# Mathematics - III

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## Lecture - 15
## Fourier Transforms

Welcome to the Lecture series on differential equations for undergraduate students. Today's lecture is on Fourier Transforms. In last lecture we had learned about Fourier sine transform and cosine transform. If do you remember for your cosine transform for any function f, we have define as 2 upon pi integral 0 to infinity f x multiplied with cos omega x integrated with respect to x, that is we are getting a function with respect to omega. Similarly, sine transform for any function f was 2 upon pi integral 0 to infinity, f x sine omega x integrated with respect to x. Today let us try to find out some important properties of these functions.

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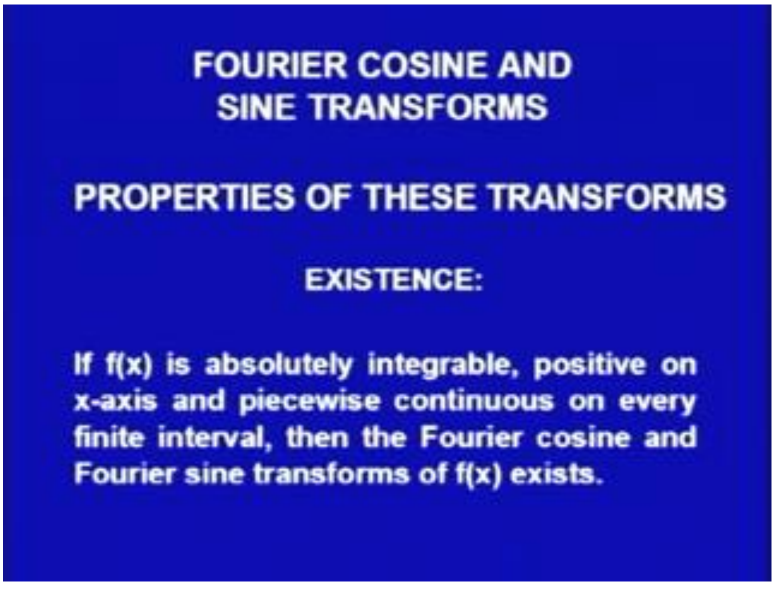


So, first existence if f x is absolutely integrable positive on x axis and piecewise continuous on every finite interval, we have seen that this property what does they mean that it is integrable, positive on x axis and piecewise continuous on every finite interval. Then Fourier cosine and Fourier sine transform of the function f x does exist, so this gives the existence of Fourier sine and Fourier cosine transform.

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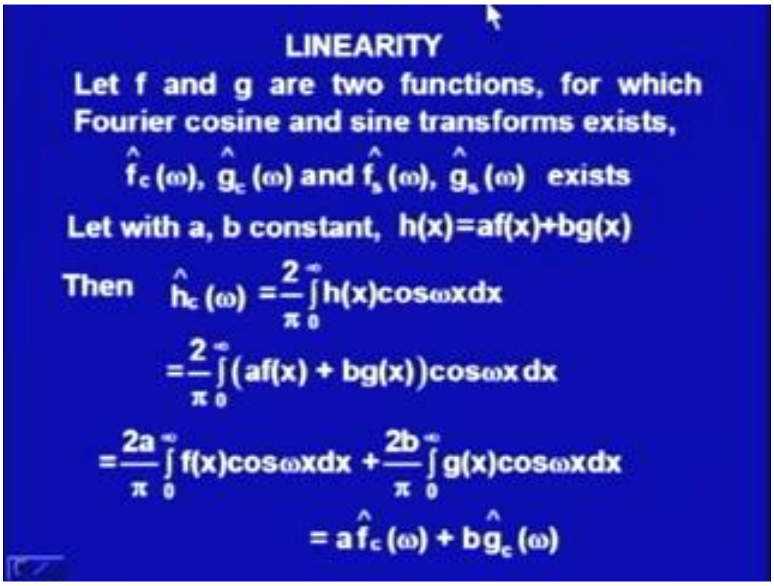


So, next property is linearity. Let f and g are two functions for which Fourier cosine and sine transform exists, that is f c hat omega, g c hat omega, and f s hat omega, g s hat omega does exists. And let for any constants a and b define a new function h as a times f x plus b times g x for every x. Then if I try to find out the Fourier cosine transform of this new function h what it would be by definition it will be 2 upon pi integral 0 to infinity h x cos omega x d x.

Now, h x is nothing but, a times f x plus b g x, so let us substitute it we get 2 upon pi integral 0 to infinity a times f x plus b g x cos omega x d x. Now using the properties of definite integral we can write this as the two integrals, 2 a upon pi integral 0 to infinity f x cos omega x d x plus 2 b upon pi integral 0 to infinity g x cos omega x d x. Now, these integrals are nothing but, the first one is nothing but, the Fourier cosine transform of f x and the second one is nothing but, the Fourier cosine transform of g, so we do have it us a times f c hat omega plus b times g c hat omega.

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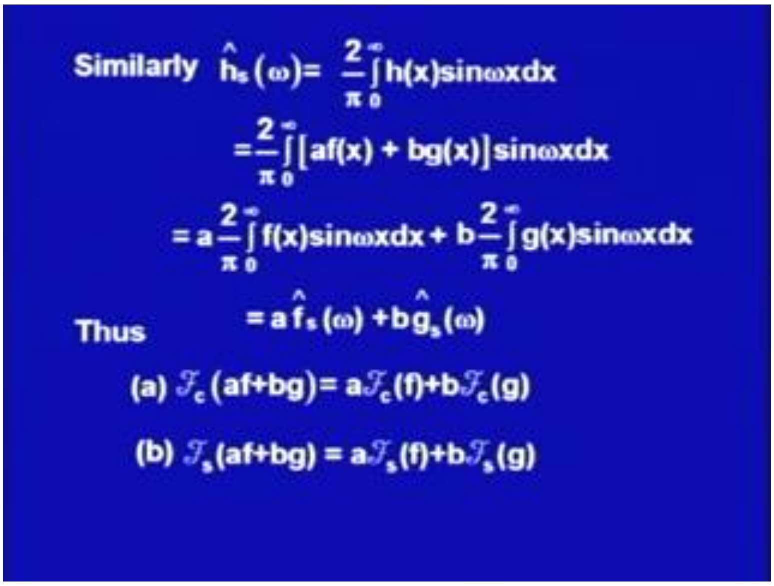


Similarly, if I try to find out the Fourier sine transform of h by definition it will be 2 upon pi 0 to infinity h x sine omega x d x. Now, substitute the value of h, that is as a f plus b g, we get it as 2 upon pi integral 0 to infinity a times f x plus b times g x multiplied with sine omega x integrated with respect to x. Now, again using the properties of definite integral break it into two integrals, one is a times 2 upon pi 0 to infinity f x sine omega x d x, another is b times 2 upon pi integral 0 to infinity g x sine omega x.

Now, the first integral is nothing but, the Fourier sine transform of the function f and the second integral is the Fourier sine transform of g, so what we have got it as, a times Fourier sine transform of f, that is f s hat omega plus b times g s hat omega. Thus what we had learnt that, if Fourier sine and cosine transform are exists, then Fourier cosine transform of a f plus b g, that is linear combination of f and g is nothing but, a times Fourier cosine transform of f plus b times Fourier cosine transform of g.

