Mathematics - III

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Lecture - 14
Fourier Integrals

Welcome to the lecture series on differential equations for under graduate students. Today's lecture is on Fourier Integrals. In some last lectures we had seen that Fourier series are very important tool in solving many practical problems involving, periodic functions. Let us try to extend this idea, to non periodic function as well, because in many practical problems, we have functions which are not periodic. So, let us try it, to extend this, in this one and let us first see with the help of example.

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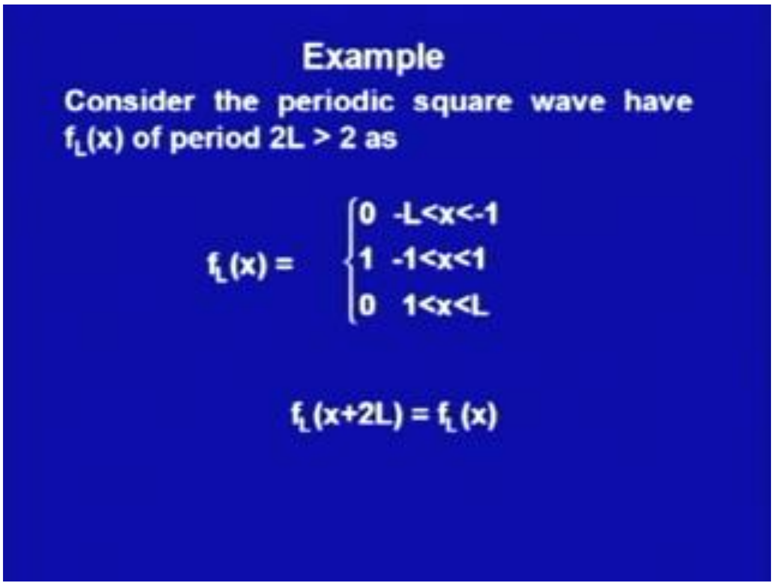

Consider the periodic square wave f L X as 0 from minus L to minus1, 1 from minus1 to1 and 0 from 1 to L. So, certainly we want that, this period 2 L has to be greater than, 2 since we want from minus1 to plus 1 it should be 1, and after that it should be 0, and we want it is periodic with period 2 L, so f L X plus 2 L should be equal to f L X for all x.

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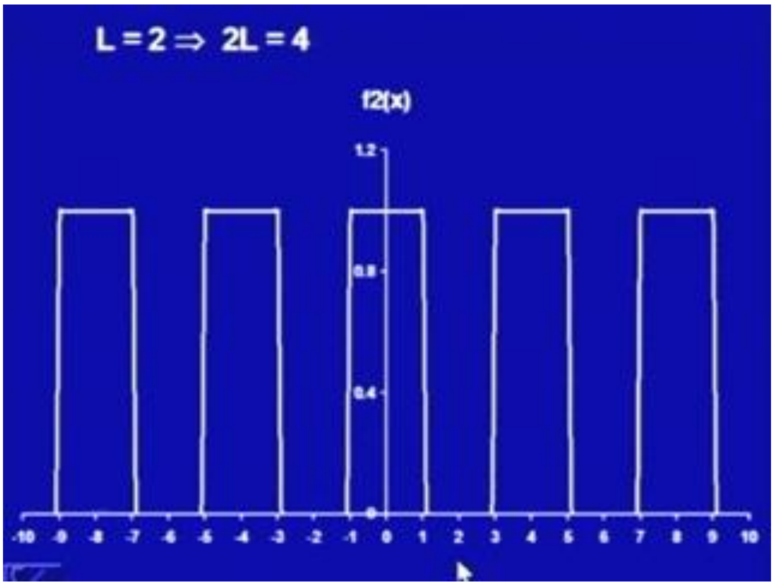

Let us see, how this function looks like, why we are calling a square wave, so if I take, L is equal to 2 that means, period 2 L would be 4, and the function would be, 0 from minus 2 to minus1, then 1 from minus1 to plus1, and then again 0 from1 to 2. And we are making it periodic from f X plus 4 is equal to f X, that is, we are just moving here, after that here from minus 2 to plus 2, and then again we are repeating, so whatever the function over here, were repeating like this, so here, we are getting is this square waves.

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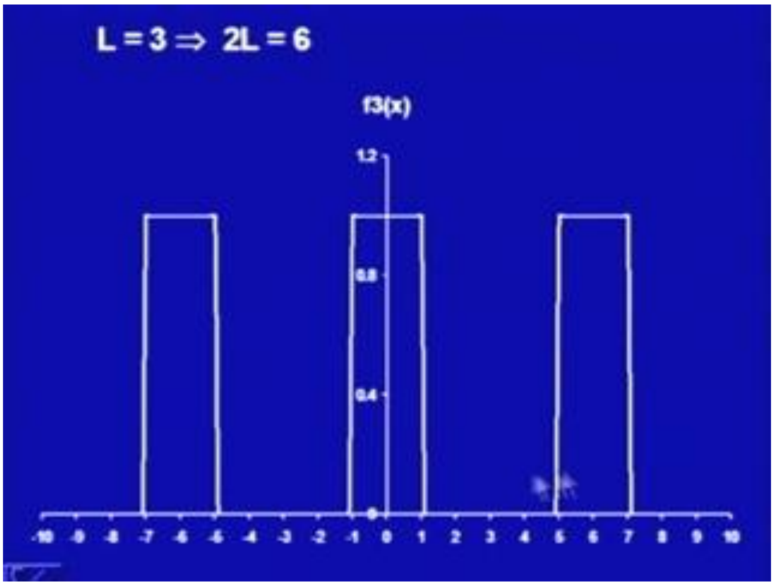

Now, if I extend this L 2 3, we would get, again that from minus 3 to minus 1 0, 1 to plus1 it will be plus 1, and then 1 2 3 it is 0, again, now the period is 6. So, we would repeat it after that 1, so we are getting, the squares a little further.

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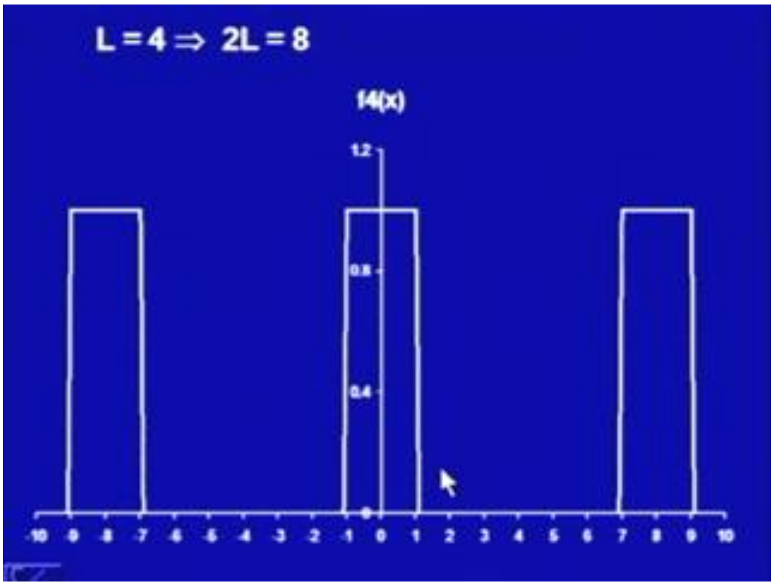

Again if I take, L is equal to 4, it would be like this, so if I move 1 like this, we would be getting is that these squares, would be recurring, at a more and more distances, let us see, if I take L to be very large.

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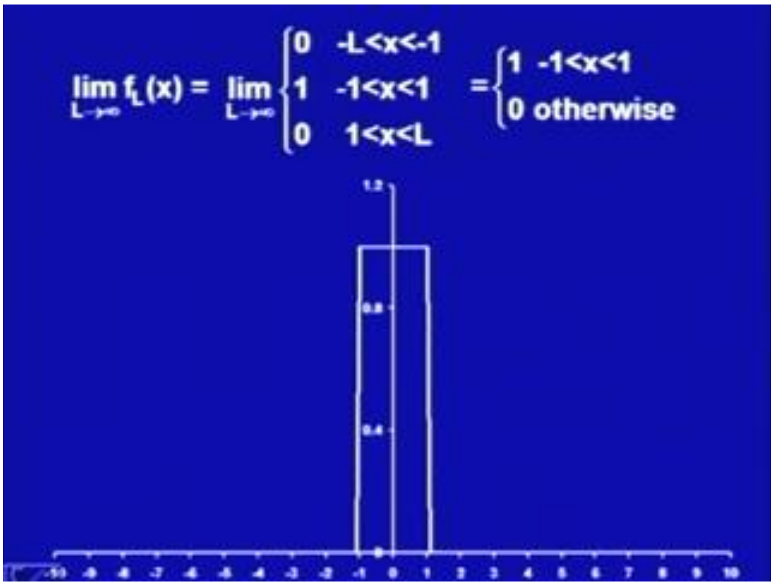

That is take the limit as, L approaches to infinity of this function, that says is, I would get the function 1, in minus 1 to plus 1, and 0 otherwise, because L we are approaching to the infinity, so from minus infinity to minus 1 it would be 0, and from 1 to infinity it would be 0. What this function would look like, 1 square in between, from minus 1 to plus1, and 0 elsewhere, now let us see, this we have said is that is, f L X is a periodic function. Now, let us see, how this Fourier series of this function, f L X will behave, when we increase, L or we change the L.

