Mathematics - III
Prof. Tanuja Srivastava
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture - 12
Fourier Series – Part - 2

Welcome to the lecture series on differential equations for under graduate students. Today's lecture is in continuation of Fourier series. In last lecture, we had learnt that for periodic functions we can find out the Fourier series of those functions, we had learnt the method to find out the Fourier series for given functions. In the examples we had seen that for certain functions, we were getting the Fourier series containing either only cosine terms or only sin terms. When it is happening? Can we answer this, we can answer it, but before that we will relook at certain properties of the function. So, first we will see even and odd functions, first odd function.

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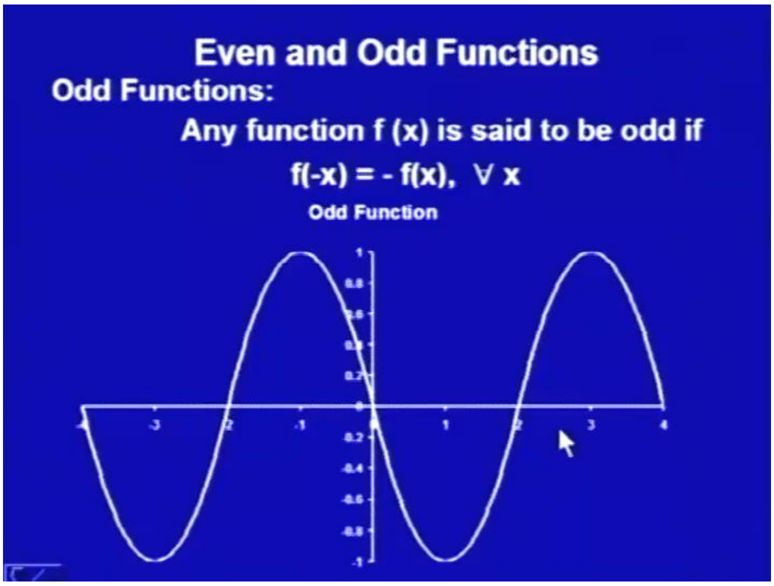

A function would be called odd, if for f of minus x is same as minus of f of x for all x, say for example, in this function, here we see this function is from 0 to 2 if we see the values are in the negative. So, let us see at 1 at the 1 value of this function is minus 1. Now if I see value at minus 1, value at minus 1 is plus 1 that is minus of minus 1 that is plus 1. Similarly, if I see any value let us say here at half or at 0.5 the value over here is coming somewhere here as 0.6 or something and when we are moving here, we would be getting this is in the negative and here in the positive one. So, we are getting is that this function is odd, that is it is changing its mode whatever it is going upside the x-axis that is towards the positive y-axis. The same thing is being repeated in the reverse order in the negative side of the x-axis or that is the y-axis.

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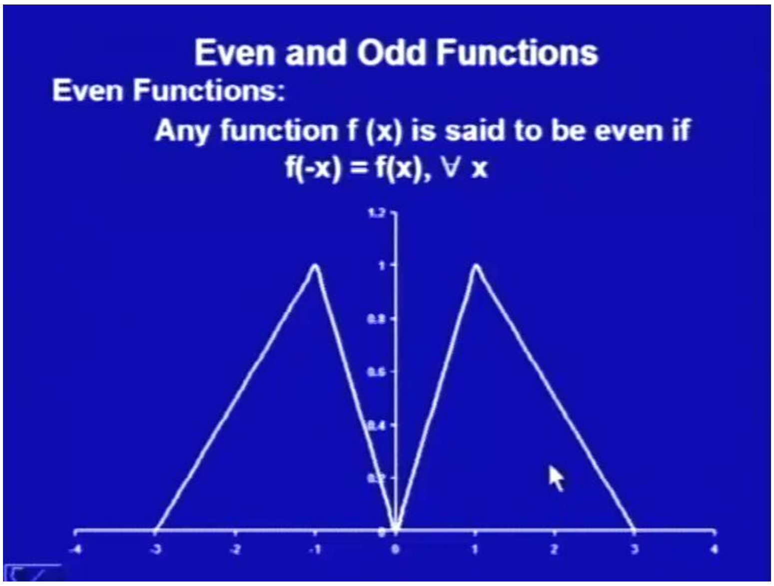

Similarly, for even function a function is called even, if f of minus x is f of x for all x that is it is not changing the sign. Let us see here, we see again here is the function, which is something a triangle kind of thing on the positive side of x-axis, if I go to the negative side of the x-axis, we are getting the similar kind of function that is if I see the value at plus 1, the value at plus 1 is somewhere here as actually the plus 1 and at minus 1 is also the value is plus 1.

Similarly, if I see value at 2 value at 2 here is coming somewhere as 0.5 while as at minus 2 if I see the value is again coming as 0.5. That the value of at the positive side of the x-axis whatever is we are getting is the f x the same thing we are getting on the negative side of the x, that is the function is repeating itself on both the sides of the x-axis, this is called the even function.

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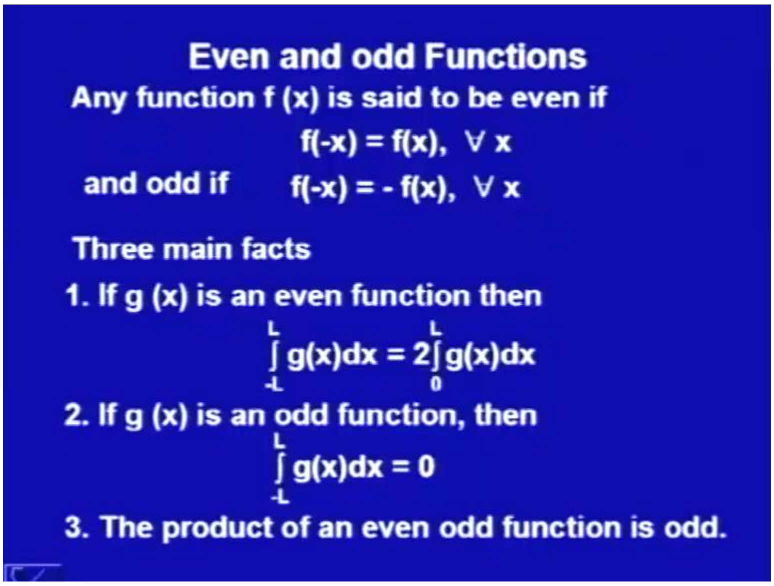

So, what we had learnt, we had learnt that any function f x would be called even if f of minus x is same as f of x for all x and odd if f of minus x is same as minus of f of x for all x. What this properties or what this even odd is telling us, let us see the three main facts were this section, first is that if g x is an even function, then if I integrate it over any interval from minus L to plus L I can write it as 2 times integral 0 to L g x d x. And if g x is an odd function, then this integral minus L to plus L g x d x is 0, this we are knowing from the properties of the definite integral. The third property which we would be using here is the product of an even and odd function is odd.

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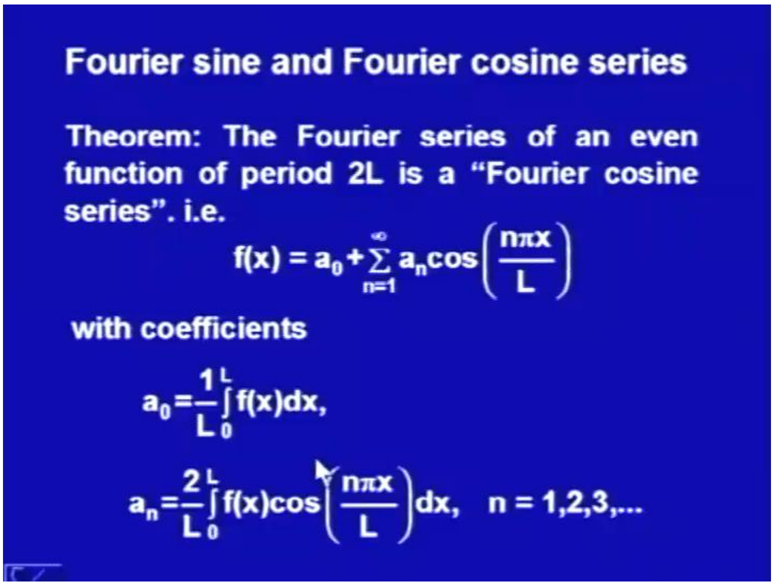

Now, let us see what is happening with this function in the Fourier series, so the result here is the Fourier series of an even function of period 2 L is a Fourier cosine series. So; that means, we will get the series f x as a naught plus summation n is running from 1 to infinity a n cos n pi x by L with the coefficients a naught as 1 by L 0 to L f x d x and coefficient a n as 2 by L 0 to L f x cos n pi x by L d x for n running from 1, 2, 3 and so on.

