Mathematics III
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Lecture - 3
Approximate Solution of an Initial Value Problem

Dear viewers, the title of my lecture is Approximate Solution of an Initial Value Problem. So far the differential equations considered by us had a general solution, in the case of an initial value problem, we obtained a unique solution by using the initial condition by at x naught equal to y naught in the general solution.

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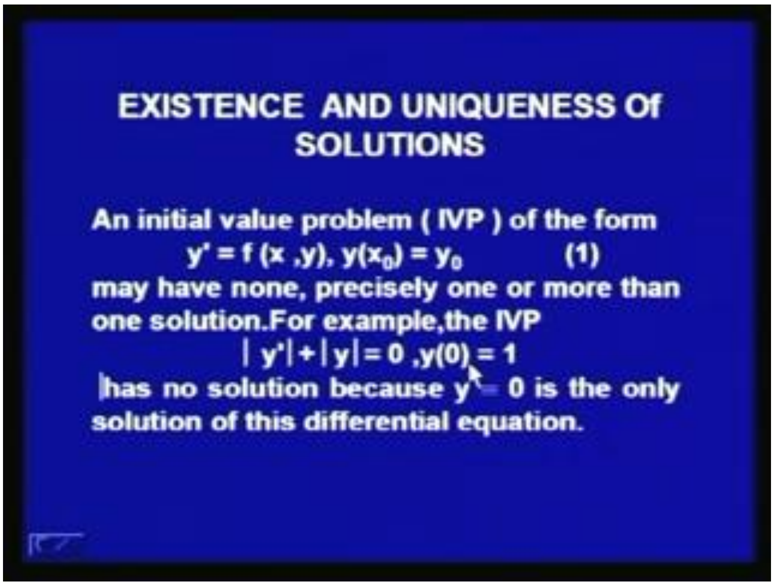

This is just one of the three possibilities that might occur in the general case, in the general case an initial value problem d y by d x equal to f x y where y x naught equal to y naught, may have no solution, precisely one solution or more than one solution. For example, let us consider the initial value problem mod of y dash plus mod of y equal to 0, where y at 0 is given to be equal to 1.

Now, this differential equation mod of y dash plus mod of y equal to 0 has only one solution that is y equal to 0, because the left hand side of this differential equation with the sum of two non negative real valued functions. So, there sum is 0 means y is equal to 0 for all x, but this does not satisfy the initial condition, because the initial condition is y at x equal to 0 is 1, so this initial value problem does not have any solution.

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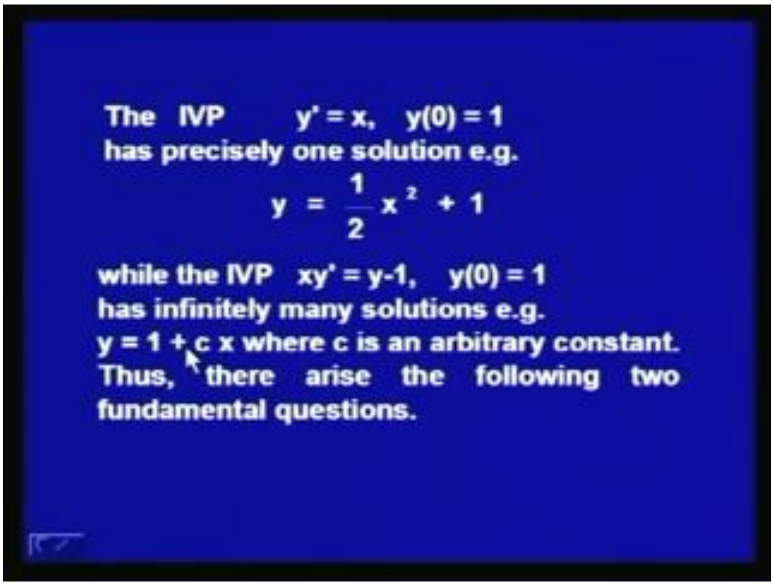

Next, let us consider the initial value problem d y by d x equal to x, where we are given that y at x equal to 0 is 1. Now, this differential equation d y by d x equal to x, we can solve by using the method of separation of variables, we may write it as d y equal to x d x, then integrate both sides we will have y equal to half of x square plus c, using the initial condition y at x equal to 0 is equal to 1. We get the value of the constant c as 1 and so this initial value problem has only one solution, which is y equal to half of x square plus 1.

If, we consider the initial value problem x y dash equal to y minus 1, where we are given y 0 equal to 1, then we can see that it has infinitely many solutions, you see if you take x equal to 0. Then y is equal to 1 follows from the differential equation x y dash equal to y minus 1 directly. So, let us take x naught equal to 0 divide by x, this differential equation and then write d y by d x equal to y by x minus 1 by x, we can write it as a linear differential equation of first order.

And then find the integrating factor itself it will turn out that the integrating factor here is 1 by x. We can multiply this equation by the integrating factor 1 by x and integrate with respect to x, we shall see that the solution is y equal to 1 plus c x for all x naught equal to 0, but y equal to 1 plus c x, also satisfy the condition y 0 equal to 1. So, we can say that y equal to 1 plus c x is the solution of the given initial value problem for all values of x, where c is an arbitrary constant.

Now, since c is an arbitrary constant, so the given initial value problem has infinitely many solutions and thus there arise the following two fundamental questions.

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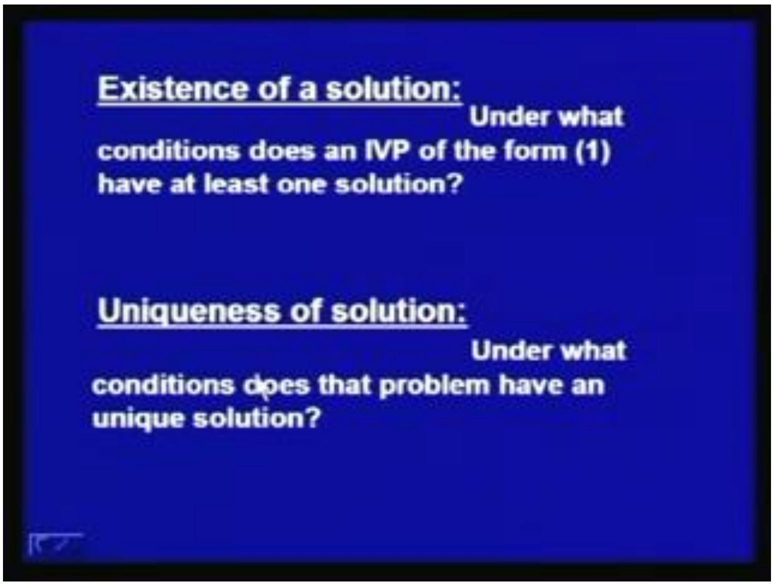

Existence of a solution under what conditions does an initial value problem of the form 1, that is d y by d x equal to f x y, y at x naught equal to y naught have at least one solution. Number 2, uniqueness of the solution under what conditions does the initial value problem have an unique solution was the theorems that answers these two questions, that is existence of the solution and uniqueness of solution are known as existence and uniqueness theorems.

The examples considered by us above of the initial value problems were, so simple that we could find the answers to the existence and uniqueness of the solutions just by looking at them or just by doing some simple calculations. But in the case of complicated differential equations that is the once, which cannot be solved by the elementary methods studied by us so far. The existence and uniqueness theorems, which we are going to study now will be of great practical importance.

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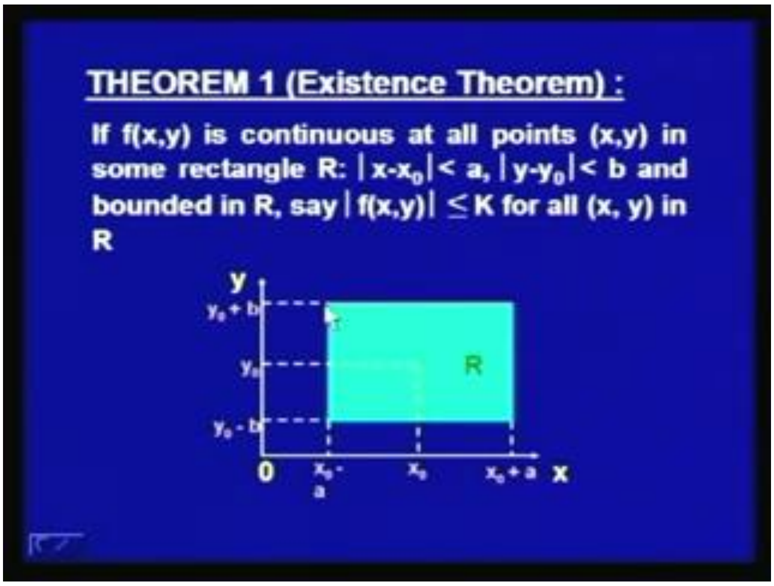

Let us look at the existence theorem first, If f x y is continuous at all points x y in some rectangle R of the x y plane given by mod of x minus x naught less than a, mod of y minus y naught less than b and bounded in R. That is to say mod of f x y is less than or equal to k for all x y in R, you can see in this picture, this is the region R in the x y plane bounded by x equal to x naught minus a and x equal to x naught plus a and y equal to y naught minus b and y equal to y naught plus b.

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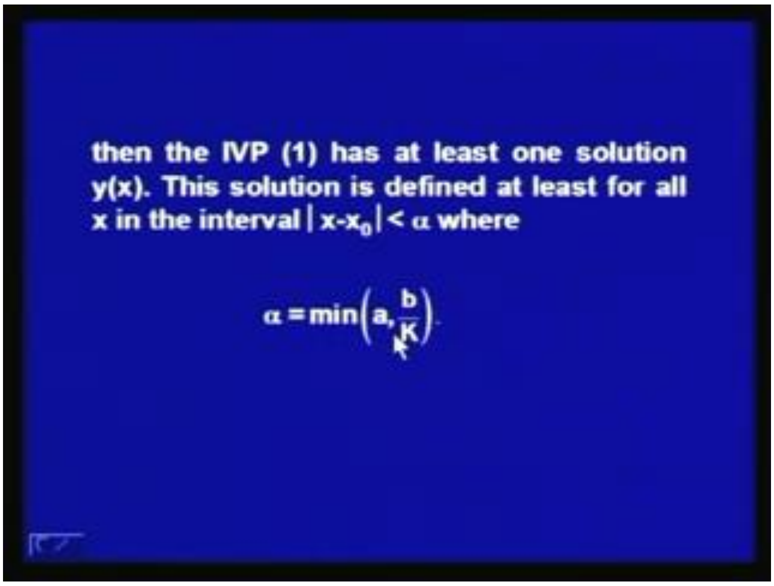

Then, the initial value problem has at least one solution y x and this solution is defined at least for all values of x in the interval mod of x minus x naught less than alpha, where alpha is the minimum of a and b by k

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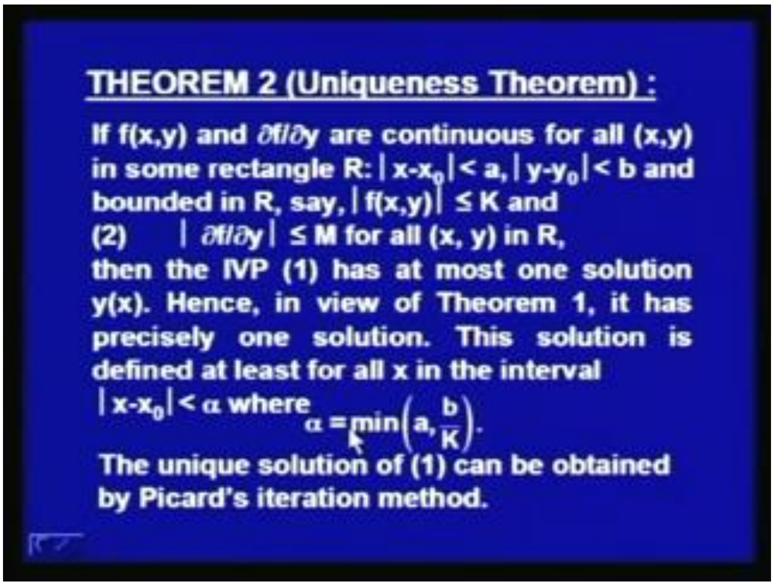

So, that is to say, we can say that if the function f x y in the initial value problem is continuous in some region of the x y plane containing the point x naught y naught, then the initial value problem has at least one solution. So, let us now talk about the uniqueness of the solution, the uniqueness theorem gives us the conditions, the subs and conditions which f x y has to satisfy in order that the initial value problem has precisely one solution.

