Mathematics - III
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Lecture - 13
Convergence of the Fourier Series

Welcome to the lecture series on differential equations for under graduate students, today's lecture is on Convergence and sum of the Fourier Series. Till now in Fourier series, we had learn that for a periodic function, we can write a Fourier series for that function, we had also learn if the function is defined only on a finite interval on the real line, we can still extend this and write the either cosine series or the sine series and we called it the half range expansions.

In the plots of those Fourier series and the functions, we had seen that, as with the partial sums we are including more and more terms, we are getting more closer to the function with the series. These things are, how we are seeing is that is why they are matching and why they are not matching, here is that now we are going to discuss this mathematical property of this Fourier series, which we are calling convergence and the sum of the Fourier series.

For the convergence, as it is very easy to find out the Fourier series, given any periodic function, we can find out the coefficients using the Euler's formula or we can extend it to the even extension and, or odd extension and get the sine and cosine series, again using the Euler's formula I found easy. But, proving that the convergence or that why it is matching that is a little bit more difficult and beyond the scope of this first level course. So, I will just quote here one result about this convergence of the series, which is due to ((Refer Time: 02:13)) this result says...

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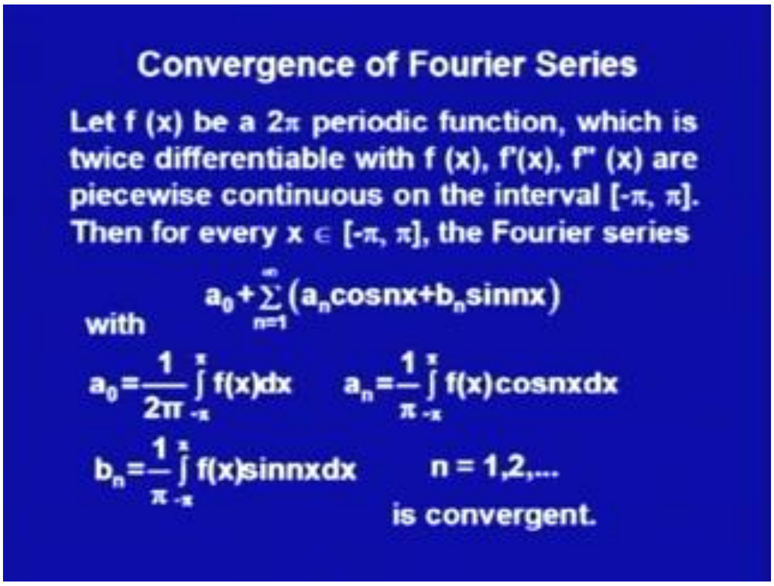

Let, f x be a 2 pi periodic function, which is twice differentiable with f x, f dash x and f double dash x are piecewise continuous on the interval, minus pi to plus pi. Then for every x, in the interval minus pi to the plus pi, the Fourier series, a naught plus summation n is running from one to infinity a n cos n x plus b n sin n x, with coefficients a naught as one upon 2 pi minus pi to plus pi f x d x, an as 1 upon pi minus pi to plus pi f x cos n x d x and b n as 1 upon pi minus pi to plus pi f x sin n x d x, for all n 1 2 3 and so on, is convergent.

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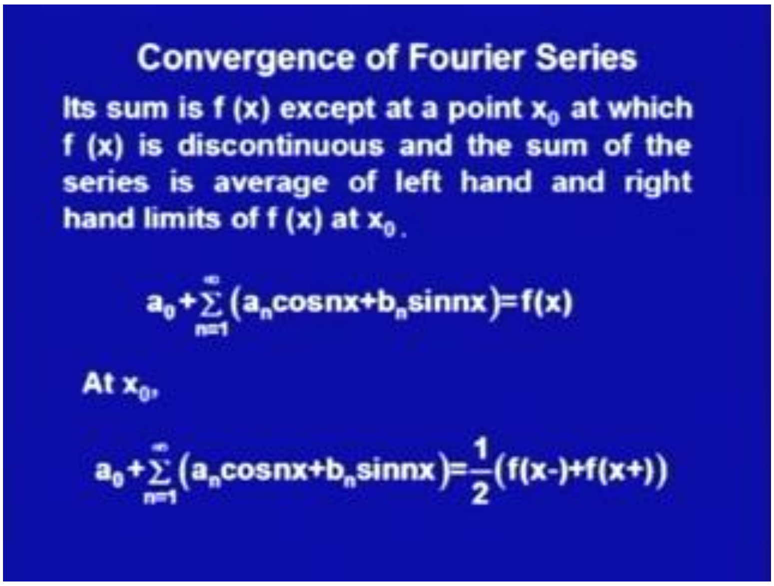

Moreover, it is sum is f x except at a point x naught at which f x is discontinuous, because we had assumed that f x is piecewise continuous and the sum of the series is average of left hand and right hand limit of f x at x naught. That is, the series a naught plus summation n is running from 1 to infinity, an cosine n x plus b n sin n x is equal to, would be seen as f x, when at all those points x where the function is continuous.

And at x naught where it is discontinuous, at that point it would be equal to half of f x minus f x naught minus plus f x naught plus that is, the left hand limit at x naught and right hand limit at x naught, you take the average and it would be equal to that. Proof of course, as I said we will not be able to do here, but we can see this result from our examples which we have done for the Fourier series. So, let us verify these results with our examples.

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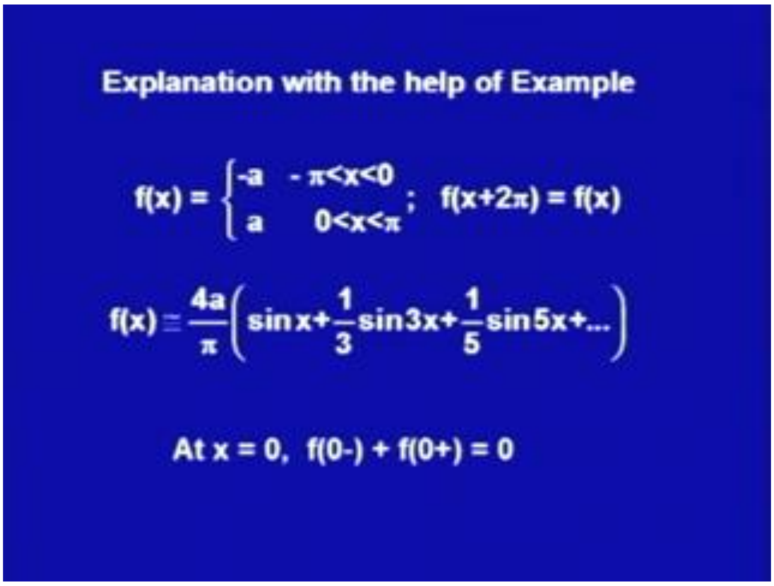

Now, let us see, we have done one example where our function was minus a in the interval minus pi to 0 and plus a in the interval 0 to pi and the function was periodic with the period 2 pi that is f x plus 2 pi is same as f x for all x, do you remember that, in the last lectures we had find out, the Fourier series of this as 4 a upon pi, sin x plus 1 by 3 sin 3 x plus 1 by 5 sine 5 x and so on.

We had seen that, at x is equal to 0 if I just see here in this particular example, the function is discontinuous at x is equal to 0, from the left hand, if I try to reach to the a 0, I would get the value as minus a, while from the right hand side, if I try to reach to the point 0, I would get the value as plus a, which is not matching with the value of the, and the function is not defined at 0. So, this function is discontinuous at 0. Now, what is the average at this point, that is f 0 minus, plus f 0 plus, a plus minus a, that would be 0. Now, see the graph which we have drawn that of the partial sums.

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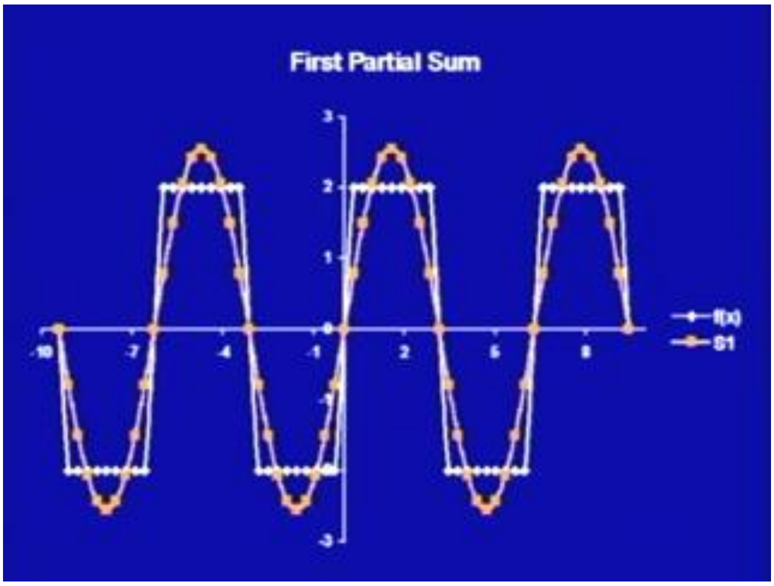

This was the first partial sum, that is where we have taken only 4 a upon pi sin x, you see here, that this is the function of, in the minus pi to plus pi, minus pi to 0 it is negative, of minus a, plus 0 to plus pi it is, plus a. And the Fourier series, if you see is, that is, this is how we are going on now, at the point 0. Let us see here, of course, the same thing because it is 2 pi periodic, so you will be getting is, minus pi plus pi and all those things as such, see here at 0, average is 0.

Now, this Fourier series we are saying is, that is at the point 0 we are getting it 0, while as, at a points here we are seeing is at some points we are matching, now you see this is my first partial sum, it is not the whole Fourier series. Let us move to the second partial sum, what we are seeing is that, as n is increasing, now this is that n is equal to 1 we are having so much deviation from the actual values, let us move to the second partial sum.

