Mathematics - III
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Lecture - 16
Partial Differential Equation

Welcome to the lecture series on differential equations for undergraduate students. Today's topic is Partial Differential Equations, like ordinary differential equations, partial differential equations are also equations in which is to be solved for unknown quantity, and the unknown quantity here is also a function but unlike ordinary differential equations. Here the unknown function could be of more than one variable and the equation could be a relationship between that unknown function and it is partial differentiation.

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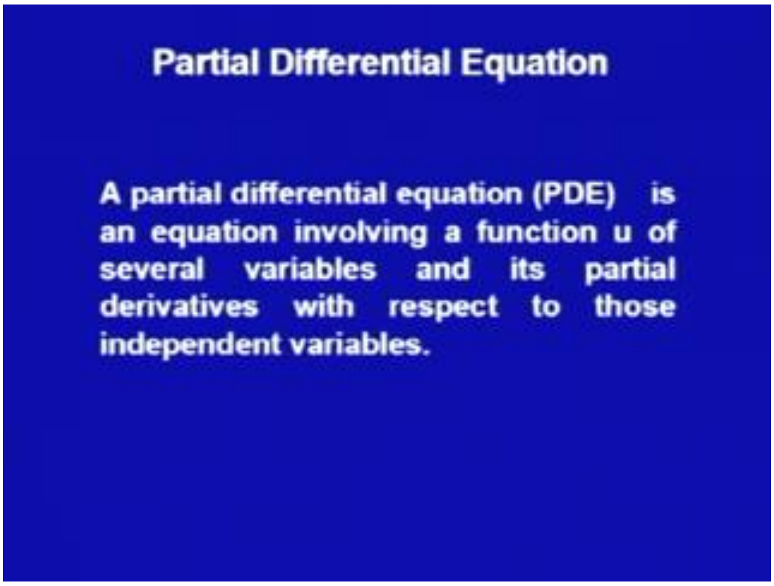

So, we could define that partial differential equation as, a partial differential equation is an equation involving a function u of several variables and it is partial derivatives with respect to those independent variables. So, as in the ordinary differential equations, we were having a function, unknown function which we had denote it by y and one variable that we denote it by x. Here, we may have more than one independent variables, that is now we will denote those independent variables as x y and t so on and the unknown function as u. Let us see some examples of partial differential equations.

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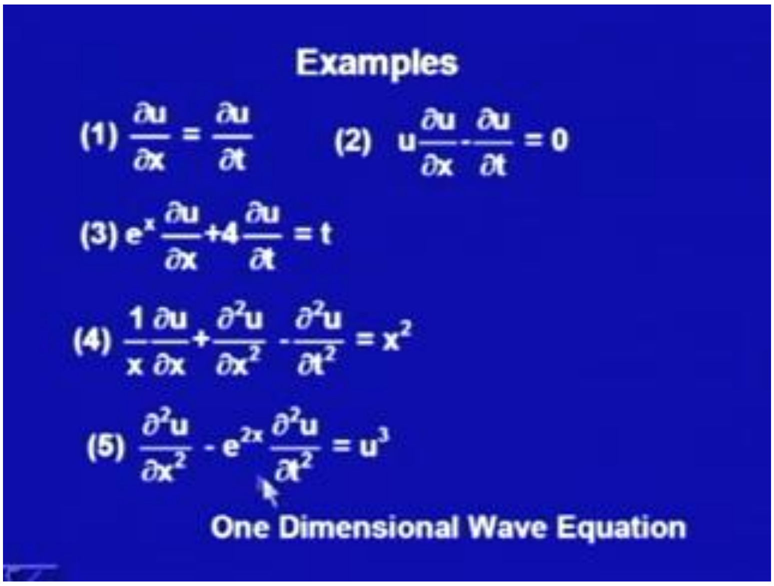

Say, del u by del x is equal to del u by del t, u times del u by del x minus del u by del t is equal to 0, e to the power x times del u by del x plus 4 times del u by del t is equal to t, 1 upon x del u upon del x plus del 2 u upon del x 2 minus del 2 u upon del t 2 is equal to x square, del 2 u over del x 2 minus e to the power 2 x del 2 u upon del t 2 is equal to u cube. So, you see here is, in all these equations, the unknown function is u.

In the first equation, this is u is a function of 2 variables, x and t and we do have a relationship with the partial derivatives with respect to x and with respect to t. Similarly, here also, here also, this is also an unknown function is of the two variables x and t, this last equation is also known as one dimensional wave equation.

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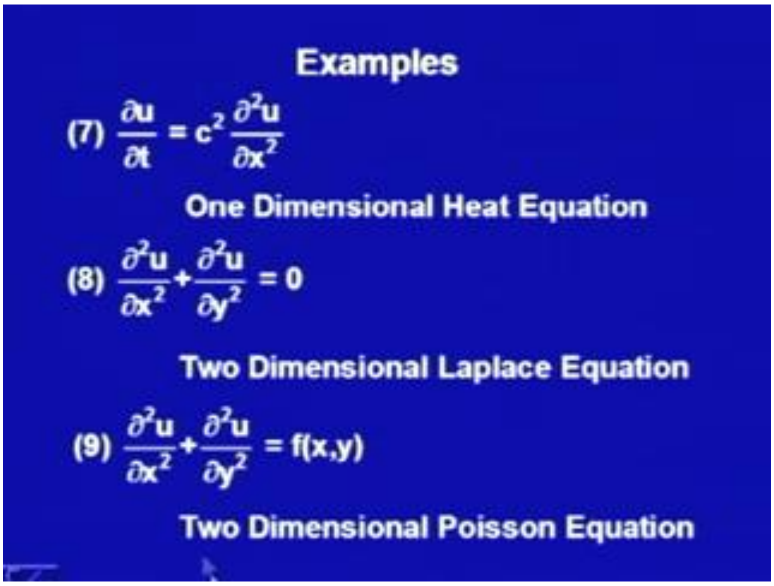

Then, we do have del u upon del t is equal to c square times del 2 u over del x 2, this is known as one dimensional heat equation, another equation known as del 2 u upon del x 2 plus del 2 u upon del y 2 is equal to 0.You see here, the independent variables are x and y and the unknown function is u, this is known as the two dimensional Laplace equation, del 2 u over del x 2 plus del 2 u over del y 2 is equal to a function of x and y, this is known as two dimensional Poisson equation. You see, here is the differences that is, here is the right hand side is 0, here is the function of x and y, this function is known function.
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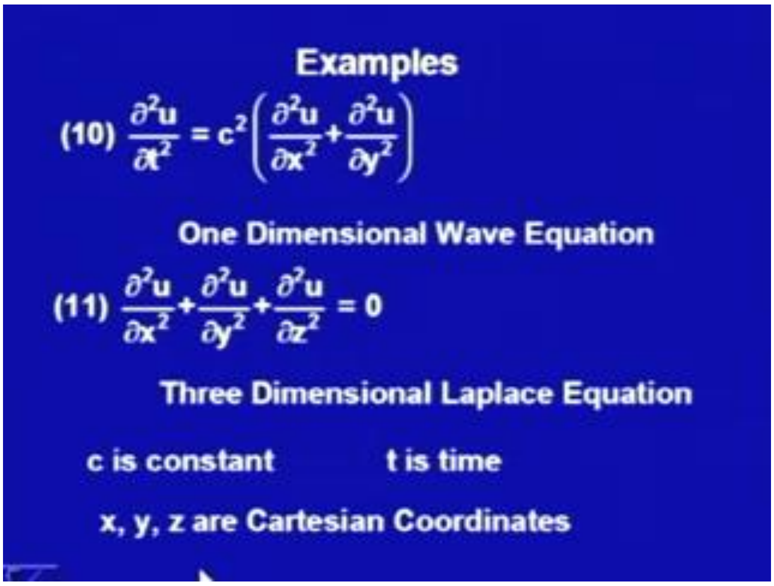

Another example is, del 2 u upon del t 2 is equal to c square times del 2 u upon del x 2 plus del 2 u upon del y 2, this is two dimensional wave equation. This is not one dimensional, this is two dimensional, you see here, the unknown function u is of the three variables x, y and t and we do have partial derivatives with respect to t, x and y. Then we do have, del 2 u over del x 2 plus del 2 u over del y 2 plus del 2 u over del z 2 is equal to 0, this is actually three dimensional Laplace equation.

Here, we do have three variables x, y and z, in all these equations, wherever we have used this c, that was used as a constant, that is a constant and we had used the variables as, one variable as t, another variable as x, y and z. So, this variable t is referred as time and this x, y ,z they are nothing but the Cartesian coordinates and you see is, sometimes you said is, one dimensional equation another times we said is two dimensional or three dimensional, they are because of these, how many variables of this Cartesian coordinates we are having.

As here we are having two Cartesian coordinates x and y, we call it two dimensional, here we are having is x, y and z, so we call it three dimensional, whenever, we are having is, del u x plus del u, del u, del u x, del u over del x plus del u over del t is equal to c kind of equation, Though we had, only x and t that is one dimensional we are calling it. All these equations which I have shown you here as an example, they are of special importance in the partial differential equations.

That is why, here I have just given a little bit, insight of those kind of equations. Let us understand some more basic terms and concepts in the partial differential equation, it is similar to the ordinary differential equations, but some more concepts you would be requiring, so basic concepts.

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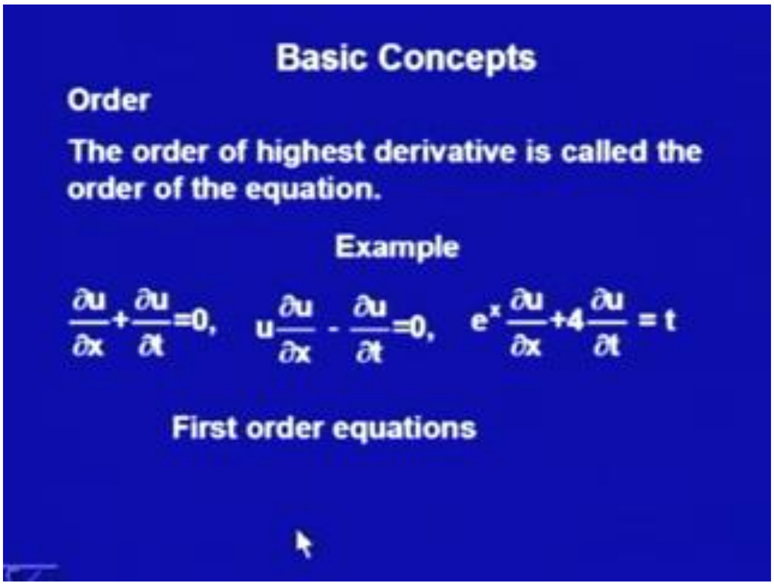

First is order, as in the ordinary differential equations, we define the order of a differential equation as the highest order of the derivative. So, here also we say, is the order of the highest derivative is called the order of the equation, say for example, del u over del x plus del u over del t is equal to 0, this has the derivative of the first order only. Similarly, u times del u over del x minus del u over del t, again both the derivatives of the first order only, e to the power x del u over del x plus 4 times del u over del t is equal t.