We see here, that the coefficients as such it was 1 upon 2 L 0 to minus L to plus L f x d x a naught. Now, we are seeing is that because, it is an even function, so integral minus L to plus L f x d x is 2 times 0 to L f x d x, so we are changing this integral from 0 to L only similarly in the a n also the integral has been changed to 0 to L.

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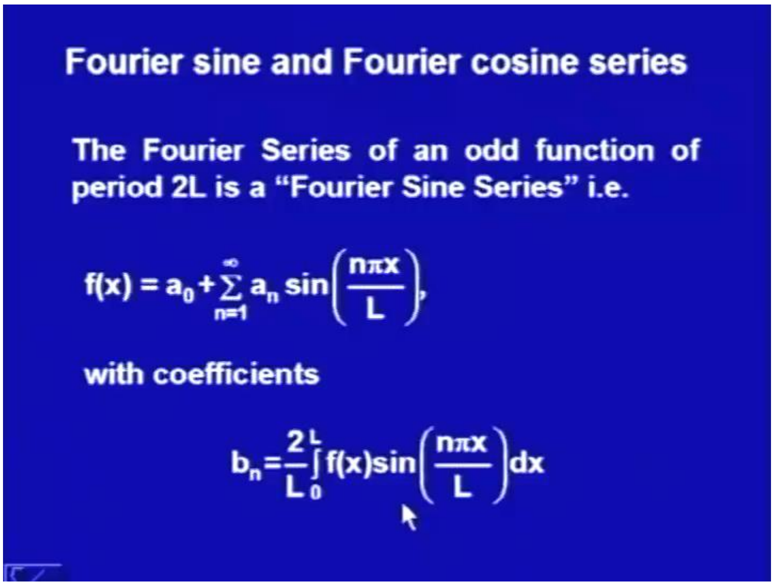

Similarly, the Fourier series of an odd function of period 2 L is a Fourier sin series, that is I would get the series as f x as a naught plus an sin n pi x by L, where a naught would be 0 and the coefficient a n would be 2 by L 0 to L f x sin n pi x over L d x. Now, since it is an odd function we would get that f x integral with this f x and sin n and x the both are odd. So, product of odd and odd function would be even, so this integral would be from minus L to plus L with change to 0 to L, while when we are writing f x into cos n pi x that would be odd function and that would result as 0.

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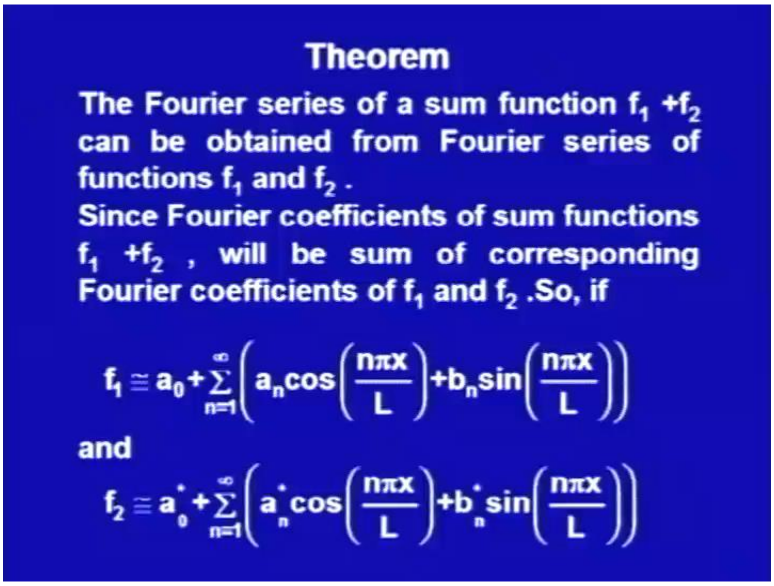

So, next we are getting that the Fourier series of a one more property, the Fourier series of a sum function f 1 plus f 2 can be obtained from the Fourier series of functions of f 1 and f 2. What it says is, it simply says is that Fourier coefficients of the sum function f 1 plus f 2 will be the sum of corresponding coefficients of f 1 plus f 2. That means; if I do have a function f 1 whose Fourier series is a naught plus summation n is running from 1 to infinity a n cos n pi x by L plus b n sin n pi x by L. And another Fourier series for the function f 2 as a naught star plus summation n is running from 1 to infinity a n star cos n pi x by L plus b n star sin n pi by x over L.

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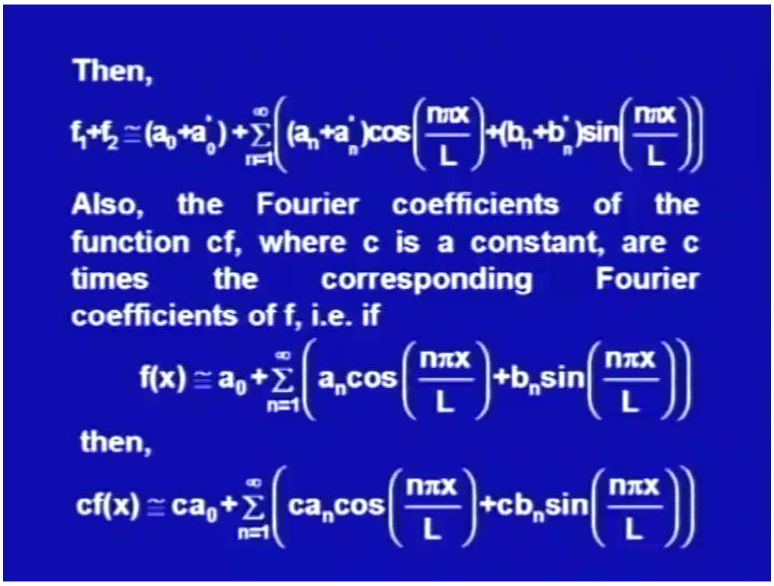

Then we will get, so Fourier series for f 1 plus f 2 as a naught plus a naught star plus summation n is running from 1 to infinity, the coefficient of cos n pi x by l would be now a n plus a n star plus the coefficient of sin n pi x by L would be b n plus b n star. So, we are getting this new Fourier series for f 1 plus f 2, moreover if the Fourier coefficient of the function of the c f where, c is a constant are the c times the corresponding Fourier coefficients of f.

That is f my Fourier series for the function f x is a naught plus summation n is running 1 to infinity a n cos n pi x by L plus b n sin n pi x by L. Then, the Fourier series for the function c f would be given as c times a naught plus summation n is running from 1 to infinity c times a n cos n pi x by L plus c times b n sin n pi x by L.

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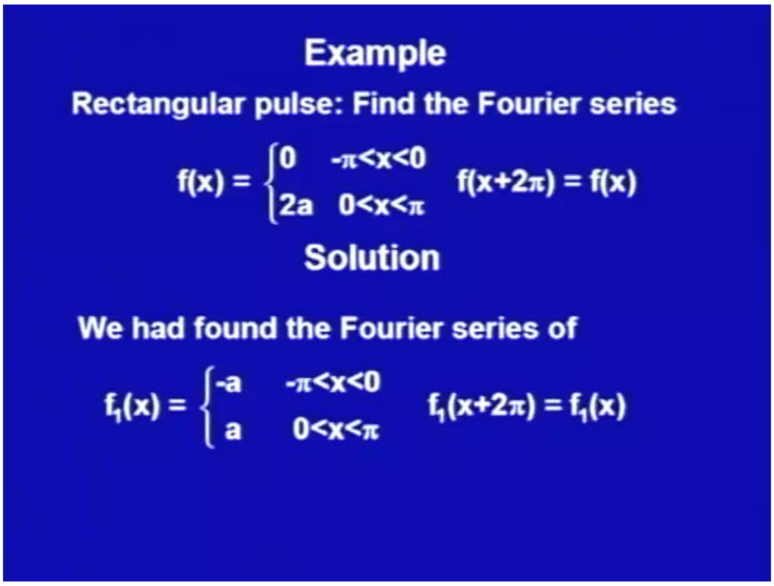

Now, we just see that is the example explaining these properties, which we have got first example is for rectangular pulse, find the Fourier series of the function f x as 0 in minus pi to 0 and 2 a from 0 to pi f x plus 2 pi is equal to f x. That is, this is a periodic function with period 2 pi, we have to find out the Fourier series for this of course, we can find out the Fourier series we do know the method how to find out the Fourier coefficients and then evaluate it, but we will just use the properties we had learnt.