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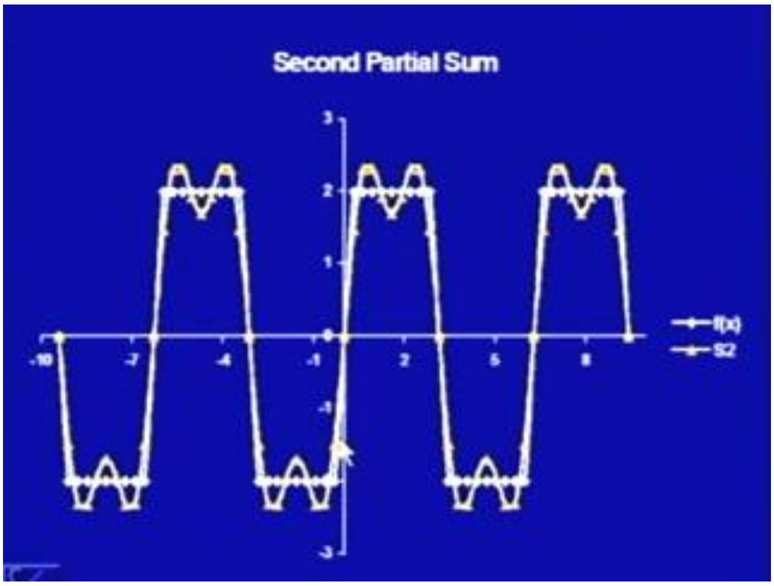

Now you see, we have reached here, now the sum is, let us concentrate with minus pi to plus pi, we are here, we are seeing is that in between these points have been less, from the previous one, you see in the previous slide ((Refer Time: 07:21)) we are having so many points in between. Now, we are having less points in between and at 0, it is 0, the function is not acting over there because here we are reaching to 0, that minus a , when we are reaching to, from this side to 0 it is plus a, plus 1, we are matching it certain more points over there. Now, if I go with the third partial sum, what will happen ((Refer Time: 07:49)).

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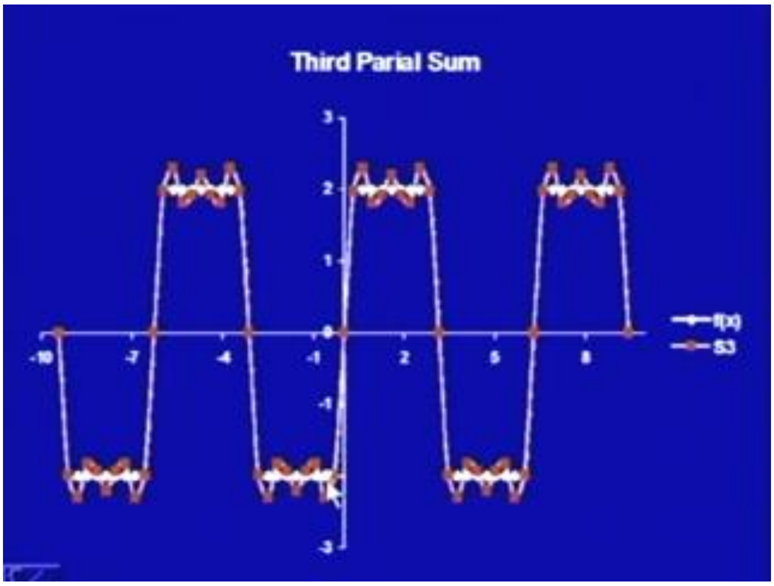

This is your third partial sum, now we are, at more points where we are, making that we are equal to the function, at 0 we are all the time matching. Now, you see in the third partial sum, the points in between all the points they have been disappeared because the function is not anywhere in between this, at 0 we have not defining the function. Then, if I move to the.

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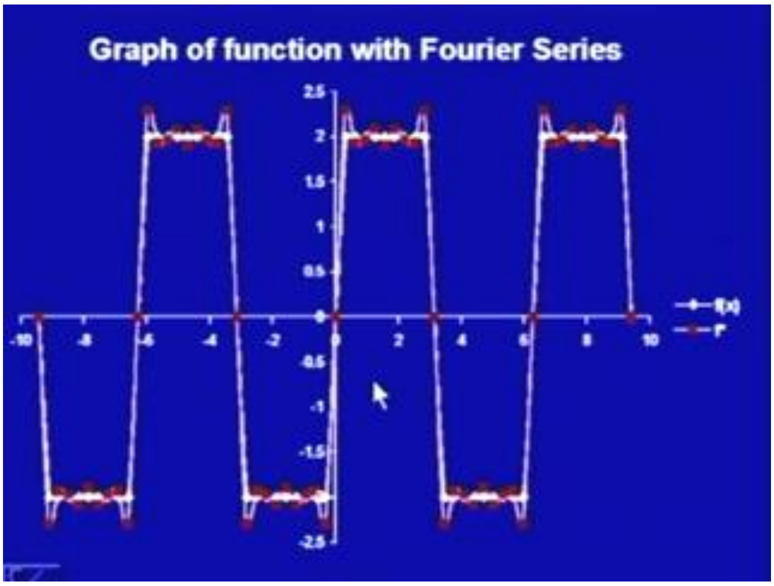

Some, five or six terms here I have taken because I cannot try it for very large number of this thing, some six or seven terms has been taken over here, now you see, we are matching at many points. So, what we are saying is, the function is continuous in the interval 0 to pi, the function is continuous in the interval minus pi to 0, the sum of the Fourier series, that is the partial sums which I am talking over here they are matching with the points.

At 0, where it is discontinuous, the value that the Fourier series, we having this is the red points are telling as the point on the Fourier series and this whites are the function values, this is 0. So, we are seeing is that, as n is increasing at a continuous points, the sum of the Fourier series is equal to the I ((Refer Time: 09:21)). Here, we could not say as actually equal to, but it is reaching towards the equal to the value of the function, while is, when it is continuous it is coming at the average of these two limit points.

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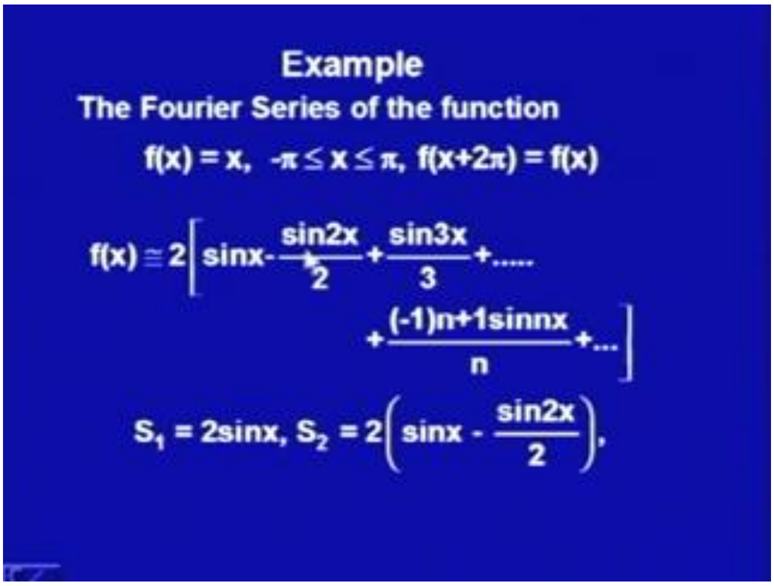

Another example, you remember that we have done this example also, minus pi to plus pi the function is x, again periodic with period 2 pi, f x plus 2 pi is equal to f 2 x, if you do remember the Fourier series of this function was two times sin x minus sin 2 x by 2 plus sine 3 x by 3 and so on, nth term is minus 1 to the power n plus 1 sin n x upon n and so on. Now, let us see it is partial sum and the graph of the function, so first term 2 sin x, second term 2 sin x minus sin 2 x by 2 and so on.

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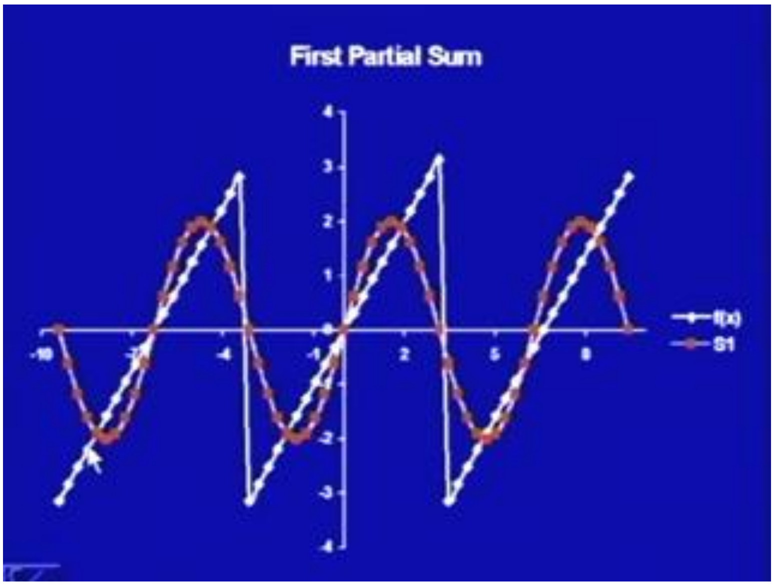

Now, with first partial sum, this is what is our function, from minus pi to plus pi it is x, then it is periodic. So, when we are reaching from minus pi to plus pi this is the function, so when I take it again this one, then it should start from here, this point which we are having is this line, this line I had purposefully put it, so that you could see is, that is here is this function is not in between here, this is not the function value, there is no point on this line. You see all the white points we are having over here, this just I am this trying to show you that is how you see now in the Fourier series, this first partial sum.

We are moving towards now, at one point only I think is we have, matching with the function and then we are not matching with the function, we are matching with the function at the two points say maximum and at 0, the function is discontinuous at the limit points that is, at minus pi to plus pi. So, when this Fourier series is changing, we are finding it out from here to here, what is the average, average is, plus pi plus minus pi that is 0.

You say is, that is at this end points the value is 0, now here is one point, we are getting some points in between also, this is the Fourier series is not going to up to here, it is moving from here because after that it is changing so, because when we draw the graph it would just take a curve like this one, this is first partial sum. Let us take the second partial sum that is, first two terms.

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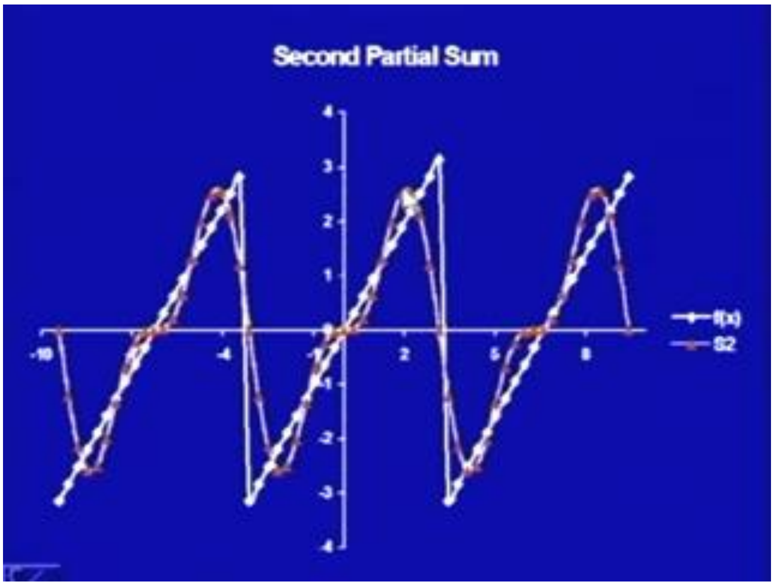

Now you see, we have matched here more than those two three points and the first one we were matching only one point here, one point here, one point here. Now you see, we have matched a little bit more points over there, you see is, that is more overlapping with the function, again the points of discontinuity or this end points. So, here also, now you see is, that is at last one we are having more points over here, now we are having less points over here and similarly over here, again. Because, when we are drawing this function, so its curving also we are not again reaching towards this ((Refer Time: 12:51)). Now, come to the third partial sum.