Similarly, for Fourier sine transform of a f plus b g is same as, a times Fourier sine transform of f plus b times Fourier sine transform of g; that is these transform Fourier sine transform and Fourier cosine transform, they are linear they contain are they take up the property, linearity property.

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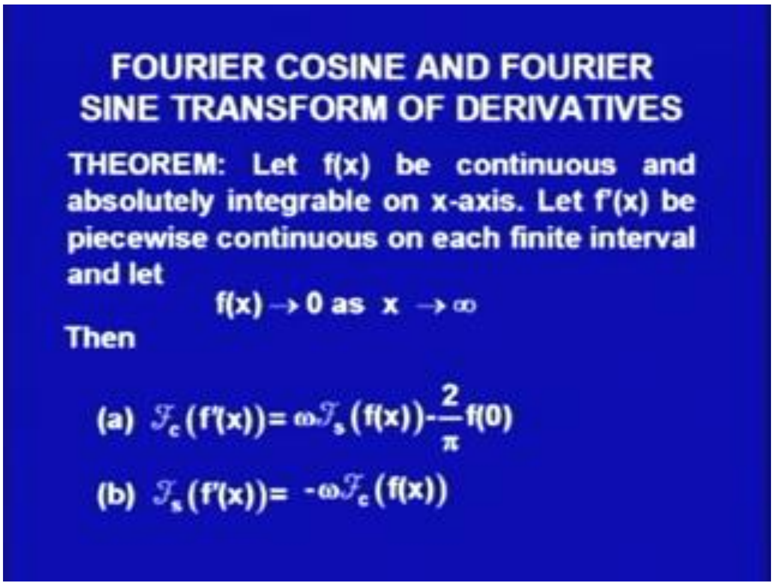


Another property, Fourier sine and cosine transform of derivatives, that is of the function is differentiable what would be the Fourier sine or cosine transform of derivatives. So, the first result is, let f x be continuous and absolutely integrable on x axis and let the derivative of f that is f dash x be piecewise continuous on each finite interval and moreover, f x approaches to 0 as x approaches to infinity.

Then the Fourier cosine transform of f dash x that derivative of f is given as omega times Fourier sine transform of f minus 2 upon pi f of 0 and Fourier sine transform of f dash x is minus omega times Fourier cosine transform of f x. Now, let us prove this result, find out that is how it is happening, so Fourier cosine transform of f dash x.

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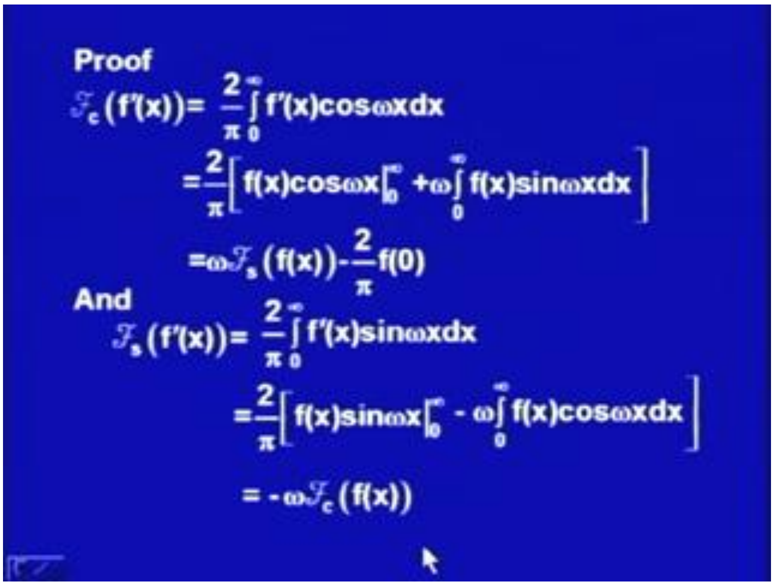


By definition it should be 2 upon pi 0 to infinity f dash x cos omega x d x, now integrated with integration by parse, taking f dash x as the integrable function and cos omega as the differentiation function second function. So, it would be 2 upon pi f x cos omega x 0 to infinity plus omega times 0 to infinity f x derivative of cos omega is sine omega and omega is outside minus sine omega, so plus 0 to infinity f x sine omega x d x.

Now, first evaluation since f x approaches to 0 as x approaches to infinity, so at infinite it would be 0, now when x is 0, f x is f 0 and cos omega x that x is equal to 0 is 1, so what we would be getting is the 2 upon pi minus f upon 0, 2 upon pi f 0 from here. And this is nothing but the Fourier sine transform of f, so omega times Fourier sine transform of f, this is the first result we have got. And Fourier sine transform of f dash x by definition it is 2 upon pi integral 0 to infinity f dash x sine omega x d x.

Again will go with integration by part, so it would be 2 upon pi f x sine omega x evaluated from 0 to infinity minus omega times integral 0 to infinity f x cos omega x d x. Now, as we do know that as x approaches to infinity f x approaches to 0, so as x approaches to infinity this function will give us 0, at x is equal to 0 sine 0 is 0, that says this particularly function or this part is not giving us anything over here it is 0. What is being left is minus 2 upon pi omega 0 to infinity f x cos omega x d x which is nothing but, the Fourier cosine transform of f. So, we have got minus omega times Fourier cosine transform of f, this is the result we have got, so we have proved this result.

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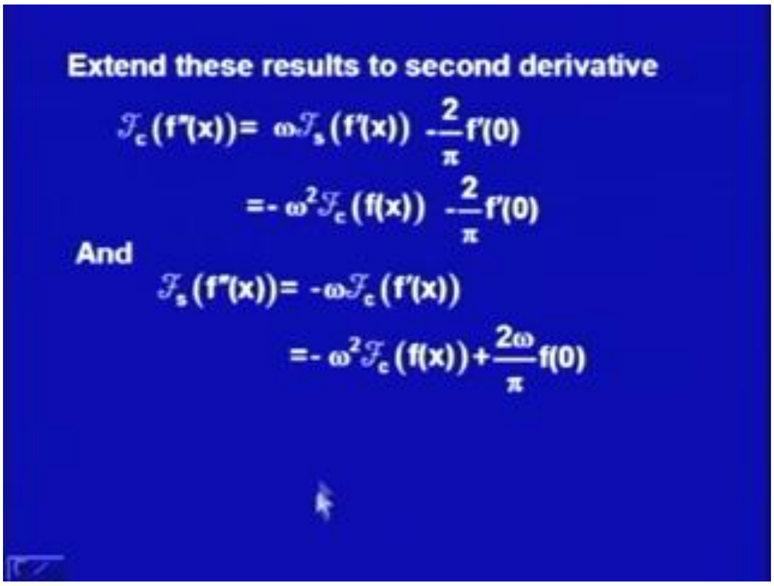


Now, let us come to extend these results to the second derivatives, this derivative results, so we are interested in finding the Fourier cosine transform of second derivative of f. Now, use the previous formula which says us that, it should be the omega times Fourier sine transform of it is integral, that is first derivative f dash x minus 2 upon pi f dash at 0. Now, what is the Fourier sine transform of f dash x, would use the formula it is minus omega f c, that is Fourier cosine transform of f.

So, minus omega and omega I have got minus omega square Fourier cosine transform of f x and the second term is as such minus 2 upon pi f dash 0. Similarly, the Fourier sine transform for f double dash x, that is f s Fourier sine transform of second derivative of f x using the formula for first for the f dash, we get it minus omega times Fourier cosine transform of f dash x.