Let us see, we have one in the last lecture one example where we had find out, the Fourier series for this function minus a from minus pi to 0 and a from 0 to pi and the periodic with period 2 pi.

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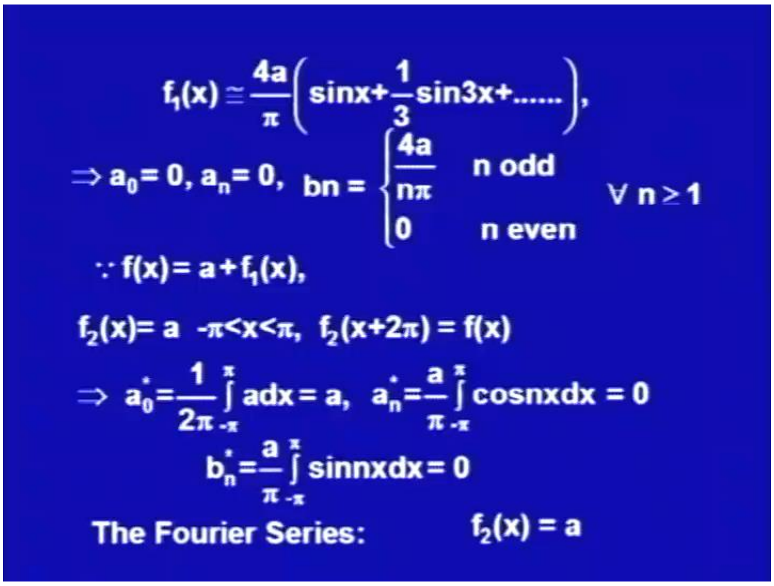

If you do remember we have got the Fourier series for this function as 4 a upon pi sin x plus 1 by 3 sin 3 x minus plus 1 by 5 sin 5 x and, so on. And now we see that this my f what we have got this is nothing, but as, so here in this one we are having is a naught and an 0 and b n was 4 a upon n pi for n odd and n even. And now this f x which we have to find out now the Fourier series that is nothing but, a plus f 1 x, f 1 x is minus a n 0 to pi minus pi to 0 and a from 0 to pi. So, if I add a I would get 0 in minus pi to 0 and 2 a from 0 to pi.

So, we are getting this 1 now, we take f 2 as a from minus pi to plus pi and it is a periodic function with period 2 pi. As we know that constant function, we can always treat with ((Refer Time: 10:23)) whatever we like, so now, for f 1 we already know what is the Fourier series for f 2 let us find out the Fourier series. So, a naught star would be 1 upon 2 pi minus pi to plus pi a d x that integral would turn out to be a, if I take an star that would be a upon pi minus pi to plus pi cos n x d x, we do know the integral minus pi to plus pi cos n x d x is 0 for all n integer n's.

Similarly, the b n star would be a upon pi integral minus pi to plus pi sin n x d x, again we know the integral minus pi to plus pi sin n x d x for any integer n is 0. So, we are getting both a n star and b n star as 0 for all n and we are getting a naught star as a, so the Fourier series for f 2 is nothing but, a only the constant. So, what we will get the Fourier series for f, Fourier series for f that we would have that coefficients of f 2 and coefficients of f 1 that has to be added up.

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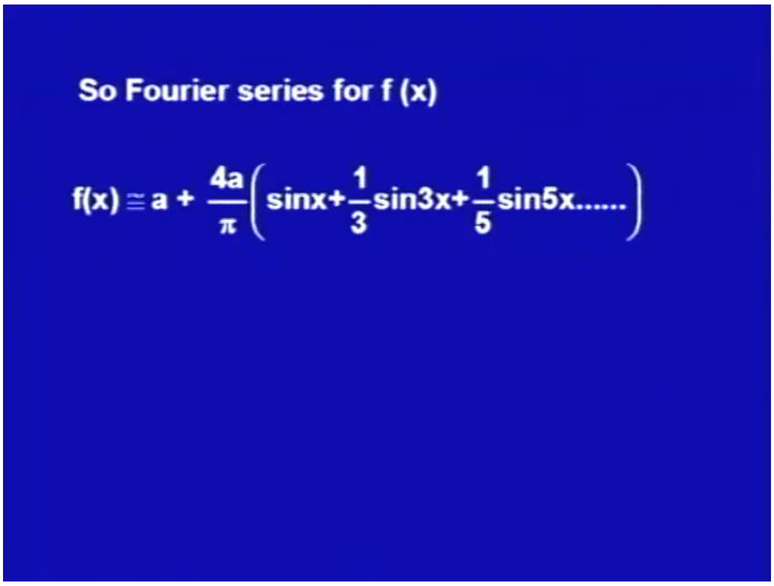

So, what we will get the Fourier series a plus 4 a upon pi sin x plus 1 by 3 sin 3 x plus 1 by 5 sin 5 x and so on. So, now you see is that is using this property of the sum of the functions we can find out the Fourier series very easily.

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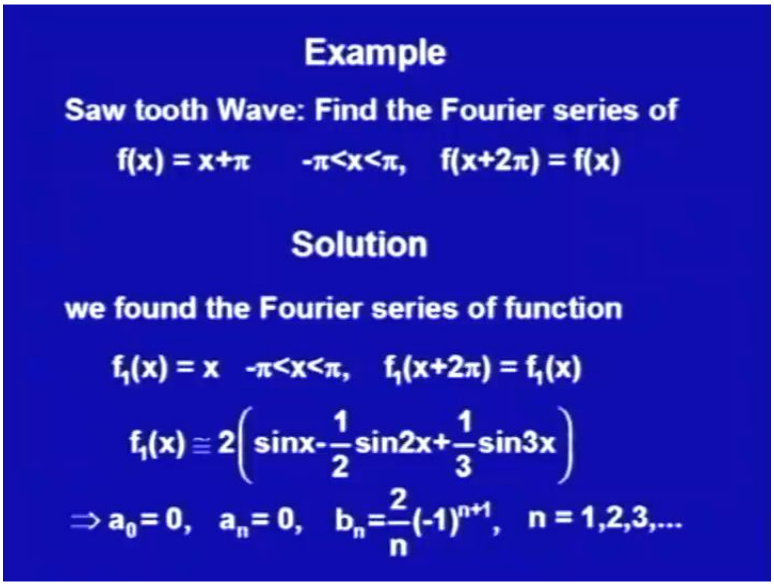

Now, let us do one more example, find the Fourier series of the function f x is equal to x plus pi minus pi to plus pi and periodic with f x plus 2 pi as f x, we have to find out the

Fourier series for this function. Now, see we had already done one more example in the last lecture, they we had find out the function f x is equal to x for minus pi to plus pi in periodic with period 2 pi.

So, now if we see my new function f x is nothing but, f 1 x plus pi and for this series f 1 x we do find out that the Fourier series was 2 times sin x minus 1 by 2 sin 2 x plus 1 by 3 sin 3 x and so on. That says is we have got a naught as 0, a n as 0 and b n as 2 upon n minus 1 to the power n plus 1, for n is equal to 1, 2, 3, and so on.

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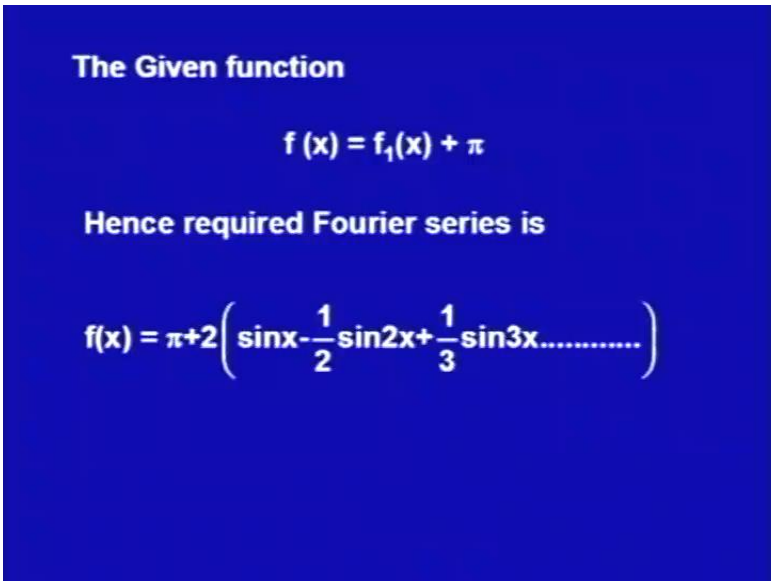

So now, we have to just find out the Fourier series pi f x is f 1 x plus pi in the last example just now we had seen when this another function was a just a constant, we have got that the Fourier series is nothing but, that constant only. So, the required Fourier series would be now f x as pi plus 2 times sin x minus 1 by 2 sin 2 x plus 1 by 3 sin 3 x and so on.