Again, we are having all the derivatives occurring are of the first order only, so the highest order of a derivative occurring are the first order and in the second equation, we do have the unknown function u also, but the derivative is only of the first order. So, all these are the examples of first order equations.

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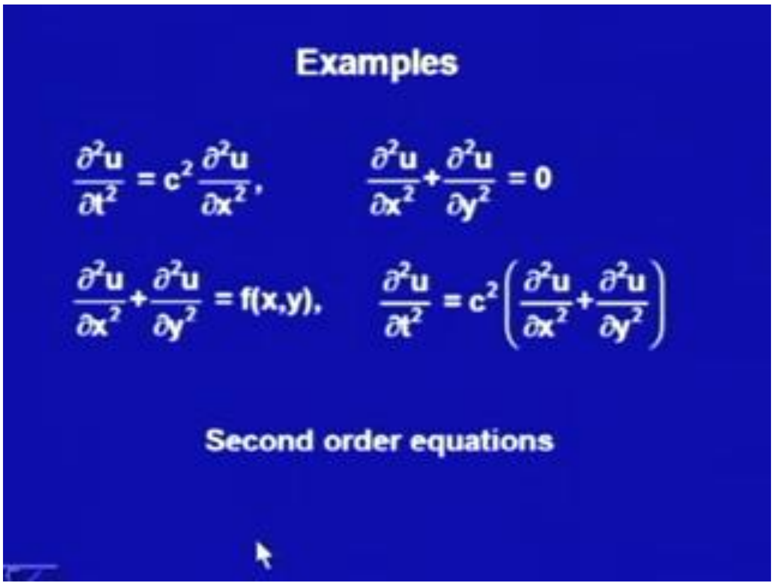

Similarly, if you see other examples, del 2 u over del t 2 is equal to c square times del 2 u over del x 2. Here, the derivative occurring is of the second order, this is also of second order, del 2 u over del x 2 plus del 2 u over del y 2 is equal to 0, here also the second order derivative, that the highest order derivative is occurring is the second order. In this also, the highest order derivative which is occurring is, of the second order, you remember that, this is our one dimensional wave equation, this is two dimensional Laplace equation, this is two dimensional Parson equation.

The highest order derivative is occurring is, of the second order, similarly in this one, this is two dimensional wave equation. So, we do have all these equations, the order of this equation, this is the second order.

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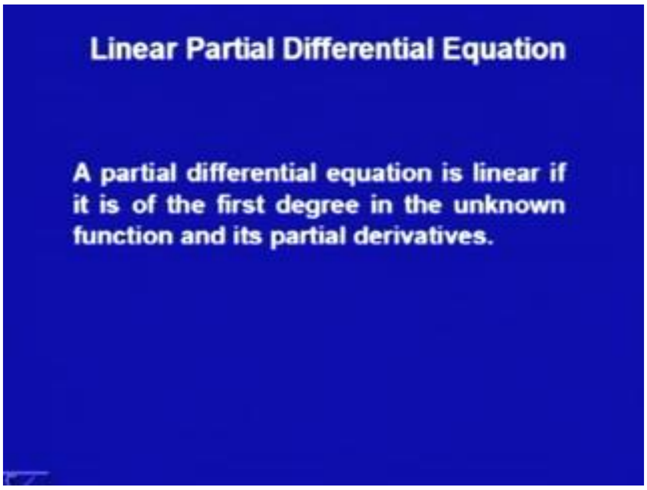

Next the term is, linear partial differential equation, as in the case of ordinary partial differential equations, we said it to be linear, if the derivatives and the unknown functions are occurring linearly. So, similarly here also, we are defining linear partial differential equation, a partial differential equation is linear if it is of the first degree in unknown function and its partial derivatives.

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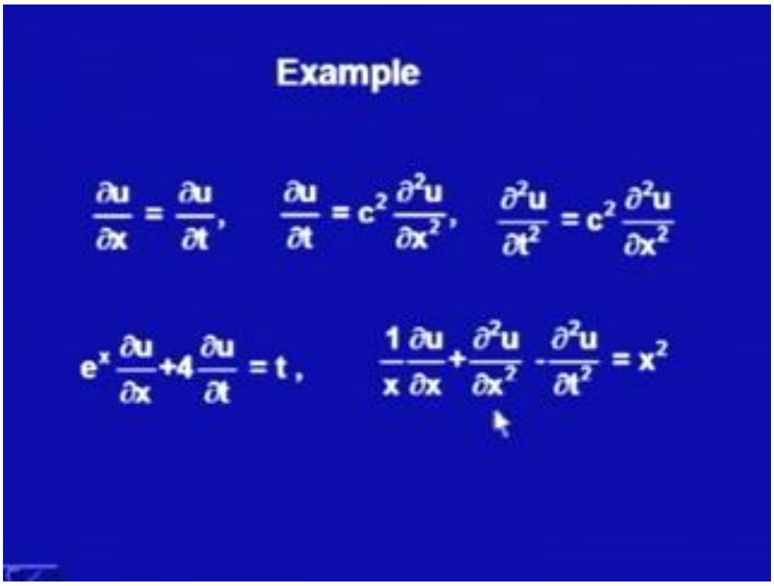

Say for example, in our examples if you see, this example del u upon del x is equal to del u upon del t, we are having is that, the derivatives are occurring in the first degree and linearly. Here also, the derivative that is second derivative, first derivative; all are occurring linearly, the coefficient is c square, c is a constant. Here also, del 2 u over del t 2 is equal to c square del 2 u over del x 2 the coefficients, a c square which is a constant, so they are occurring linearly.

In this equation, you see e to the power x del u over del x plus 4 times del u over del t is equal to t, the coefficient of del u upon del x is e to the power x, it is not a function, this is not containing u, similarly, the coefficient of this is 4 and the right hand side, this is t, this is not a u, so we are having is that, the unknown function and its derivatives they are occurring linearly, so this is also linear.

Here also, we see that derivative is the function that, the coefficient of the derivatives del u over del x that is 1 by x, the derivative of this coefficient of this del 2 u over del x 2 that is the 1, del 2 u over del t 2,the coefficient is minus 1. We are having is, that the coefficients are not having the function of u or the other derivatives of unknown function, so this is also linear. So, all these are examples of linear.

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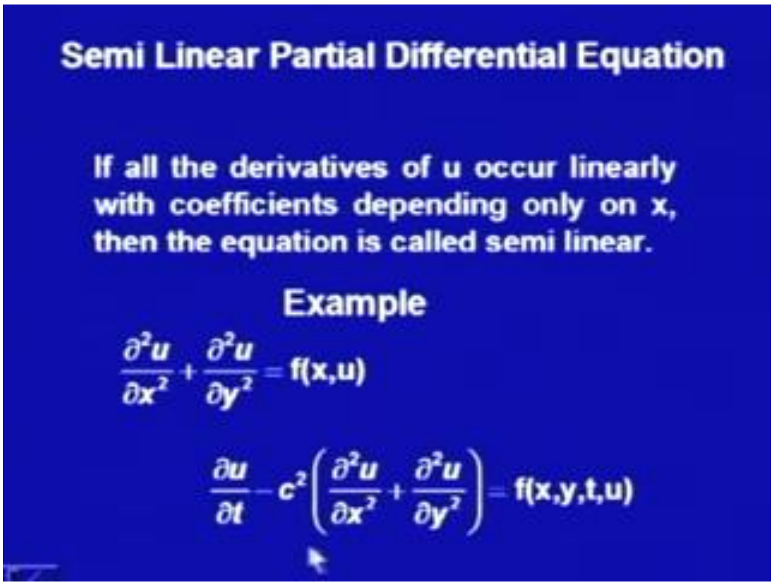

In partial differential equations, we will define some more kind of a linear equations, one we call semi linear. If, all the derivatives of u occur linearly with coefficients depending only on x, we had seen in the last examples that is, somewhere the coefficients were depending on x, somewhere the coefficients were depending on g. So, now if the coefficients are depending only on x, so when I am calling it x, it may have x, y or z depending upon if it is, of higher dimensional equation, then this equation is called semi linear.

Let us see some example of this one, if I take this example, del 2 u over del x 2 plus del 2 u over del y 2 is equal to f of x, u, this is semi linear. Here the, you are having is that, the coefficients we are not talking about the other one, we are just talking about the coefficients of the unknown function and its derivative, they are constant here. Here also, it is constant and what we are having is that is, here we are having is that the right hand side, that is the constant function, it is u.

And a function of x and u, we are saying is that function is having u linearly and its coefficient is depending only on x, its coefficients are depending on this, may be a function of this one, but the coefficients would depend only on x, this is semi linear one.

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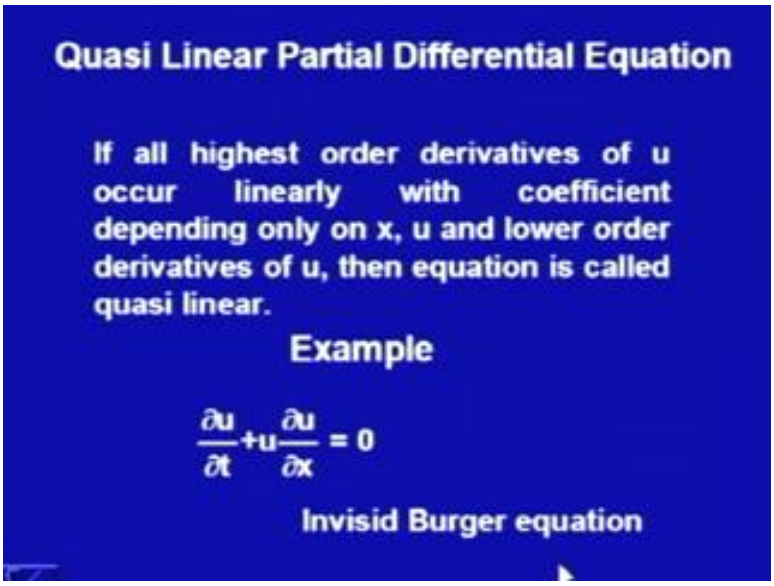

Similarly, we would define one more kind of differential, partial differential equation that we are calling quasi linear partial differential equation, what it is, if all highest order derivatives, whatever be if they second order or third order, first order; highest order we have to take in that equation, that occurs linearly with coefficients depending on x, u and lower order derivatives of u, then the equation is called quasi linear.

See for example, del u over del t plus u times del u over del x is equal to 0, here, the unknown function is u, its highest derivative over is here of the first order, its coefficient is depending on x, u and lower order derivatives because this is first order, so there is no low order derivative, unknown function u is this. So, here the derivative is, here the coefficient is u, this we call quasi linear differential equation.

In ordinary differential equations, we had only linear differential equations and there we said is, that is, if the coefficients are the constants or the coefficients are the function of x, there we have discuss only one particular case of coefficient of x and that we called quasi equation. There also we have done something is called the homogeneous and non homogeneous, in the singular manner here also we define, this particular equation is known as Invisid Burger equation.