Fourier cosine transform of f dash x is omega times Fourier sine transform of f x, so this minus omega square Fourier should be Fourier sine transform of f x minus 2 upon pi, so minus minus sign will get plus, 2 upon pi with omega 2 omega upon pi f at 0. So, we have got that is, in the first derivative for your cosine transform is changing to the Fourier sine transform, in the second derivative for your cosine transform is changing the for your cosine transform.

Fourier sine transform in the first derivative is changing to Fourier cosine transform and in the second derivative, it is changing again to the Fourier sine transform. Let us just try to use this formula for solving some problem at evaluation of some transforms.

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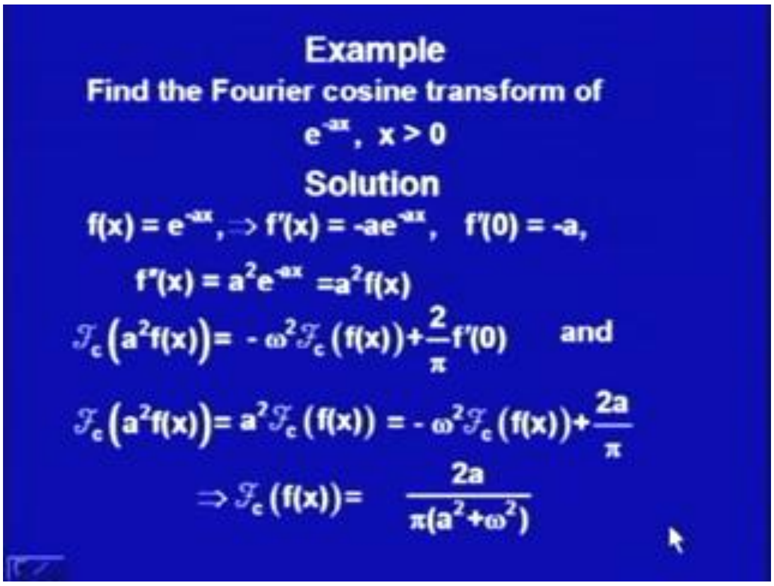


Find the Fourier cosine transform of the function e to the power minus a x for x positive, now how you are going to find it out, f x is e to the power minus a x, so f dash x is minus a times e to the power minus a x and f dash 0 is minus a. F double dash x would be a square e to the power minus a x, what we are getting is f x f double dash x we are just getting the constant difference, that is a square times f x.

So, Fourier cosine transform of a square f x, a is a constant, so by the properties of linearity it would be nothing but, a square times Fourier transform of f x and a square f x is f double dash x. Now, I use the formula for Fourier transform of second derivative of f, that says it should be minus omega square Fourier cosine transform of f plus 2 upon pi f dash 0, f dash 0 would be from here we had already calculated it is minus a.

So, what we have got, now here it is linearity property, so what we have got finally actually that a square times Fourier cosine transform of f x is same as minus omega square times Fourier cosine transform of f x plus 2 a upon pi. Now take these parts over here that is Fourier cosine transform it is a square plus omega square, so what we have got, the Fourier cosine transform of f x as 2 a upon pi divide into a square plus omega square. This term we have taken this side, so it is a square plus omega square times

Fourier cosine transform of f x is 2 a upon pi. So, we have got the Fourier cosine transform of f x as 2 a upon pi into a square plus omega square, now you see we have not evaluated any integral over here, using these properties we can find out the Fourier cosine transform of this function more easily.

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Now, come to the Fourier transforms, before coming to the Fourier transform, let us have as we have done in the Fourier series the complex form of Fourier series. Let us just move to the, because we had find out the Fourier sine transform and Fourier cosine transform when we have converted the Fourier sine series or Fourier series into the integrals. Now, let us convert the complex form of the Fourier series into integral, so complex form of Fourier integral.

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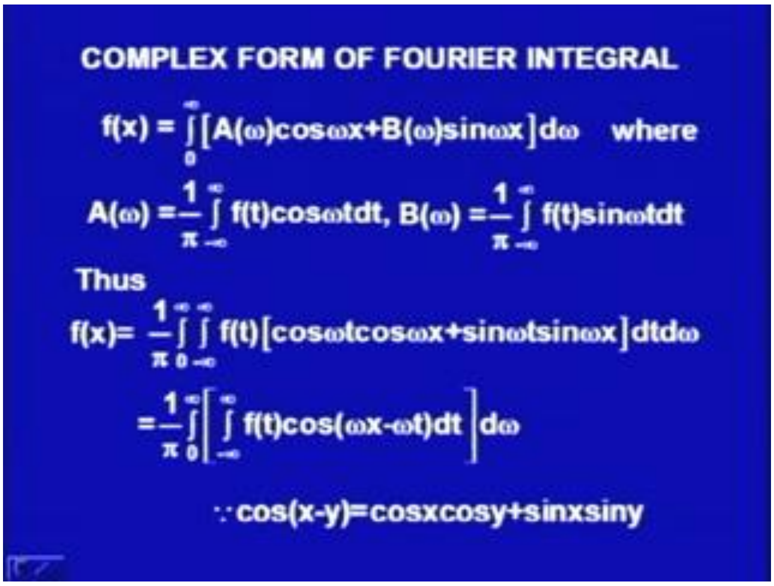


Let f x is, we do know 0 to infinity A omega cos omega x plus B omega sine omega x d omega, where we do know that A omega is 1 upon pi minus infinity to plus infinity, f t cos omega t d t and B omega is 1 upon pi 0 to infinity f t sine omega t d t, this we had done in the Fourier integral. So, f x we can write as 1 upon pi, now I am writing this A omega and B omega inside over here, so this integral 0 to infinity as such with A omega I can write minus infinity to plus infinity f t cos omega t and then, cos omega x.

Similarly, here the integral would be 1 upon pi that is again common, 0 to minus infinity to plus infinity that is I have joined the two integrals, because is the properties of definition integral this we could write as a one single integral, f t sine omega t and sine omega x. So, f t again have taken common and this term have come as cos omega t cos omega x plus sine omega t sine omega x d t d omega I would have simplified, this is 1 upon pi 0 to infinity.

Now, what is inside this integral what I have got, cos omega t cos omega x plus sine omega t sine omega x, if we see this is nothing but, cos of omega x minus omega t or you could call omega t minus omega x, because cos of minus x is cos of x, so it does not make any difference, but we will write it as cos omega x minus cos omega t. So, this is integral minus infinity to plus infinity f t cos omega x minus omega t d t d omega, now cos is as cosine is an even function, so this is the formula which we had used.