So, now we are see is that is rather than finding out the Fourier series knew that is finding out all the Fourier coefficients again doing with the integrals and so on, we can try and we can do it by new Fourier series, we can just add up the constants or thus Fourier coefficients and we can find out these Fourier series. These are the certain examples, which are very easy we have done, but if you see is that we had find out for the two functions, the Fourier series where the coefficients finding the coefficients we have done the tuff integrals.

So, for the sum function or if the coefficients are also involve, we can find out the sum function or the linear combination by the just taking the linear combination of those Fourier coefficients. Let us go with this even and odd function, now we will come to one more practical problems which we are facing, now we had learn that Fourier series we can obtain for the periodic functions, that is my function has to be defined on all whole real line and it should have it should repeat after sometime that is that the period.

Now, most practical problems we do not have this kind of functions, we may have the functions, which are defined only on a particular interval only not on the whole real line. How to find out the Fourier series for that, how to solve those problems, we can do it we can what we do, we had learn that is when we are finding out the Fourier sin series and cosine series, we are assuming is that if the function is even we would get the Fourier sin series only when the it function is even we will get the Fourier cosine series only.

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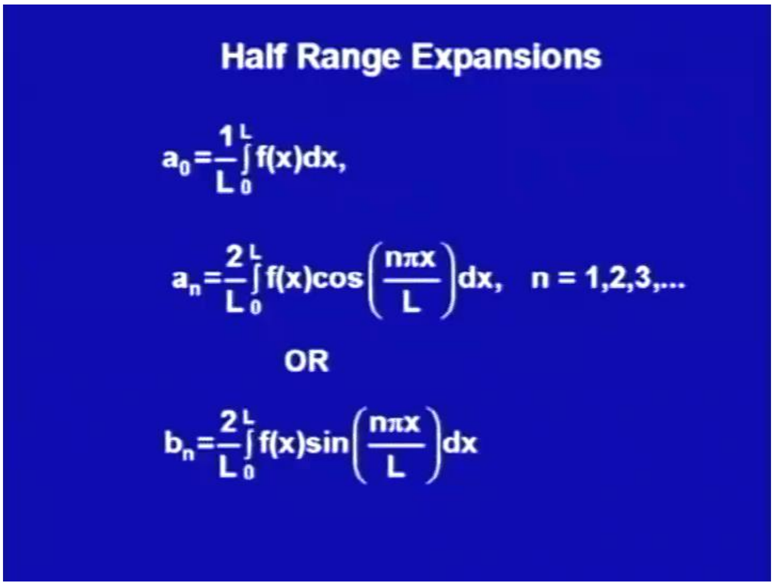

And the coefficients would be a naught as 1 upon L 0 to L f x d x and n as 2 upon L 0 to L f x cos n pi x by L d x for n is 1, 2, 3 and so on or if the function is odd we can get the Fourier sin series with the coefficients b n as 2 by L 0 to L f x sin n pi x by L d x. That says is we can find out the Fourier coefficients, only for the function defined in interval 0 to l.

But, the condition is that the function has to be either even or odd and it has to be periodic that says is that gives us an insight that for the function, which is defined only on the interval 0 to L for that also we can find out the Fourier series, only thing is we have to either extend a function to the as even extension or odd extension. Let us see, what we are meaning by these extensions, the function is let us say is defined only in the interval 0 to L here as this straight line.

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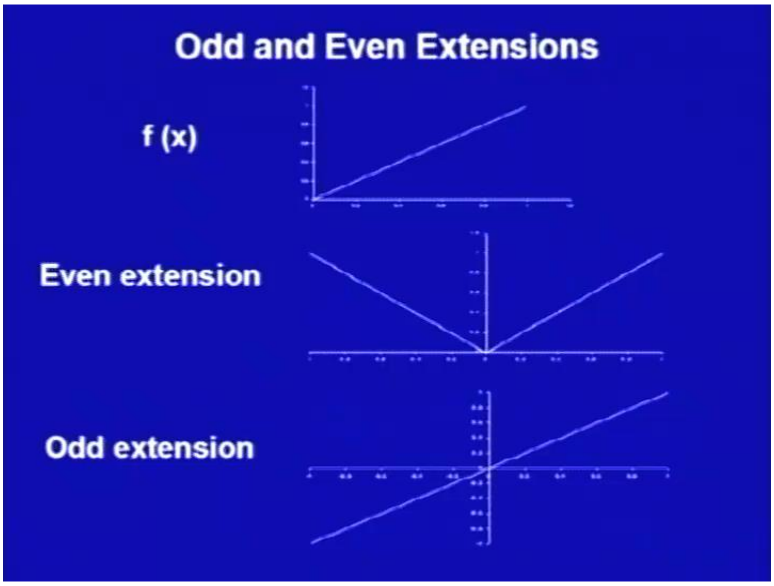

I can have it is even extension, that is I can extend it to this it should have the similar kind of function on the negative side also. Now, I can repeat this function from minus L to plus L again and that would be a periodic, similarly we can get the odd extension, we are having this function 0 to L this was defined over as a straight line for this we would just go with the negative 1 and we have this extension. Now we can ahead with here again periodically like that one. So, that says is that we can extend the function either with even function or as an odd function. And then we can have the period as 12 that is we can repeat it out and now we are getting is even or odd function and we can get the integral with the function defined only on 0 to L.

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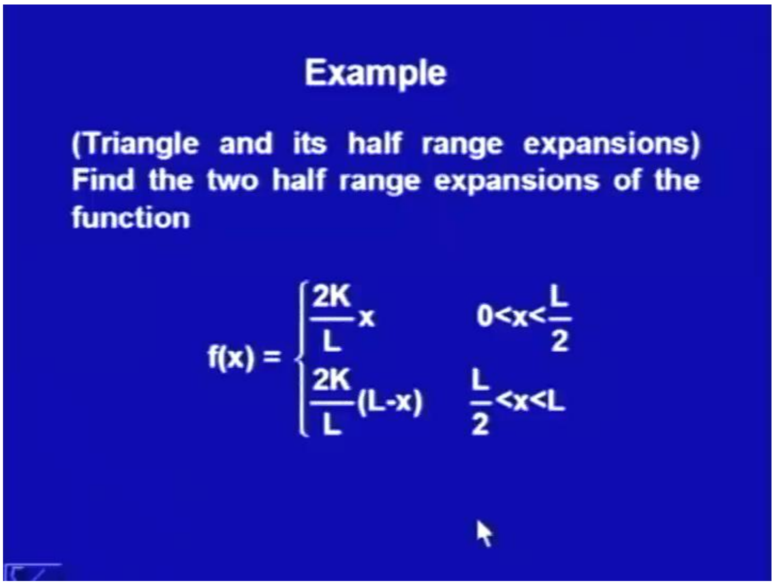

Let us see, how we are doing with the help of an example, this example says find the two half range expansions of the function f x, which is defined as 2 K by L times x in the interval 0 to L by 2 and 2 K upon L into L minus x in the interval L by 2 to L, we can see that this function is actually a triangle, we are having is that its increasing in 0 to L by 2 and then it is decreasing from L by 2 to L.

So, this is what is we are calling is triangle and it is half range expansions, what is the meaning of this half range expansion, we have learn that is if the function is defined only in 0 to L. We will use the function only if defined in the 0 to L, but what we will find out either we will find out a cosine series or a sin series, that is either we will repeat the function 0 to L same as minus L to 0 that is as an even function. And then repeat it out that is make it periodic function and find out the Fourier series or make it as odd function from minus L to 0 and then repeat it and find out a Fourier sin series. So, whatever this series we are getting for this odd extension or even extensions, those things we are calling as half range expansions.