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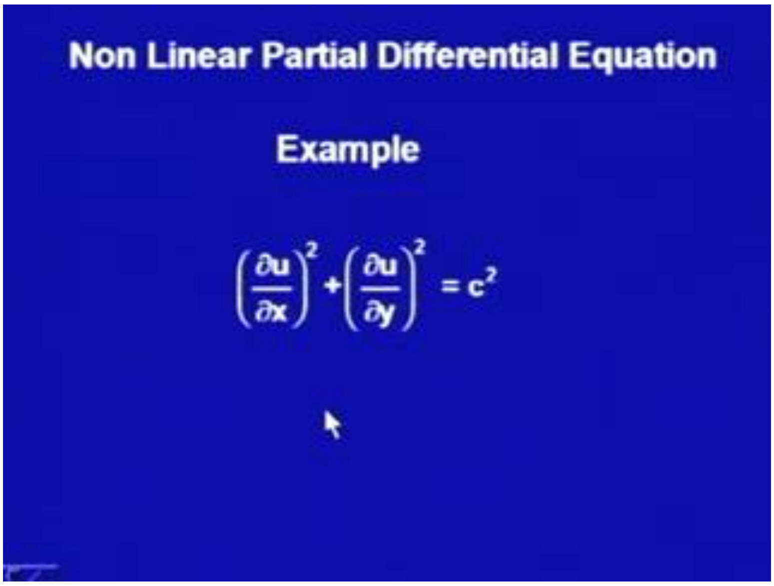

Non linear equation, let us see, if it is not linear, semi linear or quasi linear, we call it del u over del x square plus del u over del y whole square is equal to c square, you see, the derivative is del u over del x and del u over del y and we are having its degree 2. So, they are not having of degree 1, so this is non-linear, this is a simplest example I was giving, now, as in the ordinary differential equations, here for the linear partial differential equations, we again define the homogeneous and non homogeneous.

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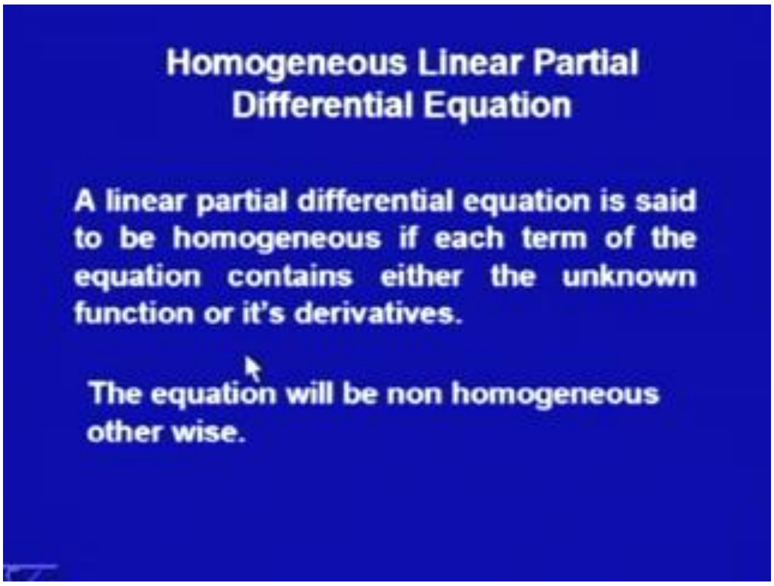

So, a linear partial differential equation is said to be homogeneous, if each term of the equation contains either the unknown function or its derivatives and otherwise the equation would be called non homogeneous. What we mean by, if you do remember, there we, when we have used in the ordinary differential equations, the linear differential equations we said is that, all the derivatives and the unknown function occurring linearly and the right hand side is 0 and if a right hand side was not 0, we called it to be non homogeneous. The same thing is being said here in the different manner that is all, that is, each term is containing either unknown function or its derivative, otherwise, it is 0.

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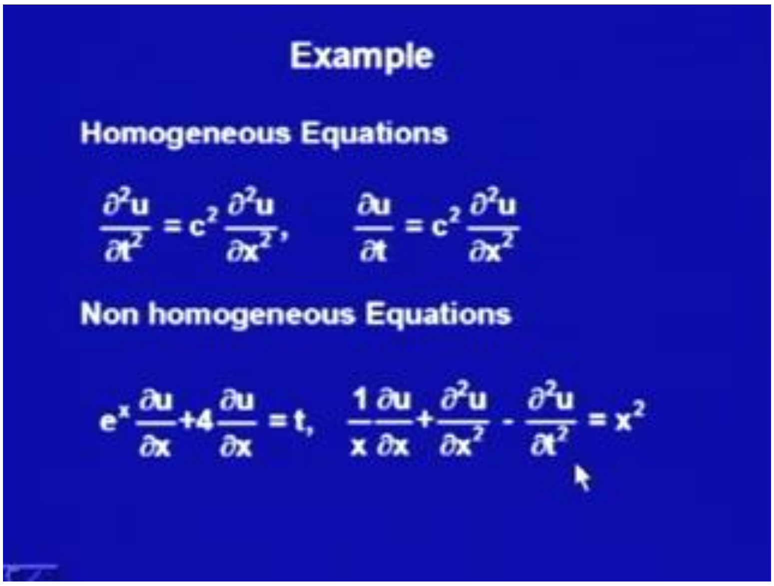

So, a homogeneous equation example, del 2 u over del t 2 is equal to c square times del 2 u over del x 2, so if I try to write in that manner, in which we use to write the ordinary differential equations, we would write it as del 2 u over del t 2 minus c square times del 2 u over del x 2 is equal to 0, so the right hand side would be 0. Similarly, del u over del t is equal to c square times del 2 u over del x 2 that is, each term contains either the unknown function or its derivatives, we are not having any function.

Any term here, which is not containing u or its derivatives, non homogeneous equation you see, e to the power x del u over del x plus 4 times del u over del t, it should be del t is equal to t. Now you see, we are having this containing the derivative of unknown function, containing the derivative of unknown function, but this is the function, this is the term which is not either containing u nor its any derivative. So, this is non homogeneous equation.

Similarly, you see here, this term is containing the derivative with respect to x, this is also containing the derivative with respect to x, this term is also containing the derivative with respect to t, but this term is not containing any of u or its derivatives first order, second order or with respect to x or t anything, so this is again a equation which is non homogeneous.

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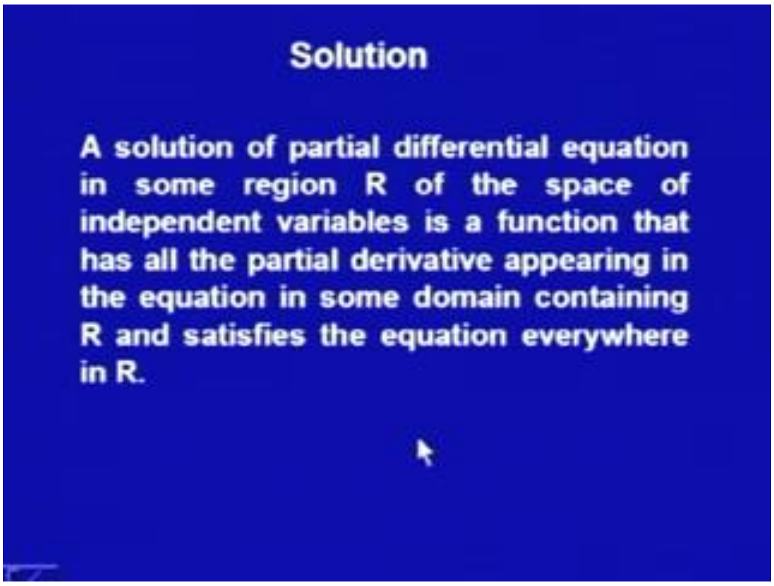

Now, as in the ordinary differential equations, we define the solution, we said is that differential equations are in unknown function u, that is the unknown quantity is a function, u, so here, the solution would also be a function u. So, which function you would call the solution, the definition says, a solution of a partial differential equation, in some region R of the space of independent variables is function that has, all the partial derivatives appearing in the equation, in some domain containing R and satisfies the equation everywhere in R.

Let see, what we are saying is, if there is an equation, partial differential equation; that means, it will contain the relationship between the unknown function and its derivatives with respect to certain coefficients. So, what we say is, a function which is continuous and have all the partial derivatives which are appearing in that equation, that is, it must have all the partial derivatives of that function and then that function and its derivatives must satisfy the equation everywhere in that region R.

Then we would call, this is solution in that region R, before seeing is, that is, what is the solution.

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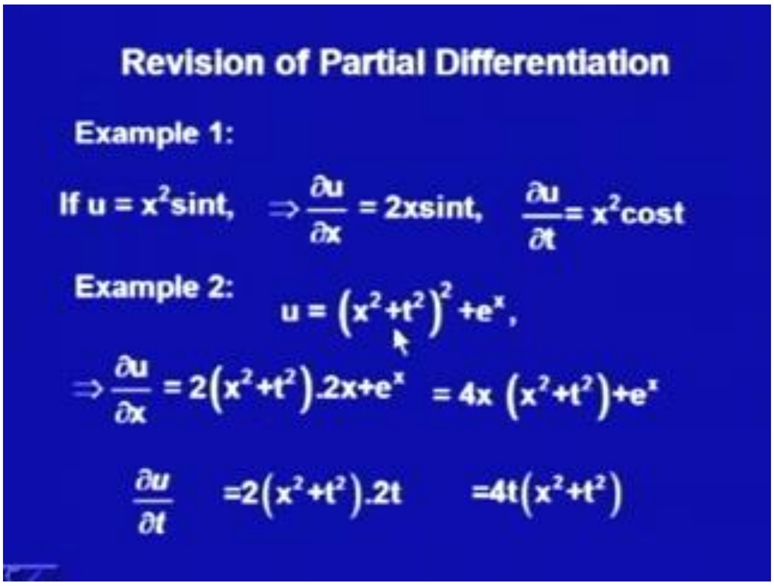

Let us, revisit our partial differentiation a little bit, so that, whatever the terms or whatever the partial derivatives you would be involving, in our understanding of this partial differential equation, I am just giving you that a little bit understanding over here. So, let us see it with the help of certain examples, now if my function u is of x and t where, function is of x square into sin t, now we do know that, we could write its partial derivative with respect to x.

That means, we would take t as constant and the derivative of x is 2 times x and sin t, that is as treated as constant. Similarly, if I want to find out the partial derivative with respect to t, I would get it as x square into cos t, why this x is just been treated here as a constant and the derivative of sin t is nothing but the cos t. Take the another example, if u is a function of x square plus t square, whole square plus e to the power x then what will be its partial derivative with respect to x and with respect to t.

So, first let us take with respect to x, this is a function of x and t, so we will just treat it as a chain rule or the function of function. So, 2 times x square plus t square and then the derivative of this would be 2 x, because t is treated as a constant plus this is the derivative of e to the power x with respect to x is e to the power x, so what we do get is finally, 4 times x into x square plus t square plus e to the power x.