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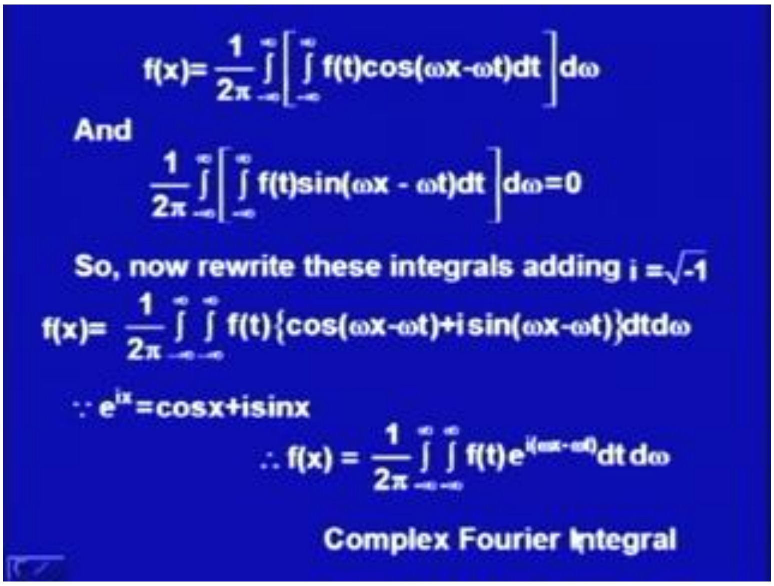


So, what we have got, we can write it as 1 upon 2 pi minus infinity to plus infinity, because it is even function, so that says it I can use the integral 0 to infinity, let us come back again to the previous slide see. ((Refer Slide: 16:22)) Here, this cosine is an even function, now if I integrate it this complete function would be actually even with respect to omega.

If we see this complete function is because f is not containing the term of omega f is with respect to t, the omega is coming in the cosine function, cosine is an even function. So, with respect to omega this integral would be a function of omega and that is an even function and using the properties of definite integral which says is, that if a function is even then integral minus a to plus a of an even function f. We can write as 2 times, 0 to infinity or 0 to a, so now I am using that property in the reverse order.

So, what we get, 1 upon 2 pi, this integral now you see is I am writing minus infinity to plus infinity, minus infinity to plus infinity f t cos omega x omega t d t, d omega, now in the similar manner, if I take this integral, f t sine omega x cos omega t d t. Now, what I have done is, this integral I have not find out from anywhere, here rather than this cosine function I had used this sine function, now this sine function in the omega this is an odd function.

So, if I write this integral, then this inside integral is an odd function of omega, using the properties of definite integral this integral must be 0. So, now since this integral value is

0 I can add a power here in this f integral, that is f x plus 0 it would not give anything, but in the integral form if I am using with now I what is we are trying to do is we are to make a complex form of Fourier integral, so I would use it with a imaginary part.

So, now rewrite these integrals adding i that is square root of minus 1, I can write f x as 1 upon 2 pi integral minus infinity to plus infinity that is outside, then integral minus infinity to plus infinity, f t in both these one it is common, here I am writing is cos omega x minus omega t plus i times sine omega x minus omega t d t d omega. So, what I have done is, I have added these two integrals with plus i sign because this is 0, so it would be same as f x. But, now what I have got inside this bracket, this is cos omega x minus omega t plus i times sine omega x minus omega t.

Now, if you do remember our complex exponential function, this can be given as e to the power i omega x minus omega t, so f x would be 1 upon 2 pi minus infinity to plus infinity f t e to the power i times omega x minus omega t d t d omega. Now, you see that we had written this function in the form of integral with complex function, so this is called the complex Fourier integral.

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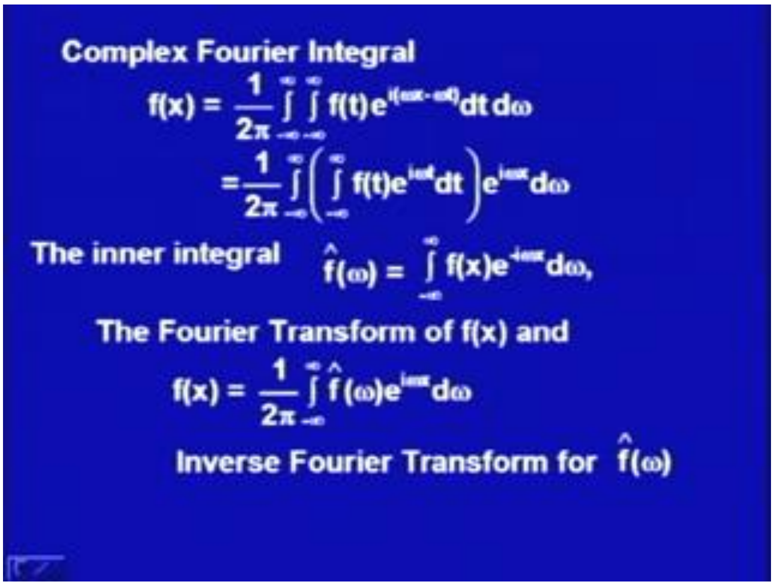


So, complex Fourier integral we had find out as 1 upon 2 pi minus infinity to plus infinity integral again minus infinity to plus infinity ft, e to the power i omega x minus omega t d t d omega. Now, let us simplify this double integral again, this e to the power i omega x minus i omega t, I can write as e to the power i omega x into e to the power minus i omega t, so we have getting it, this 1 upon 2 pi as such, 1 integral with respect to t another is integral with respect to omega.

So, first the integral because omega t both omega and t are coming over here, so you would write it here, so it is ft e to the power minus i omega, it should be minus, minus i omega t d t into e to the power now, this is the function of omega only because t has been integrated out into e to the power minus i omega x d omega. So, now what we are getting is, this inner integral, we would call because this is the function of omega, we have denote it as f hat omega minus infinity to plus infinity f x e to the power minus omega x d omega.

Now, you see is that is t I have changed with x, this is called the Fourier transform of the function f, and now write this f x as 1 upon 2 pi integral minus infinity to plus infinity this inner integral is nothing but, the Fourier transform of f. So, f hat omega e to the power i omega x d omega this gives me the function back at x. This is called the inverse Fourier transform for the function f hat omega.

So, what we have got that is, using the complex form of Fourier integral we had ((Refer Time: 22:12)) we had represented the function f as double integral here, where the inner integral was with respect to the function variable and we are getting it as a function of another variable omega. And the outer integral is with respect to that, another variable omega and giving back me the value of the function at the first variable x, now let us come to the properties of this function, first is existence of Fourier transform.

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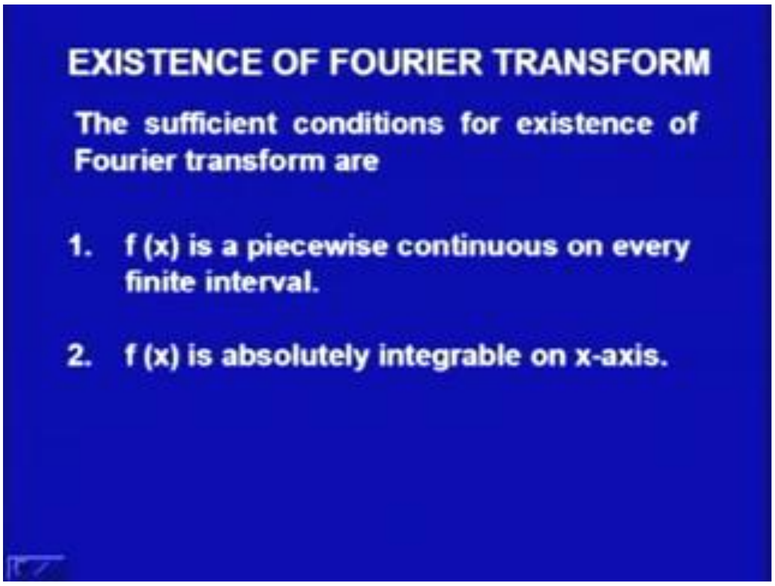


The sufficient condition for existence of Fourier transforms are, f x is a piecewise continuous on every finite interval, second is that f x is absolutely integrable on x axis. As we have seen that is the conditions for existence of Fourier integrals, the existence of Fourier series and the existence of Fourier cosine and sine transforms. What we have done when we have obtain this, all this complex form of Fourier transform, we had used many places that is interchange of the integrals, the limit that it is existing.