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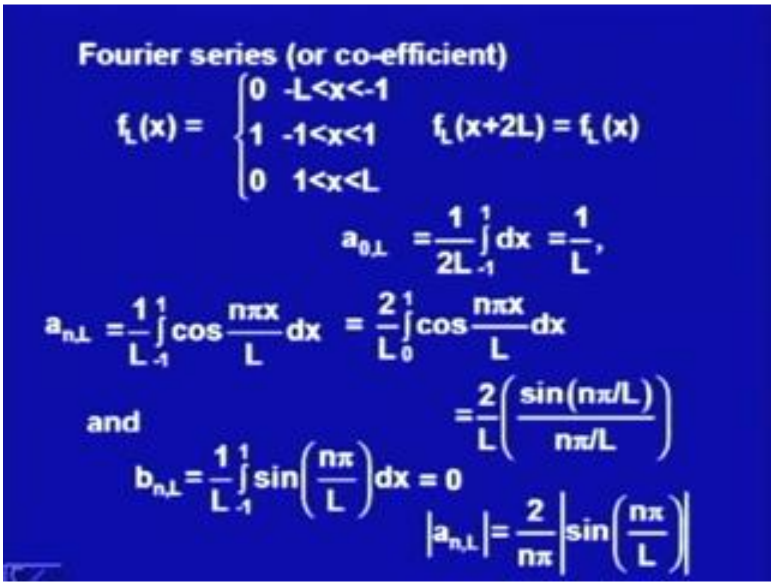

So, let us see first, for the Fourier series for f L X, this is f L X, which is, 1 in minus 1 to plus 1, and 0 from minus L to plus minus 1 and 0, from 1 to L and then periodic, with f X plus 2 L as f L X, find out what is the Fourier coefficients, that is using the Euler's formula, a naught L, L I am using, that is we are using this, particular L. One upon 2 L minus 1 to plus1 d x, because the function is 0 from minus L to minus 1 and 0, from 1 to L so we do have 1 only in the interval minus 1 to plus 1, which is nothing but 1 by L.

And what will be a n, a n would be nothing but 1 upon L, integral minus 1 to plus 1 cos n pi x by L d x, because the function f X is 1 in the interval minus 1 to plus 1. What will be it is this integral ((Refer Time: 04:34)) function, we can write as two times, so 2 by L 0 to 1 cos n pi x over L d x, it is integral is nothing but the sine n pi x by L divided by n pi by L, evaluated from 0 to1, we would get finally as, 2 by L sine n pi by L upon sine n pi L.

Now, what will be the coefficient b n, b n L would be, integral 1 upon l, minus1 to plus 1 sine n pi x over L d x, now you see this function is, odd function in this one, so this should be integral should be 0. So, we are getting finally, Fourier cos series for this function, and what will be it is amplitude, because b n is 0, we would get the modulus of a n that is, square root of the, a n square plus b n square an square b n is 0, so a n square only square root we do take only positive quantity. So, it would be 2 upon n pi sine n pi by L, now if I, just plot this one, this is called the Fourier amplitude spectrum.

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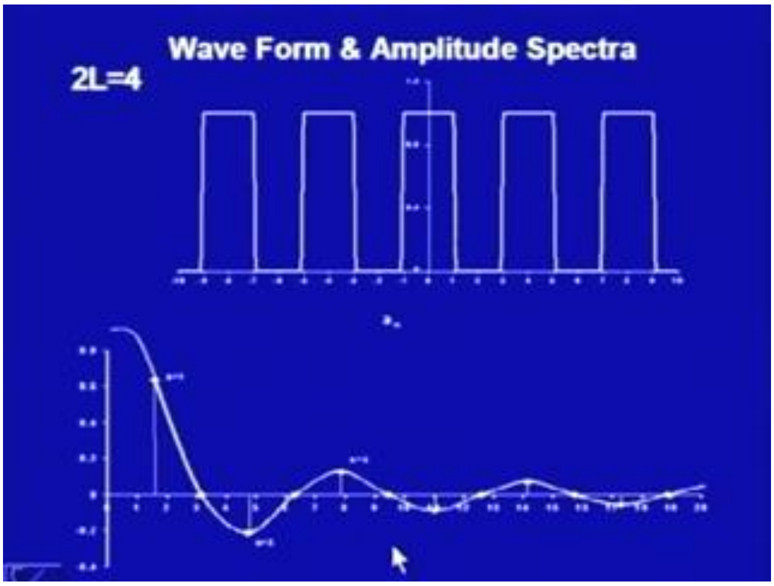

Let us see, this curve, first we have taken that 2 L is 4, so the wave forms are, a closer enough, this kind of function we were having, if I plots it is amplitude spectra that means, I am plotting, a n L versus n pi L, a n L was, 2 upon n pi modulus of sine n pi by L. Now, when L is equal to here 2, so what we would be having is 2 upon pi sine n pi by 2, now when n is1 sine pi by 2 is 1, what I would get, amplitude as 2 upon pi, and here we are plotting it with, n pi by L that is here n is 1, that is pi by 2 at pi by 2 I am plotting it here.

Like this, this is been plotted, so when n is equal to 3 I would be more less1 and like that, now we see, in this graph, we are getting this wave, then it is going down, then it is coming up. So, we call it half wave, in each half wave I am having 1 maximum amplitude.

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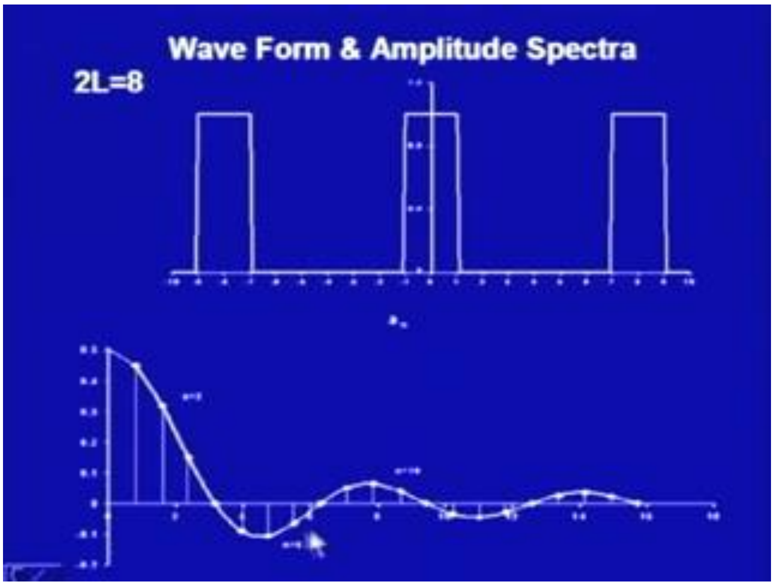

Let us see, when L is equal to 4, that is my period is now 8, this be the function type, now if I am plotting this1, the maximum value, which we would be going for n is equal to, 2 here. That, would be here, because n is equal to 1, we would be taking it pi by 4, like that, we just come here, this is value is less than your 0.5, in the first one it was less than 0.8. Now, here it is less than 0.5, and in half wave I am having, 3 amplitudes for each one, like that we are going on.

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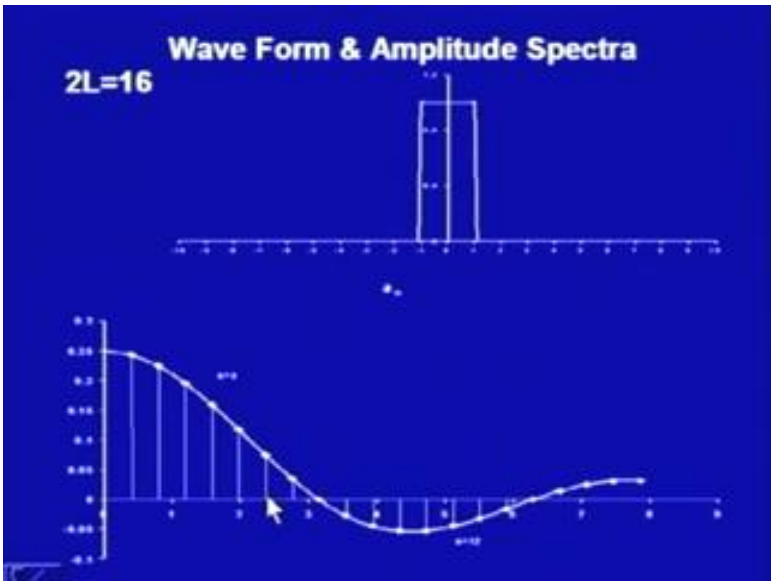

Now, if I take L is equal to16 of course, here next one has not been plotted, it would be little bit far away, we do get, these kind of one, and maximum value we are get getting is less than 0.25. Now, what we are seeing in all these graphs, where I am plotting this amplitude versus, this omega and we would call n pi by L as omega n, we are getting is, that as we are increasing my L, our amplitudes are, coming more and more in half wave, that is they are becoming dense.

Moreover, their value or this length is decreasing, so what we say is that, if I increase my L, I would get that, this will become more dense, having half wave many of them, and the length, that will decrease and you could reach to the 0. What now, what this conclusion we are going to draw from here.