So, if f x y and its first order partial derivative with respect to y, that is delta f over delta y are continuous for all x y in some rectangle R. That is mod of x minus x naught less than a, mod of y minus y naught less than b and bounded in r, that is to say mod of f x y less than or equal to k and mod of delta f over delta y less than or equal to m for all x y in r. Then the initial value problem d y by d x equal to f x y, where y x naught equal to y naught has at most one solution y x.

Now, in view of the existence theorem that is theorem number 1, it follows that the initial value problem will then have precisely one solution. This solution is defined at least for all x in the interval mod of x minus x naught less than alpha, where alpha is the minimum of a and b by k. The uniqueness solution of the unique solution of the initial value problem can be obtained by using the Picard's iteration method, which we shall study little later.

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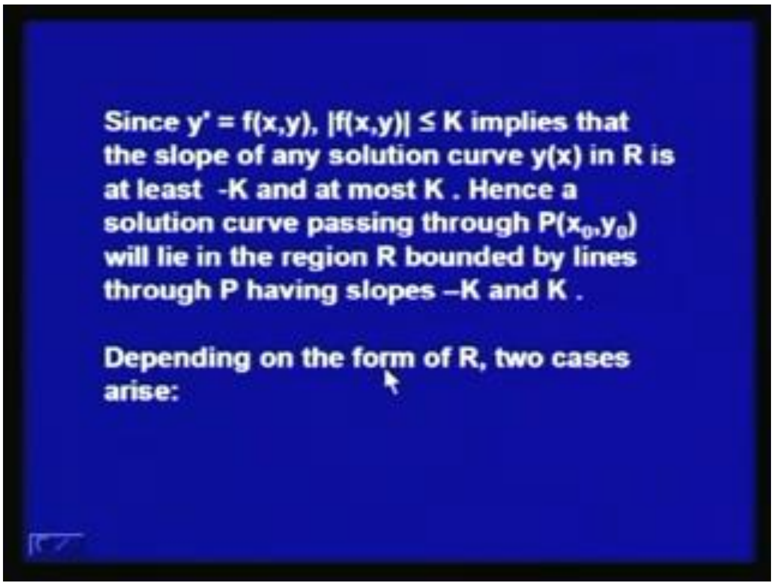

Now, since d y by d x is equal to f x y, so mod of f x y less than or equal to K implies that mod of d y by d x is less than or equal to K, that is to say that the slope of any solution curve y x in the region r is at least minus K and at most plus K. And hence, a solution curve passing through the point P that is x naught y naught will lie in the region R bounded by the lines through P having slopes minus K plus K, now depending on the form of the region R there arise two cases.

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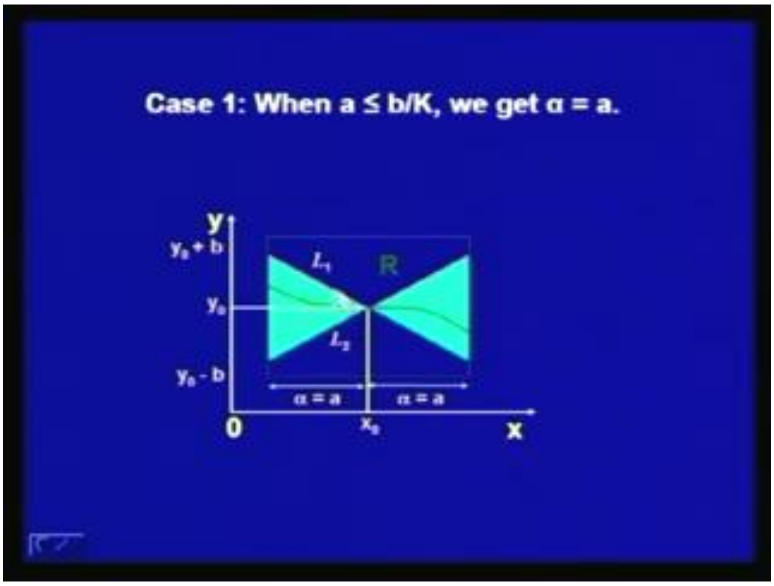

In the case 1, when a is less than or equal to b by K, we will have alpha is equal to a because, alpha is the minimum of a and b by K. So, when alpha is equal to a, in this figure you can see that this is your solution curve, the solution curve lies in the region R bounded by these two lines having slopes K and minus K, L 2 has slope of K. While, L 1 has minus K and their both pass through the point x naught y naught, the solution curve also passes through the point x naught y naught.

So here, when alpha is equal to a solution exist for all values of x in the interval x naught minus a 2, x naught plus a. And, its slope is at least minus K and at most plus K as because, it lies in the region the shaded region, which is bounded by the two lines having slopes minus K and plus K.

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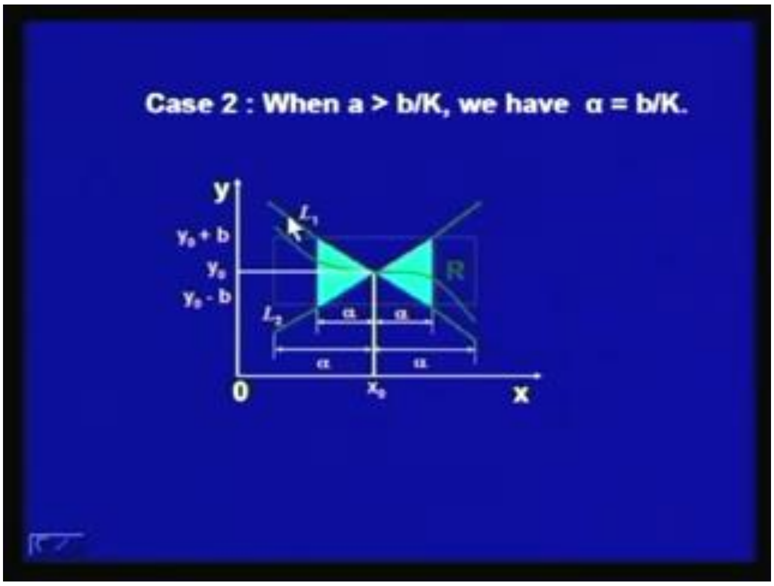

In the case 2, when a is greater than b by K, alpha being the minimum of a and b by K, will imply that alpha is equal to b by K. So, here mod of x minus x naught less than alpha gives the solution curve lies in the region shaded region, which is from x naught minus alpha 2 x naught plus alpha that is x naught minus b by K 2 x naught plus b by K. And, the slope of the solution curve is at least minus K and at most plus K, L 1 has slope minus K while L 2 has slope plus K and the solution curve is passing through the point x naught y naught.

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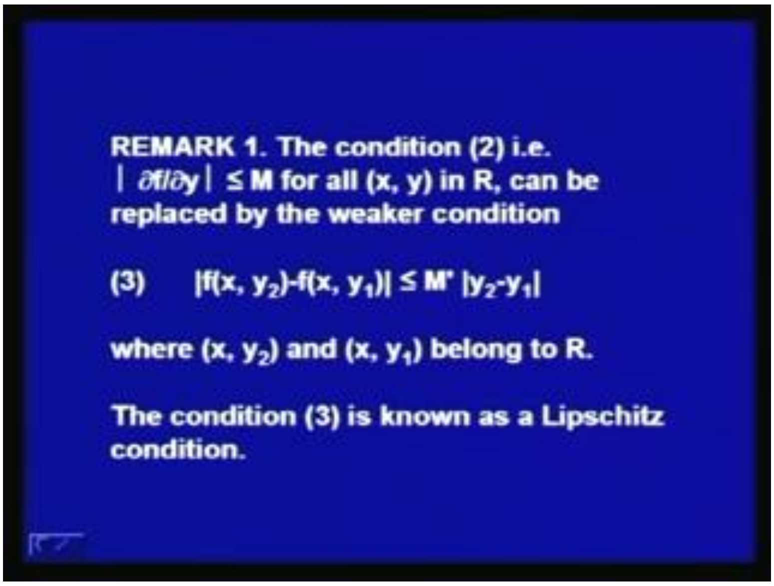

Now, let us study remark 1, the condition 2 that is mod of delta f over delta y less than or equal to 1 for all x y in R in the uniqueness theorem can be replaced by the weaker condition. Mod of f x y 2 minus f x y 1 less than or equal to m dash in to mod of y 2 minus y 1, where x y 2 and x y 1 are any 2 points belonging to the region R, but this condition 3 is known as a Lipschitz condition.

If a function f x y is such that mod of delta f over delta y is less than or equal to m, then it will always satisfy the Lipschitz condition. Because, by the mean value theorem we can write that mod of f x y 2 minus f x y 1 is less than or equal to some constant m dash times mod of y 2 minus y 1. So, thus we arrive at a stronger form of the uniqueness theorem, because there are many functions which do not possess a continuous partial derivative, but satisfy the Lipschitz condition for some constant m dash. Let us study the mod 2. If the function f x y is continuous, it is not sufficient to guarantee the uniqueness of the solution.

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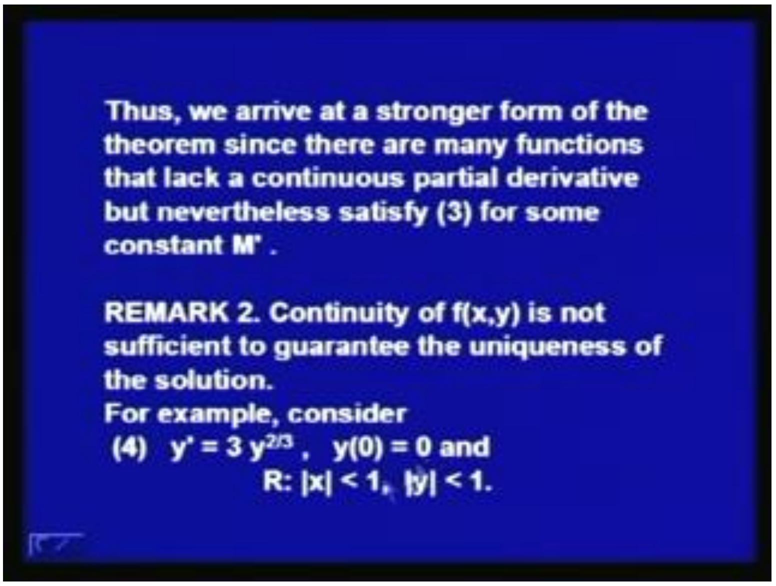

For example, let us consider d y by d x equal to 3 times y to the power 2 y 3, where we are given that y at x equal to 0 is 0 and the region R is given by mod of x less than 1, mod of y less than 1.

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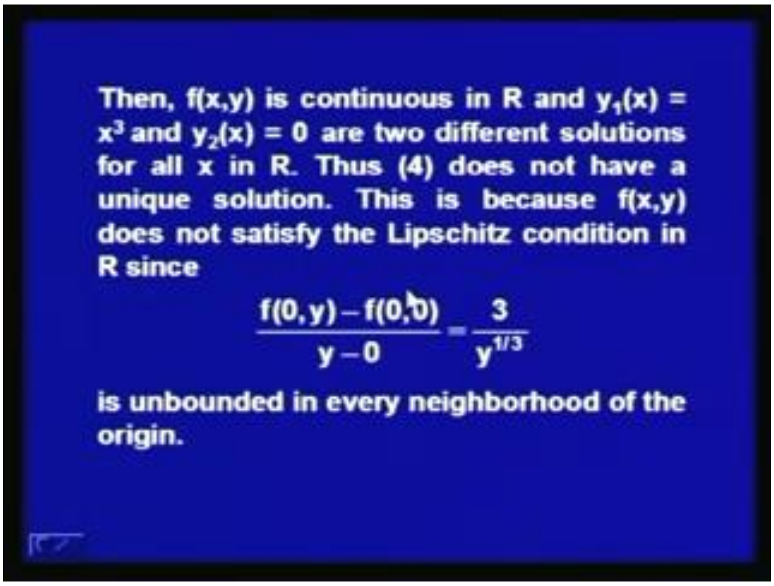

We can see here that f x y which is equal to 3 times y to the power 2 by 3 is continuous in the given region R and y 1 x equal to x cube and y 2 x equal to 0 are two different solutions of the differential equation for all values of x in R. Thus the differential equation d y by d x equal to 3 times y to the power 2 by 3 does not have a unique solution.