And so all those things are mathematically it is little bit difficult to prove and to find out those things and it would be beyond the level of this course, so I am not going to that particular things here I am giving you one result, which says us the sufficient conditions for existence of Fourier transform, that is what we want that is to exist those limits and to exist those integrals. So, first condition is that my function has to be piecewise continuous and second is that it should be absolutely integrable.

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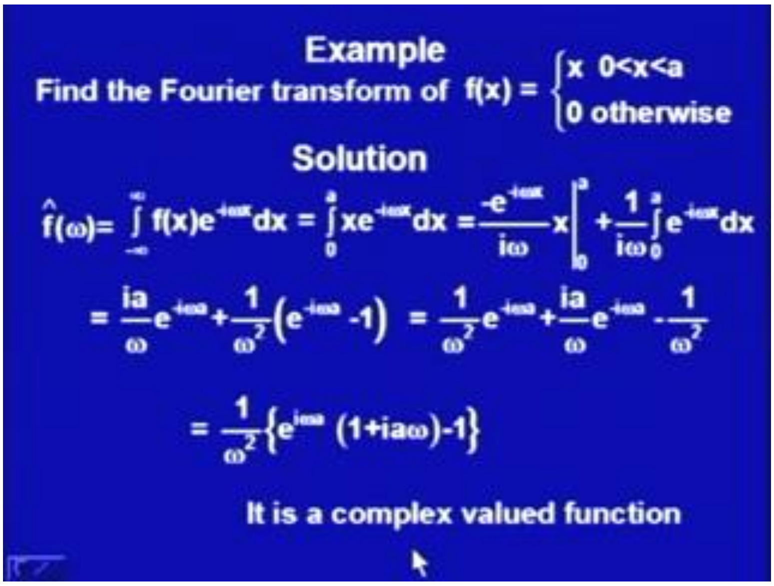


Let us try one example to find out the Fourier transform of any given function, questions is find the Fourier transform of the function f, which is x in the interval 0 to a and 0 otherwise. Let us find it out, Fourier transform of any function f hat omega by definition is integral minus infinity to plus infinity f x e to the power minus i omega x d x, let us try to find it out, here this integral function is defined in the range 0 to a, as x and it is 0 otherwise.

So, this integral would change to 0 to a, x times e to the power minus i omega x d x, let us integrate this by parts, what we get e to the power minus i omega x upon minus i omega x times, evaluated 0 to a plus 1 upon i omega integral 0 to a, e to the power minus i omega x d x.

Now, evaluate at x is equal to a, it would be e to the power minus i omega and is would be a, so i would be getting is and 1 upon i is minus i, so you would be getting is i times a upon omega e to the power minus i omega and at x is equal to 0, this would give me 0.

Plus this integral of e to the power minus i omega x is minus e to the power i omega x upon i omega, so i square omega square and i square would give me 1, minus 1, so plus 1 upon omega square evaluated at. So, e to the power minus i omega and e to the power minus i, 0 that is 1. So, we do get e to the power minus i omega a minus 1, let us rewrite it, what we have got, 1 upon omega square e to the power minus i omega plus i a upon omega times e is power minus i omega minus 1 upon omega square.

Again rewrite it in more simplified form that is here I have taken 1 upon omega square common, so e to the power i omega would be here 1 plus i a omega and this omega square outside those is minus 1. Now, you see it is a complex valued function, so the Fourier transform is a complex valued function, when we practically use we normally take thus a real part and imaginary part separately also with practical purpose but that part we will do later on when we go for the practical purposes. Here we are just doing the definition and understanding this transforms. Let us do one more example

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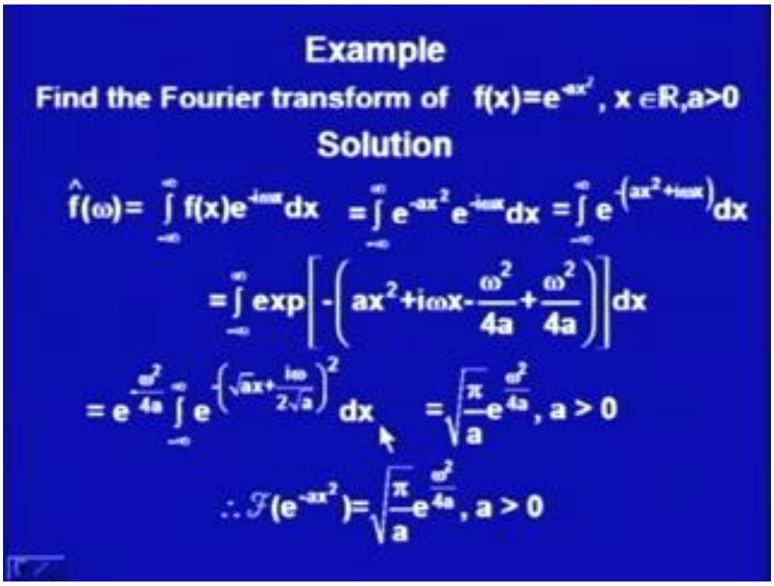


Find the Fourier transform of the function e to the power minus ax square for x in the real line and a positive, f hat omega by definition minus infinity to plus infinity f x e to the power minus i omega x d x, the function is defined on the whole real line. So, it is e to the power minus ax square e to the power minus i omega x dx ,which is rewriting it as e to the power minus ax square plus i omega x d x, now we are having in the exponent as the square functions.

So, we make it whole square for that we go, write it as minus ax square, so this minus is outside this inside 1 I have try to make it as whole square, ax square plus i omega x it would be whole square if Ii do have this i omega that should be. So, I want i square omega square by 4 and a is here. So, it should be minus omega square upon 4 a plus omega square upon 4 a because we have just subtracted, so we to added, now again simplified, what it says is e to the power minus omega square upon 4 a that I have taken outside because this integral is with respect to x.

And here it is integral of e to the power minus, now what is this function, this function has come up as square root a times x plus i omega upon 2 by root a whole square, integral it of it with respect to x, this integral is equal to square root pi upon a, e to the power omega square upon 4 a for a positive. That says us we had not evaluated this integral over here, we will see it later on that is how it is coming, what we are getting the result that is Fourier transform of e to the power minus ax square is square root pi upon a, e to the power omega square upon 4 a for a positive. Now, let us see, the evaluation of this integral, that is not very easy integral let us see that how we are evaluating this integral.