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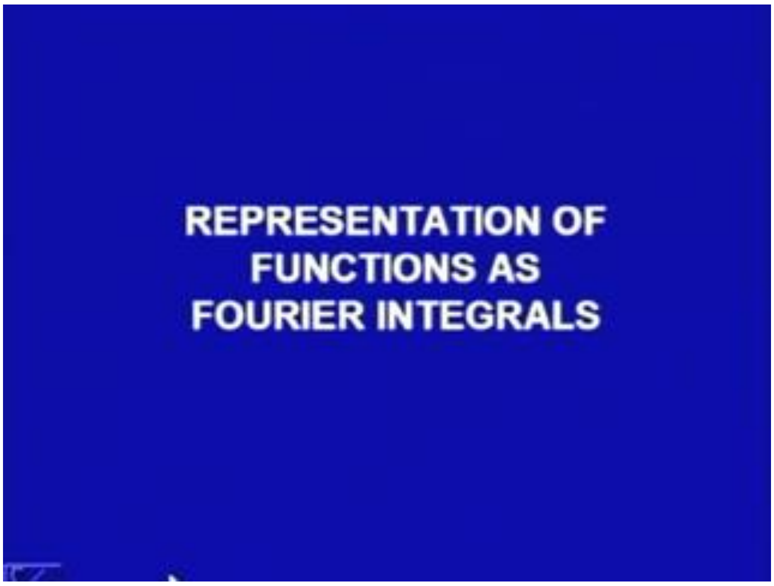

Let us see, from here we would come to this one, representation of a function, as Fourier integral, that is still now rather than having a sum, that is Fourier series, the series we would be, changing to the integral. How we are going to do it, we will take the help of the, just now whatever we had learnt from the graphs, so let us see is, that is mathematically how we are going to do it.

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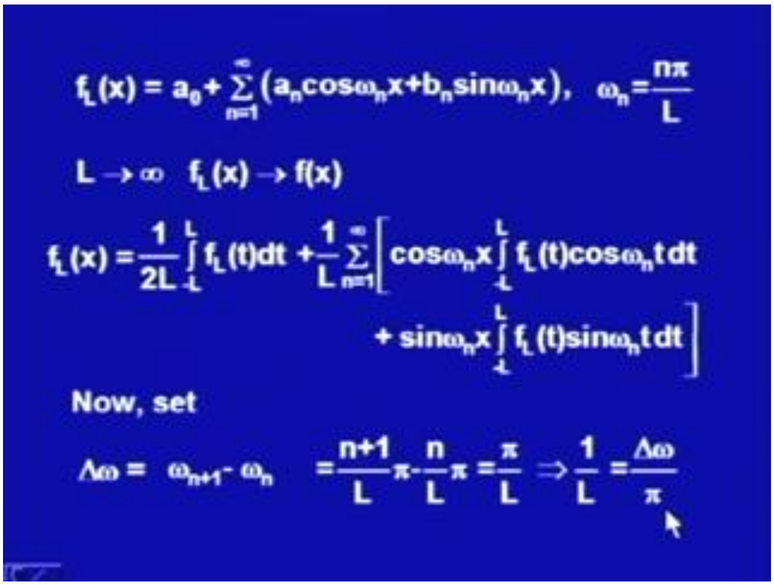

Fourier series for, function f L X would be, a naught summation n is running from1 to infinity, a n cos omega n x plus b n sine omega n x, where omega n of course, I am writing n pi by L, that is rather than writhing here, as cos n pi by L x like that, we are writing here. Now, substitute this a naught a n and b n by the Euler's formula, for that, one more thing, we would be assuming over here, that as L is approaching to infinity, f L X is approaching to f X, f X if you do remember, we had find out the function is, 1 in the interval minus1 to plus1 and 0 elsewhere, that was the limit.

Moreover we would be assuming is that, wherever is required, all the integrals, and this series convergence, is existing, that is the sum of the series and the integrals are existing. Moreover we would be assuming, certain places, that is they can be interchanged, that is the summation in integration signs can be interchanged, for that, if you do remembers, you have done, uniform convergence and that kind of results. So, those results we would be assuming here, as they are happening, so let us see.

Now, a naught is1 upon 2 L minus L to plus L f L t d t, I am changing this function x to t in the integral, because I am saying is that is f L X is being represented at X, so all the integrals and integrant, we would be changing it to the t. So, that there is no confusion between the integral and the value at the, or thus point at, which the function we are evaluating, plus1 by L summation n is running from1 to infinity, cos omega n x, a n is1 upon L, that we have taken outside, integral minus L to plus L f L t cos omega n t d t and plus sine omega n x integral minus L to plus L f L t sine omega n t d t, that is all these coefficients a naught a n and b n.

We had replaced it them by, Euler's formulae a naught this one for a n 1 upon L this one and for b n1 upon L times this one. Let us see, some delta omega, this we are saying is omega n plus1 minus omega n, omega n is n pi over L, that says is it would be, pi by L, now if L is approaching to infinity pi by L, would be approaching to 0, that is delta omega would be approaching to 0, more over 1 upon L, we could write as, delta omega upon pi.

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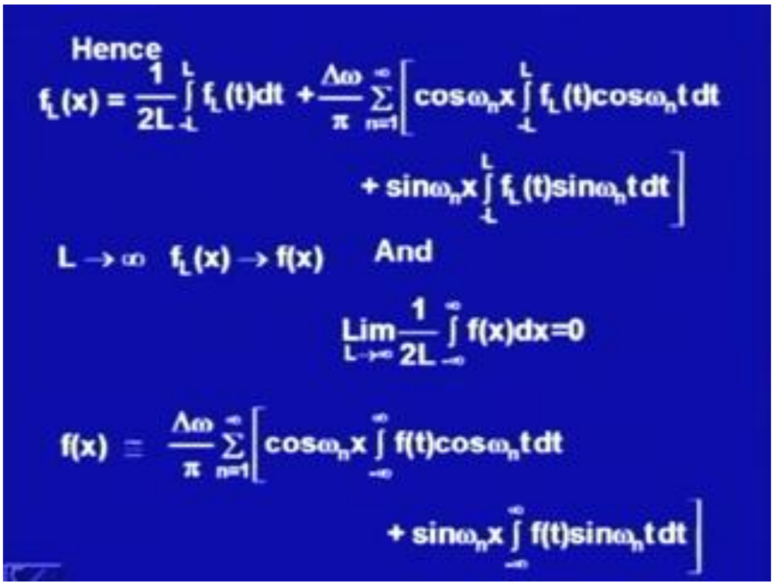

So, f L X I can write as, 1 upon 2 L minus L to plus L f L t d t plus delta omega upon pi 1 upon L, we are changing it to delta omega upon pi, summation n is running from1 to infinity cos omega n x, integral minus L to plus L f L t cos omega n t d t, plus sine omega n x integral minus l to plus l f L t sine omega n t d t. Now, take the limit on both the sides, so as L approaches to infinity, we do know that f L X approaches to f X, and as L approaches to infinity, integral 1 upon 2 L minus L to plus L actually, f X d x minus infinity to plus infinity f X d x, this is 0.

Then f X, so now, I am taking the limit on both the sides, so f L X is approaching to f X this, where it would be, this integral would be 0. I would be getting is, delta omega upon pi, summation n is running from 1 to infinity, cos omega n x integral minus infinity to plus infinity f t cos omega n t d t plus sine omega n x integral minus infinity to plus infinity f t sine omega n t d t. Now, what assumptions, we have taken over here, we see here, we have taken f L X is approaching to f X.

So, wherever we were having is f L t, that is approaching to f t, we had assumed, that all the limits integrals, and the summations they had existing, moreover the interchange is possible, what it says is, that if I take the limit over here, then I can interchange this limits with the summation sign. So, the limit am taking inside, limit with respect to L, am changing it, that is the integral minus infinity to plus infinity, moreover the limit f L t is being also changed with f t.

So, all those assumptions on the limit, integrals on summations, we had assumed over here. So, finally, we have got, f X we can approximate as, delta omega upon pi, summation n is running from1 to infinity cos omega n x, integral minus infinity to plus infinity f t cos omega n t d t plus sine omega n x, integral minus infinity to plus infinity f t, sine omega n t d t.

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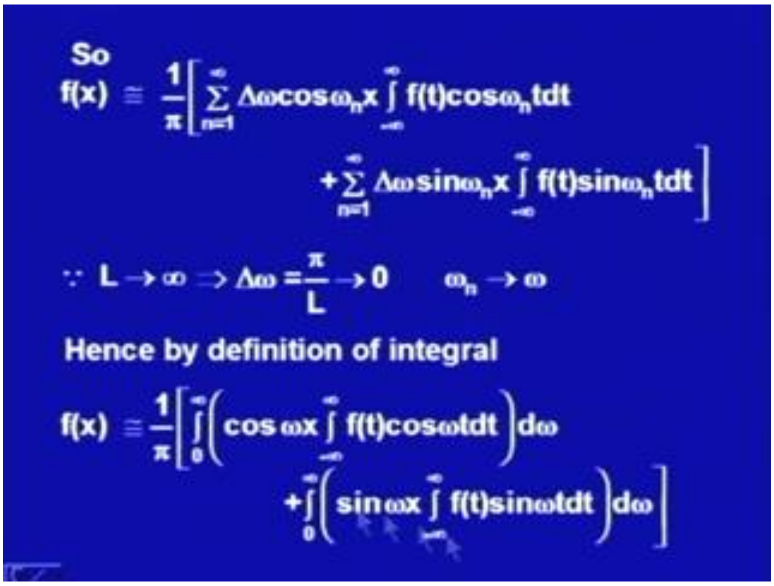

So, f X now, delta omega I am taking inside, the summation sign is also inside, 1 upon pi. Now, rewriting the same term, summation n is running from1 to infinity, delta omega times cos omega n x and then, this integral minus infinity to plus infinity f t cos omega n t d t, and again this breaking this one. So, summation n is running from1 to infinity, delta omega sine omega n x and then the integral minus infinity to plus infinity f t sine omega n t d t.

