I am using x minus c t, this gives that is my t naught would be t minus x by c.

So, we are getting is that at x is equal to 0, t has to be t naught, so u 0 t is g t. So, at t naught it should be g of t naught and t naught is nothing but t minus x upon c. So, what we have got the solution, we have got the solution u x t would be f of x minus c t, when x is greater than c t and it would be g of t minus x by c, when x is less than c t. Let us see is this actually the solution of our differential equation and is it satisfying both the boundary condition, that is really the solution of our initial boundary value problem.

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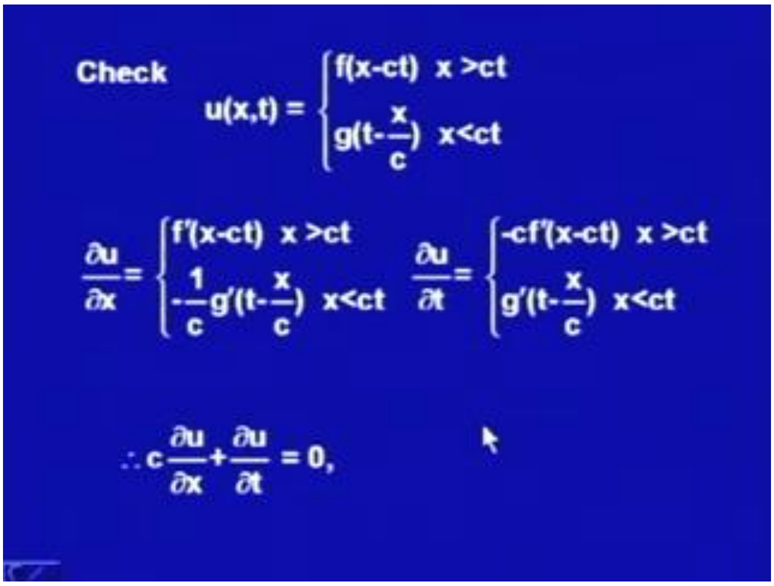

Let us try to see, this is what we have got my u, so here f and g are general, what would be del u over del x, f dash means derivative with respect to x minus c t, del u over del x means derivate of f with respect x minus c t. And derivative of x minus c t with respect to x, that is 1, this is in the reason x is greater than c t, in the reason x is less than c t, I will differentiate g. So, I am using both the places, this dash for the derivative, but you have to get the reference, that is the derivative has to be with respect to that argument.

So, here this derivative with respect to t minus x by c and the derivative of t minus x by c with respect to x, that would be minus x by minus 1 by c. Since, t is constant over here. So, it is minus 1 by c g dash t minus x by c, whenever I am using this f dash or g dash with different arguments; that it says is the derivative with respect to that argument.

Now, get del u over del t in the similar manner, the derivative with respect to x minus c t and the derivative x minus c t with respect to t would be minus c.

And the derivative of this with respect to t minus x by c is g dash t minus x by c and derivative of t minus x by c with respect to t would be 1. So, this one now we see, if I take c times del u over del x, I would get it c times f dash x minus c t and here it is in the reason x is greater than c t and the same reason del u or del t is minus c times f dash x minus c t. That says is they are, this plus, this would be 0, that is c times del u over del x plus del u over del t would be 0 in the reason x greater than c t.

Similarly, in the reason x less than c t, it would be minus g dash t times x minus c and this would be plus g dash t minus x by c, that says is again the sum of these two would be 0. Hence, what we are getting is that this solution is satisfying my differential for given partial differential equation. Moreover, when I take t is equal to 0, t is equal to 0, means x is greater than 0 and here this reason will transfer to x is less than 0.

Now, if you do remember our differential equation was for the reason x greater than 0, that is when t is equal to 0, this portion is not there or this is not the reason in which we are interested. So, at t is equal to 0, u x would be f of x, this is what is my initial condition. Then, at x is equal to 0, x is equal to 0, says is c t should be less than 0, whatever would be this c with the positive or negative t, so c is positive.

So, it says is that ct is less than 0; that says t is less than 0. So, this is redundant reason, because we are taking is t to be positive, this is the only reason, which would b f interest for us is t is positive, here when x is equal to 0, I would get only g t. So, u 0 t is g t, that is the boundary condition, we are satisfying, so this is satisfying my boundary conditions initial condition all the differential equation.

So, this is the solution of this initial boundary problem, what we have got is, if we do have initial boundary value problem for the first order wave equation. The method of characteristic said that is we could find out a characteristic for c positive, which satisfies both the conditions. Initial condition and the boundary condition and we have got the solution, so this is solution for this one.