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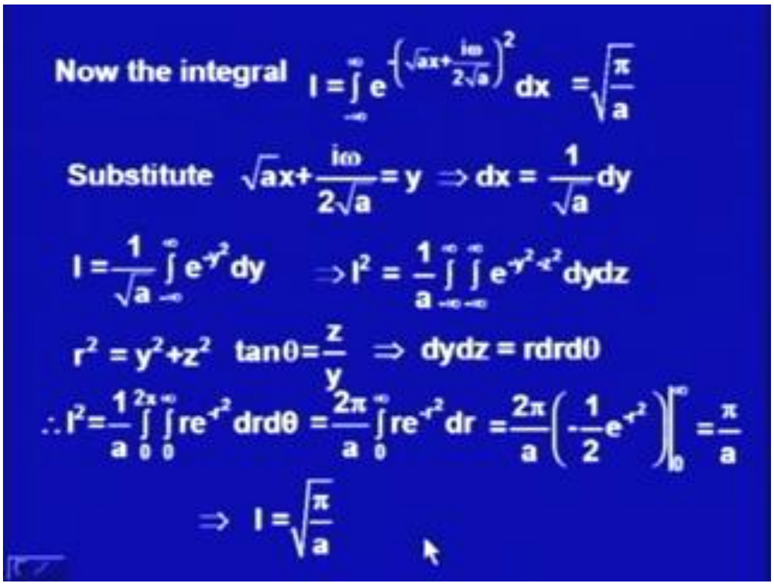


So, let us say that i is this integral minus infinity to plus infinity e to the power minus square root a times x plus i omega upon 2 upon square root a whole square d x. We want to show that is integral is square root pi upon a, let us just try with i, let us take this whatever is the quantity here that is the square root a times x plus i omega upon 2 square root a as y, then d x would be 1 upon a square root a dy.

So, i would be 1 upon square root a minus infinity to plus infinity because as x approaches to minus infinity y would also approach to minus infinity because all these thing are constant and as y approaches to plus infinity, this x approaches to plus infinity y would also approach to plus infinity. So, this integral is 1 upon square root a minus infinity to plus infinity e to the power minus y square dy, now double this integral, I make this part, that is I am multiplying i with i, multiplication of integral that simply says is I can 1 upon a square root a into 1 upon a square root a that is 1 upon a.

And the integral 0 to infinity e to the power minus y square into the integral 0 to infinity e to the power, let us say with another variable, so that there is no confusion, e to the power minus z square dz. Now, what we are taking is that this multiplication as the double integral. Now, because that simply the double integral, so what we have got 1 upon a minus infinity to plus infinity minus infinity to plus infinity integral e is power minus y square minus z square d y dz.

Now, let us change this Cartesian coordinates into the older one, that is using r square as y square plus z square and tan theta z upon y so r theta, we, do know by the change of integration that is change of variable the gives me that d y over d z would be change to r, d r, d theta. So, the i square would become now 1 upon a as such and this limit if you do remember or you could just say that is, what we are doing is we are changing this, this is on the whole Cartesian space that is on the whole space.

Now, for whole space if I try to change it to the a polar one like this one, so what we say is that is I have to take the r to be all real numbers that is 0 to infinity and my theta should take the whole orientation 0 to 2 pi. So, it is 1 upon a 0 to 2 pi, 0 to infinity r, r is coming from this d y d z is rd r d theta and this is e to the power minus r square, so r times e to the power minus r square d r d theta. Now, let us consider this integral 0 to infinity r e to the power minus r square d r because there is no function of theta. So, this we can integrate with respect to r no.

So, if I take e to the power minus r square its derivative, is actually minus 2 r e to the power minus r square, so what we could say is r e to the power minus r square is nothing but the anti derivative of the thing, so let us make it as, first if I integrate it with respect to theta also that is here I think is I have done this first integration with respect to theta. So, this integral we are going to evaluate, we see this integral, we do have this function as r e to the power minus r square, it does not contain any element of theta.

So, the first integral if I had to take 0 to infinity r e to the power minus r square d r, this can be evaluated separately and another integral 0 to 2 pi with respect to theta that can be evaluated separately with a constant, so let us first evaluate this simple integral, that is with respect to a constant. So, what it will give me, it will give only 2 pi, so we got 2 pi upon a 0 to infinity integral r e to the power minus r square d r.

Now, let us see this function, if I take e to the power minus r square and find out its derivative, it would be using the chain rule, minus 2 r times e to the power minus r square. What it says is that r times e to the power minus r square is nothing but the anti derivative of some constant times, e to the power minus r square, so what it would be this 2 pi upon a is as such , and this is the anti derivative of minus 1 by 2 e to the power minus r square, that is its integral is minus 1 by 2 e to the power minus r square, evaluated from 0 to infinity.

Now, at infinite as r approaches to infinity, r square also will approach to infinity that says is e to the power minus r square that is approaching to 0, this we do know from the limit, so as r approaches to infinity this function will give me 0, as r approaches to 0 this will give me 1 that is minus half, so what have would get it as actually, multiplied minus of and minus, minus and will cancel it out, so it is 1 upon 2 only, so it is pi upon a. So, what we got i square as pi upon a, so i would be square root of pi upon a and that is what we had substituted in the all our integral to find out the Fourier transform of the function. See you see is that is, Fourier transform of the function, this find of exponential function which have it easy but if it is square one, this is a little bit tedious to integrate.

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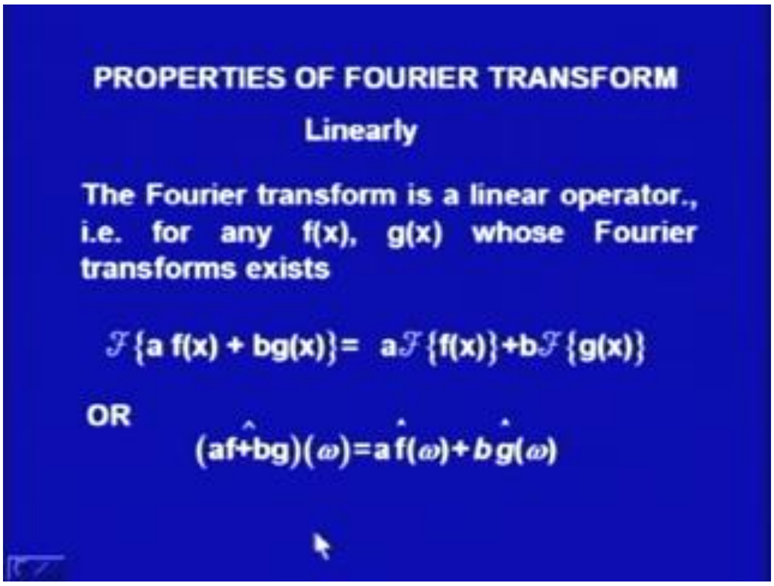

Let us see the properties of the Fourier transforms which will help us in finding out the Fourier transform as well as in finding out some other things also, first property as in the terms of a linearity as we have seen in the Fourier cosine and Fourier sine transforms. Also here the Fourier transform is a linear operator, that is for any f and g whose Fourier transforms are exist in that any two functions f and g for whose the Fourier transforms are exist, the Fourier transform of a times f plus b times g is nothing, but a times Fourier transform of f plus b times Fourier transform of g. For proving it is very easy we could see, this can be written in another notations also but this notation is says that Fourier transform of a f plus b g hat omega is same as a times Fourier transform of omega plus b times Fourier transform of g hat omega, any of these two things can be written, we just have to understand that it is linear.