Now, take the derivative with respect to t and treat x as a constant, then del u over del t would be, 2 times x square plus t square as a whole one and then the derivative of this function would be only 2 t, because x is being treated as a constant and since it is a constant. So, the derivative would occur here, so what we get, 4 t times x square plus t square, now this is the, we have used the, here the chain rule of the, ordinary derivatives.

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Now, for the partial derivatives, the chain rules says, that is if u is the function of the two variables r and s, where both r and s are also the function of two variables say x and t, that is r is a function of x and t and s is also a function of x and t. Then if I want the derivative of u with respect to x and t, while u is actually a function of r and s, then the chain rule says, that derivative, partial derivative of u with respect to x is can be given as, del u over del r into del r over del x plus del u over del s into del s over del x.

In the ordinary differentiation, what we had, we had only one variable and then that was a function of another variable. So, we had it only du over d z and d z by d x, now here because it is of the two variables, so we are adding it of, if it is more than two variables, again we would go on this adding it up in that manner. Similarly, the partial derivative with respect to t can be given as, the derivative of u with respect to r and then derivative of r with respect to t, all are the partial ones because r is also a function of x and t.

And then the partial derivative of u with respect to s, that del u over del s and the partial derivative of s with respect to t, that is del s over del t.

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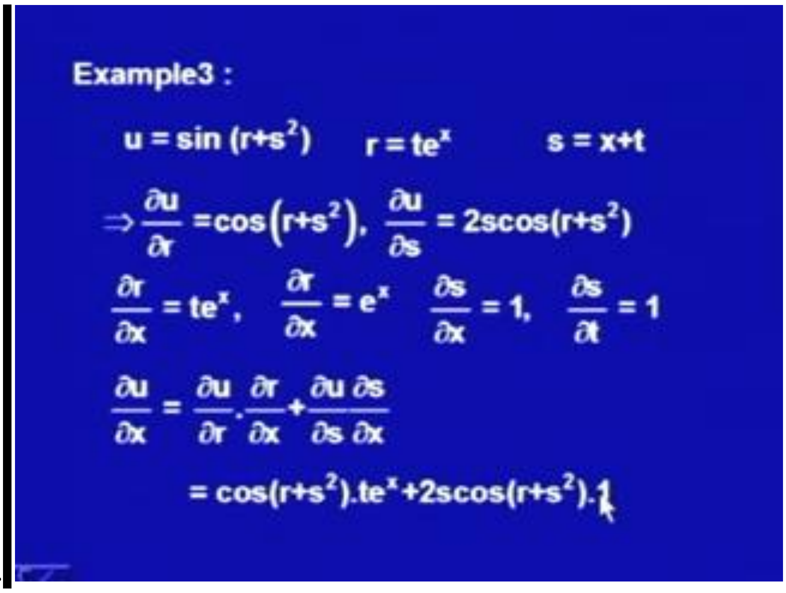

Let us see one example, suppose u is sin of r plus s square, where your r is t times e to the power x and s is x plus t, I want the partial derivative of u with respect to x and t. So, first the partial derivative of u with respect to r, del u over del r that is cosine of r plus s square and the derivative of r is 1, s is been treated constant. If I want the derivative with respect to s, then it would be cosine r plus s square that is the derivative of sin, r plus s square and then the derivative of s square that is 2 s and you would require actually del, the derivative of r and s with respect to t and x.

So, derivative of r del r over del x, treating x as the variable and t as a constant, t times e to the power x. Similarly, del r over del t would be e to the power x, t is cons[tant], t is, derivative of t is 1, del s over del x would be 1 and del s over del t would be 1. So, using the chain rule, del u over del x as del u over del r into del r by del x plus del u over del s into del s over del x, we would get cos r plus s square into t to the power e x plus 2 s cos r plus s square into 1.

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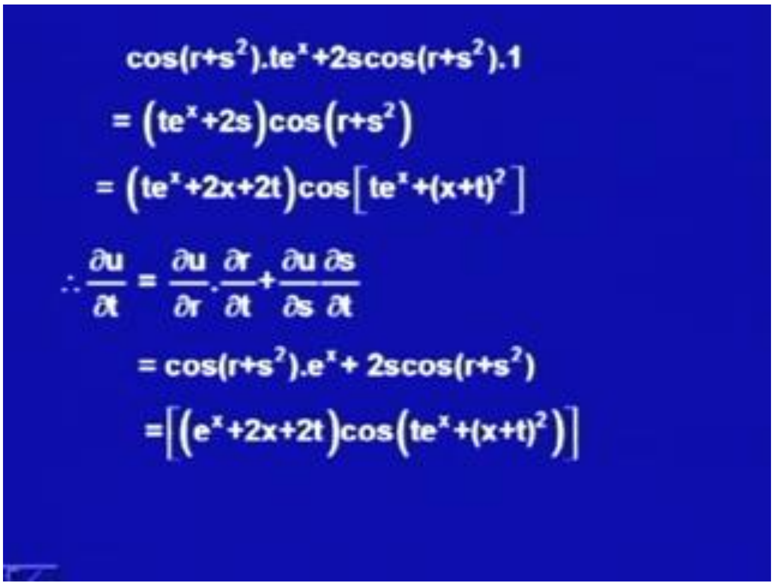

That, we can rewrite it in this form, so replacing by r and s by the function, that is r is t times e to the power x and s is x plus t, so it is, this one. Similarly, if I try to find it out del u by del t, that would be del u by del r into del r by del t plus del u by del s into del s by del t, now substitute this del u by del r, cos r plus s square, del r by del t was e to the power x del u by del s is 2 s cos r plus s square and del s by del t is 1.

So, we would be getting, this again substituting this r and s square in the terms of x and t, you would get this one, so this is what, is your chain rule. Now, after this basic concepts revisiting, let us try to find out verification of solution to the partial differential equations, that is, what we will try to see, given certain partial differential equation, we will try to see if a given function is solution of it or not, that is what, why you call it solution.

If it is having those partial derivatives which are involving that equation and if that unknown function and its derivatives are satisfying that equation. So, let us take one example.

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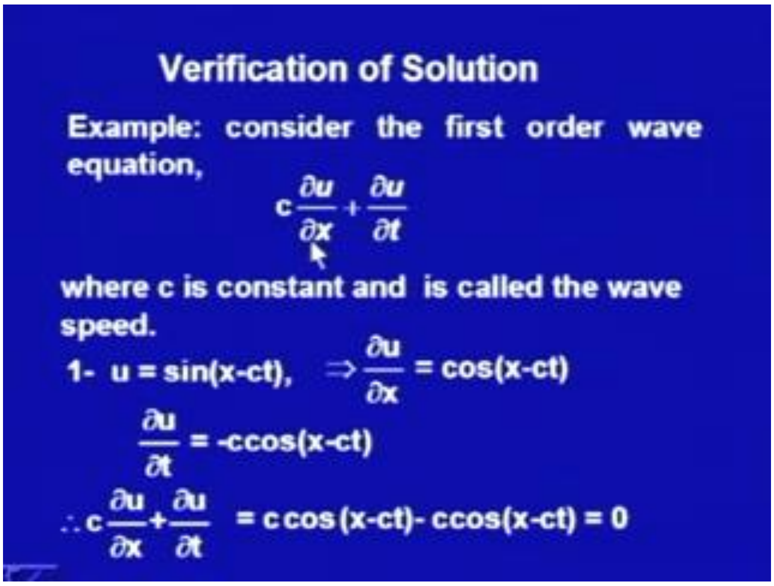

Consider this first order wave equation, this is called the first order wave equation, c times del u over del x plus del u over del t is equal to 0, where this constant, c is the constant and this is called the wave speed. Now, you see that is, in a ordinary differential equations what we had find it out that is, there was certain function, which we said is that is, they are satisfying the equation.

And we did find it out that the function was same with the constant change here, you see here, actually what will happen, that is we would have, we having many functions which would be satisfying. Now, let us see that is, with the example and then we would find it out, let us see, take the function u as sin of x minus c t, the derivative. So, we do know that is, sin is continuous and differentiable function and the derivative with respect to x and with respect to t, both would be existing.

So we have, here the first order derivatives, so we do find out, what is del u by del x that would because sin of x minus c t and the derivative of x minus c t with respect to x would be 1 and derivative with respect to t, that is again cosine x minus c t and the derivative of x minus c t with respect to t would be minus c, so it is del u over del t is minus c times cosine x minus c t.

So, now if you see is, c times del u over del x, what it would be, it is c times cosine x minus c t plus del u over del t, that is minus c times cosine x minus c t, that gives me 0, that is this equation satisfied. That means, this was the solution to the equation, c times del u over del x plus del u over del t is equal to 0.

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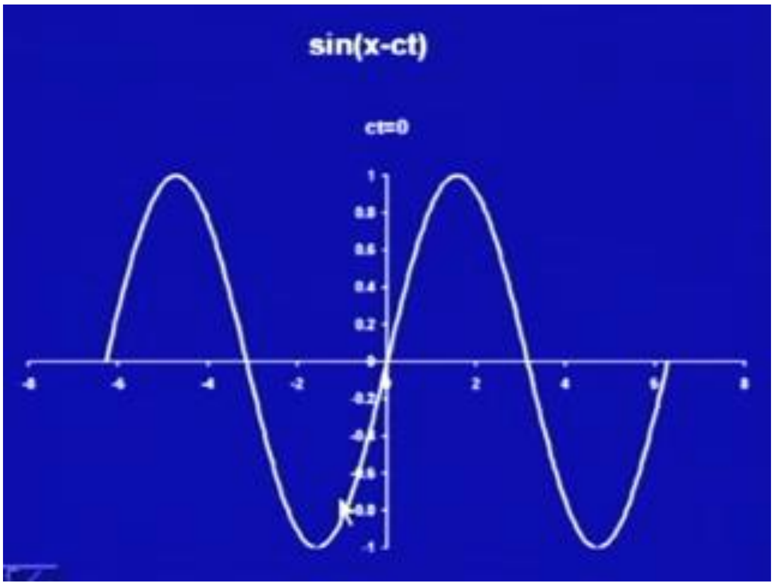

Now, you see, what it is solution, how it is making out, if I take c t to be 0, we have taken x minus c t, so c t I have taken 0, what could I say is, that c I am taking as 1 and c as 0, then this would be the solution, this function would be the solution to the given equation.