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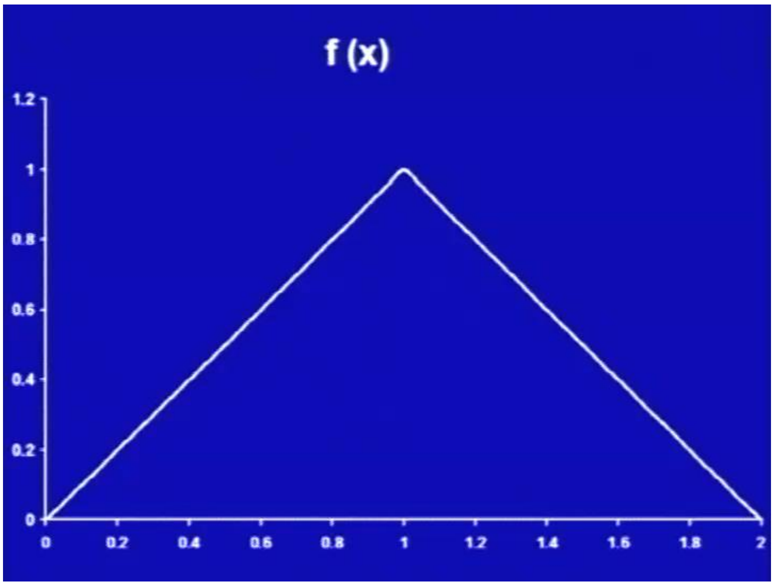

So, let us try to say that is what is this function and what is it is, so the function is this triangle, which is defined from 0 to 2.

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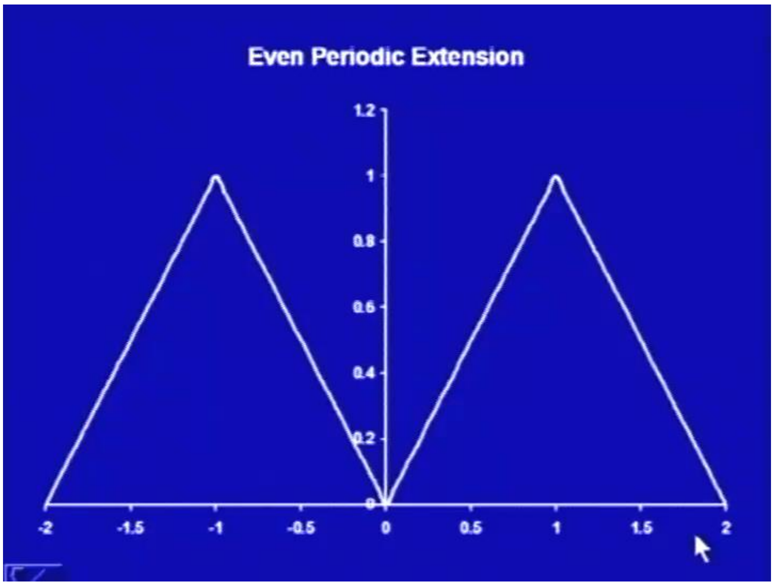

Let us first do the even extensions, that is the function we have taken from 0 to 2 even extension that is minus 2 to 0 I will repeat the same triangle over here. So, we see that the value at 0.5 is 0.5 here value at minus 0.5 is also 0.5 here value at 1 is 1 here. So, value at minus 1 is also 1, similarly value at 1.5 is 0.5 and value at minus 1.5 is also plus 0.5 that is we have just got this even extension and now you see is that is it is a periodic function. So, we would go just ahead with this function, now defined from minus 2 to plus 2 which is an even function and we will repeat it, so we can get the Fourier series.

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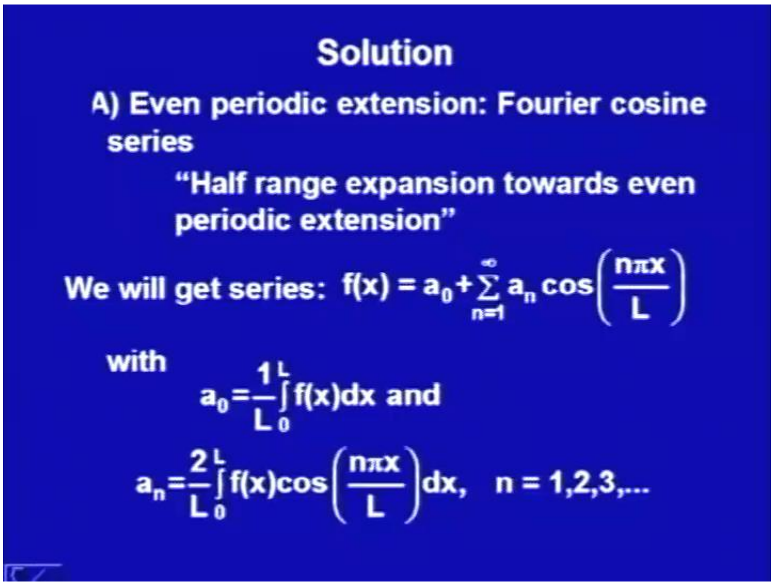

So, Fourier series first we are doing this even periodic extension, we will get the what we will called as for a cosine series half range expansion towards even periodic extension of the given function. So, we will get this series as a naught plus summation n is running from 1 to infinity a n cos n pi x by L with coefficient a naught as 1 by L 0 to L f x d x and a n as 2 by L 0 to L f x cos n pi x by L d x, let us try to find it out for this given function.

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$$\therefore a_0 = \frac{1}{L}\left[\int_0^{L/2} \frac{2K}{L} x dx + \int_{L/2}^{L} \frac{2K}{L}(L-x)dx\right]$$

$$= \frac{1}{L}\left[\frac{K}{L}.x^2\Big|_0^{L/2} + \frac{2K}{L}(Lx)\Big|_{L/2}^{L} - \frac{K}{L}.(x^2)\Big|_{L/2}^{L}\right]$$

$$= \frac{1}{L}\left[\frac{K}{L}.\frac{L^2}{4} + \frac{2K}{L}.L.\frac{L}{2} - \frac{K}{L}\left(L^2 - \frac{L^2}{4}\right)\right]$$

$$= K/2$$

So, the function is defined in 0 to L which is been braked up into two parts 0 to L by 2 and L by 2 to L. So, first the integral a naught 1 by L 0 to L f x d x. So, 1 by L integral 0 to L that we will break up into two parts 0 to L by 2, where the function is 2 K by L x d x and from L by 2 to L the function was 2 K by L into L minus x d x. Now, integrate it out we will get this integral as K by K times x square 2 x integral is x square, similarly here we will get L times x and this integral would be K by L times x square and evaluate it here it is 0 to L by 2 here it is L by 2 to L.

So, just doing these calculations we get 1 by L times K by L into L square by 4 plus 2 K by L into L into L by 2 minus K by L, L square minus L square by 4 that is at 0 the function would be 0 for all the values, which after simplification comes out to be only K by 2, so we have got a naught as K by 2.

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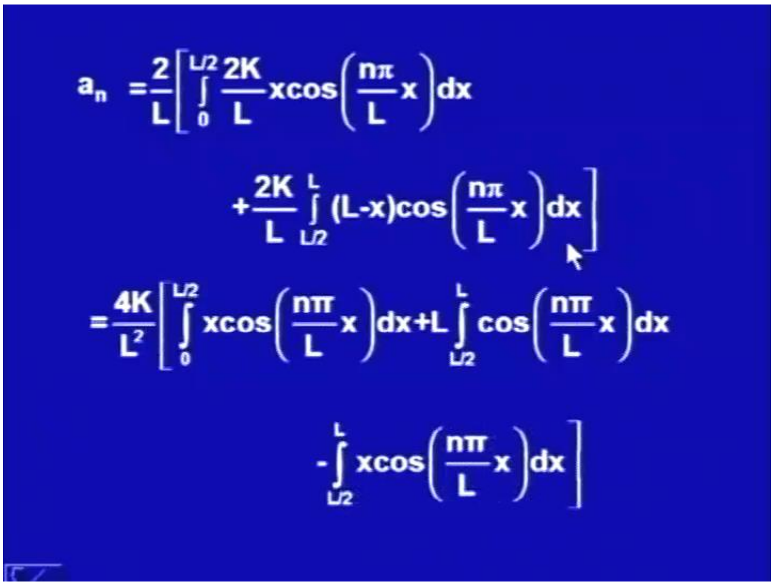

Now, what is a n the integral is 2 by L integral 0 to L f x times cosine n pi x by L d x, now the integral we will again break into two parts, that is 0 to L by 2 the function is 2 K by L into x and then multiply with cosine n pi over L x. And the other one is that is from L by 2 to L the function is L minus x 2 K by L times L minus x cos n pi by L x d x. Again we are integrating it by parts because, we do have x times cos x time kind of function, we would get it 4 K by L square integral 0 to L x cos n pi by L x d x plus L by 2 to L cos n pi x that is what we have taken the integral L times cos n pi x that would be 0.