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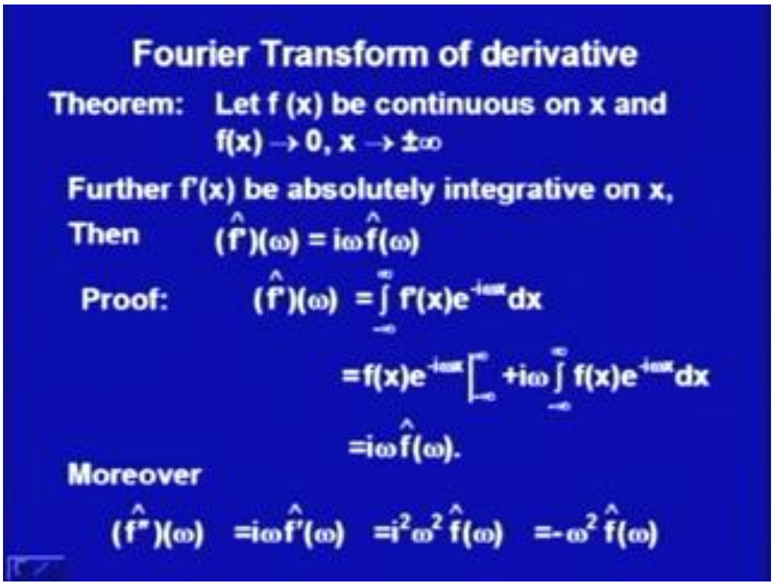


Proof was easy, so I had left for you to do by yourself, here is the Fourier transform of derivative, let f x be continuous on x and here again we are assuming that f x is approaching to 0, as x approaches to either minus infinity or plus infinity and further we are assuming that f dash x is absolutely integrable on x, then the Fourier transform of derivative of that is f hat, f dash, the Fourier transform it hat omega is nothing, but i times omega times Fourier transform of f.

Let us try to prove it, go with the definition of Fourier transform of f hat, by definition it is minus infinity to plus infinity f at x, e to the power minus i omega x d x, let us go to the integrate it by parts taking f dash x as the integrable function. So, what we get f x into e to the power minus i omega x evaluated from minus infinity to plus infinity plus f x and derivative of this is minus i omega times, e to the power minus i omega x, so here it would be plus i omega times integral minus infinity to plus infinity f x, e to the power minus i omega x d x.

Now, this is same as i times omega times f hat omega from here because i omega is as such and this is nothing but the Fourier transform of f, what happen to this part, you see we had assume that f x is approaching to 0 as a x is approaching to either plus infinity or minus infinity. So, when x is approaching to minus infinity, this is of course approaching to the plus infinity, but f x is approaching to the 0, and that give us that is if you just go that this would be, rather we are assuming that this is first going to the 0 than this is going to infinity.

So, this multiplication goes to 0, when x approaches to infinity this is approaching to 0 as well as this is also approaching to 0, so both the time it is giving as 0, that is what we have got that, the Fourier transform of the derivative is nothing but, i omega times the Fourier transform of the function. Now, let us extend this is at for the second derivative, what it says is that is if i take the Fourier transform of the second derivative, using this formula it should be i omega times the Fourier transform of the first derivative again use this formula, which says it should be i square omega square times the Fourier transform of omega. Now, i square is nothing but the minus 1, so what we got it is minus omega square times f hat omega, now you see that is how we can use this property for evaluation of some Fourier transform.

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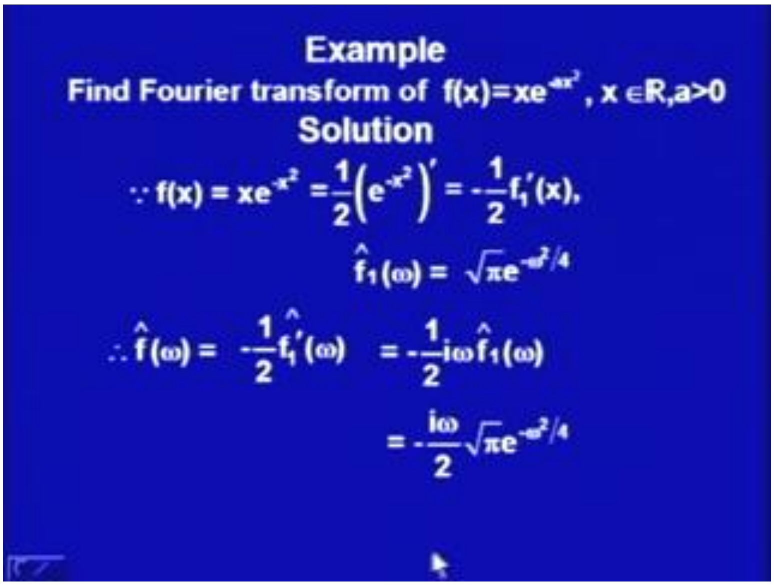


Find the Fourier transform of f x, x times e to the power minus ax square x belonging to whole real line and a is positive, you remember that is just now we had find out the Fourier transform of e to the power minus ax square, that was not very simple we have to evaluate one integral which was a little bit difficult.

Now, here we are having the function x times that one, how we are going to proceed it, will use this property just now we had obtained about this Fourier transforms. f of x is given as, x times e to the power minus x square, just when we had evaluated that integral we had find out that is if I take e to the power minus x square and its derivative, I would get minus 2 x times e to the power minus x square that says is, this is actually the derivative of e to the power minus x square. Let us write this function e to the power minus x square as f 1, so what we are getting is minus half times f 1 dash x.

Now, f 1 x is the your e to the power minus x square, just now we had evaluated the Fourier transform of e to the power minus ax square, now there if I take a to be 1, that would be the same as e to the power minus x square, what the integral we had evaluation we had find out if you do remember it that was x square root pi by a into e to the power minus omega square minus 4 a. Now, a is 1, so I would get here the Fourier transform of this f 1 omega that is Fourier transform of e to the power minus x square as a square root pi, e to the power minus omega square by 4.

Now, so Fourier transform of f is nothing but, minus half times Fourier transform of f 1 dash, now use the property of linearity and property of this derivatives, so what would be f hat omega first using the property of linearity we would get is minus half times the Fourier transform of f 1 dash. What is the Fourier transform of f 1 dash, that is just now we had find out the Fourier transform of f dash is nothing but i omega times the Fourier transform of f.

So, this minus half is as such, the Fourier transform of f 1 dash omega is i omega times f 1 dash the Fourier transform f 1 hat omega, the Fourier transform of 1 hat omega we had find out that is it was square root pi e to the power minus omega square by 4. So, put it out here, minus i omega by 2 square root pi e to the power minus omega square by 4, now you see is without doing that integration again we have come up with the solution, so that says is that is we could get, the Fourier transform easily if you use these properties.

There is one more very important and trusting property about this Fourier transform, which is called the convolution property, that is again being used to find out the Fourier transform of complex functions complex means not that complex one that is the difficult functions we can break them as a convolution and then we can find it out. So, let us we see what is call this property convolution.