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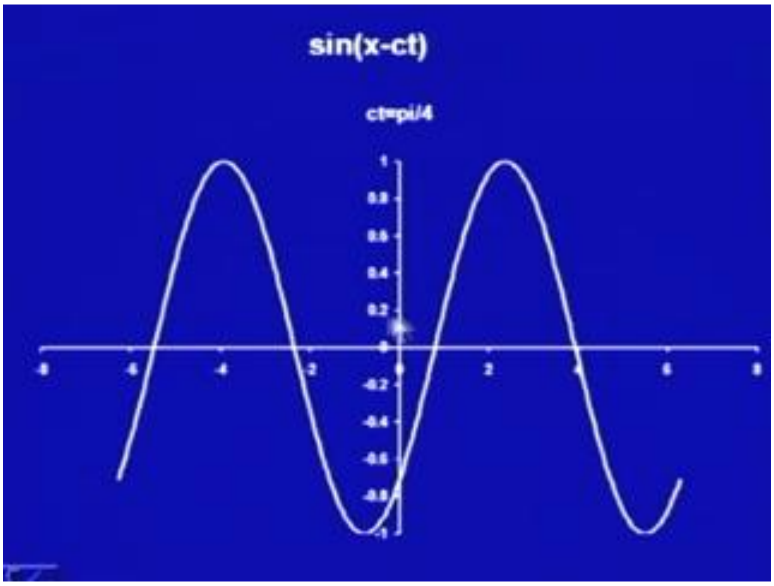

If I take, t is equal to 1 or t is equal to pi by 4, I am getting is, that the solution is little bit changing towards the right, shifting towards the right, the function is same, but it is shifting towards the right.

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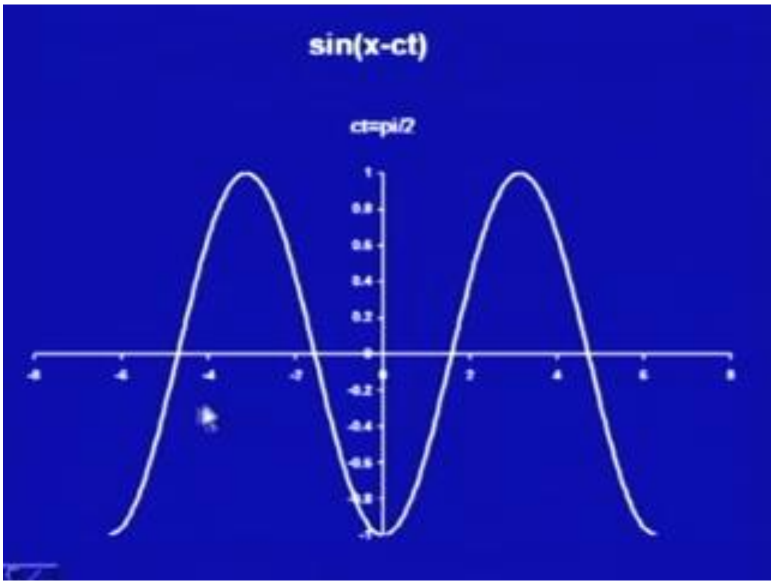

You see, at t is equal to pi by 2.

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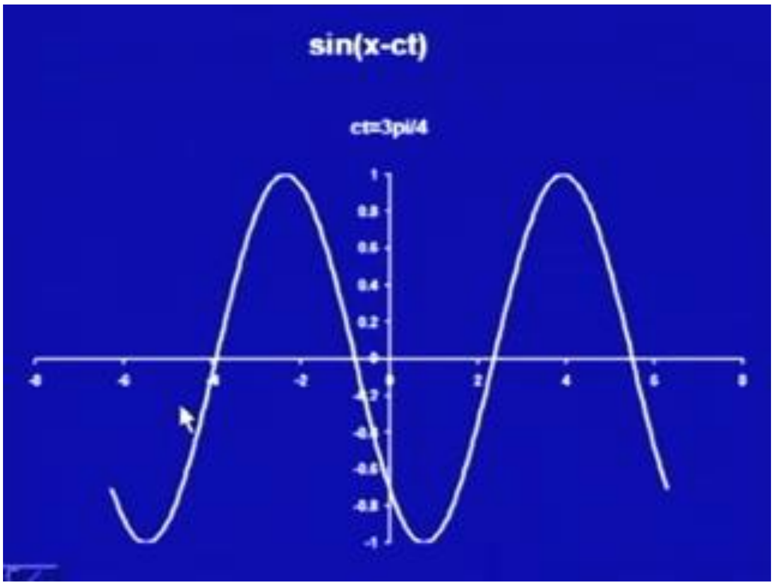

It is shifting this one, 3 pi by 4 it is again more shifting.

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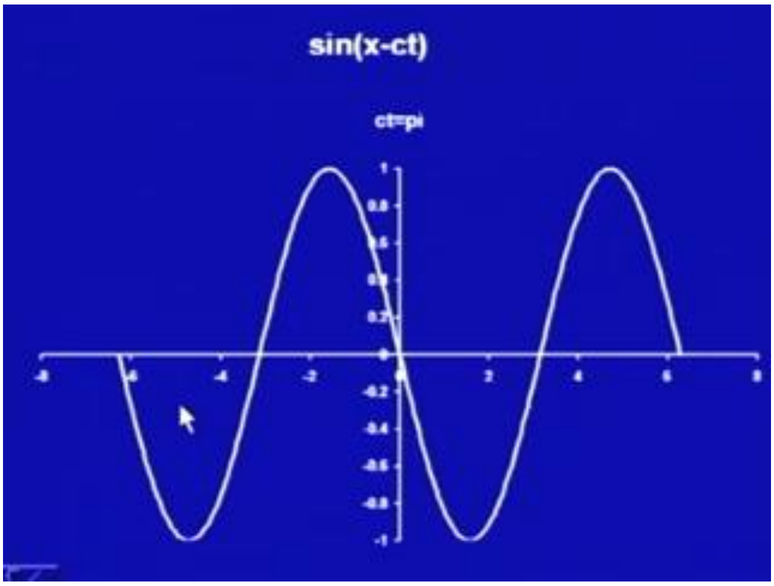

This is shifting, let us see, that is what we are have find it out.

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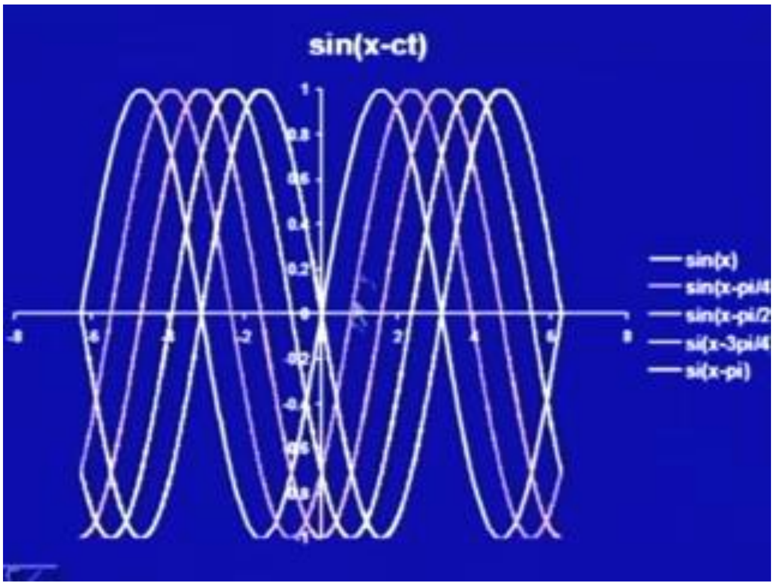

We are finding it out, that at t is equal to 0, this is the function, at t is equal to pi by 4, this is the function, at t is equal to pi by 2, this is the function, at t is equal to pi by 3, pi by 4 this is the function, at t is equal to pi this is the function. So, what we are having is, that is my function, the shape of the function is same but it is shifting towards its right as we are changing our c t.

So, either you could says that, t we are keeping fix and c we are changing or you could says, rather it should be, that is c is being fixed and t is changing, that is t as I said is, we are treating it as a time. So, with the time, this wave is shifting towards right.

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$$2\text{-}\quad u = \begin{cases} 1-(x-ct)^2 & 0 \le (x-ct)^2 \le 1 \\ 0 & (x-ct)^2 \ge 1 \end{cases}$$

$$\frac{\partial u}{\partial x} = \begin{cases} -2(x-ct) & 0 < (x-ct)^2 < 1 \\ 0 & (x-ct) \ge 1 \end{cases}$$

$$\frac{\partial u}{\partial t} = \begin{cases} 2c(x-t) & 0 < (x-ct)^2 < 1 \\ 0 & (x-ct) \ge 1 \end{cases}$$

$$\therefore\ c\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0\ \forall\ x$$

Now, let us take another function, u as 1 minus, x minus c t whole square, in the range when, x minus c t whole square is between 0 and 1 and it is 0, otherwise. Now, you see that, this is also satisfying our wave equation. See, what would be del u over del x, this function is actually we find it out, that is continuous, but we find out that del u over del x, in this region it is, minus 2 times x minus c t, derivative of x is 1 and in this region it is 0.

Similarly, the derivative with respect to t would be, the derivative of c t would be minus c actually. So, would be getting 2 times c x minus t in this region and 0. So, now, if I consider our wave equation that was, c times del u over del x plus del u over del t, we see is, that is what is, if I multiply it with c, these things are just c times del u over del x is same as minus del u over del t. In this region, this is same and in this region, that is when x minus c t is greater than one, both are 0s are, they are satisfying it, for all x.

That means, this is also a function, this function is also satisfying our wave equation. Let us see, what this function is.

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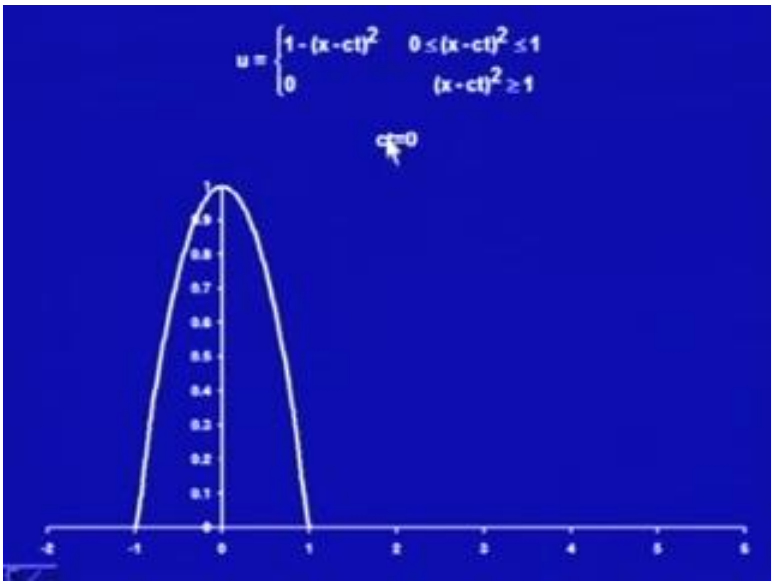

This function is, this one when I am taking is, c t is equal to 0, that is from minus 1 to plus 1, this is a wave, this kind of curve and then. So, if I move this c t to 1.