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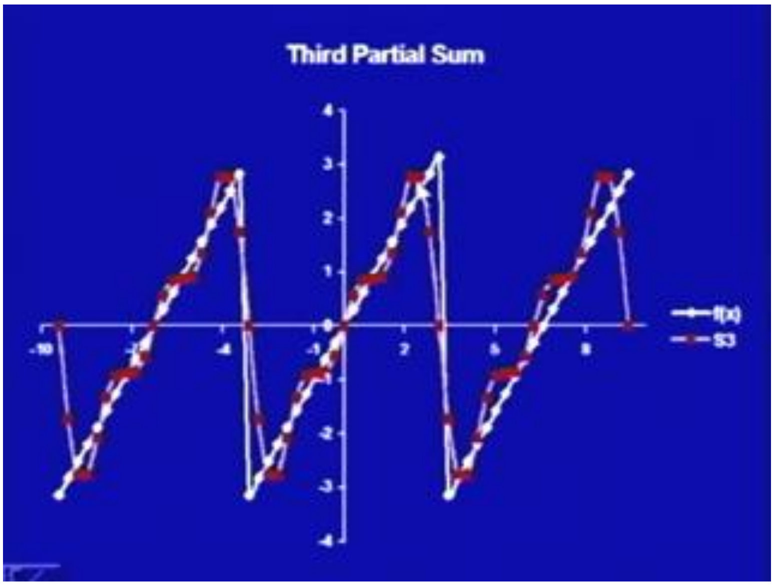

Now you see, we are matching little bit more, the function is overlapping a quite nicely over here, the only thing is that is we are having is, that you just come again discontinuity points, here the behaviour is different. We are not still moving till the end because of this discontinuity and at this point it is making this one and because of that only, because we are just talking about first three terms only. So, we are drifting from here itself.

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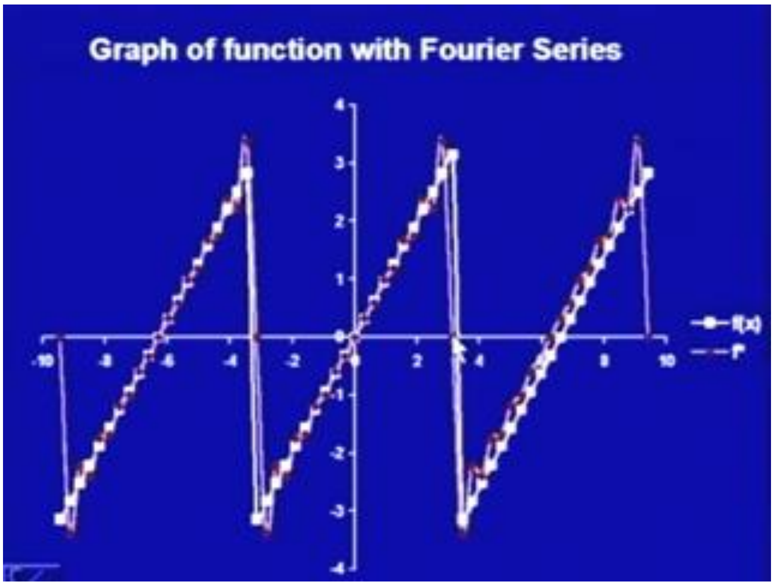

Now, let us move to the graph with some eight or nine terms, it is not more than 10, but still you can see, how we are overlapping with the function, this white beads and red beads. You just see is, that is how at the points where the function is continuous, our Fourier series is matching with this, some of the Fourier series that is the partial sum, that is because here it is not complete sum, it is partial sum.

We are matching with the sum, now you see, we are reaching towards end points as well and now in this between I do have only one point of the Fourier series which says is this average of the right hand limit and the left hand limit that point is 0, we are not having any other one, that is what we say is that is the at the point of continuous, all those points the Fourier series sum would be equal to the value of the function but at the point of discontinuity, it would be equal to the average of the limit points. Let us see one more example because those two examples were of the sin series.

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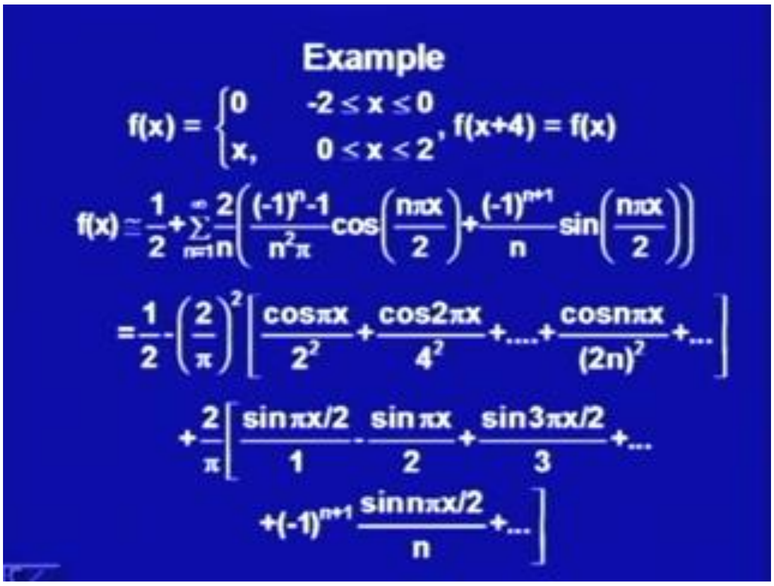

Now, here I do have this example 0 from minus 2 to 0 and x from 0 to 2, now you see, this is not periodic with the 2 pi, this is periodic with minus l to plus l and my l is here, 2 l is 4 periodic with f x plus 4 is equal to f x, this function, this examples also we have done in our last lectures. And we do know that from there, we have drawn that the Fourier series is half plus summation n is running from 1 to infinity 2 by n, minus 1 to the power n minus 1 upon n square pi cos n pi x by 2 Plus minus 1 to the power n plus 1 sin, sin n pi x by 2 or if I write it more elaborately.

I would get half minus 2 upon pi whole square outside and cos pi x upon 2 square, cos 2 pi x upon 4 square so on, cos n pi x upon 2 n square and then the sin terms 2 upon pi times sin pi x by 2 upon 1 minus sin pi x by 2 plus sine 3 pi x by 2 and so on. The nth term over in this b n series, would be minus 1 to the power n plus 1 sin n x pi by 2 over n. Now, here what we were the partial sum, when we were talking about only sin series or cosine series, we are taking only first term, second term and so on.

Now, here when I would be defining the partial sum or be talking about the n is equal to 1, n is equal to 1 means is, that is I would be taking both the terms, this cosine term as well as the sin term. So, my first partial sum would be, half minus 2 upon pi whole square cos pi x upon 2 square plus 2 upon pi sin pi x by 2 upon 1 and the second partial sum, taking up to this point and this point, it is not the else we have to take first all these points and then go all these points.

We are just moving with n, n means that is a n cos n pi x and b n sin n pi x by n. So, and this is, a naught, so half this is there

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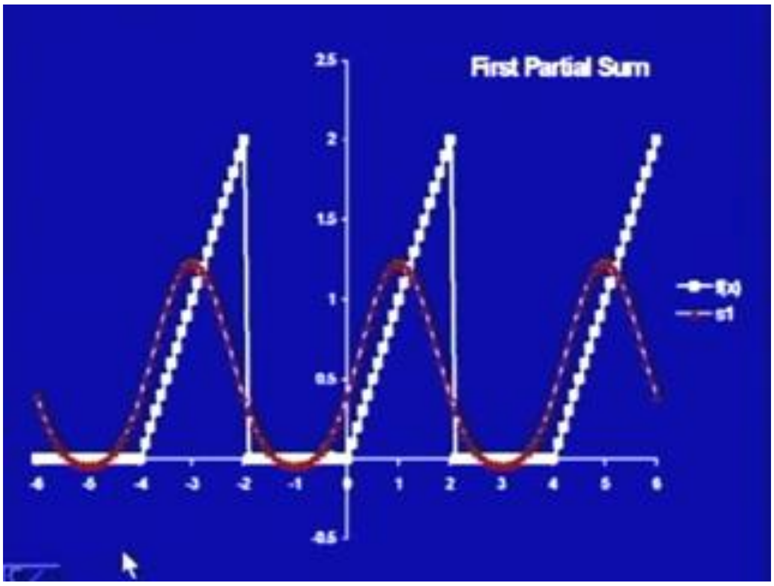

Let us see the graph, so the first partial sum, now the function is 0 from minus 2 to 0 and from 0 to 2, this is x. So, these white beads are showing you the ((Refer Time: 17:23)) points, at certain discrete points, this is the first partial sum, we do have that the function is continuous, but still it takes a step over here we are, this is first partial sum. We are at drifting away at certain points only over, here we are matching on the down side and then at certain points here we are matching. And because at end points, again at the limit points the function is discontinuous, so we are coming, drifting away from there and we are not reaching towards the end of this one.

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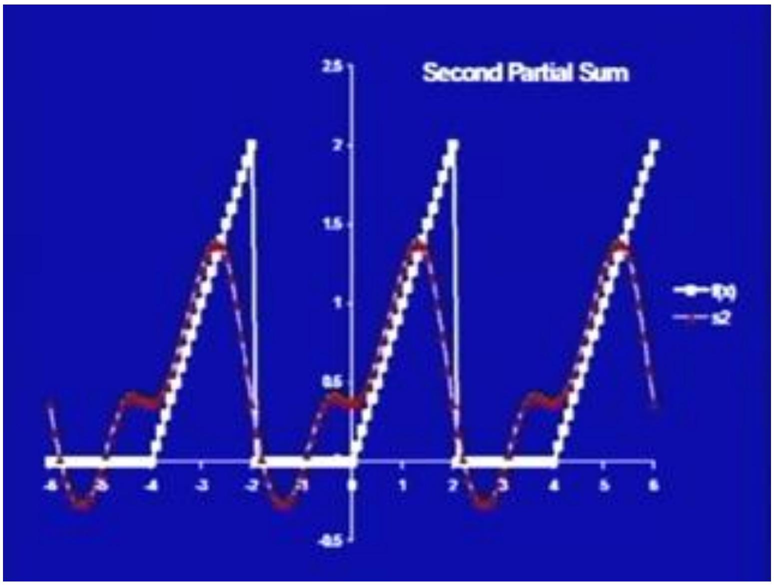

Let us move to the second partial sum little bit more better presentation, but you see, here we have drifted away and then we have drifted away over here also and then we are matching towards these points. Now you see here the points we are finding out that is continuous all right, we are coming over here, while at here we are coming this is the point of discontinuity. Now, you see it is many points in between, at a point of discontinuity it should have been the point should have been somewhere in the between that is, from 2 to 0, I should have got some where at a 1 some, somewhere the point should have been. But this is only the second partial sum, we are drifting it one, let us see some more.