Now, we will use the definition of, integral you do remember, that the definition of integral, this we are changing it to, that is Since L is approaching to infinity, delta omega would be approaching to 0, omega n, now we will take is approaching to omega. Hence, by definition of the integral, we would get, f X as1 upon pi, this summation is changing to the integral 0 to infinity cos omega x, this is the multiple of this thing. So, what we are saying is now total function, this function is fixed with respect to now our limit.

But, one thing we are taking is, omega n is approaching to omega, so we are including, it here that, this function is changing to integral f t cosine omega t d t this integral. So, now we are getting, finally 0 to infinity cos omega x into integral minus infinity to plus infinity f t cos omega t d t, whole integral with respect to omega. Similarly, the second one, integral 0 to infinity, sine omega x multiplied with integral minus infinity to plus infinity f t sine omega t d t d omega.

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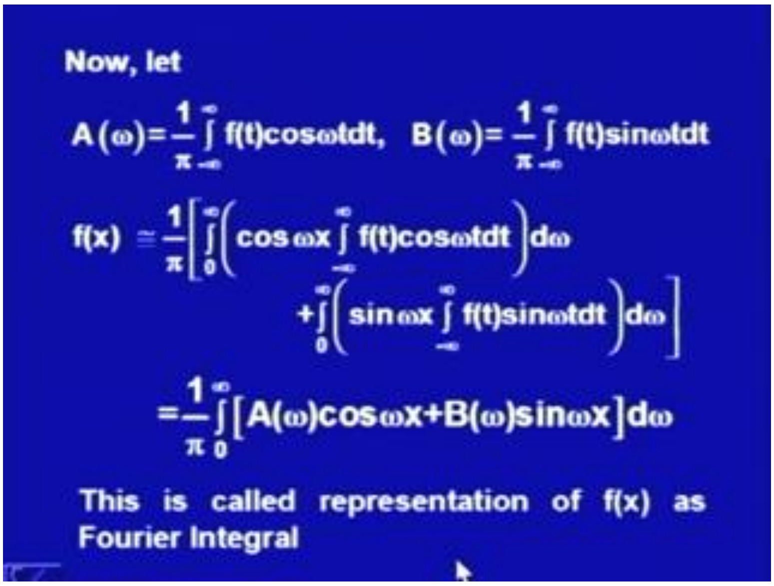

Now, rewrite it I get, assume A omega as, 1 upon pi integral minus infinity to plus infinity f t cos omega t d t, this thing we are assuming as, A omega and because this would be a function of omega, t would be integrate it out. Similarly, the other one we can assume as B omega, that 1 upon pi integral f t sine omega t d t on the whole real line minus infinity to plus infinity, then rewrite the function, then f x we could write,1 upon pi integral, this integral again am this rewriting.

So, now this inside integral, this we are changing, this is the function of omega, so A omega and this inside integral, is B omega. So, what we would get it rewrite, 1 upon pi 0 to infinity, a omega cos omega x plus B omega sine omega x integrate it with respect to omega, this is called the representation of this function f X, as Fourier integral. Why, we are using the term Fourier, if you do remember Fourier series, we said is series of sine and cosine.

Here what I am having is, I am having an integral, which has cosine term and sine term, and it has certain coefficients, which we call A omega and B omega. In the series, we if you do remember we were having is, a n and b n and then this summation, here we are having integral, and this A omega, B omega, d omega, so we are calling it Fourier integral, what it helps.

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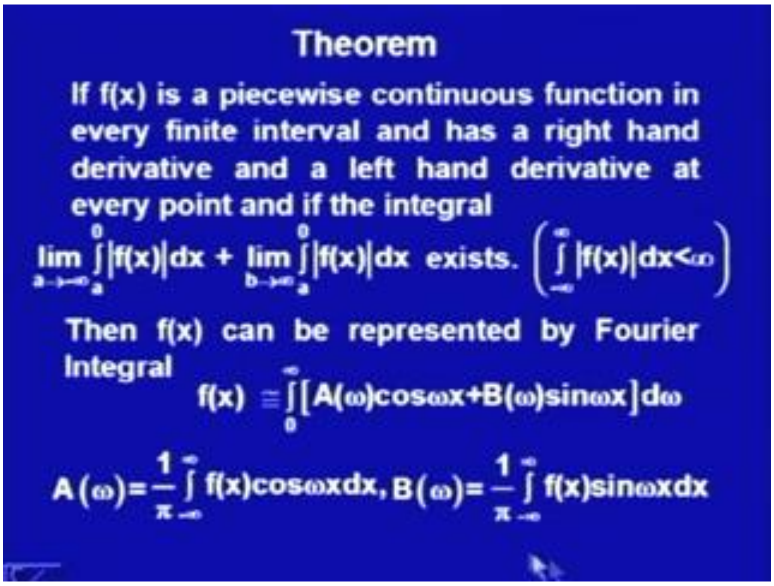

Let us see, this result, which we have find it out, theoretically what it says, if f X is a piecewise continuous function, in every finite interval and has a right hand derivative and left derivative, at every point. So, now we are not assuming, that the function is differentiable, actually we say is, that is only if the left hand derivative and right hand derivative are existing, and the function is piecewise continuous, moreover if the integral as a approaches to minus infinity of a to 0 f X d x, and b approaches to plus infinity 0, to I am sorry, it should be 0 to b, 0 to b modulus f X d x.

This must exist, this we can rewrite as, integral minus infinity to plus infinity modulus f X d x does exist, have remember it, this is call the existence of, integral of the function f X or this is simple terms we say is that the, function is integrable. Then f X can be represented by, the Fourier integral as A omega cos omega x plus B omega sine omega x integrate it, with respect omega in the interval 0 to infinity.

So, what we have got the function is, now we are not requiring the function to be, your periodic we are saying is, function is piecewise continuous, has left hand and right hand derivative at every point and the function is integrable, and the whole real line. Then, it can be represented as a Fourier integral, the form of the Fourier integral is, that integral from 0 to infinity, A omega cos omega x plus B omega sine omega x d omega, where this A omega is nothing but 1 upon pi integral minus infinity to plus infinity f X cos omega x d x, and B omega is the integral1 upon pi, integral minus infinity to plus infinity f X sine omega x d x.

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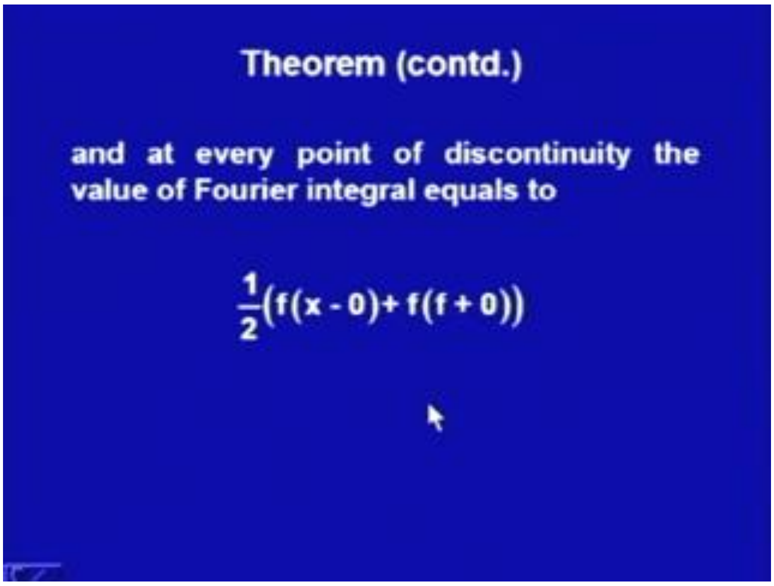

Moreover, at every point of discontinuity, the value of the Fourier integral, equals to the, average of the left hand limit and the right hand limit, this should be x plus 0 rather than f. f of x minus 0 that means, the left hand limit at x, and f of x plus 0 that is the, right hand limit at x. So, at the point of continuity, we are having is at the can be represented, the value of the function would be equal to that Fourier integral, and at the point of discontinuity, the value of the Fourier integral would be equal to the average of the right hand and left hand limit.