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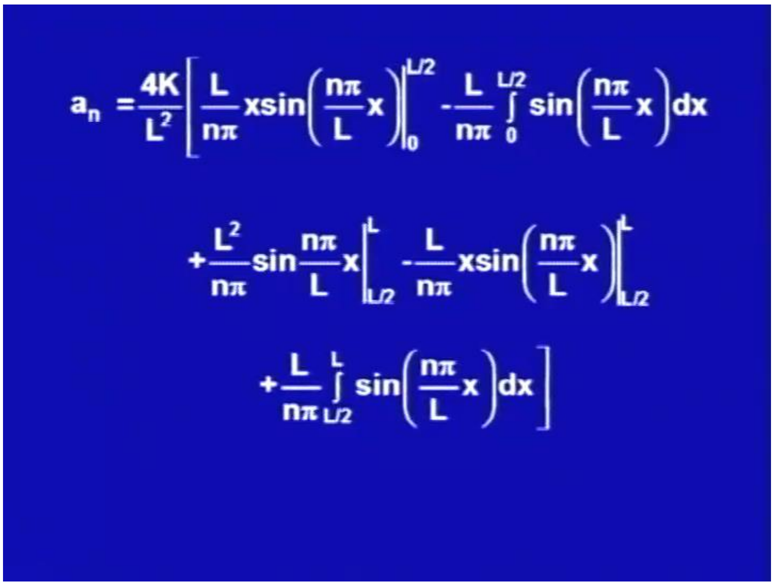

And we are just getting thus an as 4 K by L square L upon n pi x sin n pi over L x evaluated from 0 to L by 2 minus L upon n pi integral 0 to L by 2 sin n pi by Lx d x this is the first integral. The second integral would be L square upon n pi sin n pi over L x evaluate from L by 2 to L and minus L by n pi x times sin n pi over L x this evaluated from L by 2 to L and the integral for this one would be that second part plus L by 2 is integral L by 2 to L sin n pi over L x d x.

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$$a_n = \frac{4K}{L^2}\left[\frac{L^2}{2n\pi}\sin\left(\frac{n\pi}{2}\right)+\frac{L^2}{n^2\pi^2}\cos\left(\frac{n\pi}{L}x\right)\right]_0^{L/2}$$

$$-\frac{L^2}{n\pi}\sin\left(\frac{n\pi}{2}\right)+\frac{L^2}{2n\pi}\sin\left(\frac{n\pi}{2}\right)-\frac{L^2}{n^2\pi^2}\cos\left(\frac{n\pi}{L}x\right)\Bigg]_{L/2}^{L}\Bigg]$$

$$=\frac{4K}{L^2}\cdot\frac{L^2}{n^2\pi^2}\left[\cos\left(\frac{n\pi}{2}\right)-\cos 0-\cos(n\pi)+\cos\left(\frac{n\pi}{2}\right)\right]$$

$$=-\frac{4K}{n^2\pi^2}\left[1+\cos n\pi-2\cos\left(\frac{n\pi}{2}\right)\right]$$

$$\Rightarrow a_n=\frac{4K}{n^2\pi^2}\left[2\cos\left(\frac{n\pi}{2}\right)-\cos n\pi-1\right]$$

So, a n would be 4 K upon L square into L square upon 2 n pi sin n pi by 2 plus L square upon n square pi square cos n pi over L x evaluate it from 0 to L by 2 minus L square upon n pi sin n pi by 2 plus L square upon 2 n pi sin n pi by 2 minus L square upon n square pi square cos n pi over L x evaluated from L by 2 to L, which would be 4 K upon L square into L square upon n square into pi square since sin n pi by 2 here would be 0.

So, L square upon n square pi square plus into cos n pi by 2 minus cosine 0 minus cosine n pi plus cosine n pi by 2, which is coming as minus 4 K upon n square into pi square times 1 plus cos n pi minus 2 times cosine n pi by 2, we can rewrite it again as a n as 4 K upon n square pi square into 2 cos n pi by 2 minus cos n pi minus 1.

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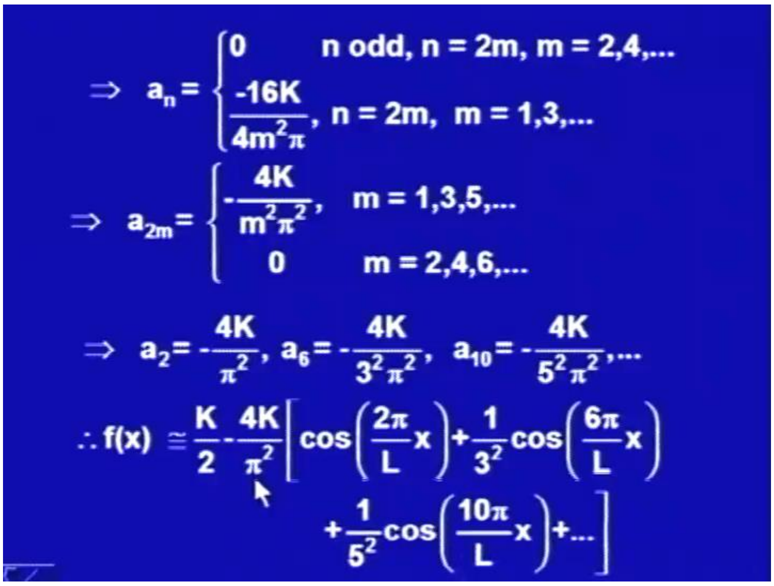

Since n pi by 2 is 0 and cos n pi is plus minus 1 to the power n we would get a n as 0 for n odd and when n is even and it is of the form 2 m with m as 2, 4 and so on that is n is equal to 4 8 and so on it would be 0. When n is even and of the form n is equal to 2 and with m as 1, 3 that is n is equal to 2, 6, 10 and so on we will get an as minus 16 k upon 4 m square times pi.

We can rewrite it again, since for all odd n a n is 0 for even n that is n is equal to 2 m we can write it as minus 4 K upon m square pi square for m taking values 1, 3, 5 and so on that is odd values and 0 when m is taking even values 2, 4 and 6. What it says is that my a 1 would be 0, a 2 when m is 1 will give me minus 4 K upon pi square, then a 3 being odd is 0, a 4 is m is 2 that would be 0, a 5 again odd is 0, a 6 we would get as m is equal to 3 minus 4 K upon 3 square pi square and so on, a 10 we will get minus 4 K upon 5 square pi square.

Thus what we will get our Fourier cosine series as K by 2 that is what is a naught minus 4 K upon pi square cos 2 pi by L x, this 4 K upon pi square with minus sign, this is common with all the coefficients, so that we have taken common, a 1 is 0, a 2 we are getting a 2 means n is 2 that is cos n pi x over L that, so n is 2 here, when the next term we would get is an as 6. So, cos 6 pi x over L 1 upon 3 square and so on we will get plus 1 upon 5 square cos 10 pi over L x and so on.

Now, the function we have taken as we defined from 0 to L with K 1 constant and L is another constant. In the example the graph of the function with h 1 I have taken L as 2 and K as 1, in that case what I will get this Fourier series as K by 2 minus 4 K upon pi square cos 2 pi by 2 x plus 1 by 3 square cos 6 pi by 2 x and so on or we will get K by 2 minus 4 K upon pi square cos pi x plus 1 by 3 square cos 3 pi x plus 1 by 5 square cos 5 pi x and so on, moreover we can keep k as 1, so I will get her 1 by 2 and here 4 upon pi square.