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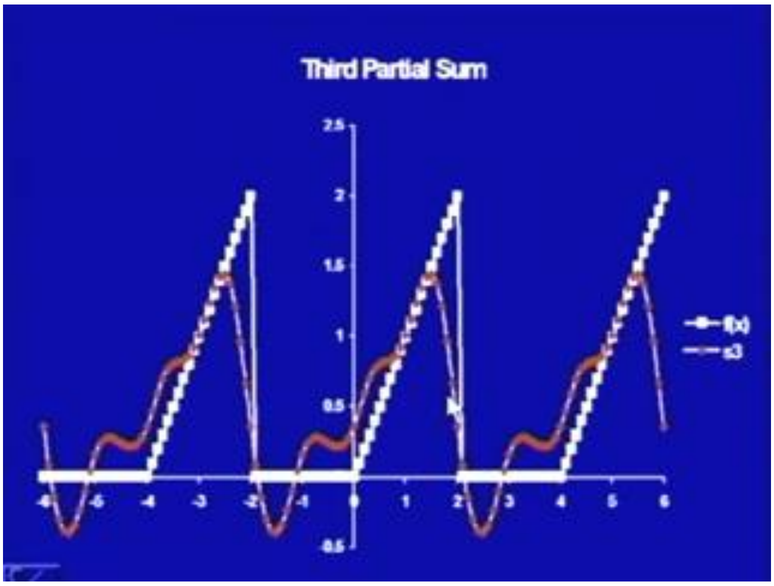

Third partial sum, we are again coming up, but because of your finding it out the function is, as we are having here is a discontinuous at a end points and here in the between also that is, we are finding it out that it is 0 and from here it is actually looking as continuous, but at this end point it becomes discontinuous. You find it out that, even when we were talking about only sin series by the time of the third partial sum we were very near to the convergence, we have able to come that is a, the my function is matching with the Fourier series. But here you are seeing is, that is even at the third partial sum, we are not having that nice kind of behaviour of the Fourier series with respect to the function. Let us move to the fourth one.

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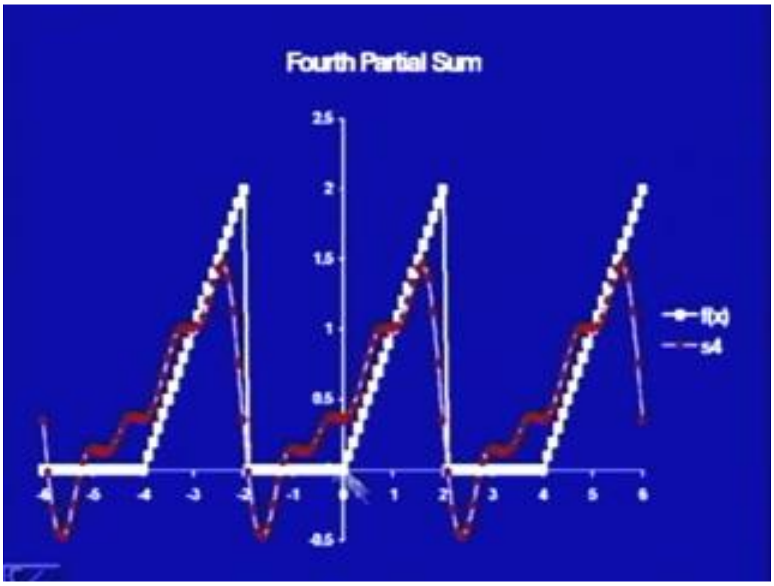

Again, we are drifting this side also to my, as we are making is that, here it is coming up very nicely. Here it is coming closer but we are drifting it away, but it is simply saying is that, here with the smaller number of n s, we will not be able to get that much nice behaviour of the Fourier series with respect to the function. Or here for, showing that the Fourier series is matching we require, more n or this convergence requires a large number of n. I think is, we do have here till fourth one only.

So, you go till some twenty terms or something, only then we would be able to see this behaviour, but theoretically it says is, that for large n, as n approaches to infinity. You would get that the sum of the Fourier series that all continuous points, would be equal to the function and at the point of discontinuity, it should be with the average of the two limit points. So, with certain simple functions we could see it very nicely, but here it is example I have taken that you could see is not happening very orderly also, that is still fourth partial sum we are not near to the, our result.

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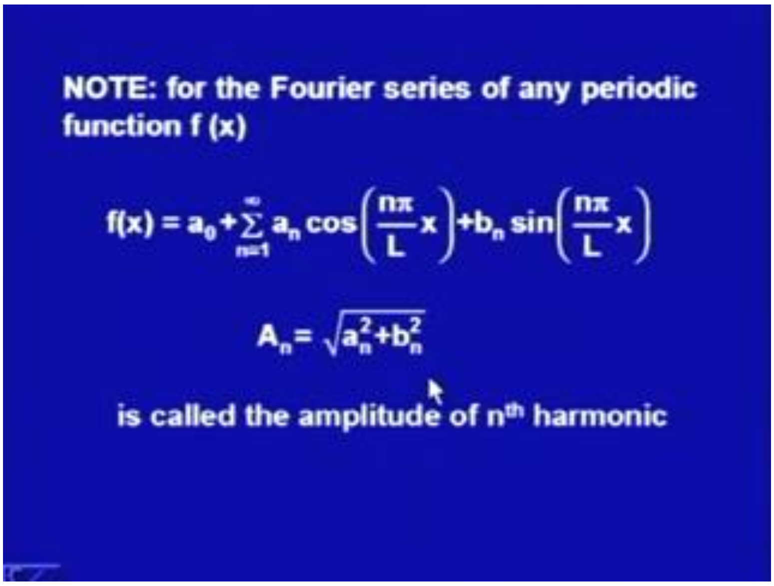

Now, one more terminology for this thing, that for the Fourier series of any periodic function f x, which we are writing as a naught summation n is running from 1 to infinity an cosine n pi x over l plus b n sin n pi x over l. This term, capital A n which is nothing but the square root of the sum of a squares of the coefficients a n and b n. This is called the amplitude of the nth harmonic, that what it says is, a in all those things what we have got is, that is the nth term, nth term means not a naught, nth term means is that is, if I take n is equal to 3, a3 cos3 pi x by l plus b 3 sin 3 pi by l x.

This third term, this would be called third harmonic. Because this is a periodic series we do know that, cosine function and sine function both are periodic. So, they would have some loop kind of things and that is called the harmonic and when this is called the amplitude of the harmonic.

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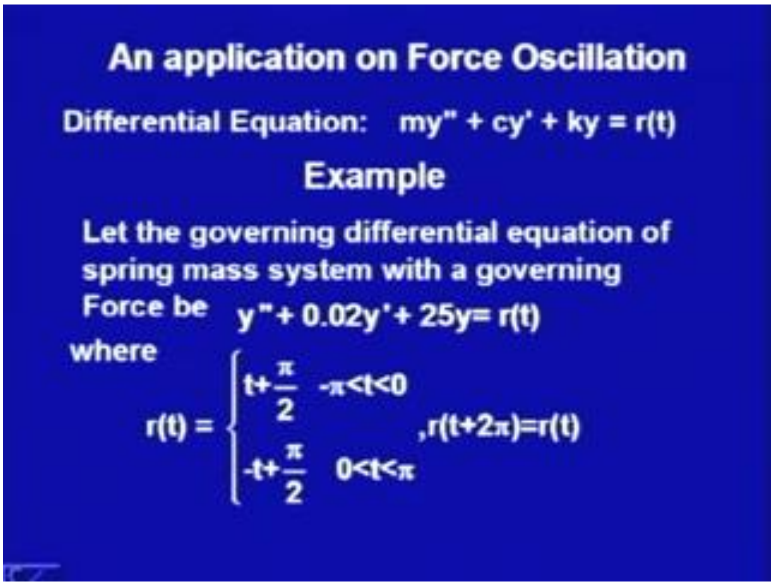

Now, let us see an application to the force oscillation, we have done such a Fourier series that is, where we are applying it in the differential equation. we have remembering that is, what we mean by the force oscillation; that means, a differential equation is governing my double dash plus c y dash plus k y is equal to r t, where m is the mass, c is the damping coefficient, k is the coefficient of the spring and r t is that your input function.

You remember, that we have already solve this kind of equations and there we have done this equation where this r t was of some special form. Let us, just try one example which we have done, that spring mass system with governing force as y double dash plus 0.02 y dash plus 25 y is equal to r t, where your damping coefficient is point 0.02, a spring constant is 25 and the mass is a 1 unit. This r t, if you do remember, we have, now in this example, now my r t is this, input function is t plus pi by 2 in a interval minus pi to 0 and minus t plus pi by 2 in the interval 0 to pi.

Moreover, this function is periodic, that is r t plus 2 pi is r t for all t, if you do remember we have done, that when r t is of the form that it is cos t or sin t or it is a some function, e to the power t x kind of things. Now, how do we find out the solution of this equation, let us just try, see what is my function r t?

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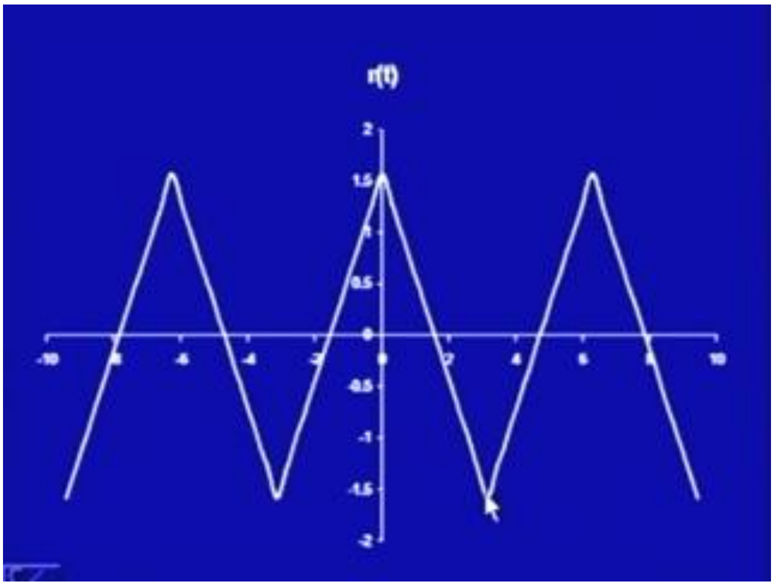

This is, t plus minus pi by 2 plus t 2 this one. So, we are getting is at 0 we are getting is plus pi by 2 at minus pi by 2 would be getting it as minus pi by 2 again at plus pi we would be getting as a minus pi by 2. If you do remember, we have done one function t.