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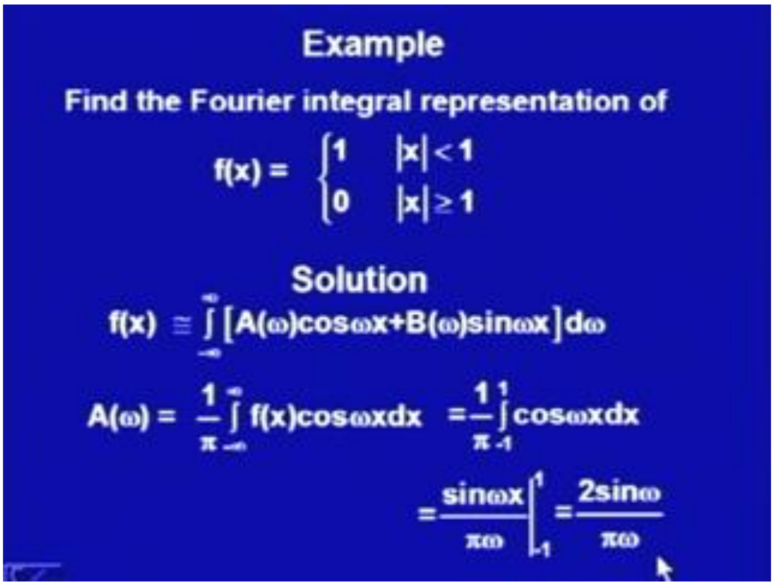

Let us see, one example find the Fourier Integral representation of the function1, which is 1 in the interval minus1 to plus1 and 0 outside, that is, this we are just having the same function. That says, we want, this kind of representation, we want an integral, which has the terms, A omega cos omega x B omega sine omega x, so we have to find out, what is this A omega and B omega, we can find out for this function f X, we could ((Refer Time: 21:43)) write it as a function.

So, let us see, A omega what it should be, just now we had find out the formula, that 1 upon pi minus infinity to plus infinity f X cosine omega x d x, now for this function f X let us evaluate it. So, it should be,1 upon pi minus1 to plus1 cos omega x d x, this is an even function, sine omega x upon pi omega, so we get 2 sine omega upon pi omega, this is what is my A omega.

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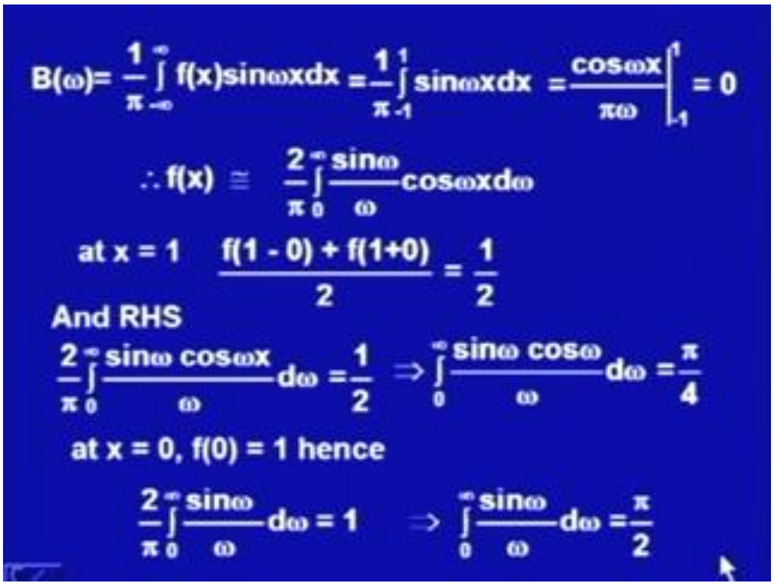

Now, B omega would be1 upon pi, integral minus infinity to plus infinity f X, sine omega x d x evaluate it, minus1 to plus1 sine omega x d x, the integral is cos omega x upon pi omega, minus1 to plus1 this is 0, we need not to evaluate, we could have seen it, that is on odd function in the interval minus1 to plus1, so it should be 0. So, what we are getting is f X as 2 upon pi, 0 to infinity, this is what a, A omega we have got 2 sine omega upon pi omega, sine omega upon omega, cos omega x d omega, this is the Fourier representation of the function f X.

The theorem says, that all the point of continuities, the value of the function, at that point would be same as the value of this integral, and at the point of discontinuity, it would be the left hand and average of left hand and right hand limit. The function, if you remember we were having is, 1 in the interval minus1 to plus1 and 0 otherwise, so at both the points minus1 and plus1, the function is discontinuous.

Let us see, at point x is equal to1, the function is discontinuous, so the left hand limit and the right hand limit 1 minus 0 means, we are reaching towards, from the left to the1, that is the function is1. So, that would be 1, and from the right if I reach, it would be 0, so it would be simply1 by 2 and the, what is this integral value, that is the right hand side, this integral we are saying is that, this function we are representing by this, Fourier integral.

What it would be, it would be nothing but 2 upon pi sine omega cos omega x, over omega d omega, so now I want x is equal to1. So, what it would be, it should be equal to half now let us write it out, it says is integral 0 to infinity, sine omega cos omega, because at x is equal to1, it should be equal to half, this is what the theorem is saying, so integral 0 to infinity sine omega cos omega upon omega d omega, this we could write as, you see, pi by 4, because this 2 would come over here.

So, what we have done is, we have actually use the theorem, which says is that, if the function is piece wise continuous, we can represent the function by Fourier integral, and from there, actually this function, this integral of this function, we had find it out, this was the point of discontinuity. Let us take, the 0.0, at the 0.0, at the 0.0 the function is continuous, and f at 0 is 1, so by the theorem, we say is that this integral, should be equal to the value of the function.

So, what is this integral at x is equal to 0, if x is equal to 0, this cos omega x I would have as 1, so this integral would be 2 upon pi sine omega, upon omega d omega, and by the result of theorem, it should be same as the value of the function, value of the function at 0 is1, so it should be equal to1. Or we can say, that the integral 0 to infinity sine omega upon omega d omega is, pi by 2, so we had, you would have used, these integrals actually at many places in practical problems.

So, we are finding out these, integrals with the help of that theorem, which we are calling the Fourier representation of, any representation of any function as a Fourier integral. Let us see, cosine and sine integral, because just now in the example, we had seen that we had got, only the cosine term not the sine term, so when it will happen. As in the case of Fourier series, we had find out there if the function is, even we were getting, only cosine series, if the function was odd, we were getting only, sine series similarly, here the things will happen.

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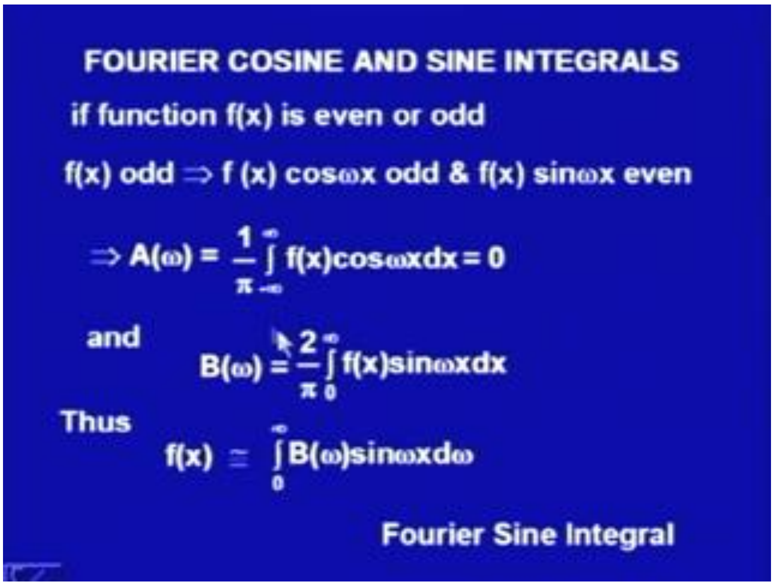

So, we say if the function is even or odd, if function is odd, then f X cosine omega x is also odd, because the product of odd and even function is odd, and f X into sine omega x, odd multiplied with an odd function, so product of an odd and odd function is even. So, f X sine omega x would be even, what it says is, if the function is odd, and the A omega is1 upon pi, minus infinity to plus infinity f X cos omega x d x. This will always be 0, because ((Refer Time: 27:05)) integral is, from minus infinity to plus infinity.

Remember that, in the definite integral property, we were having is actually minus a to plus a here, we are taking it minus infinity to plus infinity. So, we are taking it is a limit point, and we are assuming that, all those integrals, are existed that is, why we are taking it to be 0. And B omega is, 2 upon pi, 0 to infinity f X sine omega x d x, because f X sine omega x is even, so we can write it as, 2 times that integral, so1 upon pi integral minus infinity to plus infinity f x sine omega x d x would be nothing but 2 times1 upon pi, so this is B omega. And what we will get, we will get, only sine integral, so f X would be only, integral 0 to infinity B omega sine omega x d omega, where b omega is, this one, this is Fourier sine integral.

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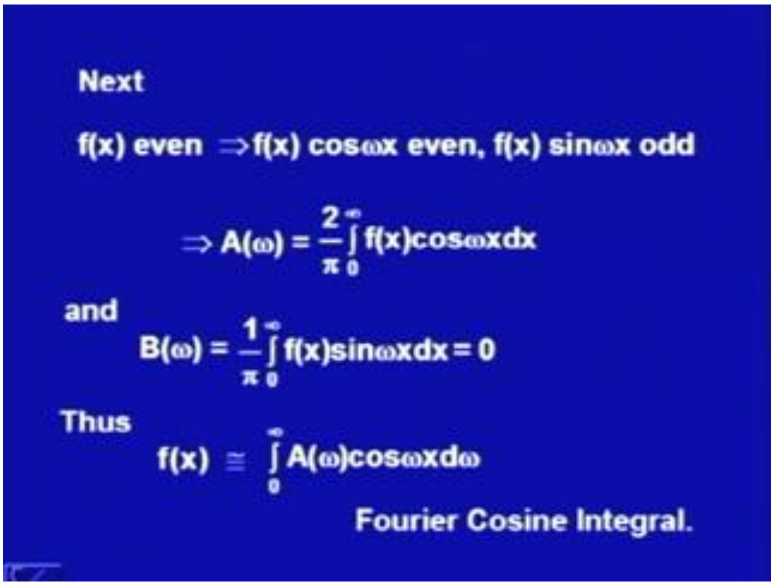

Now, if the function is, even what it says is, f X cosine omega x would be even, because the product of, even function with an, even function will give me, even function and product of, even function with odd function, sine omega x is an odd function. So, f X sine omega x, this would be odd, that says is, when we calculate this A omega and B omega, A omega the function is even. So, it would be 2 times 0 to infinity f X cos omega x d x, when it is odd, then b omega would be, simply 0, it should be minus infinity to plus infinity.