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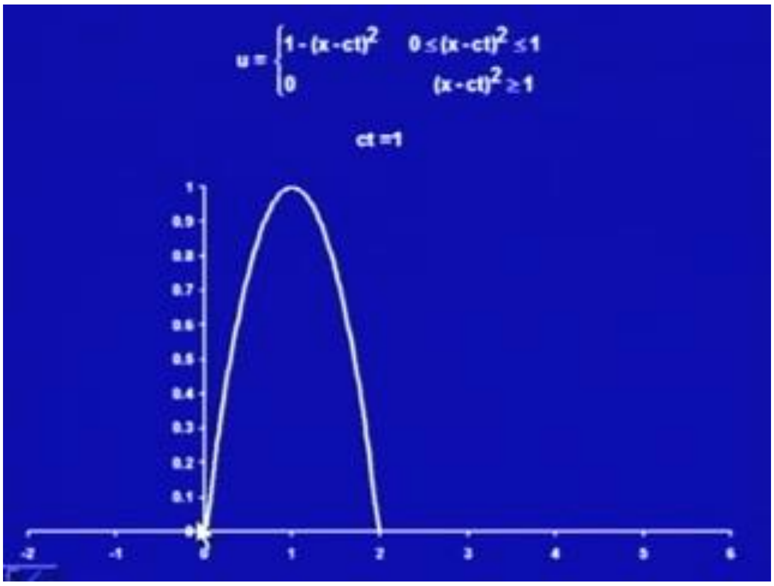

What I get is, that function is shifted from 0 to 2.

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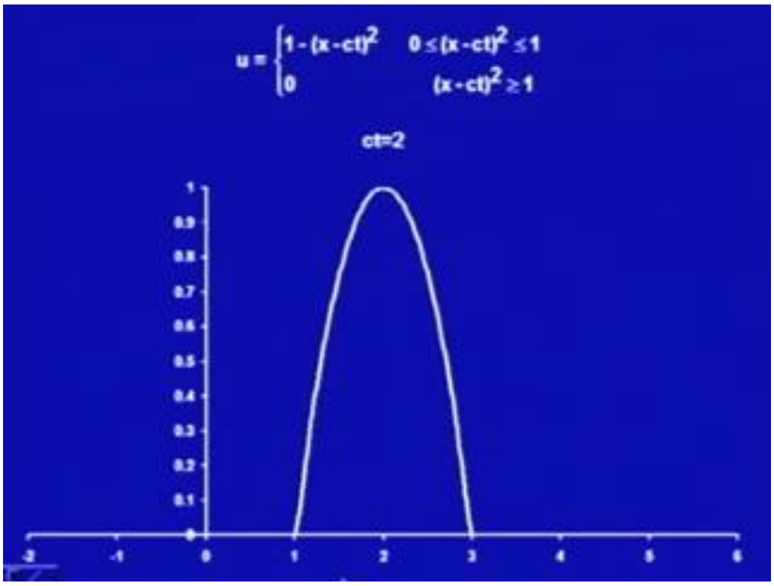

If I move it 2, 1 to 3.

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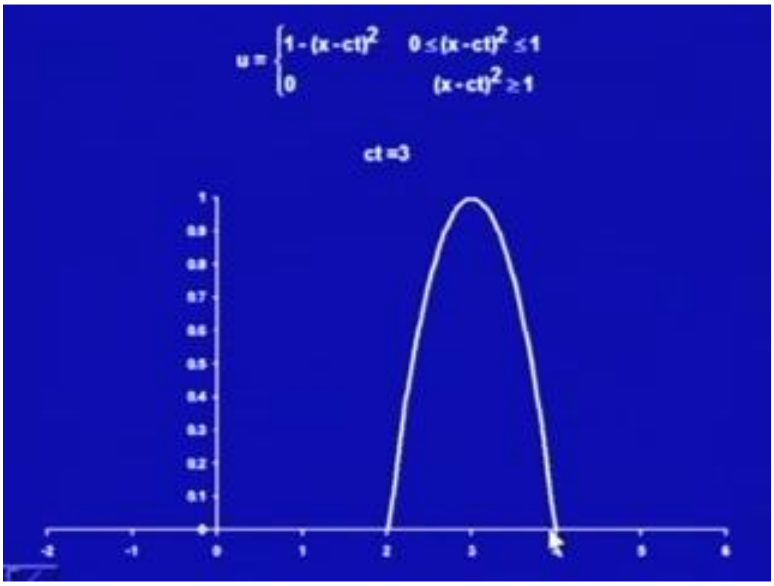

It is moved with 2 to 4 and now you see, that shape was not changing, the shape was exactly same from 0 to 1, but the function is shifting towards the right.

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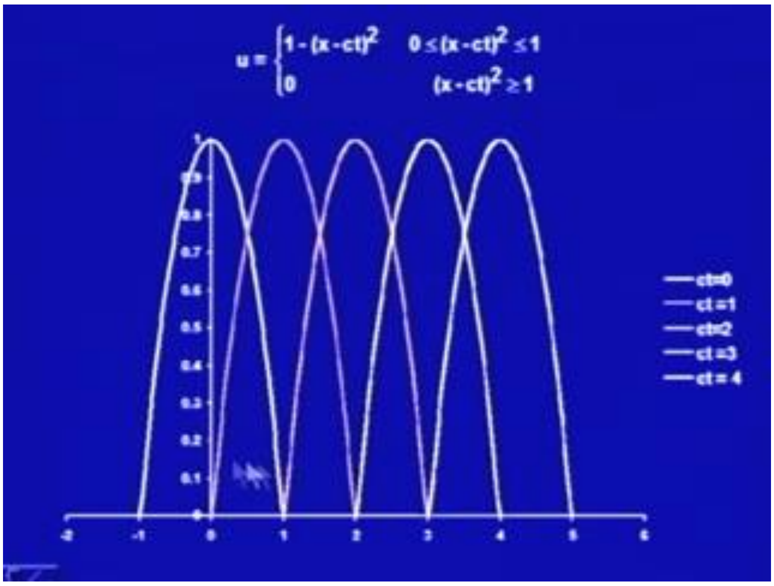

You see, this is, with 0,1, 2, 3 and 4, so what we are having is, the shape is same, only thing is that the wave, that is, this wave which has been forming that is same, but it is shifting as with respect to time, this is shifting towards its right.

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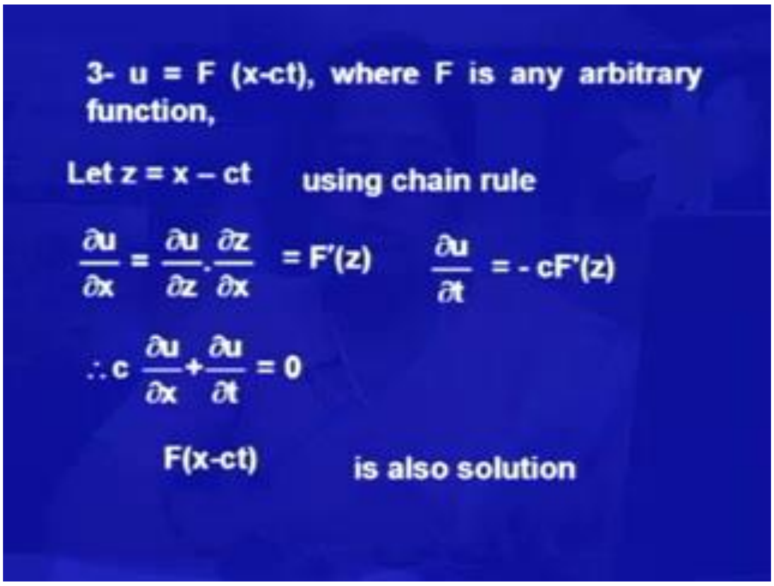

Now, let us treat another one, now here what I am taking is, this function F of x minus c t, where F is any arbitrary function, I am not giving any form of F, in previous two examples, I had given that F as sin and then F, I have given as a particular function. Now, here it is F is a general function. Now, let us see, whether if I take, treat this as a general function, which is continuous and has partial derivative with respect to x and t, will it satisfy our wave equation.

So, use thus chain rule, we get del u by del x as F dash x minus c t and del u over del t as minus c times F dash, F dash means is, that is I am taking is the derivative with respect to z, that is x minus c t. Now, what we are getting is, del u over del x is F dash z and del u over del t is minus c times F dash z, that is, if I take c times F dash z, that is same as minus del u over del t.

So, this equation is satisfied, what I was requiring is, that is F is continuous and has partial derivatives are differentiable with respect to both this one, so this is also a solution, what it says, in ordinary differential equations, if you do remember, we were use to find out a solution which involves arbitrary constants. Now, here what we are getting is, I am finding out a solution which is involving actually arbitrary function.

That says is, I do have in partial differential equations, my solutions may involve arbitrary functions and how do, do you mind this, you make solution does this partial differential equation does not have unique solution because we said is an ordinary differential equations, we do had unique solution, that there was general solution, then there was unique solutions. We said is that, there would be unique solution, which we could find out from the general solution that is giving particular values to the arbitrary constants.

If we had use certain side conditions, what was cause side conditions, we said is that they were initial conditions, initial conditions means that, if it is a first order equation, we said is initially, that is from where the function is starting or if it is second order, we had two initial conditions. First was, that is from where the function is starting, the second was that may be, that as at a first derivative is at initial point.

So, we are having is that, in ordinary differential equation we had initial conditions, for higher order in a differential equation, say for nth order, we had n minus 1, 1 x derivative, we had got the initial conditions for all that is n initial condition. Moreover, we had learned over the ordinary differential equations that is, other than the initial conditions, we were having the boundary conditions as well, when the function was define on a certain finite reason.

we said is that, on both the boundaries we had certain conditions imposed upon and in the boundary that, those problems we call boundary value problems and we had seen, that is in the boundary value problems, sometimes the solution was existing, sometimes the solution was not existing. Now, in the similar manner here to determine this unknown function f, we require some side conditions.

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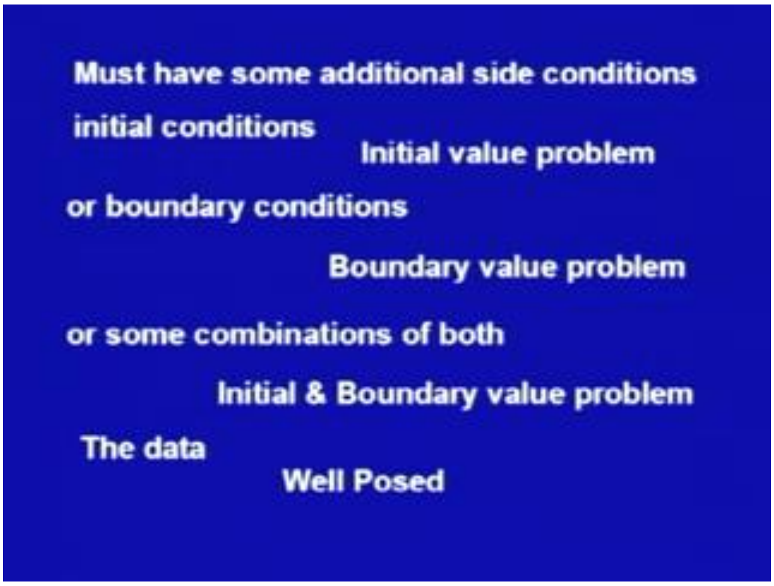

So, we call them, some additional side conditions, they could be of the form of initial conditions or they could be of the form of boundary conditions or sometimes, if a combination of both. And so we are calling them, as in the ordinary differential equation if you do remember, we were calling with the initial conditions, if they, we are solving a problem, we called it initial value problems.