And, this is because the function f x y which is 3 times y to the power 2 by 3 does not satisfy the Lipschitz condition in the region R. Since, f 0 y minus f 0 0 which will be equal to 3 times y to the power 2 by 3 divided by y minus 0 will be equal to 3 over y to the power 1 by 3, which is un bounded in every neighborhood of the origin. And origin is a point, which lies in the region R, therefore the function f x y does not satisfy the Lipschitz condition.

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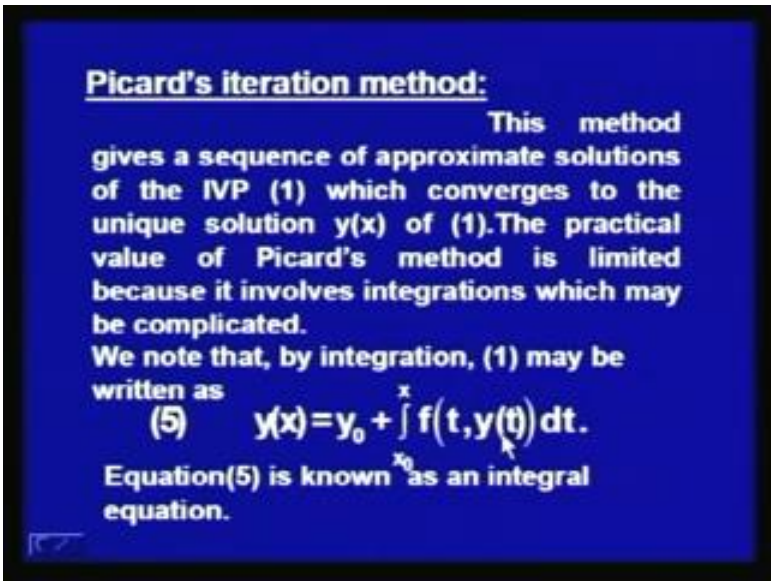

Now, let us study the Picard's iteration method to obtain a unique solution of the initial value problem. This method gives us a sequence of approximate solutions of the initial value problem 1, which converges to the uniqueness solution unique solution y x of 1, by the Picard's theorem which we shall state later on. The practical value of the Picard's method is limited, because it involves integrations which may be complicated to obtain.

We note that by integration the initial value problem d y by d x equal to f x y, where y at x naught is y naught may be written as y x equal to y naught plus integral x naught to x f t y t dt. Now, here the variable t in the integrant has been used, because x occurs as an upper limit of the integral here. Since, the unknown function y t occurs in the integrant on the right side of equation 5, equation 5 is known as an integral equation.

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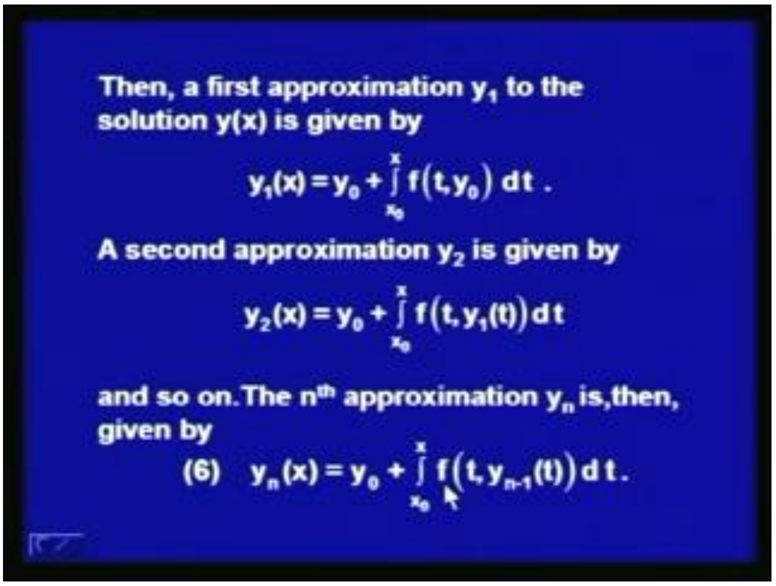

Then, a first approximation y 1 to the solution y x is given by y 1 x equal to y naught plus integral x naught to x, f t y naught t d t. So, the unknown function y t in the equation 5 on the right hand side in the integrant is replaced by the known value of y that is y naught. And, we determine the first approximation y 1 x to the solution y x of the initial value problem.

Second approximation y 2 is then obtained from y 2 x equal to y naught plus integral x naught to x, f t y 1 t d t. So, we use the value of y 1 x to determine the next approximation y 2 x and so on, we continue the nth approximation y n is then given by y n x equal to y naught plus integral x naught to x ft y n minus t d t.

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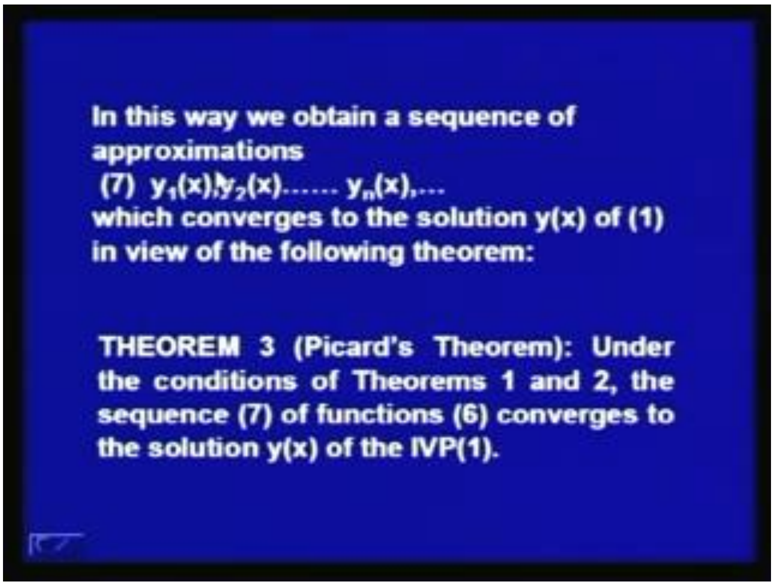

In this way, we will obtain a sequence of approximations y 1 x, y 2 x, y 3 x and so on, y n x and so on, which converges to the solution y x of the initial value problem 1 in view of the following theorem of Picard. Picard's theorem states that under the conditions of theorems 1 and 2, the sequence 7 y 1 x, y 2 x, y n x and so on of functions y x given by equation 6 converges to the solution yx of the initial value problem.

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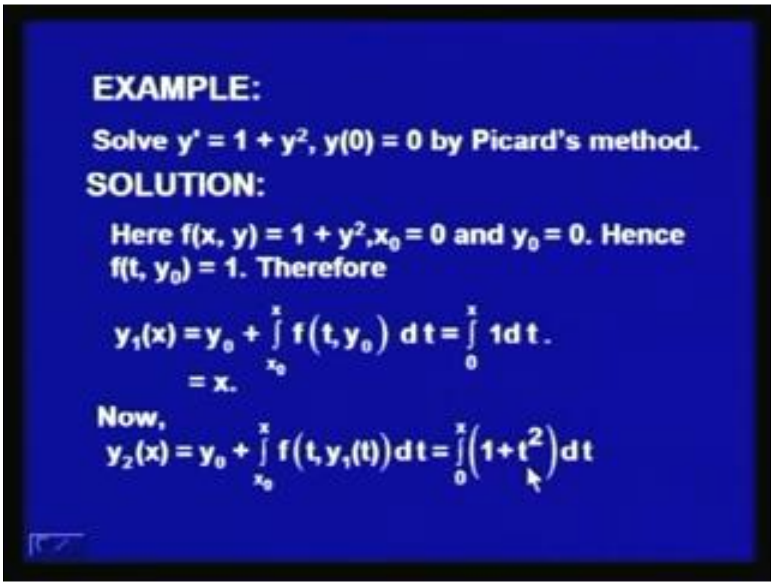

Let us solve an initial value problem using Picard’s method we are given d y by d x equal to 1 plus y square where y at x equal to 0 is 0. So, if you compare this differential equation y dash equal to 1 plus y square is the standard form d y by d x equal to f x y, you find that f x y is equal to 1 plus x square, the initial condition y at 0 equal to 0 gives us x naught equal to 0 and y naught equal to 0.

And hence, f t y naught the value of f t y naught is equal to 1 plus is equal to 1, therefore y 1 x is equal to y naught plus integral x naught to x, f t y naught d t which will be equal to integral 0 to x 1 in to d t and after integration we then have y 1 x equal to x. So, first approximation to the solution of d y by d x equal to 1 plus y square where y 0 equal to 0 is x. The next solution, next approximation y 2 x is equal to y naught plus integral x naught to x, f t y 1 t d t y 1 t is equal to t, so f t y 1 t will be equal to 1 plus t square.

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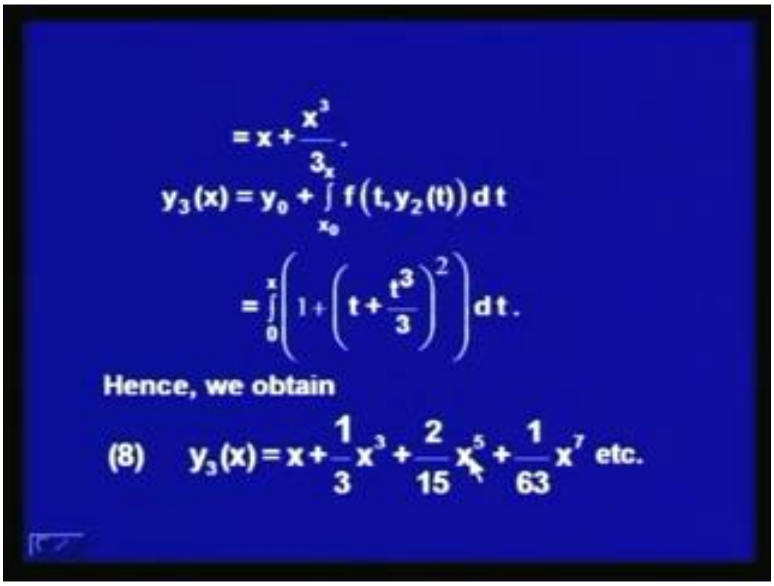

And therefore, y 2 x i the integral of 0 to x 1 plus t square d t which will give us x plus x cube by 3, so second approximation to the solution y x of the initial value problem is x plus x cube by 3. Next, we find y 3 x y 3 x is then y naught plus integral x naught to x ft y 2 t d t making use of the value of y 2 t as t plus t cube by 3, we have y 3 x as integral 0 to x 1 plus t plus t cube by 3 whole square d t. After integration, we get the value of the third approximation y 3 x as x plus x cube by 3 plus 2 by 15 x to the power 5 plus 1 by 63 x to the power 7, etcetera.