Thus, I can represent f X as 0 to infinity A omega cos omega x d omega, only cosine integral, so as in the case of Fourier sine series and cosine series we have sine, here also we are finding out, if the function is even, we get Fourier only Fourier cosine integral, and if the function is odd we get, only Fourier sine integral. Now, where we are using, so use of this Fourier integrals, in evaluation of, integrals we had seen one example, where we had integrated certain functions, which we were not able to integrate as such let us see one example.

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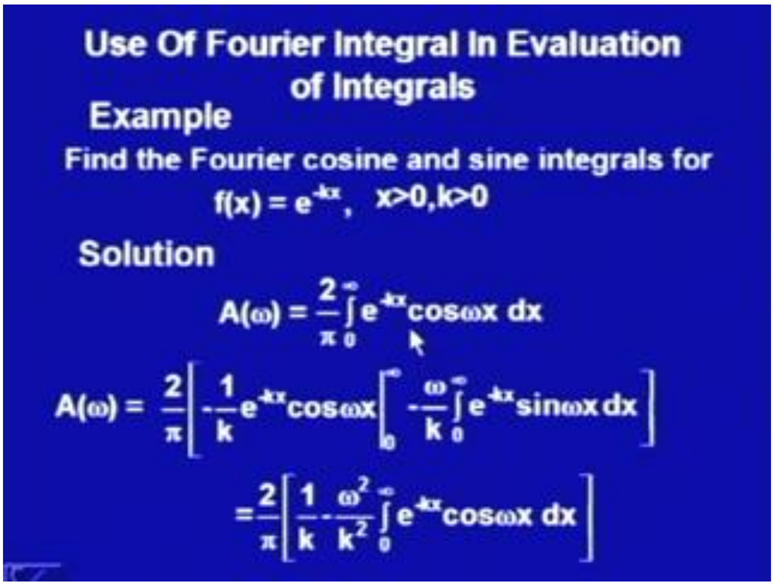

Find the Fourier cos and sine integral, for the function, e to the power minus k x for x positive and k positive, let us try it. A omega, the function e to the power minus k x for, x positive and k positive. So, we are, having this minus infinity to plus infinity, so we should start this, 0 to infinity, because the function from minus infinity to 0 is 0. So, it is 2 upon pi 0, we want cosine integral, so 2 upon pi 0 to infinity, e to the power minus k x, cos omega x d x, integrate this one.

We get, 2 upon pi, minus1 upon k e to the power minus k x, this is integration by part this function we are taking as integrating function this as the differentiation function, cos omega x, 0 to infinity, minus omega by k integral 0 to infinity, e to the power minus k x sine omega x d x. Which is 2 upon pi, simplification 1 upon k minus omega square upon k square, integral 0 to infinity, e to the power minus k x, cos omega x d x.

Now you see this is what a omega, omega was, if I just remove this constant over here, just see the what is the integrant, e to the power minus k x, sine cosine omega x, here also we are getting is e to the power minus k x cosine omega x. The coefficient 2 upon pi, that is also here, what it says is, I have again came to the same kind of integral, what it says now, I take this integral, 2 upon pi, that is it is omega square upon k square times A omega

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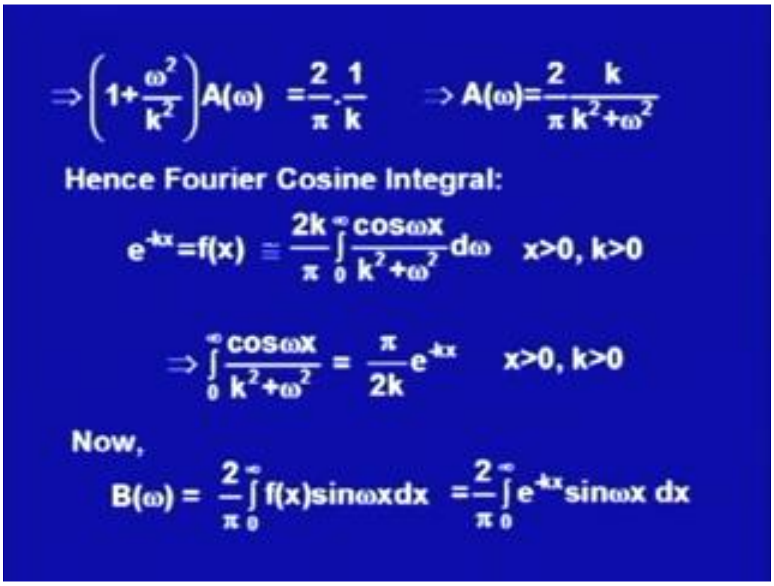

So, what we are getting is,1 plus omega square upon k square times a omega is equal to, 2 upon pi into1 upon k, what it says a omega is 2 upon pi, k upon k square upon omega square. Now, you see, we were not able to calculate, that integral e to the power minus k x sine omega x d x, directly we had calculated it, using this recurrence relation you could say, similarly, so Fourier cosine integral of this function, e to the power minus k x, we could write, 2 k upon pi, integral 0 to infinity, cos omega x, upon k square omega square d omega.

Why, A omega cos omega x d omegas, so A omega is 2 k upon pi times k square plus omega square. So, pi is here, k square plus omega square inside, because the integral is with respect to omega, e to the power minus k x f X is 2 k upon pi integral 0 to infinity cos omega x upon k square plus omega square d omega, for x positive and k positive, from here, what we are getting, we are getting that, integral 1 to infinity cos omega x upon k square omega square d omega, is equal to pi upon 2 k times e to the power minus k x, so we are evaluating, this integral with the help of this Fourier cosine integral.

So, now let us move to the Fourier sine integral, so for this, we require B omega, what is the formula for B omega, 2 upon pi integral 0 to infinity f x sine omega x d x, now my f X is, e to the power minus k x. So, we are having is 2 upon pi integral 0 to infinity, e to the power minus k x sine omega x d x, let us evaluate this integral by parts.

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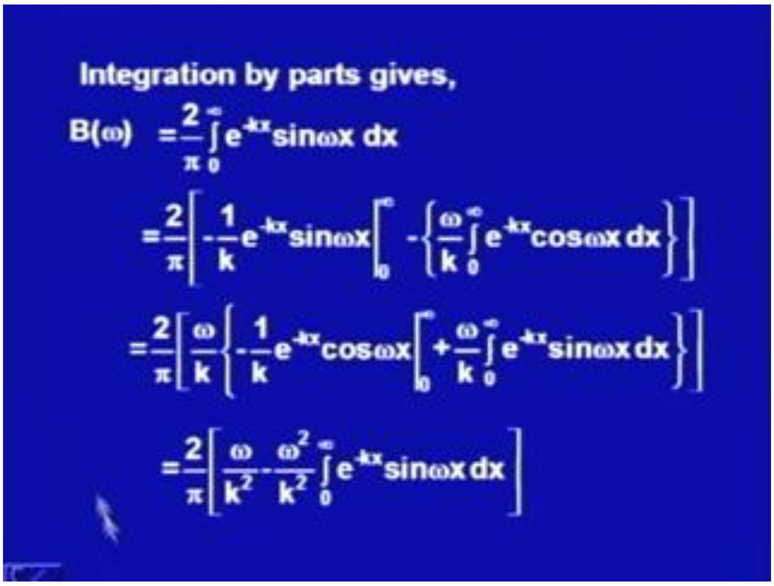

So, B omega, is this integral, we are going to evaluate it by parts, we will take, e to the power minus k x as the, integrating function and sine omega, x as the differentiating function. So, first integral, minus1 upon k, e to the power minus k x, sine omega x evaluated from 0 to infinity, minus omega by k, integral 0 to infinity, e to the power minus k x, cosine omega x d x. Now, when we are evaluating it put, x as approaching to, infinity e to the power minus k x, would have pushed to 0.

So, this value would be 0, at 0 sine omega x, that is sine 0 would be 0, so, this would give us 0, and what is being left is, 2 upon pi omega by k, minus1 by k, integral this again evaluating by the parts, e to the power minus k x cosine omega x, evaluated from 0 to infinity, plus omega by k integral 0 to infinity, e to the power minus k x sine omega x. Now, you see, we had again came to the, same integrants, which was in the B omega, the coefficient is 2 ((Refer Time: 34:24)) 2 upon pi is outside, so we can again, take this as B omega.

Let us evaluate, this function, as x approaches to plus infinity, e to the power minus k x will approach to 0, and as x approaches x is 0, cos 0 is1, e to the power minus k 0, that is this is also1. So, what we are getting is, 2 upon pi, omega upon k square, minus omega upon k square, 0 to infinity e to the power minus k x sine omega x d x, this integral with 2 upon pi is d omega.