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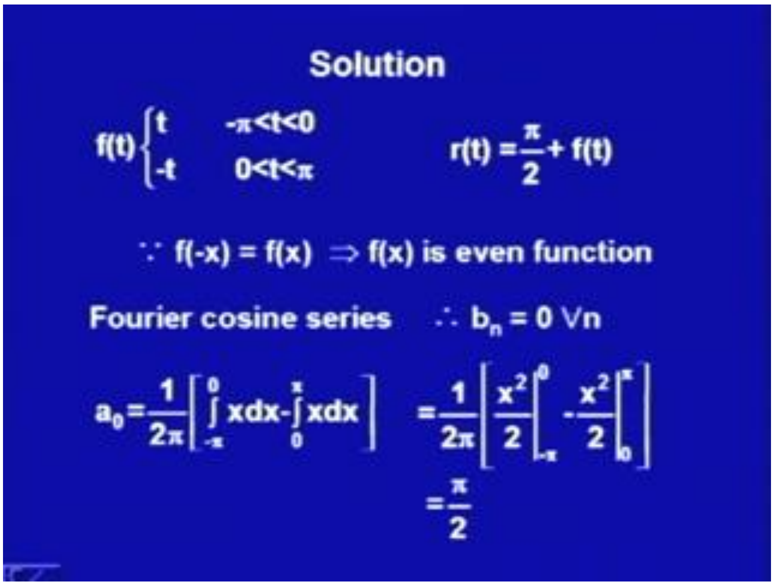

In the interval minus pi to 0 and minus t in the interval 0 to pi and my r t is nothing but, pi by 2 plus f t. Now, you remember that we have done one result in the Fourier series which says is that, if we do have the two functions, periodic functions for which the Fourier series can be written, then the Fourier series of the sum function is nothing but the sum of those Fourier series. So, now here we see is, this is the constant one, this kind of examples we have done, so we just want the Fourier series for this function, f t.

Let us find it out, see this function f t, we are getting is f of minus t would be same as minus of, would be same as f of t, that says is that, it is an even function. So, we would get only the cosine series; that means, all the coefficients b n would be 0, what will be now a naught, 1 upon 2 pi integral minus pi to plus pi f x d x, f x here is or this, if you replace this t by x is a x in the minus pi to 0 and minus x in 0 to pi. So, just break in to the two parts, get this integral solved, here it would be x square by 2, this integral and you just put the values, minus pi to 0 and 0 to pi, you get the answer as pi by 2. So, this is your a naught.

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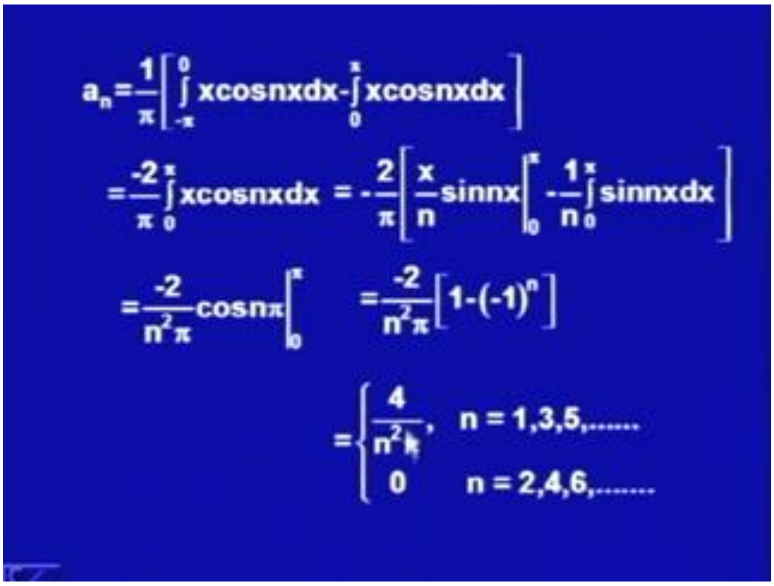

a n, 1 upon pi, minus pi to plus pi f x cos n x d x, f x again we are breaking into two parts minus pi to 0 it is x and 0 to pi it is minus x, this is x cos n x. So, we will just go with this integration by parts, the first integral would be minus 2 upon pi because the function is now x is odd and this cosine x is even, we are getting is that, minus 2 upon pi 0 to pi x cos n x d x, which is minus 2 upon pi x by n that is again integration by part sin n x evaluated 0 to pi minus 1 upon n, 0 to pi sin n x d x, which is minus 2 upon n square pi cos n x, 0 to pi.

Evaluation and we get finally that, minus 2 upon n square pi, 1 plus, minus 1 to the power n which we would be, that is for n 1 3 and 5, it would be 4 upon n square pi for n is equal to 2 4 6 it would be 0, now that is what the Fourier series.

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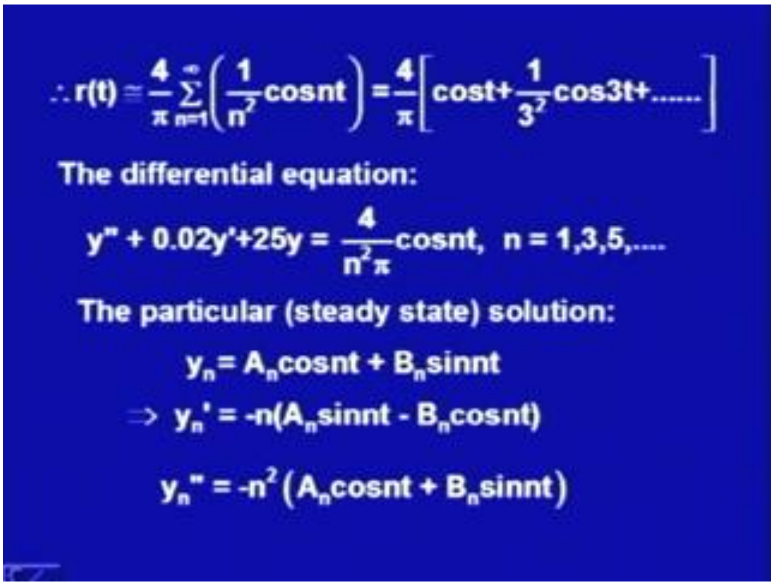

We have got, r t as 4 upon pi, summation n is running from 1 to infinity, 1 upon n square cos n t, which is 4 upon pi cos t plus 1 upon 3 square cos 3 t and so on. Now, solving the differential equation, you come to this, modify our differential equation, y double dash plus 0.02 y dash plus 25 y is equal to 4 upon n square pi cos n t, that is nth term we have written from here, our original differential equation was with r t, rather than solving it with r t.

Now, we are solving it with this nth term, that says is since what we had got just now that is from the convergence of the Fourier series, that the value of the function at continuous points is same as the sum of this series. So, now we are using that idea, where just solving at nth point, that is nth term, now if I solve it, now this is of a special form, we do know how to get the particular solution or that if you do remember we have called it a steady state solution.

The particular solution will be of the form, A n cos n t plus B n sin n t, how to get this coefficients A n B n, we just substitute y n dash y n double dash in this given equation and equate it to the right hand side and find it out, so usual method, y n dash would be minus n times a n sin n t minus b n cos n t and y n double dash would be minus n square, a n cos n t plus b n sin n t.

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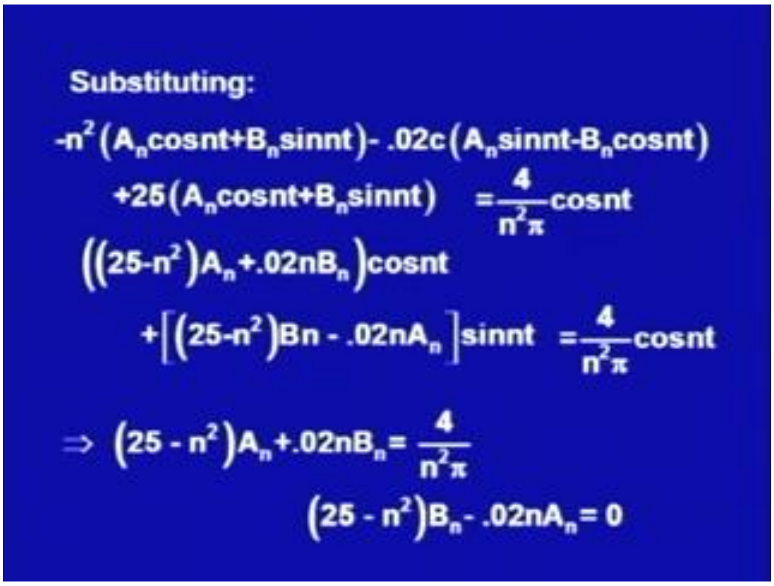

Substitute it in given differential equation, we would get minus n square A n cos n t plus B n sin n t minus 0.02 c times, that so c is actually 0.02 A n sin n t minus B n cos n t plus 25 A n cos n t plus b n sin n t, this is should be equal to 4 upon n square pi cos n t. Now, simplify the things, we do get, 25 minus n square times a n plus 0.02 n b n cos n t plus 25 minus n square B n minus 0.02 n A n sin n, is same as, it should be equal to 4 upon n square pi cos n t. Now, equate the coefficients of cosine n t and sin n t on both the sides, we would get the two equations, one is 25 minus n square A n plus 0.02 n B n is equal to 4 upon n square pi, the coefficient of cosine n t have been equated. Second term is that, coefficient of sin n t, 25 minus n square b n minus 0.02 n A n, right hand side it is 0. So, it should be 0, now these are 2 linear equations solve them.