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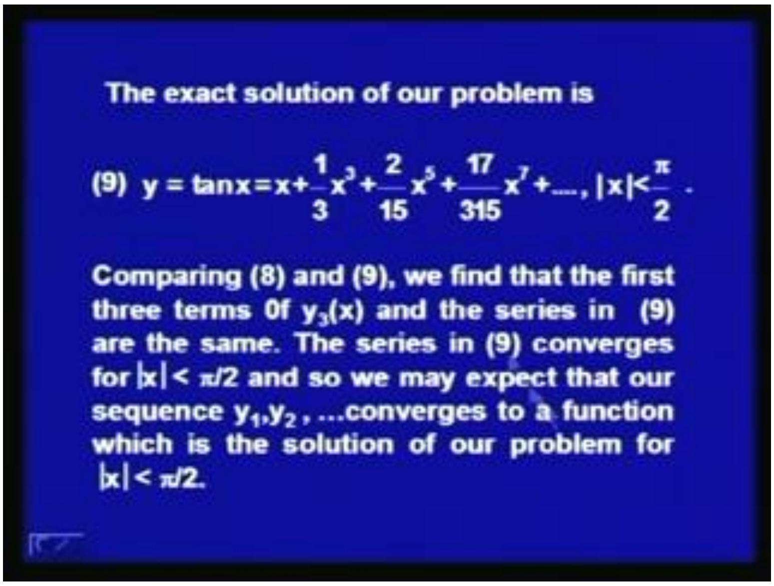

If the find exact solution of our initial value problem, it tells out that the exact solution of our initial value problem is y equal to tan x, see our initial value problem is d y over d x equal to 1 plus y square. So, we can use the method of separation of variables and write the differential equation as d y over 1 plus y square equal to d x integrate and use the initial condition y at 0 equal to 0.

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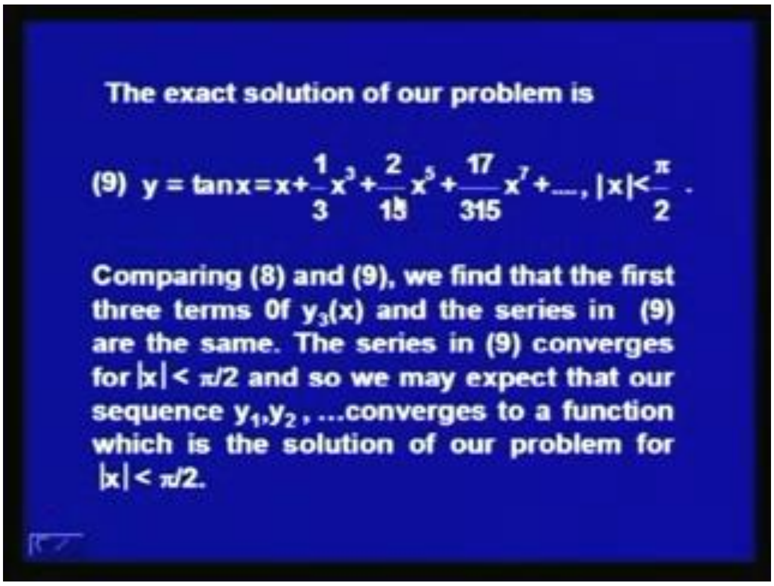

We can see that y equal to tan x is the exact solution of our problem and we know that the McLaren's series expansion of tan x is x plus x cube by 3 plus 2 by 15 x to the power

5 plus 17 by 315 x to the power 7 and so on which is valid in the region mod of x less than pi by 2. That is the series converges in the interval minus pi by 2 less than x less than pi by 2.

Now, we compare the third approximation y 3 x with the series that occurs in 9, we find that first 3 terms of y 3 x and the series in 9 are the same. The series in 9 converges for mod of x less than pi by 2 and so we may expect that our sequence of approximations y 1 y 2 and so on converges to a function which is the solution of our problem for mod of x less than pi by 2.

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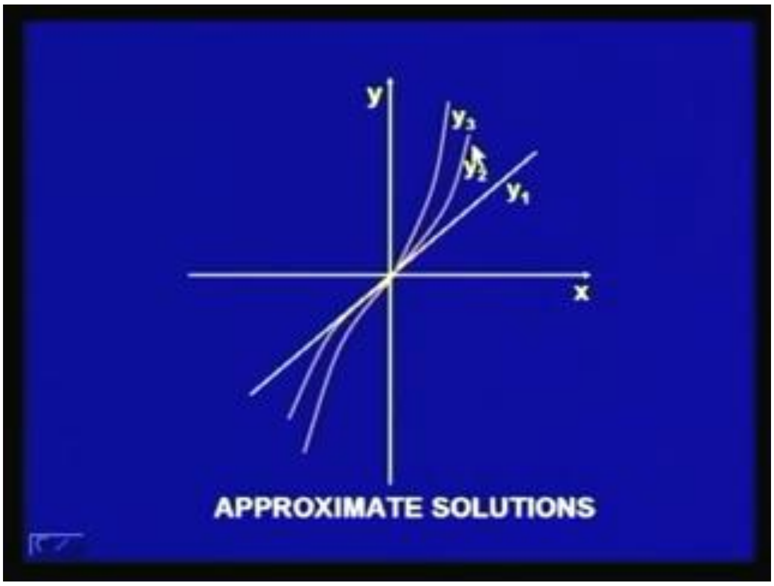

These are the approximate solutions of our initial value problem y 1 x is equal to x, so we have this curve y 1 x equal to x y 2 x is equal to x plus x cube y 3 and then we curve for y 3 x. So, these are approximate solutions of our initial value problem, which converge to the exact solution y equal to tan x of our initial value problem.

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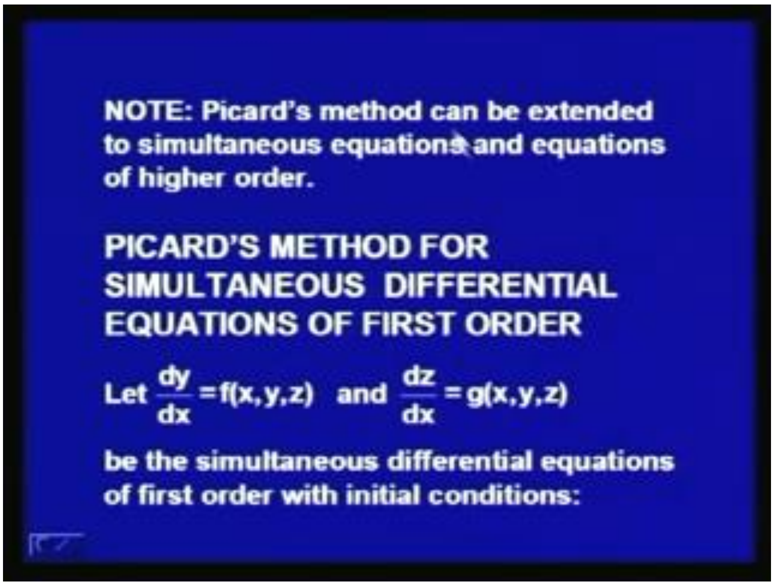

Picard's method can be extended to simultaneous equations and equations of higher order, let us first discuss Picard's method for simultaneous differential equations of first order. Let us consider the two differential equations of first order d y by d x equal to f x y z and d z by d x equal to g x y z.

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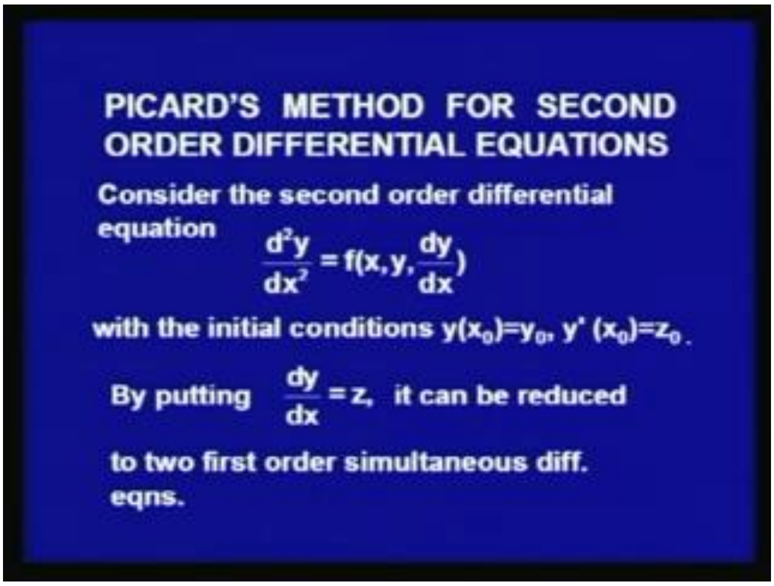

Next, we discuss Picard's method for second order differential equations, let us consider the second order differential equation d square y over d x square equal to f x y d y b y d x with the initial conditions y at x naught equal to y naught and d y by d x at x naught equal to z naught. Now, if we put d y by d x equal to z then the second order differential equation d square y over d x square equal to f x y d y by d x gives rise to two first order simultaneous differential equations.

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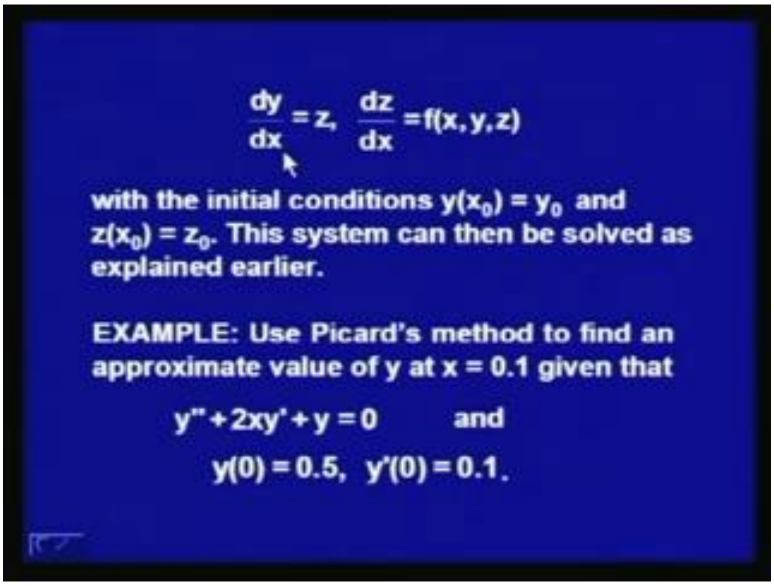

They are d y by d x equal to z and d z by d x equal to f x y z the initial conditions will be y at x naught equal to y naught and d y by d x at x naught equal to z naught gives us then d z at x naught equal to z naught. For this system of simultaneous differential equations of first order now can be solved as by the method which we have done explained earlier. Let us take an example on this method let us use Picard's method to find an approximate value of y at x equal to 0.1, where we are given that y double dash plus 2 x y dash plus y is equal to 0 and y at x equal to 00 .5 y dash at x equal to 0 is 0.1.

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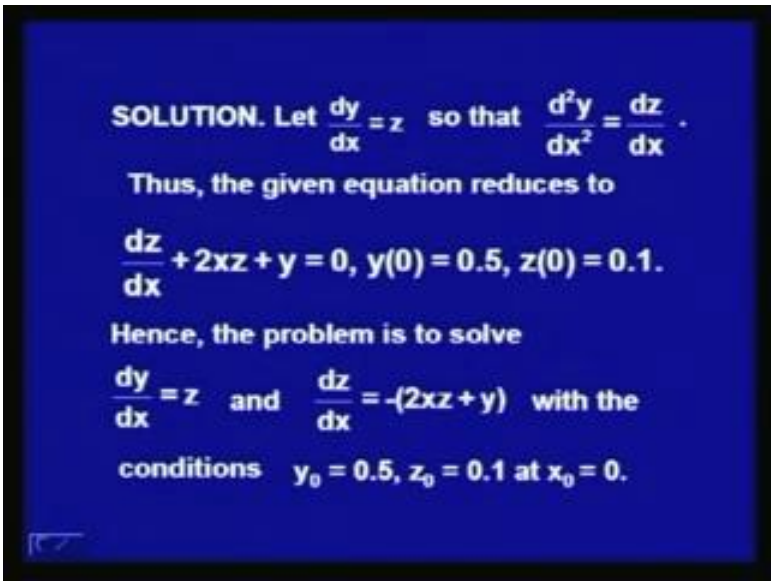

So, let us now discuss the solution of this problem, let us take d y by d x equal to z, so that d square y over d x square is equal to d z by d x and thus the given equation reduces to d z by d x plus 2 x z plus y equal to 0, where y at x equal to 0 is 0.5 and z at x equal to 0 is equal to 0.1. And thus, we have the problem to solve d y by d x equal to z and d z by d x equal to minus of 2 x z plus y, with the initial conditions y naught equal to 0.5 z naught equal to 0.1 at x naught equal to 0.