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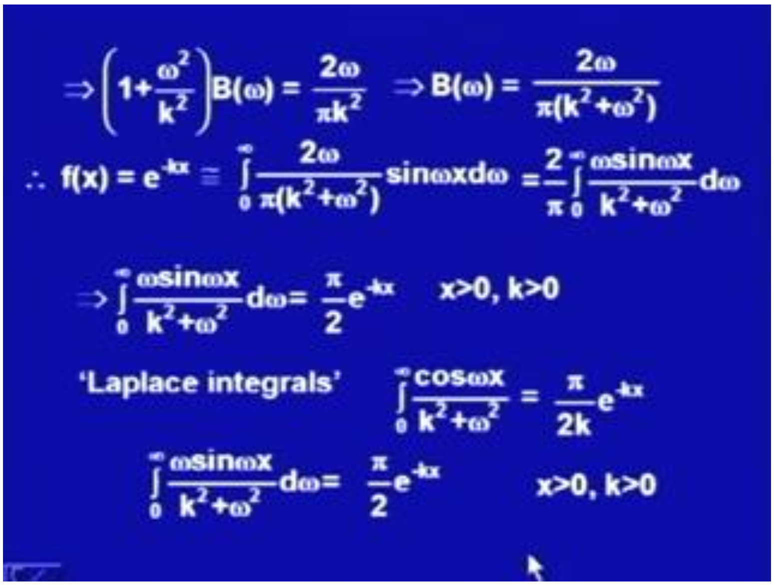

So, we can get it as, 1 plus omega square upon k square b omega is equal to 2 omega upon pi k square, this says is B omega is, 2 omega upon pi k square plus, omega square thus, what we are getting f X, which is same as, e to the power minus k x that can be, represented as a Fourier sine integral 0 to, integral 0 to infinity 2 omega upon pi times k square plus omega square sine omega x d omega. And, from here, what we are getting is, 2 upon pi omega, sine omega x upon, k square plus omega square, d omega, so this integral is equal to, pi by 2 times, e to the power minus k x, again for x positive and k positive.

So, we had evaluated, this integral with the help of, Fourier sine integral, these two integrals, just now, which we had evaluated, they are called actually, Laplace integrals. So, what we are calling Laplace integrals, the integral 0 to infinity, cosine omega x upon k square plus omega square is equal to pi by 2 k for into, e to the power minus k x, and integral 0 to infinity omega sine omega x upon k square plus omega square d omega is equal to pi by 2 e to the power minus k x, both these integral are valid for, x positive and k positive, these are known as Laplace integrals.

So, we had seen that Fourier integral, Fourier sine and cosine integrals and we had used them for evaluation of, certain integrals. Now let us, move to the today's topic, that is the transform, Fourier transform, we are saying, now before moving to the Fourier transform, let us first move to the, Fourier cosine and sine transforms.

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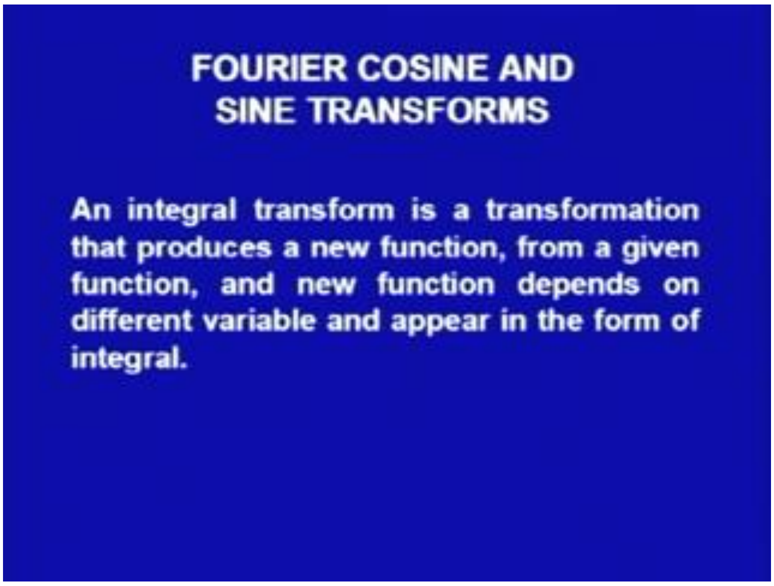

what is the transform, and integral transform is a transform, that produces a new function, from a given function, and new function, depends on the different variable and appear in the form of, integral. You had already done, one transform called the Laplace transform, here we are moving to this, Fourier cosine and sine transform.

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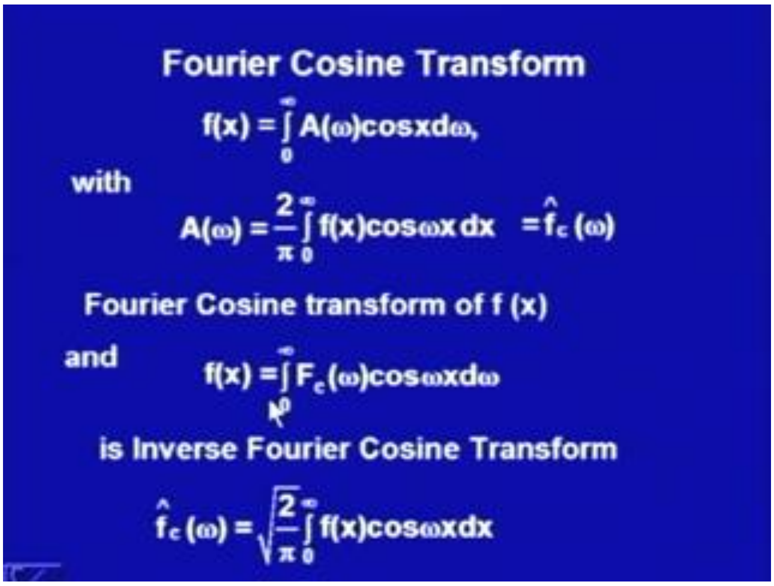

So, first let us see, Fourier cosine transform, we want the function should be represented as the integral, of something and that, should come out to be as another, function in another variable. So, f X we do know by Fourier integral, can be given as, 0 to infinity, integral 0 to infinity, A omega cos omega x d omega, when f X is even. Now, with A omega as, 2 upon pi, 0 to infinity f X cos omega x d x, this A omega.

Now you see, we have got, for this function f X, we have got one integral, which is again a function, in some other variable, this we are calling, Fourier cosine transform, A omega and moreover, what we are getting is, if I again integrate this, A omega with multiplication with this function cos omega x d omega, with respect to omega, we would get back my function f X. So, this a omega, this is called the Fourier cosine transform of f X, and f X as integral 0 to infinity, f c omega cos omega x d omega, this f c is not this capital F c, this is actually this f c hat omega is, the inverse Fourier cosine transform.

So, what we are getting is, in ((Refer Time: 39:08)) read this, you have already done this Laplace transform, in the Laplace transform you had not find out, what is the inverse Laplace transform, inverse Laplace transform, you had find out. That is, if the function, if a function is a Laplace transform, of this function, then that function you call the inverse Laplace transform, of the new function.

But here, we do have a formula that is given this function, this Fourier cosine transform, without moving that is, which from, which function we had obtained this function, we could get the, inverse Fourier cosine transform that is again a transform in the two definition of the transform. So, another variant is, here in transform I had used, that is cosine transform, I had used the coefficient, 2 upon pi this constant, and the inverse Fourier transform, I have not used any coefficient, but some variant is, that is both the places, they coefficient is being used as, square root 2 upon pi, that does not matter. In practical application, either you use this, constant at both the places that is in the cosine transform and the inverse cosine transform, or you use only 2 upon pi in the cosine transform, and do not use any coefficient in the inverse transform.

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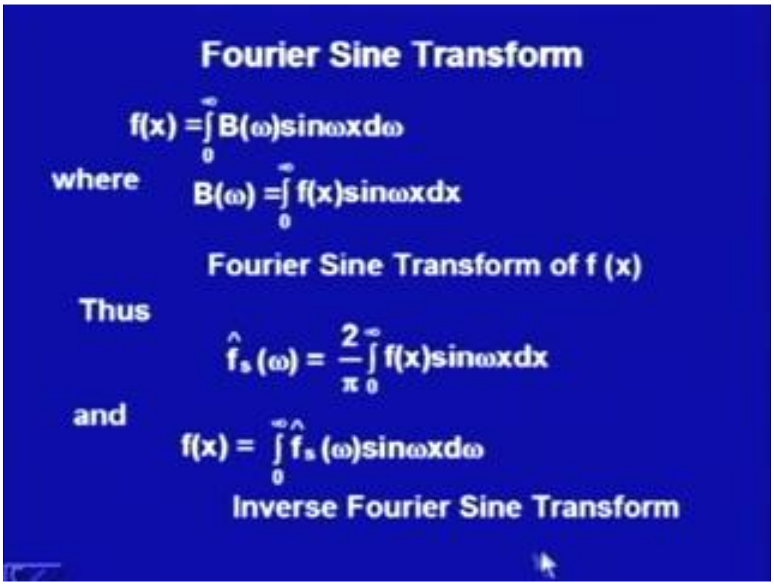

Similarly, Fourier sine transform, is been obtained using the Fourier sine integral 0 to infinity B omega sine omega x d omega, where d omega is nothing but integral 0 to infinity f X sine omega x d x. This is Fourier sine transform of f X, and we would denote it by, f s hat omega, and right 2 upon pi 0 to infinity f X sine omega x d x, and f X which is, now B omega, I am writing it as f s hat omega f s hat omega sine omega x d omega, this is inverse Fourier sine transform. So, one function we have got in omega, from the function n x, then function in omega I was having, and I got the function in x. So, both are the transform sine transform and inverse Fourier sine transform.