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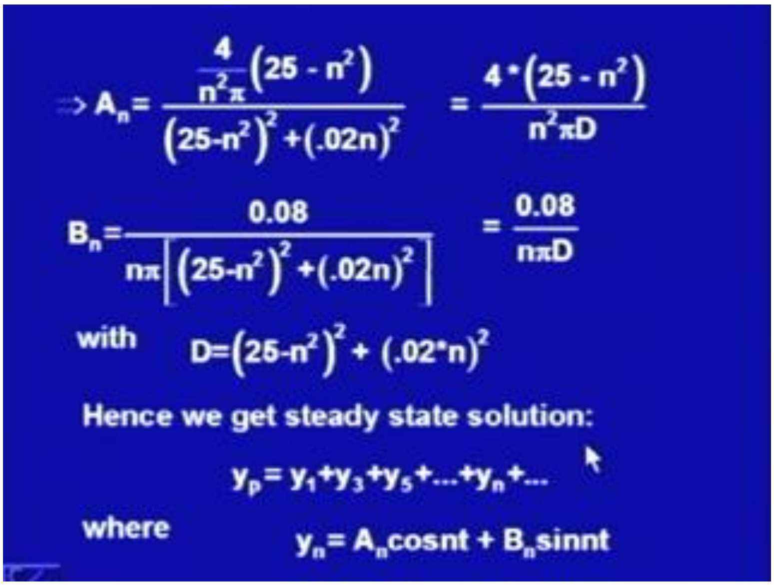

We get the solution as, A n as 4 upon n square pi, 25 minus n square upon 25 minus n square whole square plus 0.02 n whole square and B n as 0.08 upon n pi into 25 minus n square whole square plus 0.02 n square. Now, we see the denominator in the both the A n and B n same, lets substitute it as D, then what we could write A n as, 4 into 25 minus n square upon n square pi D and B n as 0.08 upon n pi D.

So, we get the steady state solution for this modified differential equation as, or actual equation that is for this modified one we would get is that the y p is, we are getting where this coefficient is A n and B n are been given by these values and the solution would be, A n cos n t and B n sin n t. Now, since we are talking about the linear equations and the linear equations again we do know that the solution of linear equations if their two solutions for linear equations then, y 1 plus y 2 were also be a solution of the linear equation.

Now, what we are having is that, we said is that rather than that function f r t, we had substituted the nth term of the Fourier series, corresponding Fourier series into the term. So, our what we are getting is that, Fourier series we are getting the terms with 1 3 5 and so on, all the even terms were 0. So, now, for our original differential equation, the particular solution or what is called the steady state solution, I could add up as y 1 plus y 3 plus y 5 plus y n, where my y n is nothing but, A n cosine n t plus B n sin n t, with A n as this and B n as this.

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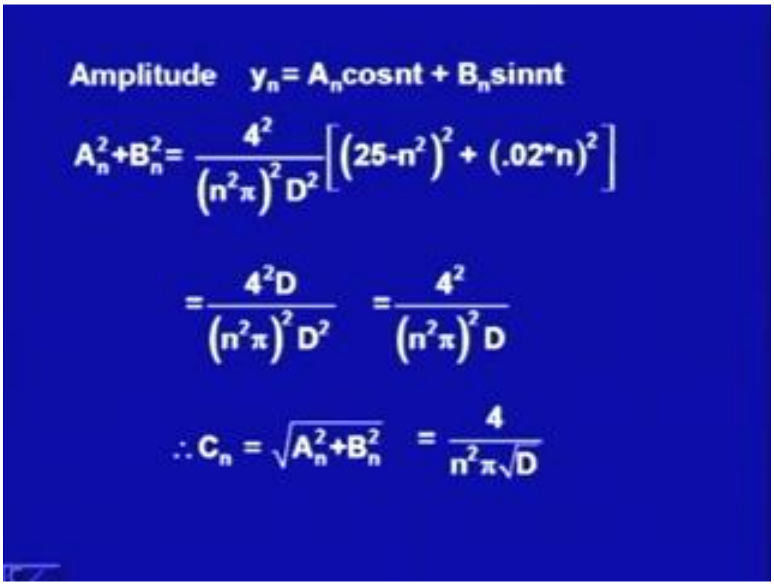

So, we will get the amplitude as, for the y n that is the solution part for this differential, modified differential equation, the y n is A n cos n t plus B n sin n t, amplitude for this would be, A n square plus B n square, if I do calculate you have seen that is in the last one, that is A n was 4 upon n pi, n square pi D and then some terms for 25 minus n square, whole square and so on, and then I had added up the second B n square, what we do get if you see, this term, this is again your D.

So, what we have got, 4 square D upon n square pi square D square, that could be just simplified as 4 square upon n square pi square, whole square D. So, what will be the amplitude, square root of A n square plus B n square that is C n and now we are calling this amplitude as C n, for this nth differential equation you could call, as a square root of A n square plus B n square that would be 4 upon n square pi, square root D. Now, put the values for different values of n, so that, we could see that what would be the amplitude of the different solutions.

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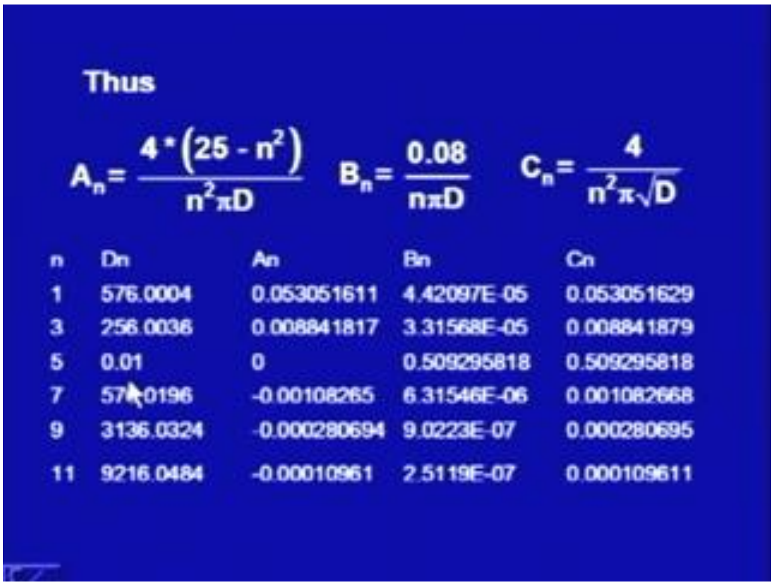

A n is 4 into 25 minus n square upon n square pi D, B n is 0.08 upon n pi D and C n we have got as 4 upon n square pi root D. Let us evaluate it for certain values of n, so here, I have done some calculations, I am showing you here, you can do it this calculations by yourself as well. Let us see here, If n is 1 that is if I am having only the first term, then D n is this much, A n is this much, B n is this much and accordingly C n is this value, just go and see in this table.

We are getting our D n, that is D n means is the D corresponding to this nth partial one, that is we are getting 576, 256 then 0.01 then again we are coming from large values then again we are going to the large value, what we are finding it out, at the n is equal to 5, this D n is very small, making it very small here because of this approximation what I am getting is, that A n is approximated as 0, B n is some value, so C n is contribution of the, in the C n is only of B n, that you see is that C n is same as B n.

But of course, they would be little bit contribution, it depends upon that is how much calculation you could do, but whereas, what a final conclusions you can do. Now, what it says is, that in the solution of the differential equation or the steady state solution, which are we were having y p as y 1 plus y 2 plus y 3 and so on, this term y 5 that is ruling because of this one. Let us see that is what, I am saying is and how it is working.

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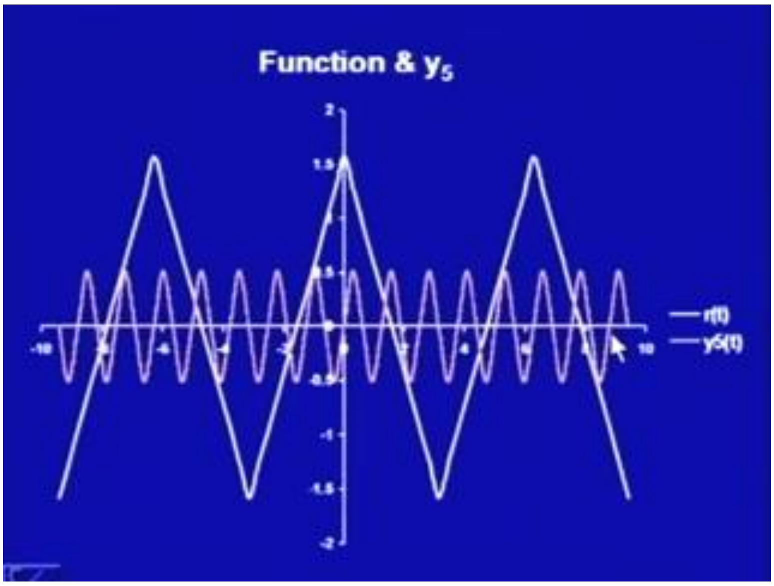

Here in the function and y 5, if you do remember this was my function r t and if I just make this graph y 5, you are getting this graph.

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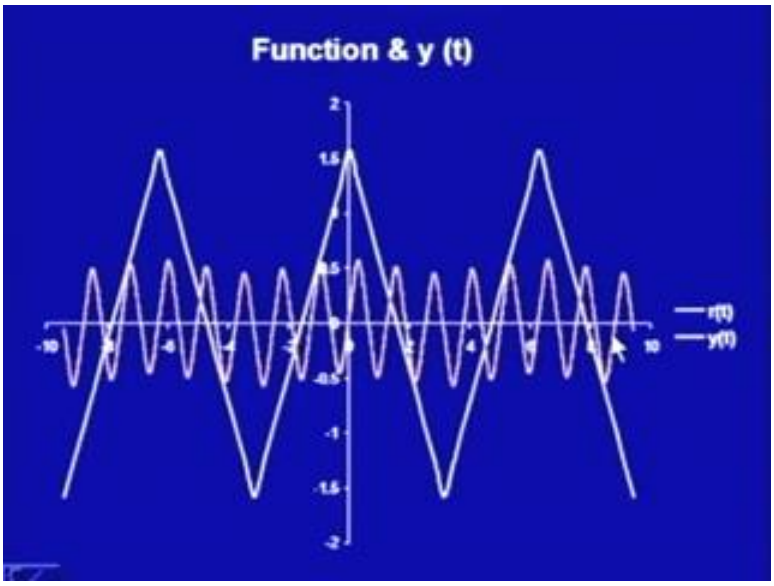

Now, let us come to this function and y t, now my y t is actually y 1 plus y 3 plus y 5 plus y 7 and I think is in the table, I have taken till y 11, so here are also I think this is the term till y 11, I have added it up. Now, you see, all other terms are not giving or they are not dominating the nature of the solution, as this y 5 is dominating, you see, this is the term till 11th term I had seen and you see here, only y 5 with fifth term or we could say is y 1, y 3 so y 5, so the third term.

You see is this, whatever is this nature oscillation which we are having here that is dominated by this y 5, rather than with any other one. So, this is very much dominating, this is what we are seeing is that, amplitude is deciding that which one is much better approximation for that function. So, rather than using that solution of all those once and solving it and then seeing is again, you see here, the convergence cannot be achieved in a small number of ns, for achieving the convergence you require a large once, this is your input function. This is actually, this y as not actually the Fourier series for your r t, this is actually input r t and this is your output y t, but this output is dominated by one function only that was, y 5.