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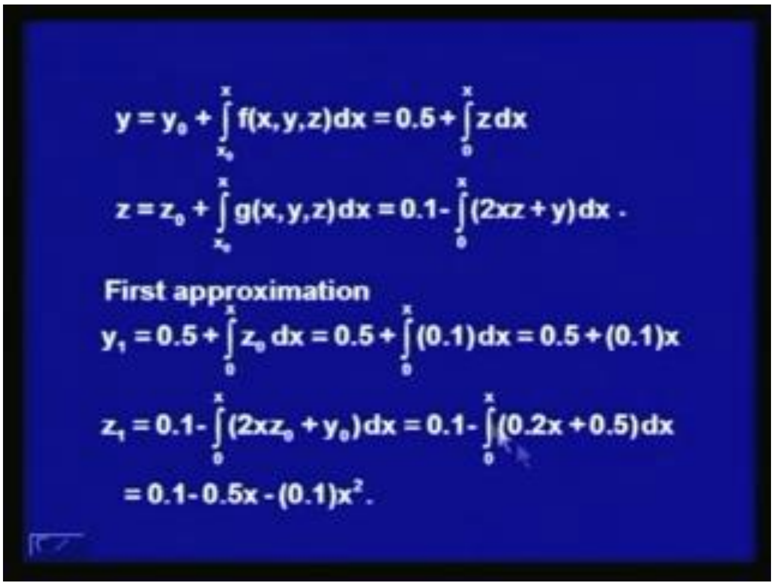

Now, by the Picard's theorem, we have y equal to y naught plus integral x naught to x f x y z d x which is equal to 0.5 plus integral 0 to x z d x f x by z is equal to z here. And z is equal to z naught plus integral x naught to x g x y z d x z naught is equal to 0.1 minus integral 0 to x 2 x z plus y because the value of g x y z is minus of 2 x z plus y d x.

And the first approximations to y and z are then given by y 1 equal to 0.5 plus integral 0 to x z naught d x. We replace z by z naught here which is equal to 0.5 plus integral 0 to x the value of z naught is 0.1, so 0.1 d x this will give you y 1 equal to 0.5 plus 0.1 in to x. And, z 1 will be equal to 0.1minus integral 0 to x 2 x z plus y will become 2 x z naught plus y naught d x. So, this will be equal to 0.1 minus integral 0 to x 0.2 in to x because z naught is 0.1. So, 0.2 in to x y naught is 0.5. So, we have the integral of 0.2 in to x plus 0.5 d x and after integration we will get the value of z 1 as 0.1 minus 0.5 in to x minus 0.1 in to x square.

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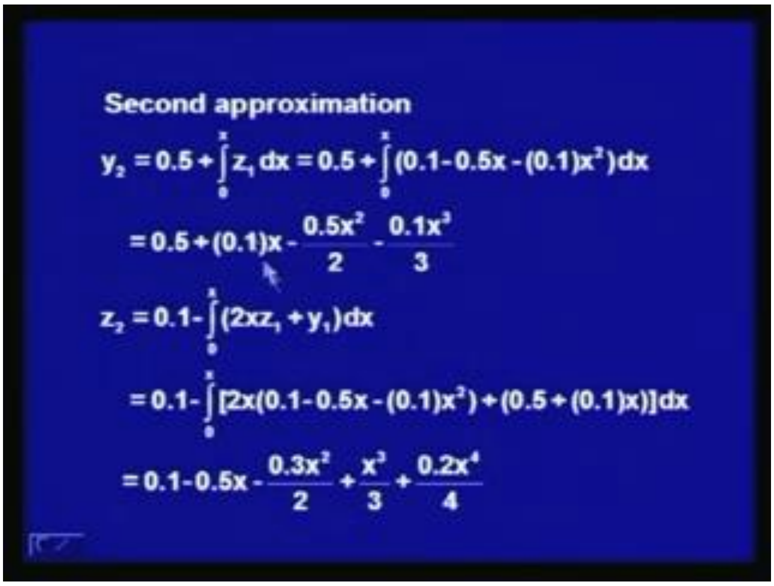

Second approximations are y 2 equal to 0.5 plus integral 0 to x z 1 d x which is equal to 0.5 plus integral 0 to x. The value of z 1 came out to be 0.1 minus 0.5 x minus 0.1 x square in to d x, which is equal to 0.5 plus 0.1 in to x minus 0.5 in to x square by 2 minus 0.1in to x cube by 3.

Z 2 is equal to 0.1 minus integral 0 to x 2 x z 1 plus y 1 d x substituting the value of z 1 and the value of y 1 and integration. After integration, we shall have the value of z 2 as

0.1 minus 0.5 in to x minus 0.3 in to x square by 2 plus x cube by 3 plus 0.2 in to x to the power 4 by 4.

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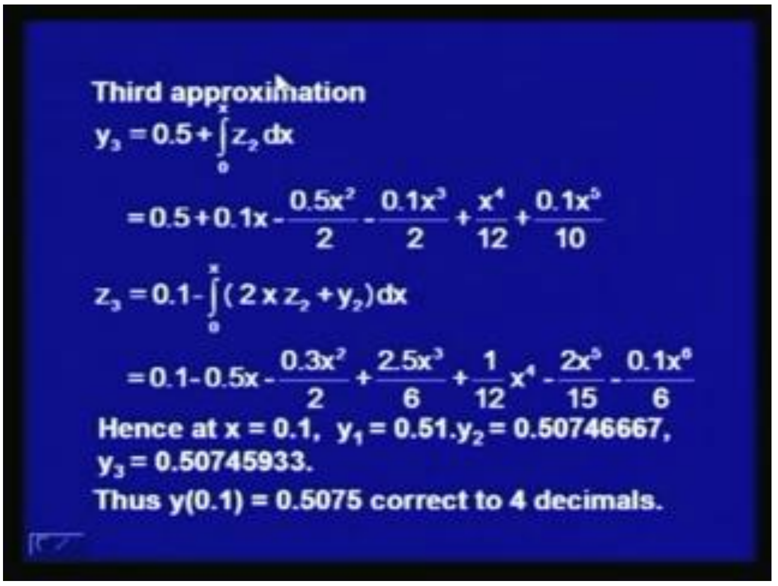

Similarly, we can find third approximations y 3 and y z 3, y 3 is equal to 0.5 integral 0 to x z 2 d x substituting the value of z 2 and integrating with respect to x. We will give us the value of y 3 as .5 plus .1 x minus 0.5 x square by 2 minus 0.1 x cube by 2 plus x to the power 4 by 12 plus 0.1 x to the power 5 by 10.

And z 3 as 0.1 minus integral 0 to x 2 x z 2 plus y 2 d x substituting the values of z 2 and y 2 here. We will get after integration 0.1 minus 0.5 in to x minus 0.3 x square by 2 plus 2 point 5 x cube by 3, 6 plus 1 by 12 x to the power 4 minus 2 x to the power 5 by 15 minus 0.1 x to the power 6 by 6. And hence at x equal to 0.1the values of y 1, y 2 and y 3 are y 1 is 0.51, y 2 is 0.507466 67, y 3 0.50745933 and thus y at 0.1 is equal to 0.5075 correct up to 4 decimal places.

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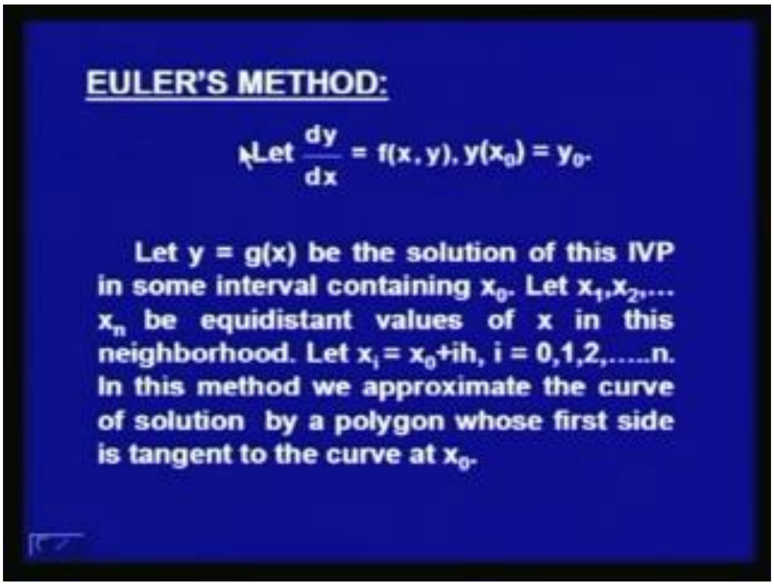

Now, we will discuss some numerical methods to find an approximate solution of an initial value problem. The differential equations that occur in the practical problems are so complicated that the methods which we have discussed earlier may not be applied to them or they are if even, if we find solutions of those differential equations by the known methods are the elementary methods. The formulae are so complicated that one often prefers to find numerical solutions of the differential equation.

So, we will be discussing the two methods numerical methods for finding an approximate solution of the given initial value problem given initial value problem. We shall assume that the initial value problem has a unique solution in an interval containing the point x naught. Let d y by d x be equal to f x y y at x naught equal to y naught the methods that we are going to discuss that is Euler's method and the improved Euler's method are known as step by step methods.

We start with an initial value of y, that is y naught at x equal to x naught and then find an approximate value of y, that is y 1 at x equal to x naught plus h that is x 1. In the second step, we find an approximate value of y that is y 2 at x equal to x 2, which is x naught plus 2 h, h is the step size and at each step we use the same formula to determine an approximate value of y, so these methods are called step by step methods.

Let us say y equal to g x be the solution of this initial value problem, in some interval containing x naught and let us say x 1 x 2 x n be equidistant values of x in this neighborhood. Let us take x i equal to x naught plus i h, so these values are equally spaced with step size h, then in this method we approximate the curve of solution of the initial value problem by a polygon, whose first side is tangent to the curve at x equal to x naught.

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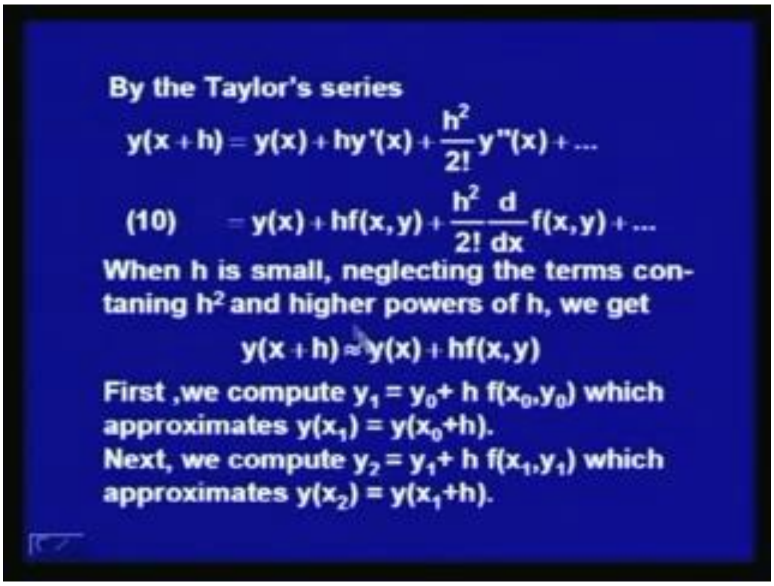

By the Taylor's series y at x plus h is equal to y x plus h y dash x plus h square by 2 factorial, y double dash x plus and so on, which can be written as y x plus h in to f x y. Because, d y by d x is equal to f x y and so we have h in to f x y plus h square by 2 factorial d square y by d x square that is y double dash x, it becomes d by d x of f x y and so on.