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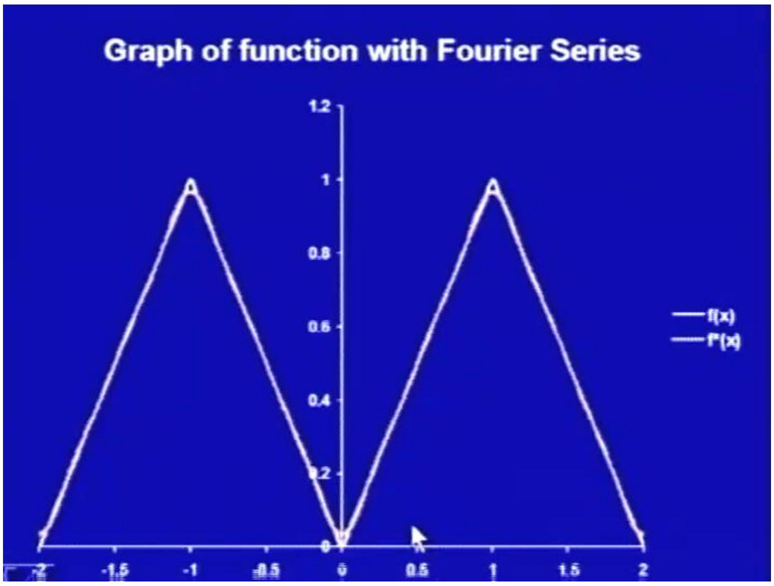

Let us see, the function and the Fourier series, so the function I have taken L I have taken as 2 and K I have taken as 1. So, the function is this triangle with the height 1 and this base as 0 to 2, we have taken it is even extension, so we have taken this minus 2 to plus 2 for this periodic function if the Fourier series I am just whatever I have find it out if I am just plotting it is just you see is that overlapping with this one with this pink line is the Fourier series.

The only point is where it is having a gap at is the peak at that one point that is where it is changing the shape of the function at that point figure having this little bit half of the function. So, we are getting the Fourier series for the even extension of the function which is as defined only from 0 to L, now let us move to the odd extension.

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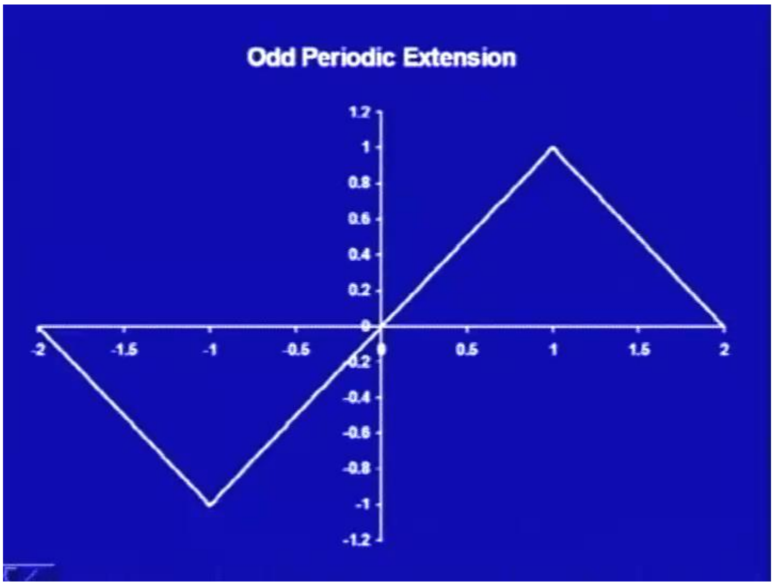

The function was defined from 0 to L as a triangle odd extension means is I have to take minus L to 0 the function value f of minus x should be minus of f of x. That is whatever value here we are having positive in this side the I have to have that value as negative, So; that means, this has been inverted over here, so you see at 1 the value of the function was 1 while at minus 1 I took it to be minus 1 at 0.5 the value of the function was 0.5 at minus 0.5 I were require it to be minus 0.5 and so on. So, this is what is our odd extension with the function minus L to plus L and then we can extend it as make it a periodic function with this is odd periodic extension find out it is half range expansion.

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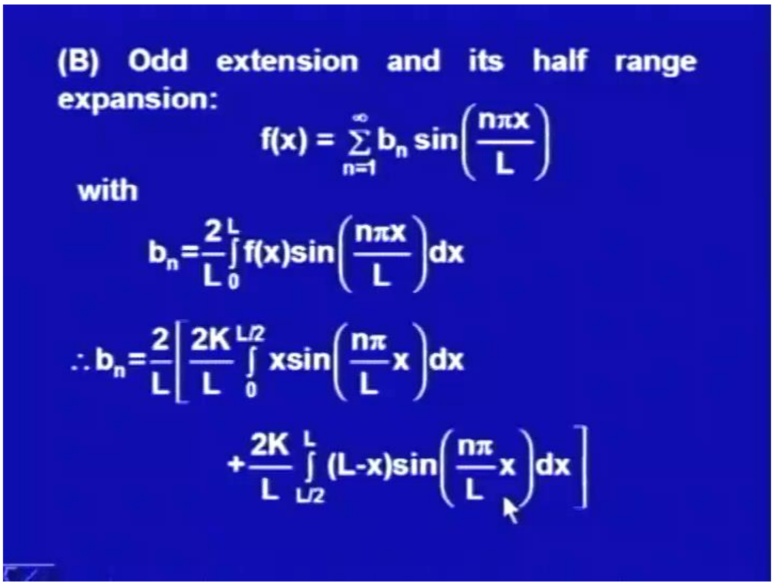

This would be because, the function is odd it should be Fourier sin series that is I would get the series of the summation n is running from 1 to infinity b n sin n pi x over L with b n as 2 upon L 0 to l f x sin n pi x over L d x. So, evaluate for our function what is my b n that would be 2 by L 2 k by L integral 0 to L by 2 x sin n pi x by L d x, since our function is defined from 0 to L by 2 and then L by 2 to L.

So, the next integral would be 2 K by L integrate from L by 2 to L L minus x sin n pi x d x, now evaluate these integrals where are we are having this function x sin x we would go for integral by part. This second integral we can break again into 2 integral 1 integral with L times sin n pi and another integral minus x times sin n pi x, so the second integral will again go by parts.

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$$b_n = \frac{4K}{L^2}\left[\frac{Lx}{n\pi}\cos\left(\frac{n\pi}{L}x\right)\Big|_0^{L/2} + \frac{L}{n\pi}\int_0^{L/2}\cos\left(\frac{n\pi}{L}x\right)dx - L.\frac{L}{n\pi}\cos\left(\frac{n\pi}{L}x\right)\Big|_{L/2}^{L} + \frac{L}{n\pi}x\cos\left(\frac{n\pi}{L}x\right)\Big|_{L/2}^{L} - \frac{L}{n\pi}\int_{L/2}^{L}\cos\left(\frac{n\pi}{L}x\right)dx\right]$$

So, b n would be 4 K upon L square L X upon n pi cos n pi x by L evaluated from 0 to L by 2 plus L upon n pi 0 to L by 2 cos n pi over L x d x minus L times L upon n pi cos n pi x over L evaluated from L by 2 to L plus L upon n pi x cos n pi x over L L by 2 to L evaluation minus L upon n pi integral L by 2 to L cos n pi over L x d x.

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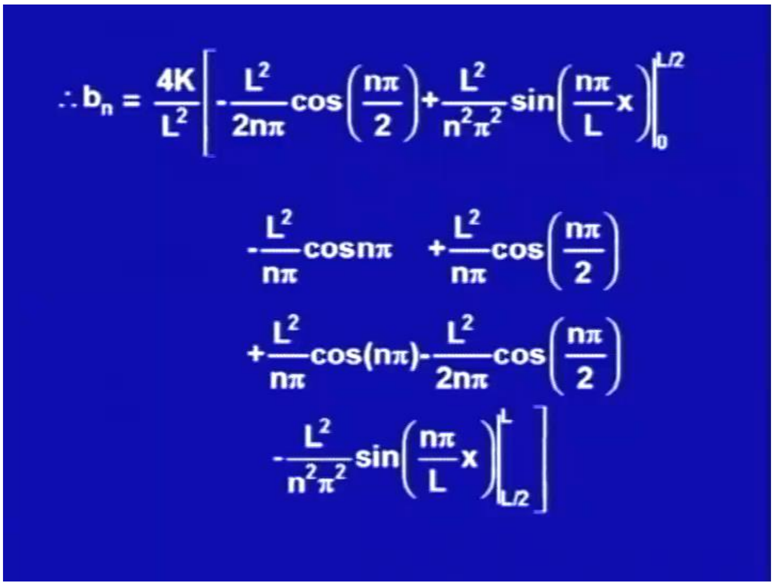

4 K upon L square into minus L square upon 2 n pi cos n pi by 2 plus L square upon n square pi square sin n pi x by L evaluate it from 0 to L by 2 minus L square upon n pi cos n pi plus L square upon n pi cos n pi by 2 plus L square upon n pi cos n pi minus L square upon 2 n pi cos n pi by 2 minus L square upon n square pi square sin n pi x by L evaluated from L to L by 2.