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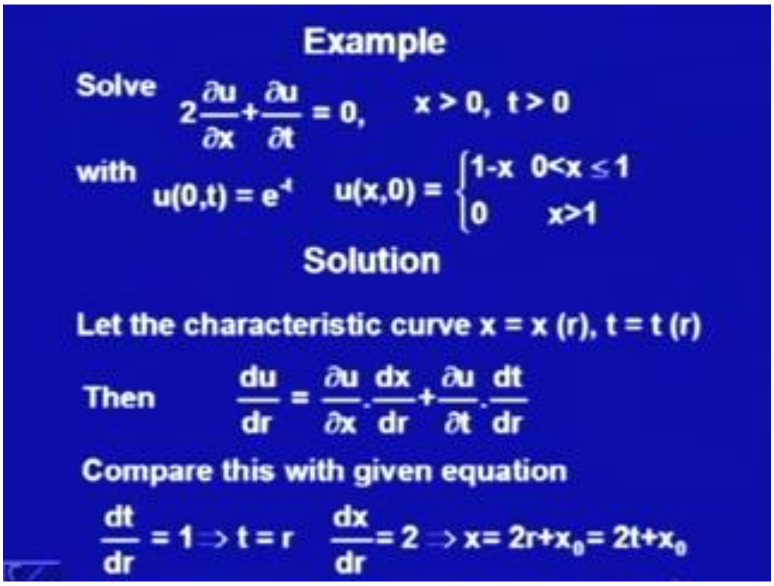

Now, let us do one example, solve this partial differential equation 2 times del u over del x plus del u over del t is equal to 0. For x positive and t positive with boundary condition u 0 to t as e to the power minus t and initial condition u x comma 0 as 1 minus x in the reason 0 to 1 and 0, for x greater than 1.

We will go with the solution as usual first find out the characteristics of this equation, so that we could find out, so for characteristic curve, again we would use as x r and t r. So, the change del u over du over d r as del u over del x times d x over d r plus del u over del t times d t over dr. Compare this with given equation, I would get d t by d r is equal to 1, implying t is equal to r, d x by d r is 2 gets x is equal to 2 r plus x naught or 2 t plus x naught, this is the constant x naught. So, the characteristic curve we have got x is equal to 2 t plus x naught, that is again you see is that this characteristic curve of this equation, we had already find out in one example.

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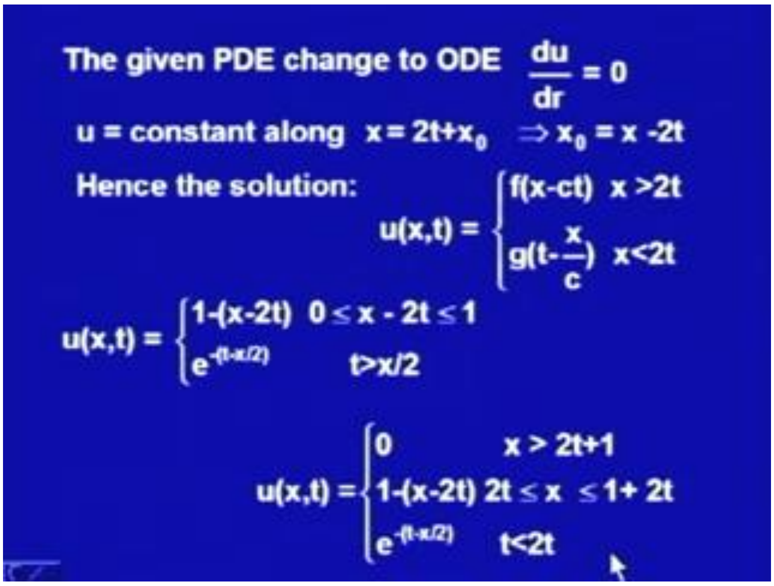

Now, given partial differential equation is changing to the ordinary differential equation du over d r is equal to 0, which gives that u would be constant along the characteristic x is equal to 2 t plus x naught. Now, up to find out this characteristic, we would use that is, now rather than again going with the solution, that is x is equal to x minus 2 t is dividing my entire plane into two reasons, in one reason, the characteristic would be originating from the x axis, that is I would use t is equal to 0.