Now, when h is the step size h is small, we can neglect the terms containing h square and higher powers of h and we thus get an approximate value of y at x plus h as y x plus h f x y. So, using this approximate formula, we can compute the first approximate value of y at x naught plus h that is x 1.

We compute the approximate value y 1, y, y naught plus h f x naught y naught, which approximates y at x 1 that is y at x naught plus h. Next, we compute y at x naught plus 2 h, that is we compute y 2 from y 1 plus h, f x 1 y 1 which approximate y at x 2, that is y at x 1 plus h or you can say y at x naught plus 2 h.

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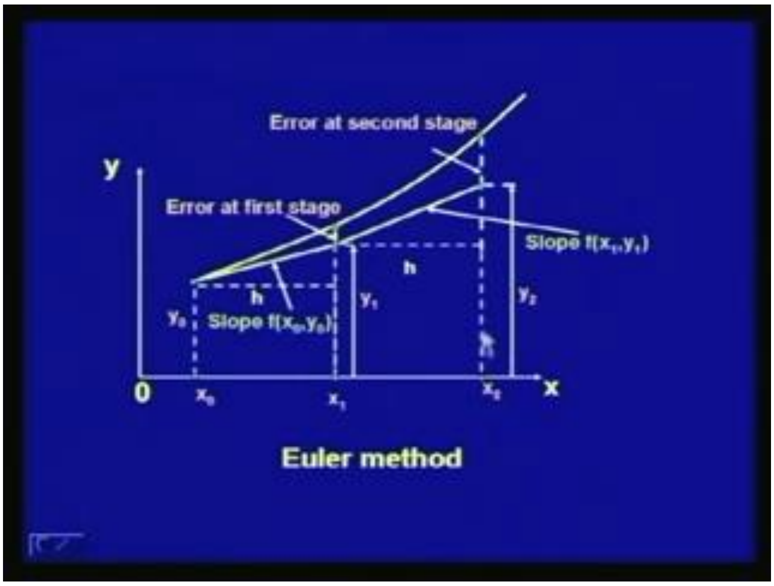

Now, this is geometrical interpretation of this method, you can see here that this is solution curve of the given initial value problem at the point x naught y naught this is the tangent to the curve, whose slope is given by f x naught y naught. We have d y by d x is equal to f x y, so the slope of the tangent is at x naught is f x naught y naught and this is your step size h at the point x equal to x naught plus h, that is x 1 the value of y 1 is given by this y 1 and the actual value of y at x equal to x 1 is this 1.

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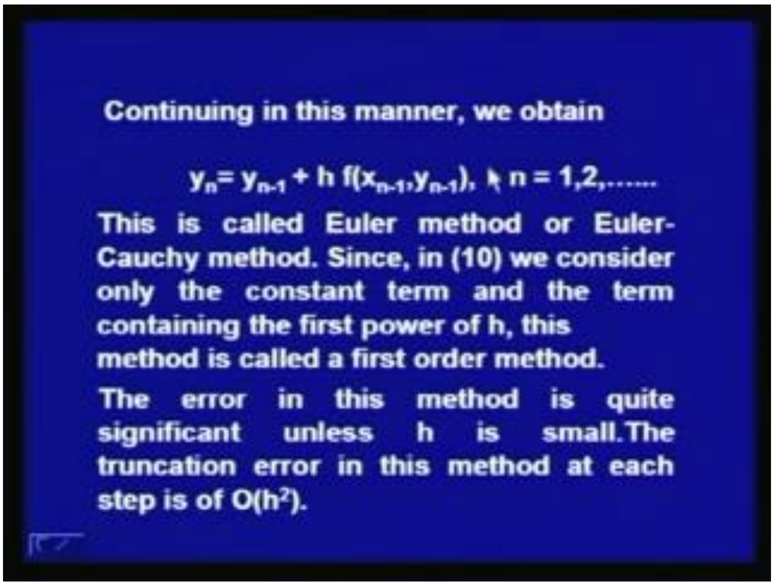

So, there is an error here, this is the error at the first stage y at x naught plus h minus y 1 that is the error here. And then in the next step x 2 we then continue along the straight line which passes thought the point x 1 y 1 and has slope f, x 1 y 1. So, then when we compute y 2 from y 1 plus h times f x 1 y 1, this is our y 2 by the actual value of y at x equal to x 2 is this, so there is an error here origin this is the error in the next stage.

Continuing in this manner, we then obtain y n equal to y n minus 1 plus h f x n minus 1 y n minus 1, where n is equal to 1 2 3 and so on, this is called Euler method or Euler-Cauchy method. Now in the Taylor series which occurs in 10, we consider only the constant term and the term containing the first power of h, because we have neglected all the terms which contain h square and higher powers of h, so this method is called a first order method. The error in this method is quite significant unless h is small, since we have neglected all terms containing h square and higher powers of h, the truncation error in this method is at each step is of order capital order h square.

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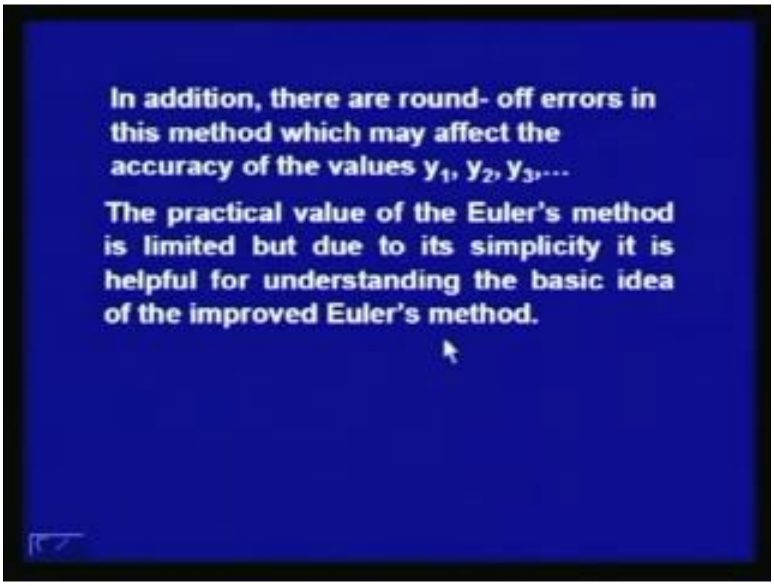

Now, in addition to this truncation error, there are round of errors as well in this method which may affect the accuracy of the values y 1, y 2, y 3 and so on. The practical value of the Euler's method is limited, but due to its simplicity, it is helpful for understanding the basic idea of the improved Euler's method, which we shall be discussing now.

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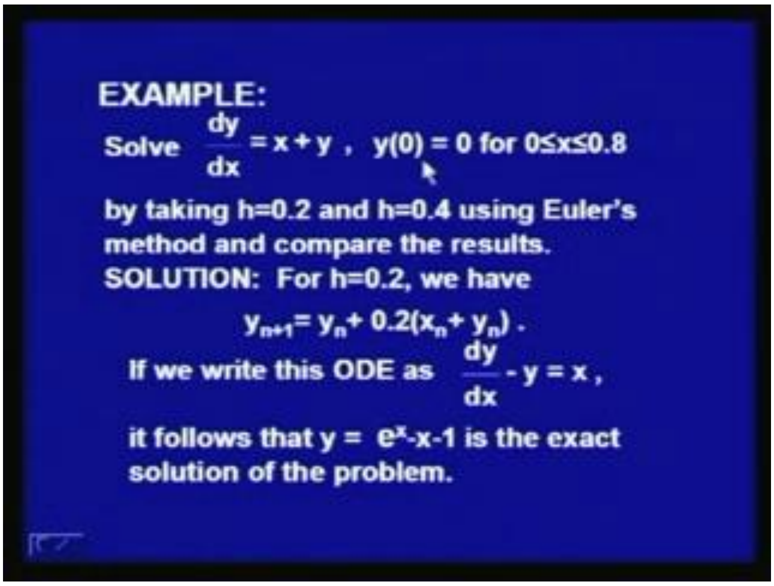

Let us, before we discuss improved Euler's method, let us take an example on Euler's method, say let us solve d y by d x equal to x plus y, where y 0 equal to 0 for 0 less than or equal to x less than or equal to 0.8. By taking h equal to 0.2 and h equal to 0.4 uses using Euler's formula method and let us compare the results, for h equal to 0.2. We have the formula Euler's formula y n plus 1 equal to y n plus 0.2 in to x n plus y n, that is the value of f x n, y n, f x, f x y here is x plus y, so f x n, y n is x n plus y n.

Now, if you write this odd ordinary differential equation as a d y by d x minus y equal to x, you can see that this is the first order linear differential equation. And, we can then find its integrating factor and multiply this by the integrating factor and integrate with respect to x. It follows that y equal to e to the power x minus x minus 1 is the exact solution of this problem making use of the initial condition y 0 equal to 0.

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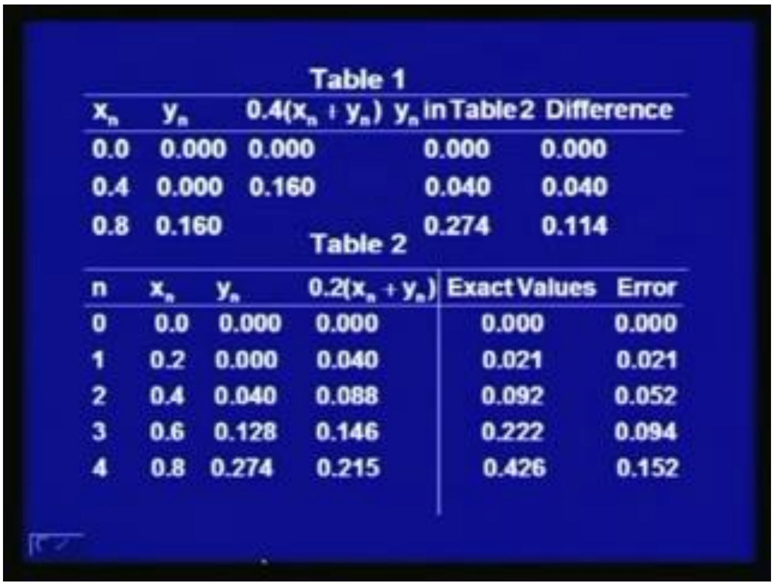

Table 1

x_n	y_n	$0.4(x_n + y_n)$	y_n in Table2	Difference
0.0	0.000	0.000	0.000	0.000
0.4	0.000	0.160	0.040	0.040
0.8	0.160		0.274	0.114

Table 2

n	x_n	y_n	$0.2(x_n + y_n)$	Exact Values	Error
0	0.0	0.000	0.000	0.000	0.000
1	0.2	0.000	0.040	0.021	0.021
2	0.4	0.040	0.088	0.092	0.052
3	0.6	0.128	0.146	0.222	0.094
4	0.8	0.274	0.215	0.426	0.152

Now, here we are tabulated the values of y 1, y 2, y 3, y 4 for x equal to 0.2, 0.4, 0.6, 0.8, when we take h equal to 0.2. And then we have also tabulated the values of y for x equal to 0.4 and 0.8, where we have taken h to be 0.4 and then we have compared the errors in the two cases.

Now, you can see that when x, when n is 0 that is we have x naught 0.0 y naught is equal to 0.00 and the value of 0.2 into x n plus y n is this and then we get using the Euler's formula the value of y at x equal to 0.2 as 0.000. Next, we can calculate 0.2 in to x 1 plus y 1, which is 0.040 we can get the value of y 2 that is y at 0.4 as 0.040.

Similarly, y 3 that is y at 0.6 we get as 0.128 and y at 0.8, we get as 0.274 and when we calculate the exact values of y, from the exact solution of the initial value problem y equal to e to the power x minus 1. We see that y at x equal to 0 is given by 0.000 y at 0.2 is 0.021 y at 0.4 is 0.092 y at 0.6 is 0.222 and y at 0.8 is 0.426.