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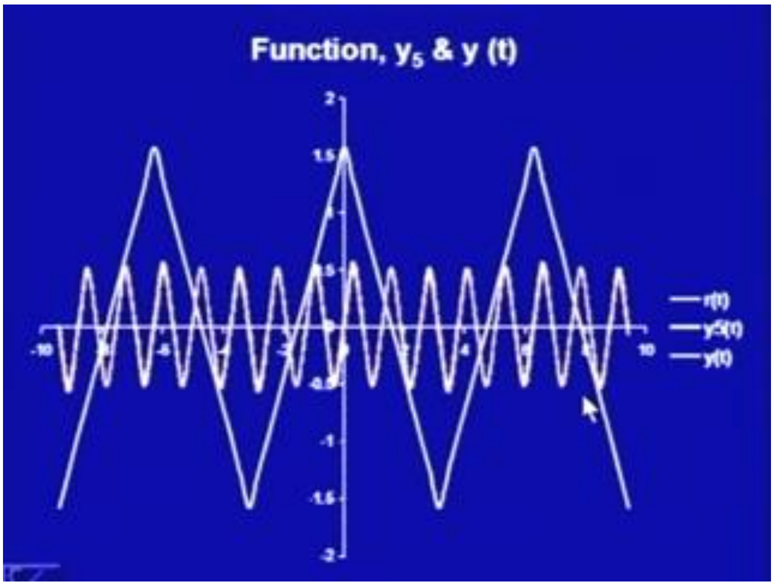

So, you see here, we are having 3 1 I think is, not very much missing now, I could see because you had seen this separately now, here I am superimposing all the 3, this is my input function that is very clear. The output you are finding it out that, both the lines are matching actually, the only point of the differences is, here little bit which if you could see it out, that is your this pink colour line is coming as, your till here and this another colour this one is, it is more or less white one, this is coming till here.

So, you see is that y 5 is actually dominating because of the amplitude only, now we have learnt this Fourier series, it was very easy for any periodic function or even if it is non periodic to get half a range extensions, to get Fourier coefficients approximate it as a Fourier series. Again we had learn, that is with some partial sums or the Fourier series sum is being equal to the value of the function mathematically, practically also we had seen, that for certain functions, very orderly with this third or fourth term only we are able to reach to the required place.

For some other function, this is takes little bit longer, that is in the value has to be some 20 or 25, but you do find it out that is we are reaching over that function. Then we had seen, in solution in the force oscillation where we, where you are going to solve only for functions which are of that form of cosine or sin kind of functions, but where we said is, if input function is any periodic function rather than the sin or cosine, is still we could give the output function and it is again oscillating.

There we had also in our particular example we had seen that it is, was not necessary that whole, all the terms of the Fourier series would be dominating, but only we would be getting is that is one or two terms depending upon that, this harmonics and this amplitude we could find it out, that which term is dominating and only with that we can go ahead with the analysis. There is one more topic which I want to cover over here that is called complex Fourier series, is anything different over here, no.

Complex Fourier series when we are saying, it is simply I am using this, as it simplifies our many calculations. So, you could say is that, this complex Fourier series we are using many times, where the calculations are much simplified than the real Fourier series and you will find it out it further classes that, this is how the things we are doing it and when we want the real Fourier series, we just go with the real components. The logic or basis behind this is, the complex exponential function, do you remember it.

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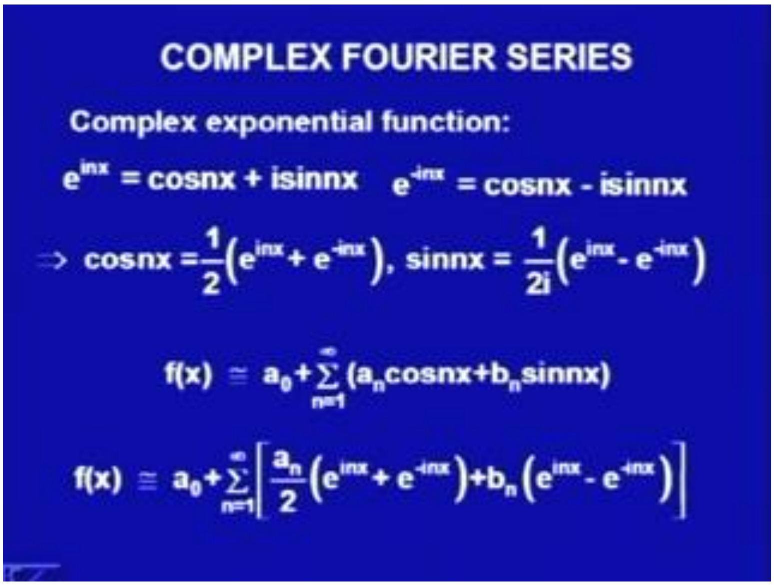

It is, if you do remember or not, e to the power i n x, this is cos n x plus i sin n x, this is again as called as Euler's formula and e to the power minus i n x as cos n x minus i sin n x. Now, from here what we could get, cos n x I can write as half of e to the power i n x plus e to the power, minus i n x and sine n x as 1 upon 2 i, e to the power i n x minus e to the power, minus i n x, what is this i, remember that we are talking about the complex. So, this i is nothing but the square root of minus 1 that, imaginary value.

Now, let us see the Fourier series for any function, we are writing as, a naught plus summation n is running from 1 to infinity a n cos n x plus b n sin n x, now change this cos n x and the sin n x with these, just now obtain the results in the terms of complex exponential functions, what I would get, substitute it. As, a naught plus summation n is running from 1 to infinity, a n cos n x have been substituting e to the power i n x plus e to the power minus i n x by 2 and with b n, again here, by 2 is missing. So, it should be b n by 2, e to the power i n x minus e to the power minus i n x.

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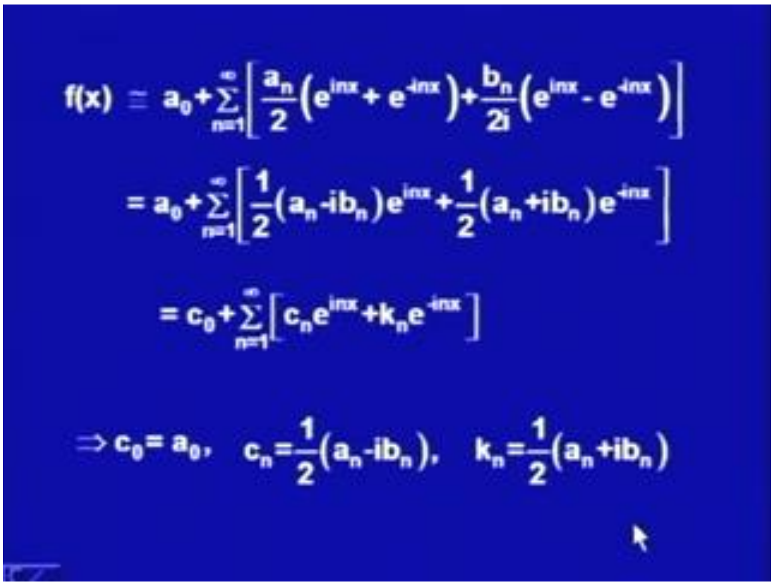

So, we are getting this series, now this is the correct one, a naught summation n is running from 1 to infinity, a n upon 2 e to the power i n x plus e to the power minus i n x plus b n upon 2 i e to the power i n x minus e to the power minus i n x. Do little bit simplification, I can write it as 1 upon 2, a n minus i b n, e to the power i n x plus half times a n plus i b n times e to the power minus i n x. That is, I have collected the coefficients of e to the power i n x and e to the power minus i n x.

Rewrite it, as c naught plus summation n is running from 1 to infinity, c n e to the power i n x plus k n, e to the power minus i n x. Obviously, c naught is nothing but a naught, c n is nothing but 1 by 2 a n minus i b n and k n is nothing but 1 by 2 a n plus i b n. So, now we have got a new Fourier series, new series where my terms are not cosine x and sin x rather than, I am having e to the power i n x and e to the power minus i n x. Now, the coefficients this c n, k n and c naught, we have derived it from this original a n, b n and a naught. So, we can use the Euler’s formula over here.

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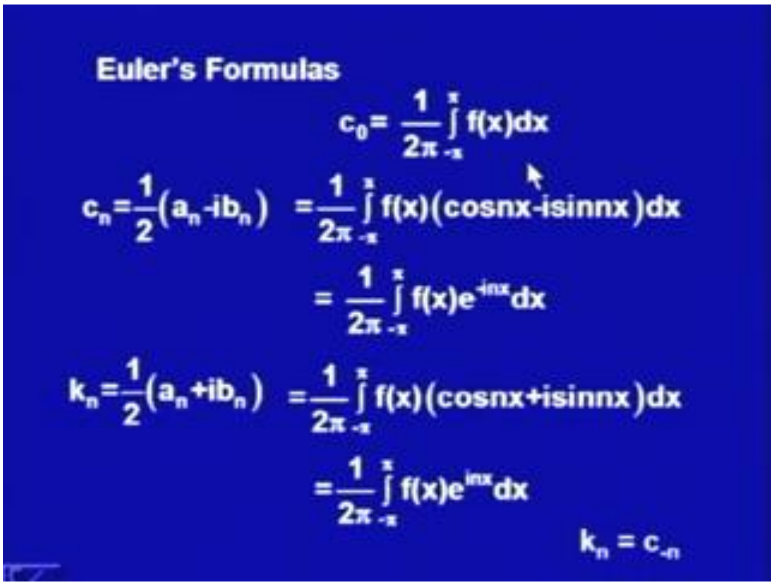

Where, c naught is same as a naught, so it is same as 1 upon 2 pi, minus pi to plus pi f x d x, c n would be nothing but as, is 1 upon 2 a n minus i b n. So, if I write the Euler's formula for a n and b n, I would get it as 1 upon 2 pi, minus pi to plus pi f x cos n x, that is for an and f x sin n x, that is for b n. So, now here minus i b n, so that is, how we have got f x into cos n x minus i sin n x whole integral with minus pi to plus pi into 1 upon 2 pi.

Similarly, we can now, if you do remember e to the, this is nothing but, e to the power minus i n x, then go for the k n, half a n plus i b n. Again using the same kind of formula I would get it as 1 upon 2 pi, minus pi to plus pi, f x cosine n x plus i sin n x d x. Now, I can write this as, e to the power plus i n x, so I am getting is 1 upon 2 pi minus pi to plus pi f x i n x d x. Now you see, c naught I am getting f x d x, c n I am getting f x e to the power minus i n x, this constant 1 upon 2 pi is same, the range of the integral is same minus pi to plus pi, the difference is here.