In another reason, the character x t would be crossing from the originating from the t axis, that is I have to use this x is equal to 0 condition to determine at particular characteristic. So, we do not know that is, how to do it, so we are getting is x minus 2 t, so the solution, if you do we just in the last slides, we had find out that is f of x minus c t for x greater than c t. So, this c would change here to 2 c is 2 actually and g of t x minus c for x less than c t and again c is changing to 2.

So, let us put up in our 1 what is my f our solution was would be our u x 0 was 1 minus x in the reason 0 to 1 and 0 elsewhere. So, we have got it, when x is greater than 2 t, that is 1 minus x minus 2 t and the boundary condition was u 0 t was e to the power minus t, so we would be getting e to the power minus t minus x by 2 this x by c. So, x by 2 for t greater than x by 2 or that is x is less than 2 t, so I have I am writing it as t is greater than x by 2.

So, let us see that is, what is this function is actually x minus 2 t is lying between 0 and 1, that says is 2 t is less than x is less than 1 plus 2 t and it says is x is less than 2 t, there some reason is this still left. So, what we have been left for the reason x greater than 2 t plus 1, it would be 0 for the reason x line between 2 t to 1 plus 2 t, the solution would be 1 minus x minus 2 t.

And for the reason x is less than 2 t, this is not t this is x is less than 2 t, it should be e to the power minus t minus x by 2. We can check that, this also satisfies our partial differential equation and moreover it also satisfying both the initial condition and the boundary condition.

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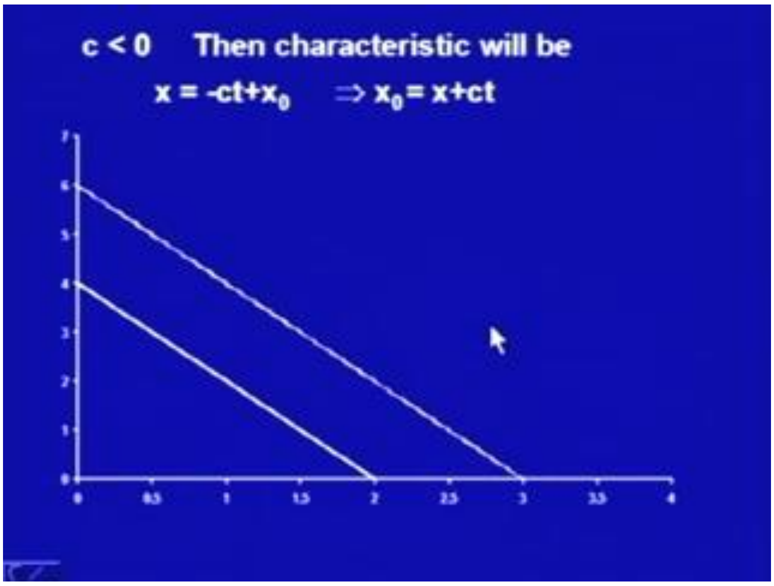

Now, let us take the case of this initial boundary value problem, when c is less than 0, the characteristic would be the same x is equal to c t plus x naught now c is negatives. So, I have taken is minus c, so that we can make it more visible, so when c is negative, I am taking it minus c, so c is positive and it is minus c, what will be my characteristic, my characteristic would be x plus c t.

Let us see this characteristic, what it is doing in our x t plane, in my x t plane; this is the characteristic x plus c t first term constant x naught. So, you see is that is here this if take t is equal to 0, this is 4 and at so we are getting is that is 1 characteristic, this is another characteristic.

Now, my characteristic is moving in this manner along the x axis, they are moving in this manner parallel lines, what they are doing is, this characteristic is also crossing the t axis. Characteristic is also crossing the t axis, that is the same characteristic is at the x axis.

Now, when c was positive, we have got that clearly, it was divided like this one, so we were having is that the characteristic is either crossing the x axis or crossing the t axis and that says is I can use either initial condition or the boundary condition. Here, what I want, I want along this One, I want x naught and t naught, such that both the initial condition and boundary condition is satisfied, what it says is, if I take t is equal to 0, then my x should be x naught and if x is equal to 0.

Then, my t should be t naught, but I be could not find out any of these things, that is any line which is satisfying, what t is equal to t naught, what we would be getting is that is my function should be f x. At this point on g t at this point, which is not possible that the same one because when we are saying it is a constant, the same constant could be with a value of the two functions at two different points.

So, thus for c negative, there is no solution for initial boundary value problem, where the partial differential equation is the first order wave equation and we are using both the initial condition and the boundary condition at x is equal to 0. Let us see, the method of characteristic in semi linear equation, again will explain with the help of an example.