These are the errors in which occur, when we compute the values of y by using the Euler's formula taking h equal to .2. So, at x equal to 0 the error is 0 at x equal to 0.2, the error between the approximate value and the actual value is 0.021, the error in y 2 is 0.052, the error in y 3 is 0.094. The error in y 4 is 0.152 and when we take h equal to 0.4 we see that y at 0.4 comes out to be 0, while y at 0.8 comes out to be .160.

These are the values of y which we have found for y at 0.4 from the table 2, y at 0.4 is 0.040 and y at 0.8 is 0.274 and the errors the difference between the two values of y here, the value of y at 0.4 is 0 here it is 0.040. So, the difference is 0.040 and here will be difference in the values of y is 0.2 7 4 minus 0.160 that is 0.114.

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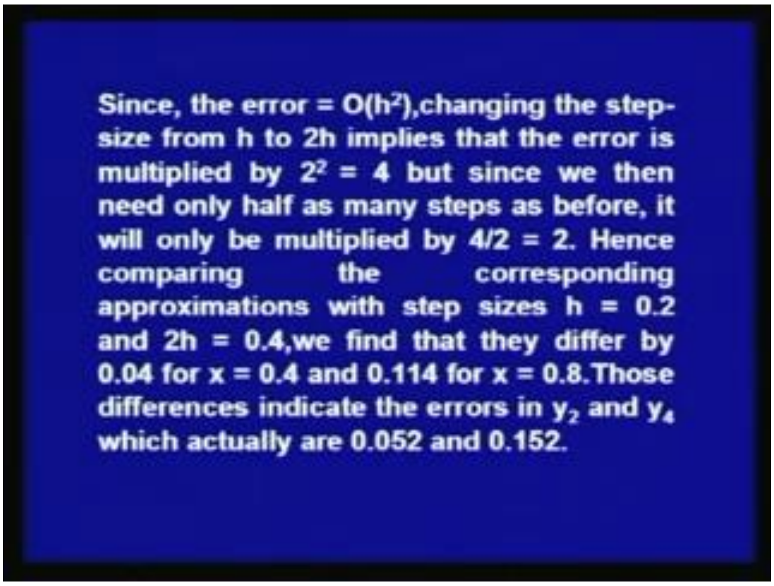

Now, since the error is equal to capital order h square, in this method changing the step size from h to 2 h, implies that the error is multiplied by 2 square that is 4, see here we have taken first h equal to 0.2 and then we have taken h equal to 0.4. So, we have change the step size from h to 2 h, now changing the step size from h to 2 h means that the error is multiplied by 2 square that is 4. But, since when we take the step size 2 h, we need only half of the steps that when we take the step size as h. The error when only be multiplied by 4 by 2 that is 2 and hence comparing the corresponding approximations with step size as h equal to 0.2 and 2 h equal to 0.4, we find that they differ by 0.04 for x equal 0.4 and 0.11 4 for x equal to 0.8. While the errors in y 2 and y 4, actually are 0.052 and 0.152.

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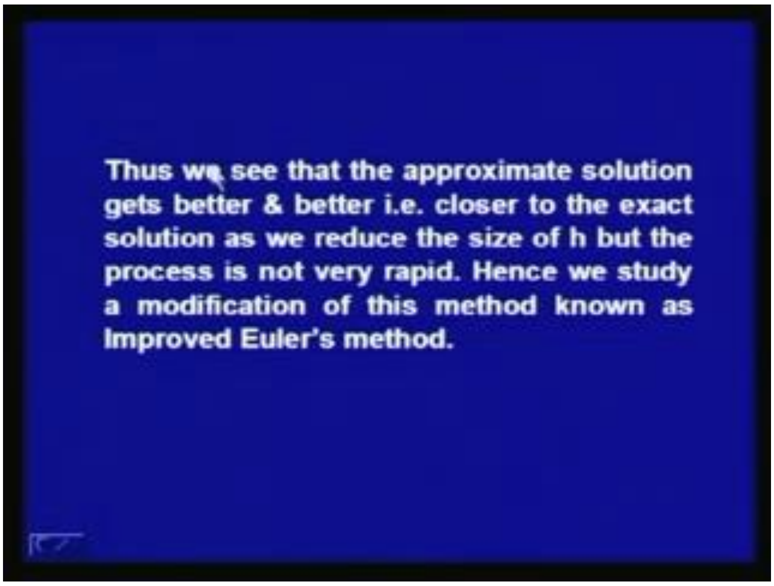

And thus, we see that the approximate solution gets better and better that is closer to the exact solution as we reduce the size of h, but the process is not very rapid. And hence, we study a modification of this method known as improved Euler's method.

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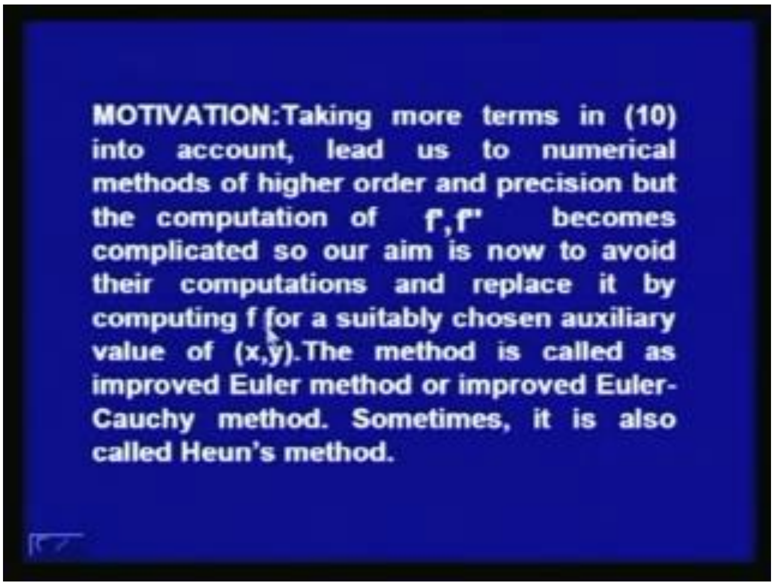

Now, if we take more terms in the Taylor series in the equation number 10 in to account, it lead us to numerical methods of higher order and precision, but the computation of f dash and f double dash becomes complicated. So, our aim is now to avoid their computations and replace it by computing f for a suitable chosen auxiliary value of x y, the method is called as improved Euler method or improved Euler-Cauchy method sometimes it is also called as Heun's method.

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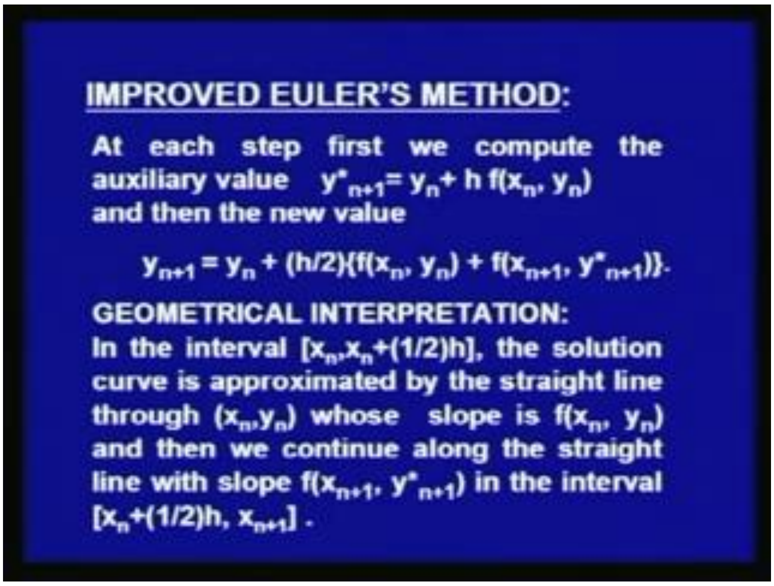

So, now let us study improved Euler's method, in this method at each step first we compute the auxiliary value y n plus 1 star equal to y n plus h f x n y n. And then we determine the new value of y that is y n plus 1 the formula y n plus h by 2 in to f x n y n plus f x n plus 1 y n plus 1 star, now let us study geometrical interpretation of this method.

In the interval x n to x n plus half h, the solution curve is approximated by the straight line through x n, y n whose slope is f x n, y n and then we continue along the straight line with slope f x plus 1, y n plus 1 star in the interval x n plus half h to x n plus h. You can see that in this formula y n plus 1 equal to y n plus h by 2 in to f x n, y n plus f x n plus 1 y n plus 1 star, we can direct it in to 2 parts, y n plus h by 2 f x n, y n.

So, that is the value which we get by taking this straight line through x n, y n whose slope is f x n, y n in the interval x n to x n plus half h at x n plus half h. The value of y will be from this stated line will come out to be plus h by 2 f x n, y n and then we continue along the straight line, whose slope is f x n plus 1 y n plus 1 star in the interval x n plus half h to x n plus 1 and that will give us the value of y n plus 1.

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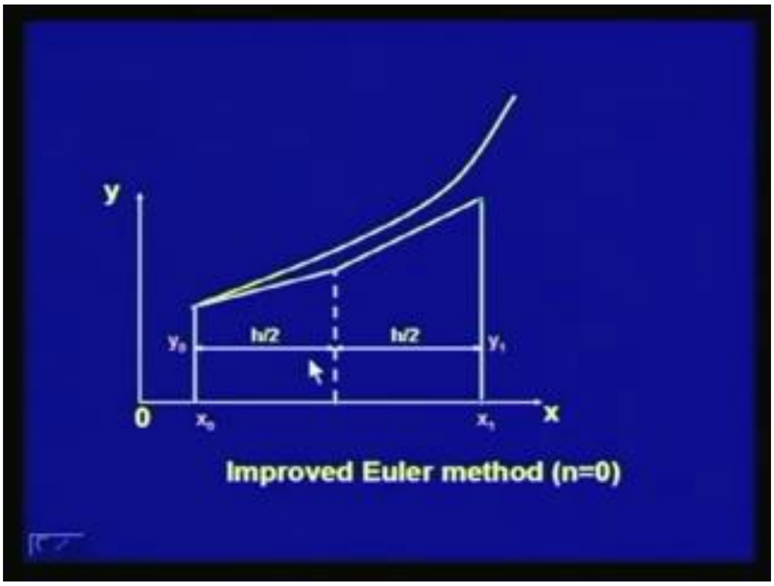

Let us, see this figure for n equal to 0, this is the solution curve of the initial value problem and this is the point x naught y naught. This is the tangent to the curve at the point x naught y naught because the slope of the tangent to the curve at the point x naught y naught is d y by d x at x equal to x naught which is f x naught y naught.

So, in the interval x naught to x naught plus h by 2, we get the value of y 1 star from here this is y 1 star and then we continue along the straight line this one, whose slope is at f x 1 y 1 star. In the interval x naught plus h by 2 to x naught plus h that is x 1 and then at x equal to x 1 this gives us the value of y that is y 1. So, there is an error in the improved Euler's method in computing the value of y at x equal to x naught plus h.

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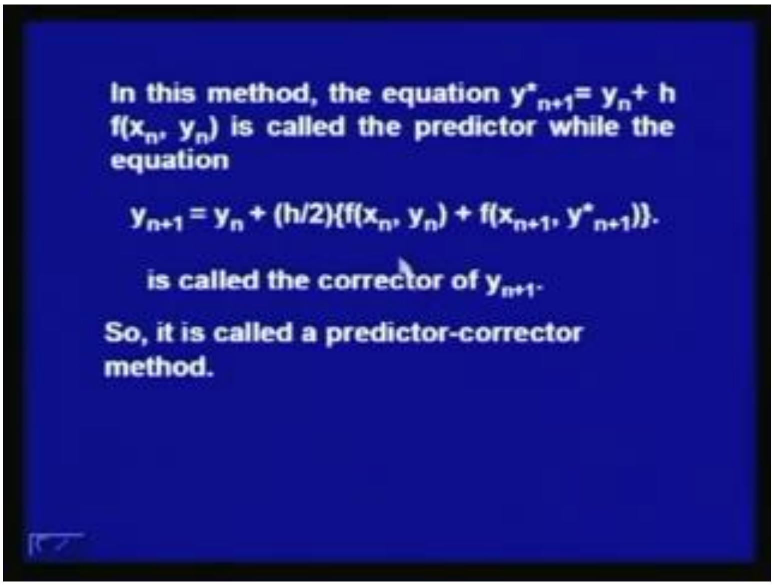

In this method the equation y n plus 1 star equal to y n plus h f x n, y n is called the predictor while the equation y n plus 1 equal to y n plus h by 2 in to f x n, y n plus f x n, plus y n plus 1 star is called the corrector of y n plus 1. So, it is called a predictor corrector method.