Then, when I am coming at k n, I am getting the difference as e to the power i n x, now, if I write k n as nothing but c of minus n, that is, if I replace here n with minus n, you will get that, my k of n is nothing but, c of minus n and when I replace n is equal to 0, I would get it as 1 and that would be c naught. So, now you see what I have got.

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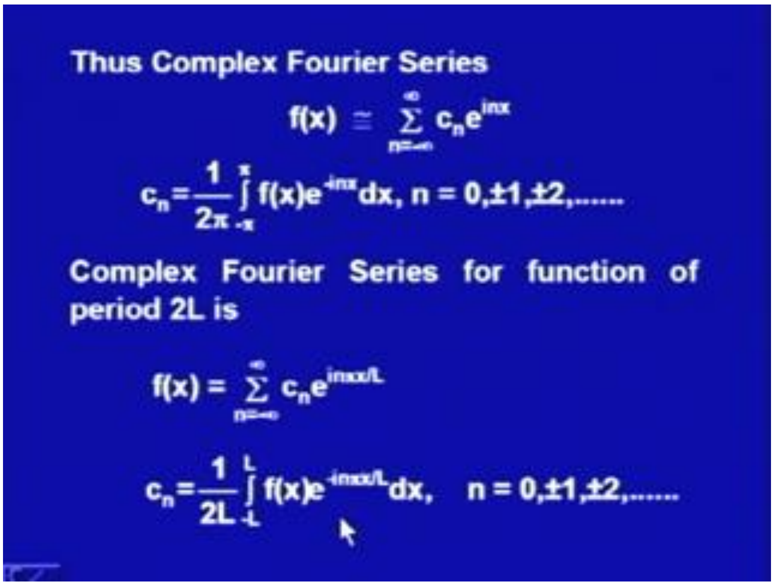

I have got the Fourier series, minus infinity to plus infinity, c n times e to the power i n x, because k n is nothing but c of minus n, c 0 is nothing but, c n ,where n is replaced by 0. So, now you find it out that, this is very compact form I can write and that is what I said, is it, is easy in the calculation. Where c n can be given as the formula, 1 upon 2 pi minus pi to plus pi f x, e to the power i n x d x, for n, 0, plus minus 1, plus minus 2 and so on or you could write in another manner that is, minus 3 minus 2 and so on, 0 1 2 and so on.

This is called the complex Fourier series for function of, this is we have obtained for function f period 2 pi. Now, on the similar trend, we can write for any function of the period of any 2 L, what I have to change is to you, do you remember that is there it, you use to be cosine n pi by x by L and so, kind of things. So, I would get it, minus infinity to plus infinity c n, i n pi x by L where c n is nothing but 1upon 2 L integral minus L to plus L, f x e to the power minus i n x pi L, n pi x by L d x for n is taking value 0, plus minus 1, plus minus 2 and so on.

So, this is, now you see till now I was writing the Fourier series which has containing such a space and then you have to the three formulae's 1 for a naught, 1 for a n and another for b n. Now, we are writing only 1 formula this is easy and even the calculation is easy, I have to integrate only 1 integral and most probably 2. Let us see, that is with the help of an example.

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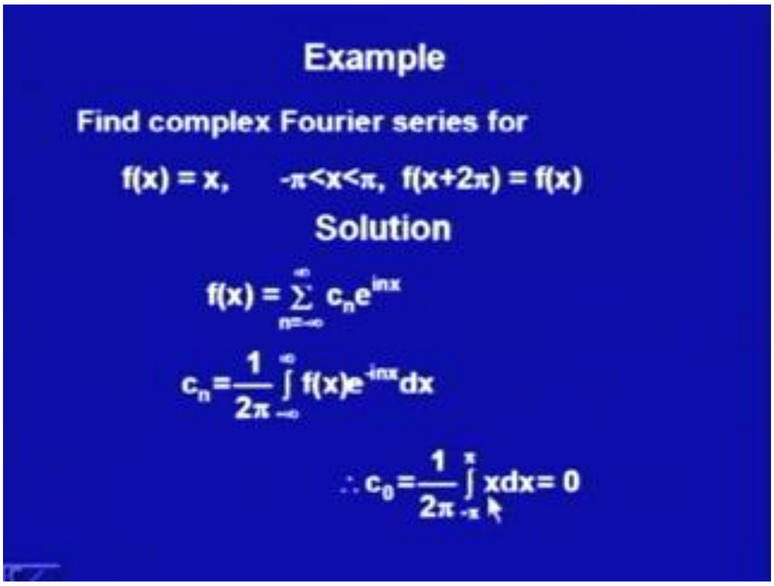

Find the complex Fourier series for f x is equal to x, from minus pi to plus pi and its periodic with f x plus 2 pi is equal to f x. Remember, we have already find out the Fourier series for this function, let us see, how we are doing it in the complex one and is it really making our calculations easy. So, let us see, we want a Fourier series of the form, summation n is running from minus infinity to plus infinity c n, i n x where c n should be of the form 1 upon 2 pi, minus infinity to plus infinity f x e to the power minus i n x d x.

Now, calculate this one, if I do have here, this n is equal to 0, I will not be having the function and the function is x and if I am having n naught 0, then I would be having x times e to the power minus n x. We do know this exponential function, that we can integrate and all those things, but if there is some multiple of the x, then we have to go with the partial one. So, we will break this into two parts, one is when n is 0, that is c naught, it would 1 upon 2 pi minus pi to plus pi x d x or this an odd function it should be 0.

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$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} x e^{-inx} dx = \frac{1}{2\pi}\left[\frac{x}{-in} e^{-inx}\Big|_{-\pi}^{\pi} + \frac{1}{in}\int_{-\pi}^{\pi} e^{-inx} dx\right]$$

$$= \frac{i}{2n\pi}\left[\pi e^{-inx} + \pi e^{inx}\right] - \frac{1}{i^2 n^2 2\pi} e^{-inx}\Big|_{-\pi}^{\pi}$$

$$= \frac{i}{2n}\left[e^{inx} + e^{-inx}\right] + \frac{1}{2n^2\pi}\left[e^{-inx} - e^{-inx}\right]$$

$$= \frac{i}{n}\cos n\pi - \frac{i}{\pi n^2}\sin n\pi = \frac{i}{n}(-1)^n$$

$$n \neq 0$$

Now, c n for all n not 0, 1 upon 2 pi minus pi to plus pi, x times e to the power minus i n x d x. Integrate by part, taking x as the first function that is the differentiating function and e to the power minus i n x as the integrating function, I will get, 1 upon 2 pi, x upon minus i n, e to the power minus i n x evaluated from minus pi to plus pi plus 1 upon i n minus pi to plus pi integral e to the power minus i n x d x. The first evaluation, when we keep x is equal to pi and x is equal to minus pi, we would get it as 1 upon minus i is the i.

So, i upon 2 n pi, pi e to the power i n minus i n pi plus pi, e to the power plus i n pi minus this integral 1 upon i square, n square 2 pi, e to the power minus i n x, evaluated from minus pi to plus pi. Again, evaluate it, we get it as i upon 2 n, this pi I am taking inside e to the power i n x plus e to the power minus i n x plus 1 upon 2 n square pi, because 1 upon i square is minus 1, e to the power minus i n x minus e to the power minus i n x.

Again simplify, we get i upon n cos n pi, minus i upon pi n square sin n pi, what the simplification we have done, that is cos i n pi plus, sorry e to the power in pi x plus e to the power minus i n pi by 2, that is cos n pi. Similarly, here this divided by 2 i would be, if I am dividing it by 2 i, then I would have here minus i it would be, sine nx, so now, you see we have got a complex function or a complex series. Since, it is cos n pi is, minus 1 to the power n and sin n pi is 0 for all n, we are getting i upon n minus 1 to the power n the coefficient c n for n not 0.

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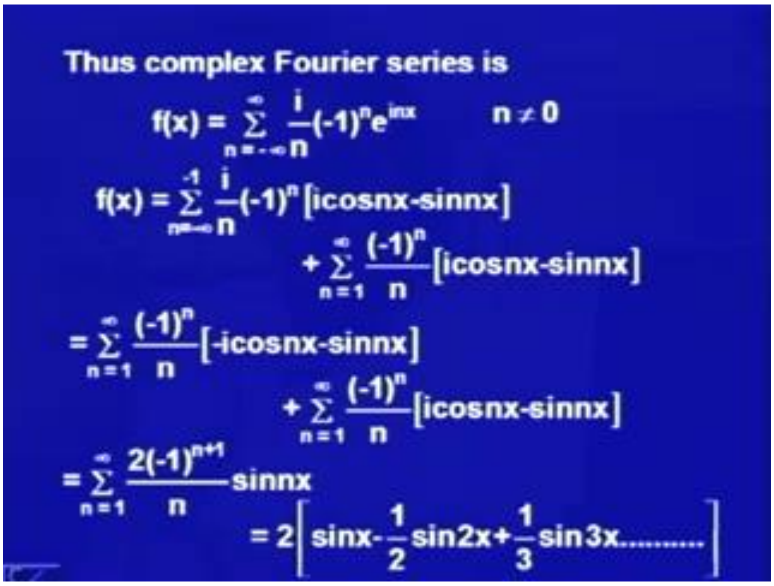

So, you see, what is the complex Fourier series, minus infinity to plus infinity i upon n, minus 1 to the power n, e to the power i n x, where n should not be or n should be not 0. So, if I expand it, it would be minus infinity to minus 1, i upon n minus 1 to the power n, i cos n x minus sin n x and e to the power and from summation n is running from 1 to infinity because n is equal to 0 is 0, minus 1 to n, power n upon n i cos n x minus sin n x.

Finally I would be getting it as, n is running from 1 to infinity, minus 1 upon 1 to the power n upon n minus cosine n x minus sin n x, which we could write like this one. Hence, finally we would get, summation n is running from 1 to infinity, two times minus 1 to the power n plus 1 upon n, sin n x. If you do remember, that, this was our Fourier, real Fourier series which we had obtained in our earlier example. So, here if a thing is only that, calculations I do not have to go through that, two kind of different integrals.

Here this, you could not have guess that is very simple one because it was giving us only sin series. But if it is giving both sin and cosine terms then, the terms becomes real easy and the calculations are easy, that is why we are using this complex Fourier series and from there, we get what is the real Fourier series. That is the all in, this today’s lecture about the Fourier series.

So, we had learnt the Fourier series, we had learn that is what is the convergence of the Fourier series and the terms, that the sum of the Fourier series is equal to the value of the function at a point of continuity and at a point of discontinuity, the sum of the Fourier series is equal to the average of the limits. That is, left limit and right limit, then we had learn one more thing, the complex Fourier series, the complex Fourier series is not anything else, only thing is, that is using this complex Fourier series many times the formulation becomes easy, the calculations becomes easy and compact form and from there we can come back to our real Fourier series. So, that is all in this Fourier series and today's lecture.

Thank you.