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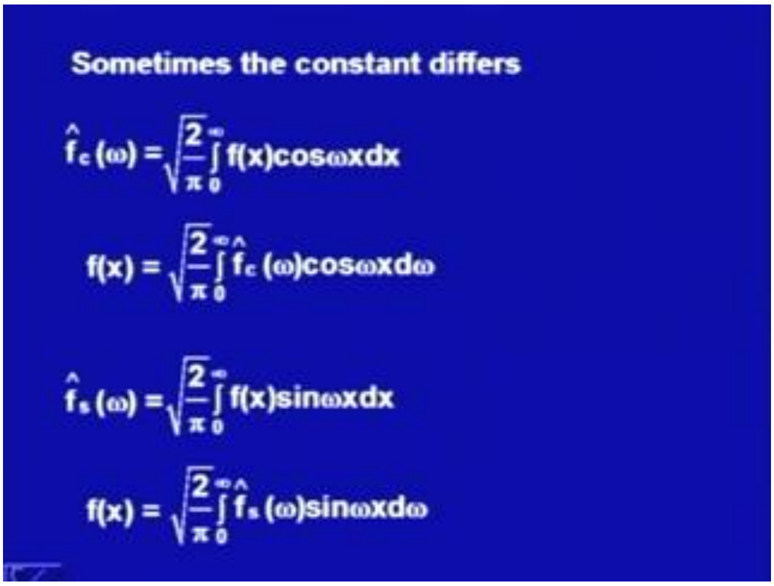

Let us see, some examples of these kinds as, I said is sometimes this constant differ, so let me just write, it out these constants, in nice definitions I had used, 2 upon pi in the transforms, and I have not used, any coefficient in the inverse transform. Sometimes both the places this, coefficient is square root 2 upon pi is being used, so that, similarity can be maintained.

So, we do get, f c hat omega, that is the Fourier cosine transform as, square root 2 upon pi, integral 10 to infinity f X cosine omega x d x and inverse Fourier cosine transform as, f X as square root 2 upon pi 0 to infinity f c hat omega cosine omega x d omega. Similarly, for sine transform, square root 2 upon pi 0 to infinity f X sine omega x d x and inverse Fourier transform, again the coefficient, square root 2 upon pi integral 0 to infinity f s hat omega sine omega x d omega.

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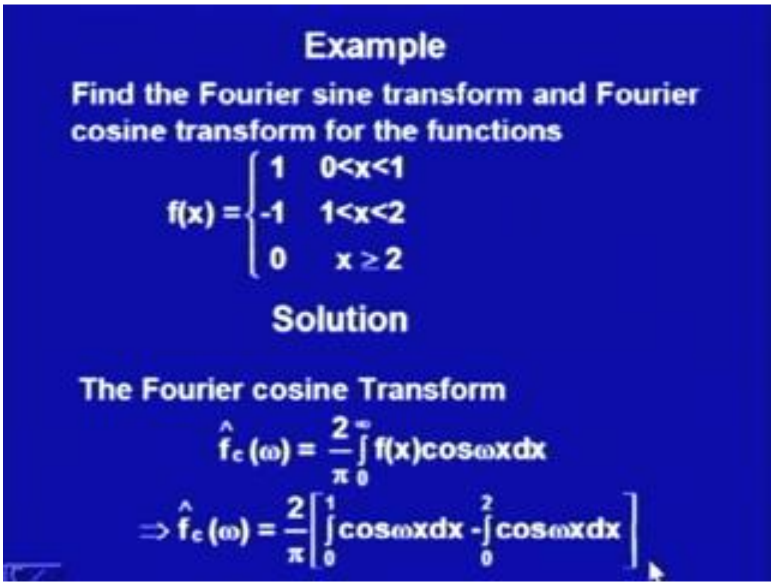

Come to the example; find the Fourier sine transform and Fourier cosine transform, for the function, which is1 in the interval 0 to1 minus1 in the interval1 to 2 and 0 for x greater than 2. So, now find out the transform, first Fourier cosine transform, I am using the formula, where the coefficient I am using as, 2 upon pi, so f c hat omega would be, 2 upon pi integral 0 to infinity f X cos omega x d x.

Now, f X is this function, which is for, which we have to calculate, this says is, f c omega f c hat omega would be, 2 upon pi integral 0 to1 the function is1. So one time, that is cosine omega x d x and in the interval1 to 2 it is minus1, so minus sign I have taken outside, it should be1 to 2, not 0 to 2, it should be1 to 2, 1 to2 cos omega x d x.

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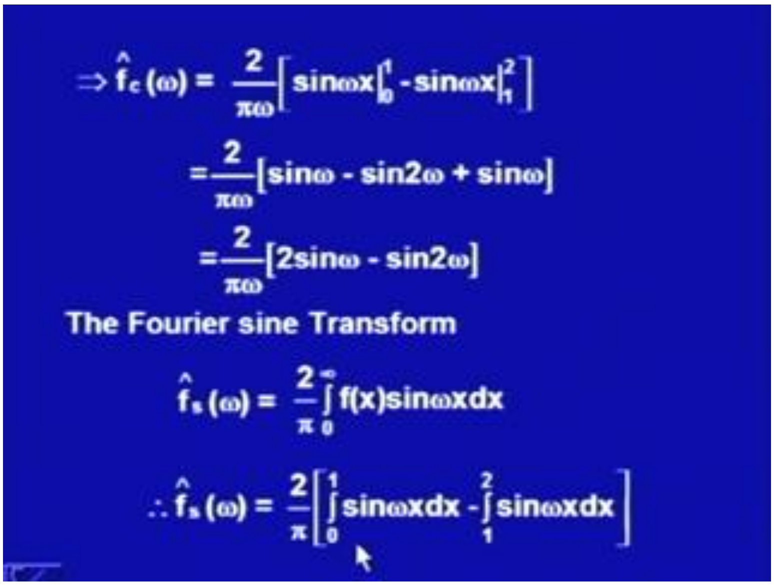

Now, evaluate these integrals, 2 upon pi omega sine omega, because integral of cos omega x is, sine omega x upon omega, evaluated 0 to1, minus sine omega x, evaluated 1 to 2, this omega, we have taken outside. We get, sine omega sine x is 0, that is 0, sine 2 omega plus sine omega, it says is 2 upon pi omega times, 2 sine omega minus sine 2 omega, this is what is my Fourier cosine transform, now find out Fourier sine transform, again I would use the coefficient 2 upon pi.

So 2 upon pi integral 0 to infinity f X sine omega x d x, so put the function is minus, function is 1, in 0 to1 and minus1, in1 to2. So, break this integral 0 to1, sine omega x d x minus integral1 to2 sine omega x d x it is 0 elsewhere, so we are assuming, that the integral of 0 function is always 0.

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$$\Rightarrow \hat{f}_s(\omega) = \frac{2}{\pi\omega}\left[-\cos\omega x\Big|_0^1 + \cos\omega x\Big|_1^2 \right]$$

$$= \frac{2}{\pi\omega}[1 - \cos\omega + \cos 2\omega - \cos\omega]$$

$$= \frac{2}{\pi\omega}[1 - 2\cos\omega + \cos 2\omega]$$

$$= \frac{2}{\pi\omega}\left[1 - 2\cos\omega + 2\cos^2\omega - 1\right]$$

$$= \frac{4\cos\omega}{\pi\omega}(\cos\omega - 1)$$

Evaluate this integral, integral of sine omega x is minus cos omega x upon omega, both the places am having the integrant as sine omega x, so omega, just write it, write outside, minus cosine omega x, evaluate it 0 to1 plus sine omega x, evaluated from1 to 2. Now, evaluate it, 2 upon pi omega cos first it is minus sign, so first I am taking 0, cos 0 is1, at x is equal to1, it is minus cos omega, then cos 2 omega minus cosine omega. So, what we have got finally, 2 upon pi omega, 1 minus,2 cos omega plus cos 2 omega, this is what is our, Fourier sine transform of the given function.

We can simplify little bit more, cos 2 omega, we can write as 2 cosine square omega minus1, so we get1 minus 2 cos omega plus 2, cos square omega minus1, we could write it as, 4 cos omega upon pi omega, cosine omega minus1, simplified it that is all. So, we are getting this as the, Fourier sine transform of the, given function, so today we had, learn how to represent, any function, which is not periodic, as Fourier integral, or we said is, that is representation of a function, as a Fourier integral, then we had learn, that is depending upon the property of the function, that is the function is odd or even.

We can find it out whether, it is or we can represent it, as a Fourier sine integral or Fourier cosine integral, and we can find out the transform, then we had came to the transform, and we had learn the, Fourier sine transform and cosine transform. What is the use of this representation of function as integral or these transforms, for integrals we had used it, for evaluation of certain integrals, which as such we were not able to evaluate.

Fourier transforms, we had started with cosine and sine transform, and next lecture we will continue with it, and we will see some properties and then we will move to the, Fourier transforms. So, that is all for today's lecture.

Thank you