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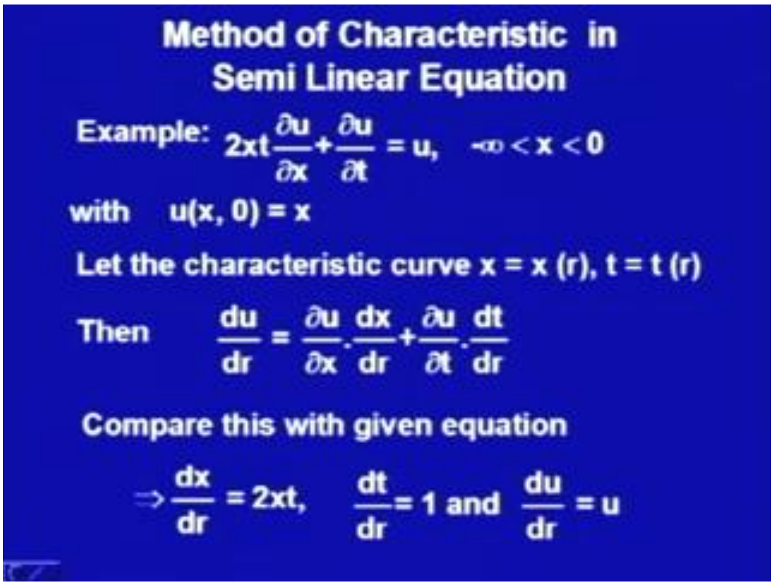

So, this is a semi linear equation; that means, the equation whose coefficients are involving only x and t. So, 2 x t del u over del x plus del u over del t is equal to u, a reason is this one with u x 0 is x now. So, again try to find out the characteristic, so using the same method, that is characteristic curve on the x t plane with x r and t r using the chain rule du over d r will be del u x del u over del x d x by d r del u over del t d t by d r. If, I compare with this one, what I would get it as from here, that d x by d r is 2 x t, d t by d r is 1 and du by d r is u.

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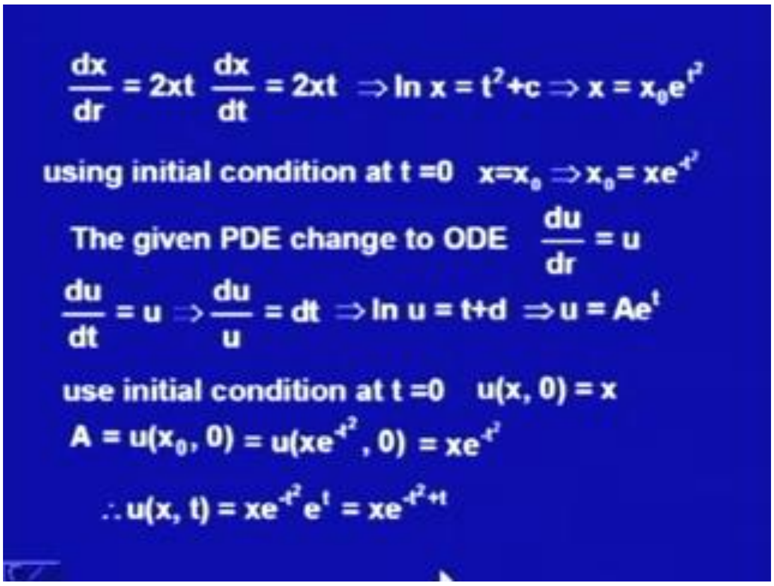

So, what we are getting from here that d x by d r is 2 x t, d x by d t is 2 x, so t is equal to r that is d t by d r is equal to 1 is giving. So, we would get it d x by d t is equal to 2 x t, try to find out the solution of this we would get d x by x is equal to 2 t d t, that says is differentiating on both the sides, integrating on both the sides log x would be t square plus c. This says is x is equal to x naught times e to the power t square, again I have taken it x naught just to keep the similar manner as we are using in the line equation.

Of course this is some constant, which is e to the powers c, so x naught we have taken because, we want the curve in x and t. So, we are taking it is in x naught. Now, if I use the initial condition at t is equal to 0, I would get that x is equal to x naught, because t is equal to 0; e to the power t square would be 1. So, x would be x naught, that says I want x naught, so x naught, I would get from this equation itself, x naught would be x times e to the power minus t square.