So, here also, we would call the initial value problem, where boundary conditions imposed upon, we call them boundary value problem and when, we are having the combination of both, that is my partial differential equation may be, having both the kind of side conditions, initial conditions as well as boundary conditions and we call them initial and boundary value problem or simply, initial boundary value problems. Now, with these, if you do remember in the boundary value problem in the ordinary differential equations.

We had find it out with certain boundary conditions, the solution is existing and with certain boundary conditions, the solution was not existing. Why, in initial value problems, you find it out that is, always the solution was existing and that solution was unique. Here, again with these side conditions, whatever the conditions, that is the data is given, that is the function at initial value is some particular value or some particular function actually, here it would come out.

We are saying is that, those values and is called data, in this partial differential equation and with respect to those data, if some initial value problem or boundary value problem or initial and boundary value problem. If unique solution is existing, because in boundary value problems, in the ordinary differential equations we had find out that, all the times solution was not existing. So, here what you would say, if with respect to these side conditions, whatever the side conditions are, that we are calling the data with respect to that data.

If, for any partial differential equation, the unique solution is existing, we will call that problem as, well posed problem, otherwise it is called ill posed problem. Now, So, come back to our example, we had find out that is, we were talking about the first order wave equation and we had find out that is, a general function f of x minus ct could be solution of that equation.

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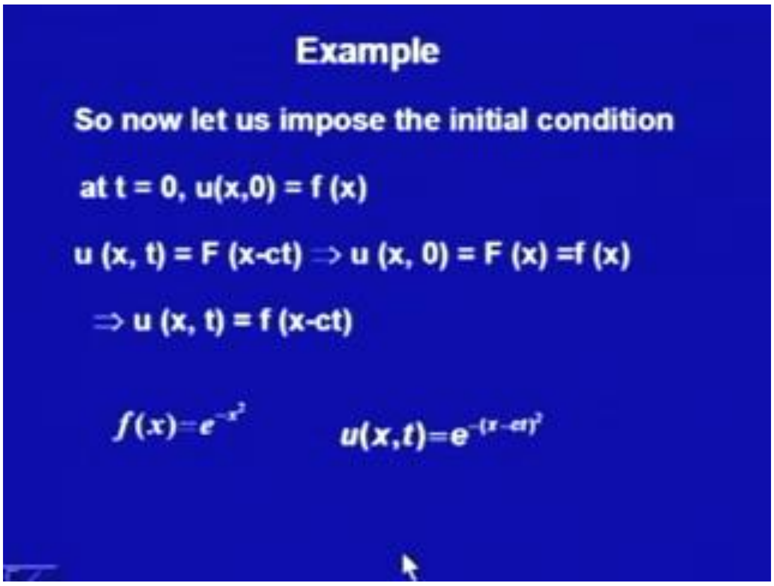

Now, we want to find out unique solution, so let us impose first, the initial condition, what initial condition here it would be, we say, at time t is equal to 0. The solution should be of the two variables u as x and t, so we are saying is at, t is equal to 0, u x is a function of x. Now, here f could be of any particular function, now in general I am telling, suppose f is a known function. Then u x t, we had find out that the solution is f of x minus c t.

Now, if I use t is equal to 0, what it would give, it would give me, that is at t is equal to 0, here I would get u x 0 and here if I put t is equal to 0, I would get capital F of x. Now, this capital F of x is same as, my initial condition says is, u at x 0 is equal is f x, that says is this capital F x must be, this known function f x. So, what we are saying is that, this initial condition with set, that function u x is a function of x, with known function, my solution is actually terms of to be that function f itself.

So, what would be my solution, my solution would be f of x minus c t, that is with initial condition, I would get the unique solution with reference to that initial condition that is function known function. So, say for example, I take f x as, e to the power minus x square then what will be my solution, my solution u x t would be, e to the power minus x minus c t whole square.

Let us see, is it a solution to this, our equation, if I take its derivative with respect to x, there would be getting it as, e to the power minus x minus ct square into minus 2 times x minus c t and the derivative with respect to t would be, e to the power minus x minus c t square into 2 c times x minus c t, so you can just check that this is satisfying our solution actually, we had already shown that it is satisfying with any f.

So, whatever this f, I am keeping it, it would satisfy this solution, if I put t is equal to 0, I would get here e to the power minus x square, that is my initial condition is also satisfy, how this function is looking like.

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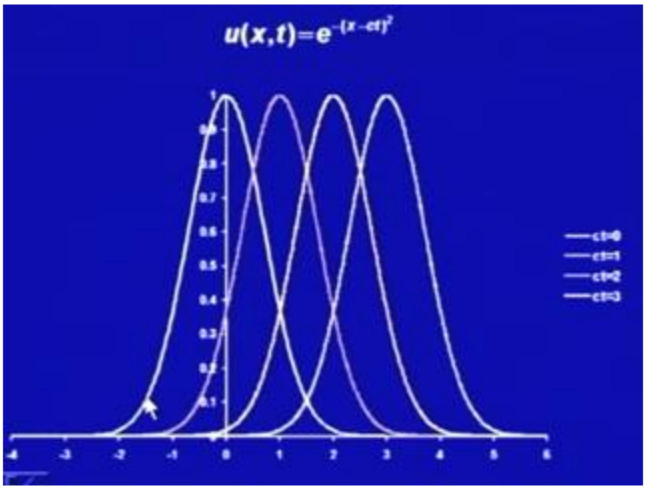

You see here, again I have taken with different values of ct, when ct is 0, this is the function, when ct is equal to 1, this is the function and when ct is equal to 2, this is third one is the function, when ct is equal to 3, this is the function. Again, we are getting is the shape of the function is same, that is, this wave is same, the thing is, that is it is changing towards or shifting towards right, so this is a solution.

So, solution what we are getting is, that solution is unique solution, we can get for one dimensional wave equation with initial condition, where initial condition has been posed at the time t is equal to 0, the unknown function takes form of a known function. So, the solution would be in the form of that known function, because the shape of the function is not changing, the function remains same only, thing is that with time, t it is changing its place, that it is shifting towards right here.

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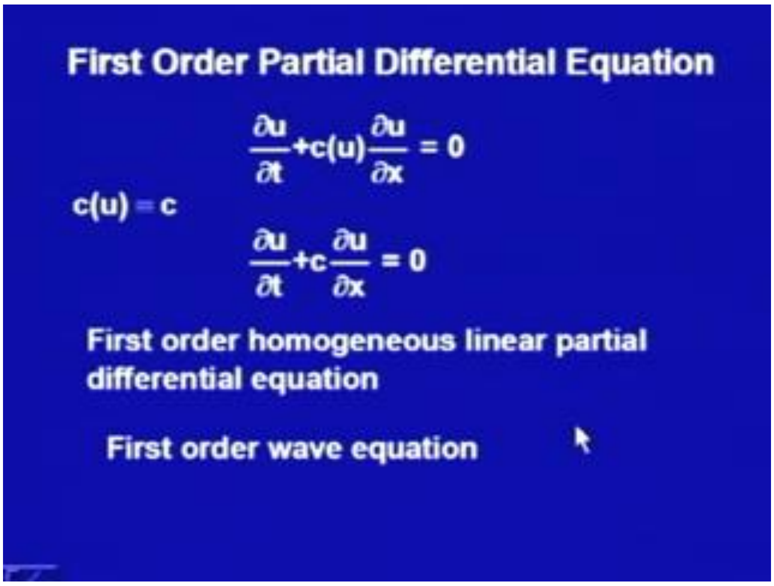

Now, let us come to the first order partial differential equation, first order partial differential equation, I take the simplest example of first order partial differential equation, del u over del t plus c u times del u over del x is equal to 0, c u that is the coefficient of del u over del x, it may be a function of u also. Now, if this c f u is just a constant, that is it is not involving any unknown function or any x or t, it is just a constant, let say c.

Then, this equation is the equation, just now we had seen, the one dimensional wave equation, del u over del t plus c times del u over del x is equal to 0. We see that, this is first order because the order is one only, it is linear, the derivatives are occurring linearly, the coefficient is constant, right hand side is 0, that is we are not having any term which is not involving u or its derivative, so it is homogeneous linear partial differential equation.

We had already seen that, we had called it first order wave equation, with our examples for the solutions, we had seen that all the solutions, in general, actually we had find out the solution f of x minus c t and if I use this f to be different values of the function. Now, different functions we had seen, we had always find it out, that the solution of this is giving is that is, a graph of a function which is shifting towards, in all those examples we had, that it was shifting towards its right, that is, it is changing the place only and we, that is why we do call it, the transport equation also, because it is transporting the data.

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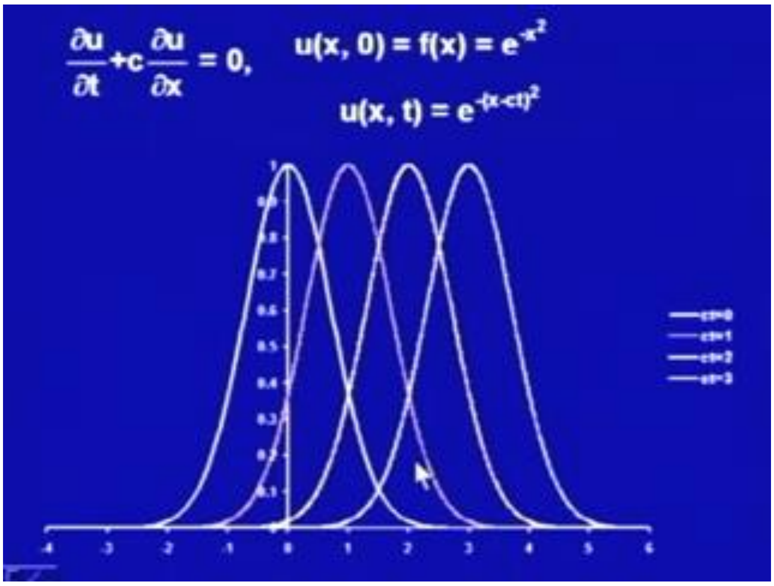

So, you see is, that is why we are calling transporting, say if we do have this equation, we see is, that is, if the function is my, e to the power minus x square, it is shifting towards right, that is, it is transporting the function towards its right, because at the speed what we are saying is, it is transporting at a speed c, with this c speed it is transporting towards, whatever be the c now, here c, all the time I have taken it to be positive values. So, it is shifting towards the right.