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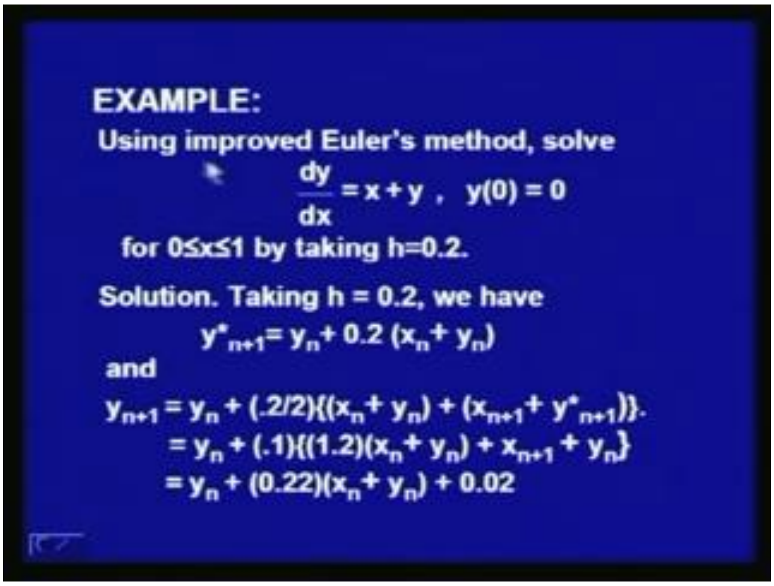

Let us consider an example on this method using improved Euler's method solve d y by d x equal to x plus y, y 0 equal to 0 for 0 less than or equal to x less than or equal to 1 by taking h equal to 0.2. Taking h equal to 0.2 we will have y n plus 1 star equal y n plus 0.2

x n plus y n here f x y is x plus y. So, f x n, y n is x n plus y n and y n plus 1 will be y n plus h by 2 that is .2 by 2, f x n, y n that is x n plus y n, f x 1, y n plus 1 star that is x n plus 1 plus y n plus 1 star.

And then we have y n plus .1 and then x n, y n plus 1 is star is y n plus 0.2 x n plus y n. So, we put that value here and we see that we have 1.2 in to x n plus y n and then x n plus 1 plus y n, x n plus 1 is x n plus h that is x n plus .2. So, we get y n plus 1 as y n plus 0.22 in to x n plus y n plus 0.02.

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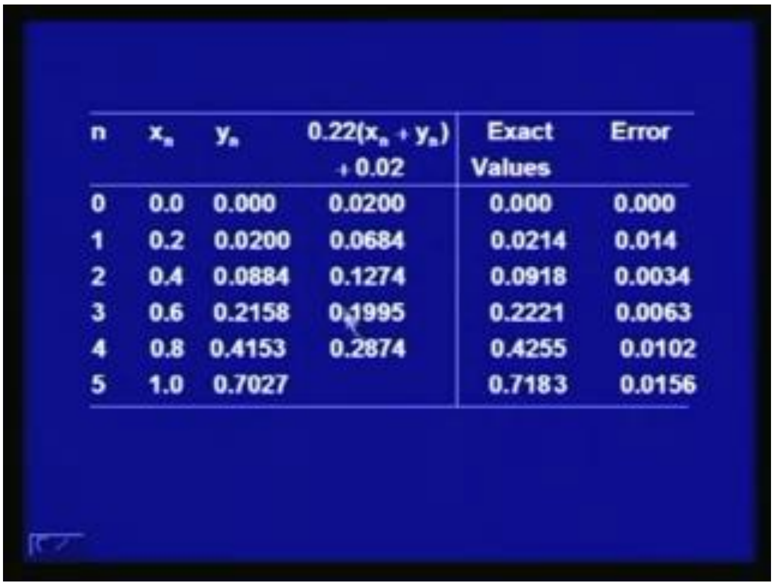

n	x_n	y_n	$0.22(x_n + y_n) + 0.02$	Exact Values	Error
0	0.0	0.000	0.0200	0.000	0.000
1	0.2	0.0200	0.0684	0.0214	0.014
2	0.4	0.0884	0.1274	0.0918	0.0034
3	0.6	0.2158	0.1995	0.2221	0.0063
4	0.8	0.4153	0.2874	0.4255	0.0102
5	1.0	0.7027		0.7183	0.0156

This table gives us the values of y corresponding to the values of x with the step size h equal to 0.2. When n is 0 x naught is 0 y naught is 0 and we compute the value of .22 x naught x naught plus y naught plus 0.02 as 0.0200. We then get the value of y at x equal to 0.2 as 0.0200, y at 0.4 comes out to be 0.0884, y at 0.6 comes out to be 0.2158 y at 0.8 comes out to be 0.4153 and y at 1.0 comes out to be 0.7027.

The actual values of y are 0.214, 0.918, 0.2221, 0.4255, 0.7183, the error that occurs in computing the values of y using this method are 0 here this is 0.014, 0.0034, 0.0063, 0.0102, 0.0156. So, you can see from the error that occurs here in computing the values of y that this method is better than the Euler’s method that we have discussed earlier.

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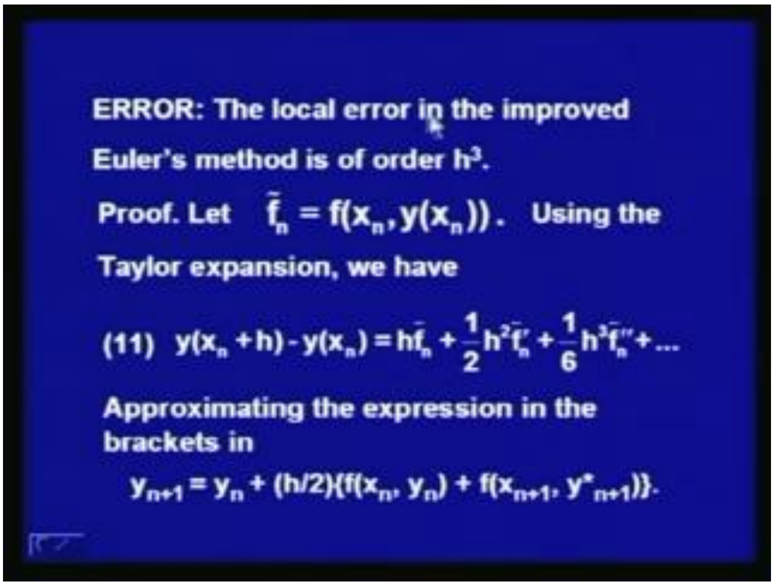

Now, let us discuss the error in this method, the local error in the improved Euler's method is of order h cube to prove this let us say f n cap is equal f x n, y at x n. If we use the Taylor's expansion, we will find that y at x n plus h minus y at x n is equal to h in to f n cap this is f n cap, because h in to d y by d x at x n is f at x n by x n. So, we will have f n cap here plus half h square f n dash cap plus 1 by 6 h cube f n double dash cap and so on.

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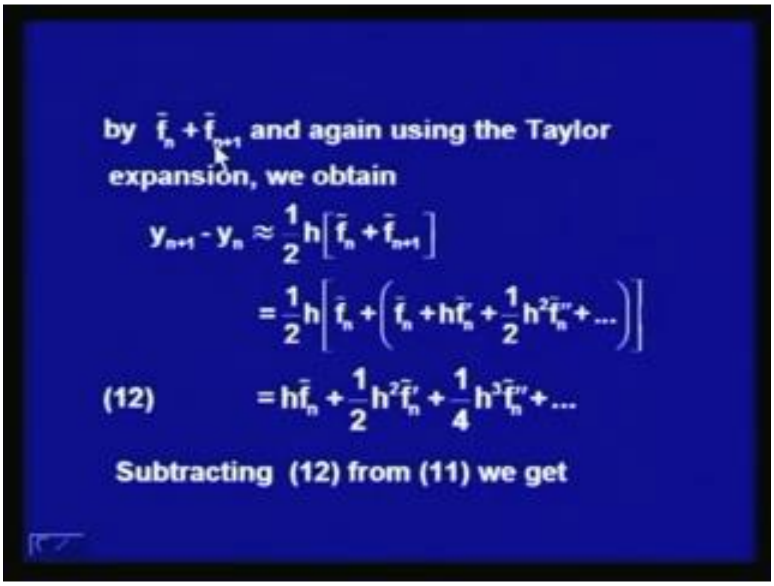

Approximating, the expression in the brackets in y n plus 1 equal to y n plus h by 2, f x n, y n plus f x n plus 1 y n plus 1 star. If, we approximate this expression inside the brackets by f n cap plus f n plus 1 and again using the Taylor expansion, we find that y n plus 1 minus y n is approximately half of h in to f n cap plus f n plus 1 cap. And when we use Taylor's expansion here we will get half h f n plus f n cap plus h f n dash plus half h square f n double dash cap and so on, which will be equal to h f n cap plus half h square f n dash cap plus 1 by 4 h cube f n double dash cap and so on.

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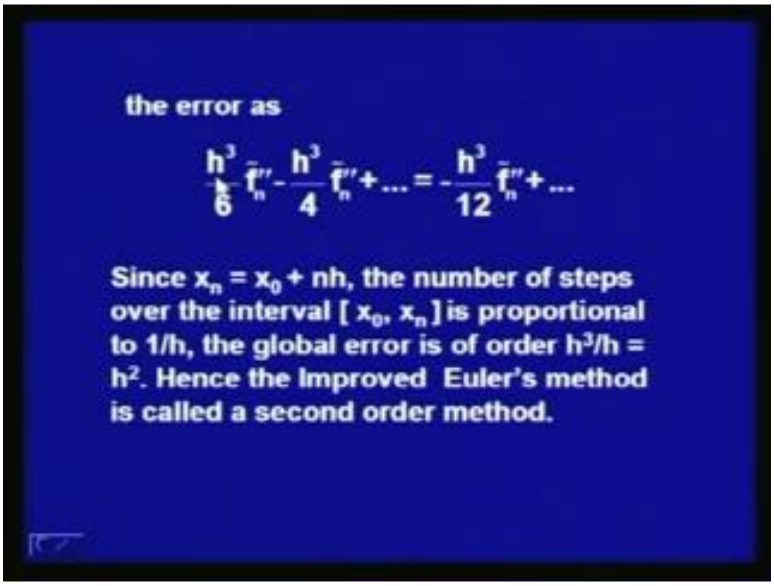

Now, we subtract this equation 12 from the equation 11 we will get the error as h cube by 6 in to f n double dash cap minus f cube by 4, f n double dash cap and so on, which is equal to minus h cube by 12 f n double dash cap and so on.. So, this is the error here is of the order of h to the power 3. Now, we have x n equal to x naught plus n h and thus the number of the steps over the interval x naught to x n, which is n is proportional to 1 by h. And therefore the global error in this method is of order h cube by h that is h square and. So, the improved Euler's method is called a second order method.

Now, in our lecture today we have discussed the numerical solution of an initial value problem, we discussed two methods here Euler's method, improved Euler's method in our next lecture. We shall discuss power series method for finding the solution of a homogeneous differential equation and then we will discuss the particular cases there that of the Linder's equation and the Vermin's equation that is all.

Thank you.