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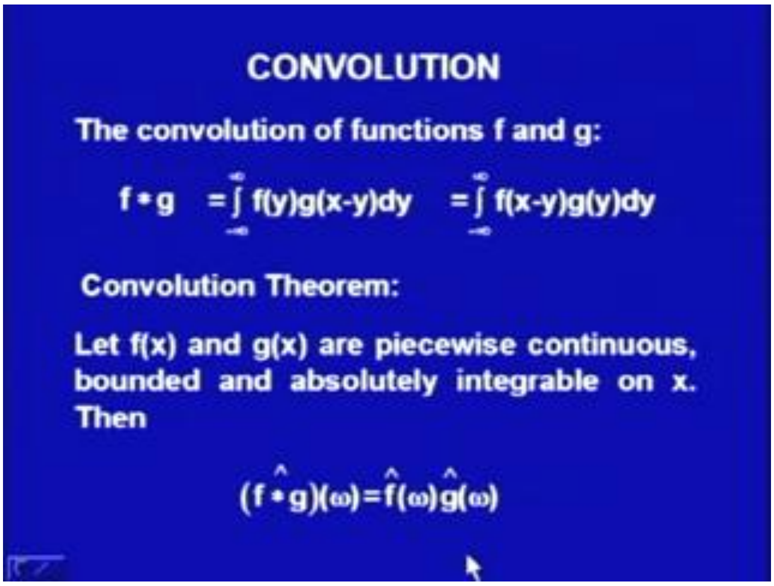

The convolution of the function f and g is defined as denoted by f convolution g, here this is star is the notation for convolution, this is defined as minus infinity to plus infinity, f y, g x minus y, d y this is at point x, we are integrating with respect to y. So, what one function we are taking as f y another function we are taking is g x, g at x minus y, now if I interchange the change of variable, if I take that is if I assume that g is equal to x minus y, I can write this as, integral minus infinity to plus infinity f of x minus y, g y, d y.

What it says is this convolution is actually, we are commutative whether you are writing f first or g first that does not matter, f convolution g is same as g convolution f, this is actually integral of the first function at some point multiplied with the another function at the point x minus that first point and evaluated with respect to that point, then we are evaluating it at function at that point x. what is the convolution theorem which we are calling this for the Fourier transform, you see.

First assuming that f x and g x are piecewise continuous bounded and absolutely integrable on x, you see why these conditions we are writing, because we want that for Fourier transform to exist we do have two condition that is was it should be piecewise continuous on every finite interval and it should be absolutely integrable on x. So, these two properties are simply give me that is of Fourier transform does exist, then the Fourier transform of f convolution g.

Now you see f convolution g is this function the integral can be given as multiplication of Fourier transform of f and Fourier transform of g, so here what is in the function one this is integral that is convolution, in the Fourier transform, that is in the omega one, it is changing to the only the multiplication of the Fourier transform, that says is it would be very easy. Now, let us see the proof of this theorem, that is how it is seeing is.

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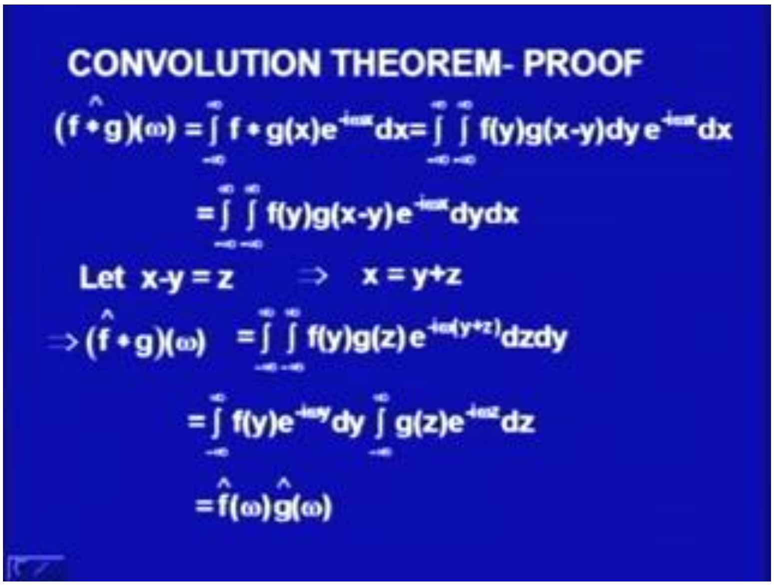


Let us start for this, the Fourier transform of f convolution g, by definition it should be the integral minus infinity to plus infinity f convolution g at x, e to the power minus i omega x d x, now let us write this f convolution g at x, that is again another integral. So, it should be first integral is as such minus infinity to plus infinity, this is integral minus infinity to plus infinity f y, g at x minus y, d y, e to the power minus i omega x d x.

Now, let us write it this as double integral minus infinity to plus infinity another one also minus infinity to plus infinity, f y, g x minus y into e to the power minus i omega x, d y, d x. Now, let us make the change of variable, as x minus y as z this says is x would be y plus z, so now we are just changing this x minus y to z, so x is changing to the y plus z, so d x would be nothing but d z that is how we are doing it, why we are keeping a constant when we are transforming only with respect to x.

So, why we are not treating as a another variable, we are just changing only one variable x, so as x is approaching to minus infinity z will also approach to minus infinity as x is approaching to plus infinity z will also approach to plus infinity, for any fixed y. So, this integral is minus infinity to plus infinity, insight function minus infinity to plus infinity integral, f y this g x minus y that is z, e to the power minus i omega x is being changed to y plus z d z dy.

Now, minus infinity to plus infinity, this now we are getting is this two integrals as, here is one function is with respect to y only another is with respect to z only, here also we can break it into as e to the power minus i omega y and e to the power minus i omega z into multiplication that is we can break this double integral into after the product of two integrals. What are those, minus infinity to plus infinity f y e to the power minus i omega y d y and integral minus infinity to plus infinity, g z e to the power minus omega z dz.

Now, let us see about these integrals are, if we see this first integral this is nothing but the Fourier transform of f and this is nothing but the Fourier transform of g, now these transform or these integrals exist because we had already assume that f and g are piecewise continuous integrable and bounded that says us that this Fourier transform does exist or these integral does exist. So, this is nothing, but f hat omega into g hat omega.

So, what we have got, the Fourier transform, this actually function is here you would see is that is in application how it is a very useful thing, that is rather than solving the problem in the actual space if we go to the another variable that is the transform function and there we can solve the problem and then we can come back because in the Fourier transform we had also seen is that how to find out the inverse Fourier transform.

If Fourier transform for this any function f is integral, minus infinity to plus infinity, f x e to the power minus i omega x d x then the Fourier inverse Fourier transform of f hat omega was 1 upon 2 pi integral minus into infinity to plus infinity f hat omega e to the power i omega x d omega. So, once we get the this function, we can go back and find out the its Fourier inverse transform and then we can come back to the our f convolution g, rather than evaluating that integral we may be some time may be more useful for getting a solution of the problems.

So, we have learnt in these lectures, a technique called the Fourier series are given by the Fourier, which said is that is any function, we can make it out as that represent is as a some of the series which we called some of the series of the containing the terms of sine and cosine and we called it Fourier series. Then we have come up with some properties of those series then sine series, cosine series for the properties defined for those functions, some conditions for the functions are satisfying.

We have learnt that the Fourier series was applicable or we could find out only when the functions were periodic then, we want that is all the practical times the functions may not be periodic we had extended that idea and we have come to the transforms. So, the first we had learnt the Fourier sine transform and then Fourier cosine transforms, then we had learnt Fourier transforms as well. We will use this transforms and this series in solution of differential equations that would be in coming lectures you would learn about these things. We had already learnt the Laplace transform like that, so that is all for today's lecture.

Thank you.