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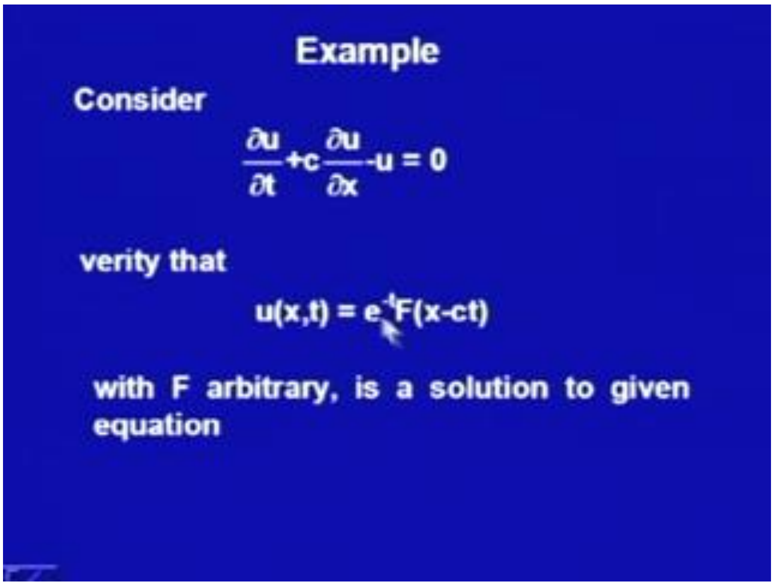

Now, how to find out the solution of this first order partial differential equations, we require to know that is, what is the method. Before knowing this method, we had seen only one example of first order wave equation, let us write one more example and see that is, does this also has more than one function as its solution. Now, we see is, that this is now not an, this is also an homogeneous equation, linear one, we do have del u over del t plus c times del u over del x minus u is equal to 0.

Now, verify that function, e to the power minus t f of x minus c t is a solution of this equation, where this f, again you see this is an arbitrary function. So, I am not treating it as a general, I am treating now with the arbitrary function, so the form of the solution we are taking here is e to the power minus t f of x minus c t, let us try to see.

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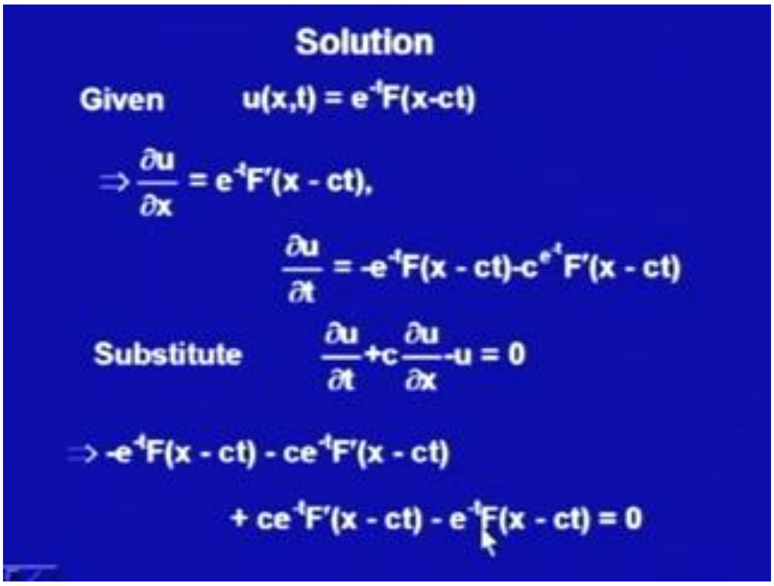

We have been given the function as, e to the power minus t f of x minus c t, we require the derivative with respect to x and with respect to t. So, first find out the derivative of this with respect to x and t, so where this, dash is denoting the derivative with respect to x minus c t, so del u over del x would be, e to the power minus t times f dash x minus c t because the derivative of x minus c t with respect to x would be 1, del u over del t would be e to the power minus t.

First, this function and then this f of x minus c t as such, then the derivative of this with respect to t is, that c times e to the power minus t, f dash x minus c t, dash means the derivative with respect to x minus c t. Now, if substitute in the given equation, the equation was del u over del t plus c times del u over del x minus u is equal to 0, what we have got that, e to the power minus t f of x minus c t minus c times e to the power minus t f dash x minus c t.

This is what is my del u over del t, plus c times this one, c times e to the power minus t f dash x minus c t minus u, u is my e to the power minus t f times x minus c t. Now, you see, this is cancelling this one, this is cancelling this one, so I am getting it is equal to 0.

So, that was satisfying the equation, again what we have got is actually, that f here is, in this one f is arbitrary. Let us see one more example.

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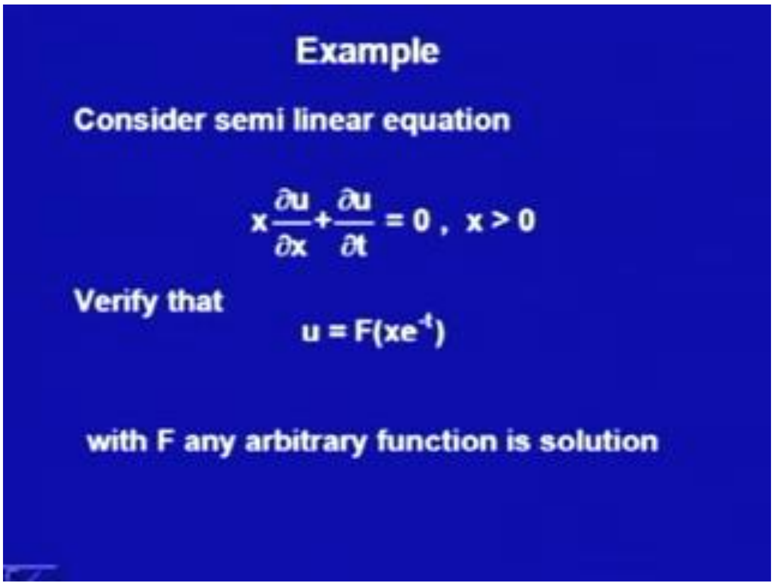

Let us consider this semi linear equation, where the coefficient is x, x times del u over del x plus del u upon del t is equal to 0, for x positive. Now, we are verifying that is, F of x times e to the power minus t is the solution, where f is again any arbitrary function. Let us try to see it.

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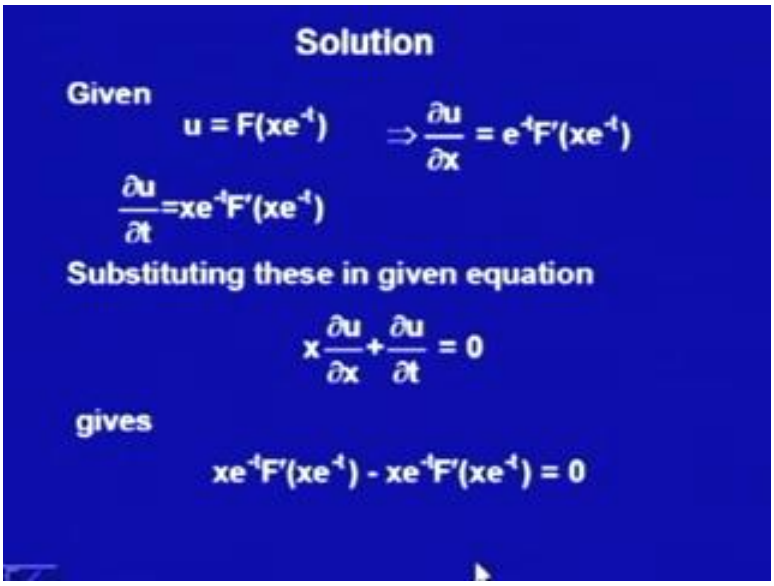

We are been given, that the function as F of x into e to the power minus t, so this we will take as z and use the derivative with respect to x and t, partial derivative with respect to x, it would be F dash x times e to the power minus t and derivative of x times e to the power minus t with respect to x would be, e to the power minus t, dash means the derivative with respect to this whole function as one variable. Similarly, del u over del t is x times e to the power minus t, F dash x e to the power minus t.

Now, substitute this in the given equation, equation was x time del u over del x plus del u over del t, so del u upon del x, x times is x times e to the power minus t, F dash x time e to the power minus t, this is same as, here it, this is with respect to t, so it should be with minus sign actually, this e to the power minus t, so it should be with minus sign, so we do get it with minus is 0, that is, it is also satisfying the equation. So, we had got that is, not only that first order wave equation, but we had different kind of equations, all are the first order equations.

We had find out, that is the function F was arbitrary in all those things, but one more interesting thing, did you upset here, when we had first order wave equation, my argument, that was x minus c t, that is unknown function F, but the argument was x minus c t and when I taken the another one, there I was having is that my argument was different and here also we were having some multiplication, that is, e to the power minus t and sometimes, something like that one.

When I have took this example of a semi linear equation, I have got that F is of course unknown, but this form, this argument is x times e to the power minus t. What we had observed, that in all these examples, the function remains unknown and we had find out that with, initial condition in the wave equations we had seen, that the initial condition was deciding what should be my F, if initially at time t is equal to 0, I give that the solution is taking this particular form of the function.

Then my function is being decided from there, but the argument is changing now, it says is that, in ordinary differential equations what we had, we had that my solution was involving arbitrary constants, so we try to find out the solution in general with arbitrary constants and then we using those side conditions, we try to find out the value of those arbitrary constants and we call them, the unique solution and the solution with the arbitrary constants, we call them the general solution.

Here, what we are having is, we are having is, that is, solution is involving arbitrary function, but in these three examples we had observed that, argument of the function is changing. Now, the question comes is, how to find out this argument, is there any method which says is, seeing the equation, can I obtain the argument or what should be the argument of the function, of course arbitrary function because the general solution we must get in the terms of arbitrary function.

But that arbitrary function is with respect to some other functions or the arguments are also functions, can I determine those functions or those arguments. So, we have learnt today, what is called partial differential equation, certain basic concepts, definition of partial differential equations, certain concepts which we are using in the partial differential equations, they are more like ordinary differential equations, but somewhere we had changed according to the partial derivatives.

We had learnt some more concepts or more terms, other than partial differential equations, we call some semi linear equations, some quasi linear equations, we had learnt that solution of partial differential equations are having unknown function, as in the form of arbitrary function. We had seen that, in ordinary differential equation, one function was defining or is giving the solution of differential equation.

In that function, we can only change with the constant, but here what we had find it out that, the function itself could be arbitrary, we had seen that with certain side conditions as, in the ordinary differential equations, we call them initial value problems or boundary value problems. Here also, we can use certain side conditions to determine the function, uniquely and those we called initial value problems, boundary value problems or we could call them initial and boundary value problems.

Of course, we will come later on, on all these particular things, but the first thing which we had learn in the first order partial differential equations that, the function has to be arbitrary, that has to be determined from the side conditions, what we do get that, argument of that function is changing, as the equation is changing. So, we would get the solution of the partial differential equation, that is or try to find out the methods to find out the solution of partial differential equations.

As, first method we would learn about determining the argument of the arbitrary function, so in the next lecture we will go, how to solve the first order partial differential equations. So, today we would be closing the lecture here, in next lecture we will go for the solution of first order partial differential equations.

Thank you.