So, this is what is my characteristic now, so you are getting is that is when we are change this equation, I have got the characteristic as x times e to the power minus t square. Now, given partial differential equation was changing to the ordinary differential equation, that is d u by d r is equal to u. Now, r is same as t, so I would be writing it as d u by d t is equal to u, again this is first order ordinary differential equation, we can find it out using the variable separable method and we can just integrate on both the sides.

So, that says is d u over u is d t integrating on both the sides, log u is t plus d some constant, that says e is equal to A times e to the power t, where is some constant, how to determine this constant. For this again, we have to use the initial condition, initial conditions is at t is equal to 0, u would be at initial condition t is equal to 0, u x comma 0 was x, that was our initial condition was given to us for this 1.

So, at t is equal to 0, x is x naught and u x 0 is given as x, so use it A would be u at x naught comma 0, x naught is nothing but x times e to the power minus t square. So, it would be u of x e to power minus t square 0, which is given as x, so it should be x times e to the power minus t square. So, this is what we have got the constant A, we could determine hence finally, what would be our solution, our solution would be u is equal to x times e to the power minus t square into e to the power t.

So, we would get u x t as x times e to the power minus t square e t or x rewrite it as x times e to the power minus t square plus t, so this is the solution of our given semi linear partial differential equation with initial condition. So, this initial value problem does have this unique solution, we can check that, this is actually the solution of our given equation.

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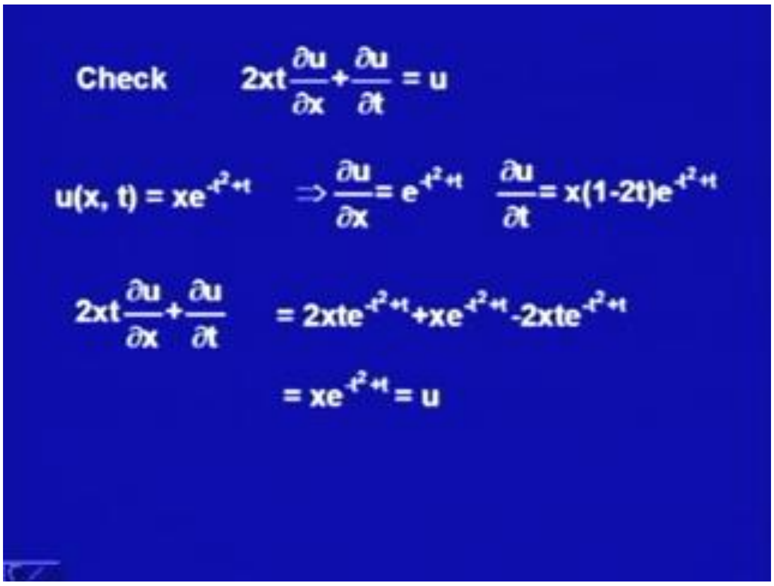

So, what we are been given, we are getting the equation is 2 x t del u over del x plus del u over del t is equal to u, we have got the solution as x times e to the power minus t square plus t. This function is we do know is continuous differentiable and actually it is continuously differentiable that is it is derivatives are also continuous, so this would be actually classical solution.

So, let see find out del u over del x, so that is derivative with respect to x of this function, so treating t as a constant, so here is only x, so it would be 1. So, it is e to the power minus t square plus t, then the derivative of this u x t with respect to t, so x is constant that is as such e to the power minus t square plus t, it is derivative is with respect to minus t square plus t would be e to the power minus t square plus t.

The derivative of t minus t square would be 1 minus 2 t, so we are getting is x times 1 minus 2 t e to the power minus t square plus t. Let us, substitute these derivatives in our given equation 2 x t times del u over del x that is e to the power minus t square plus t plus del u over del t, that is x times e to the power minus t square plus t minus 2 x t times e to the power minus t square plus t.

This and this come get cancel, what has been left is, x times e to the power minus t square plus t, that is nothing but my u. So, it is satisfying our equation, more over it is satisfying our boundary conditions as well.

So, today we have learn solving the method of first order partial differential equation, we had learn that is in the linear first order partial differential equations, we can find out the characteristic. By taking that characteristic, means is that is there is a curve in x t plane or in the plane of independent variables along which we take that the function has to be constant.

So, we are changed our equation to ordinary differential equation and from there we try to find it out the characteristic and that characteristic we had find it out using the either initial condition or both initial and boundary value conditions. We have learnt the method for the linear differential equations, linear partial differential equations, semi linear partial differential equations. Then one method is been left is in the quasi linear partial differential equation, the next lecture will consider this one. And then we will move to the second order partial differential equations, so that is all for this lecture.

Thank you.