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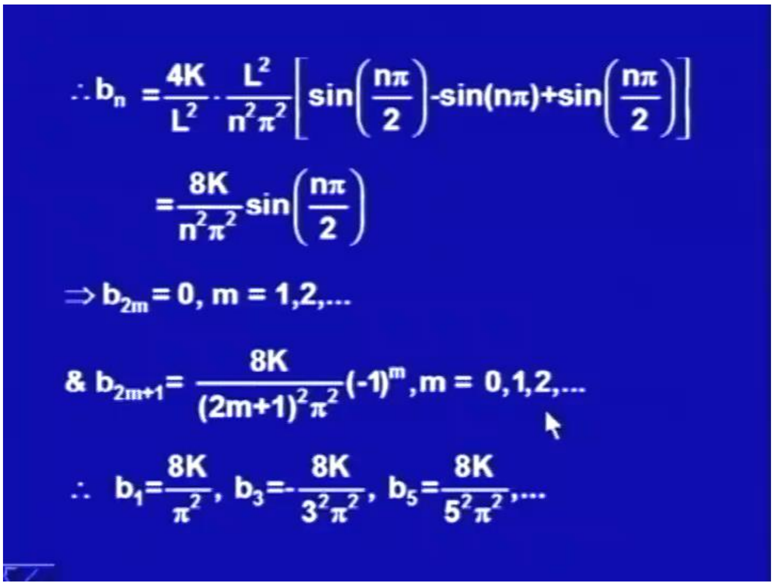

Now, do all the evaluation and we get b n as 4 K upon K square minus K square upon n square pi square into sin n pi by 2 minus sin n pi plus sin n pi by 2. Now, since sin n pi is

0 sin n pi by 2 we would get twice sin n pi by 2 that is 8 K upon n square pi square sin n pi by 2, which says is for n odd it would be plus or minus 1 for n even it would be 0. So, b 2 m would be 0 for n is equal to 1 2 and so on, and b 2 m plus 1 would be 8 K upon 2 m plus 1 is whole square pi square into minus 1 to the power m for m is taking values 0, 1, 2, and so on.

Thus, what we have got that b 1 would be 8 K upon pi square, b 3 would be minus 8 K upon 3 square pi square, b 5 would be 8 K upon 5 square pi square that is we are alternately changing the sin and getting the values that is n square pi square downwards. And in the denominator and in the numerator 8 K is common and we are alternately changing the sin for the odd ones for the even ones it is 0.

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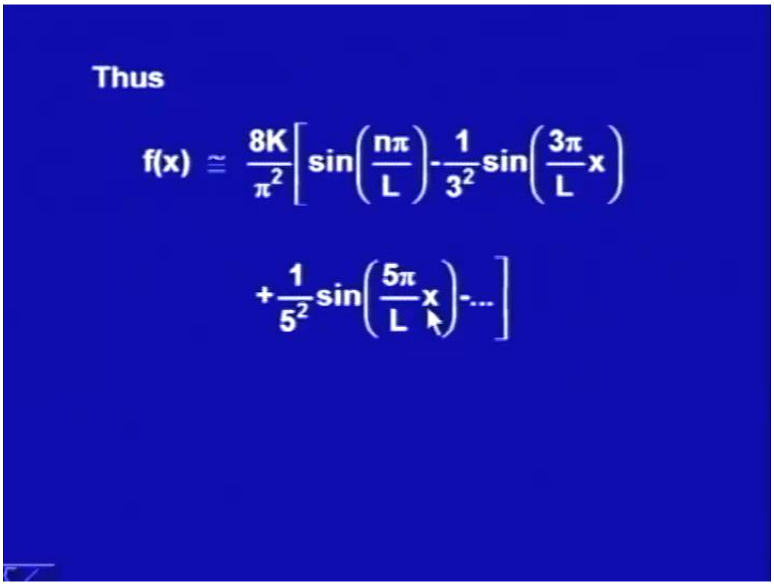

So, what we will get the Fourier series as 8 K upon pi square sin n pi by L minus 1 by 3 square sin 3 pi by L x plus 1 by 5 square sin 5 pi by L x and minus and so on. Now, in our example I have taken L as to be 2, so I will get at actually 8 K upon pi square sin n pi by 2 minus 1 by 3 square sin 3 n 3 pi by 2 x of course, here also is x 1 by 5 square sin 5 pi by 2 x and so on.

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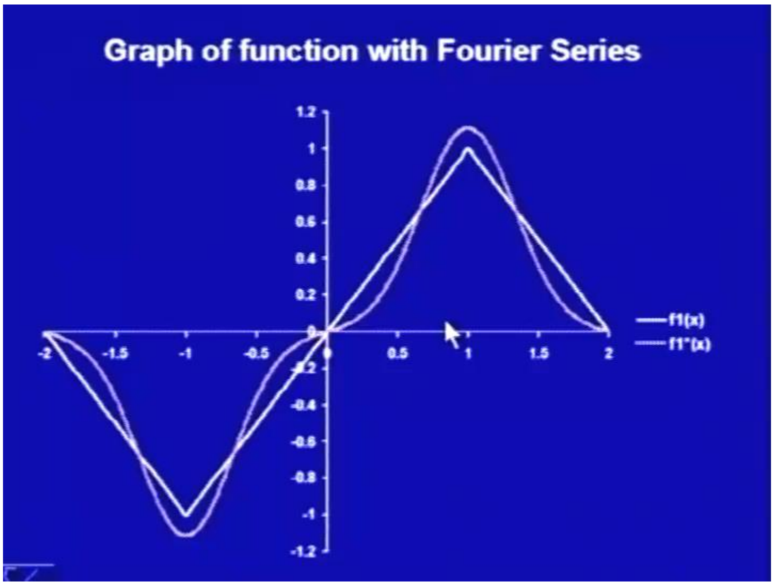

Let us see that is how it is approximating the function, so this is what was our odd extension from minus L to plus L. Now, this is the Fourier series this just we have got with the sin function this is what, now we are seeing is that this is not that much nicely approximating the function, some of things are of course, too fall because, I have taken a small number of functions as number of terms. But, is still we are getting is it is far away or that even extension we were getting much nice kind of approximation rather than this one, let us see one more example, what is happening?

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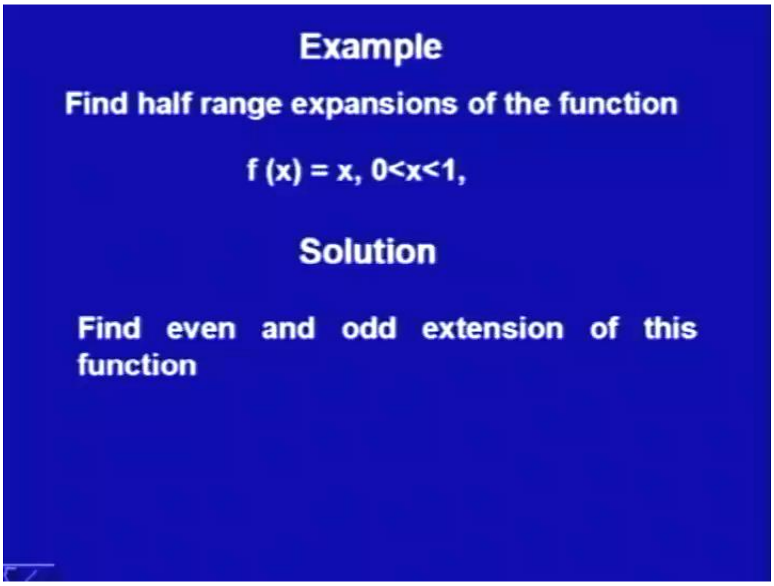

So, see find the half range expansion first the function f x is equal to x, which is defined from 0 to 1 only. Now, the function is defined from 0 to 1, so L is 1 here and the function is the straight line from 0 to 1, what we have to find out the half range expansion that is both even and odd extension, so let us first say what will be this even and odd extension of this function.

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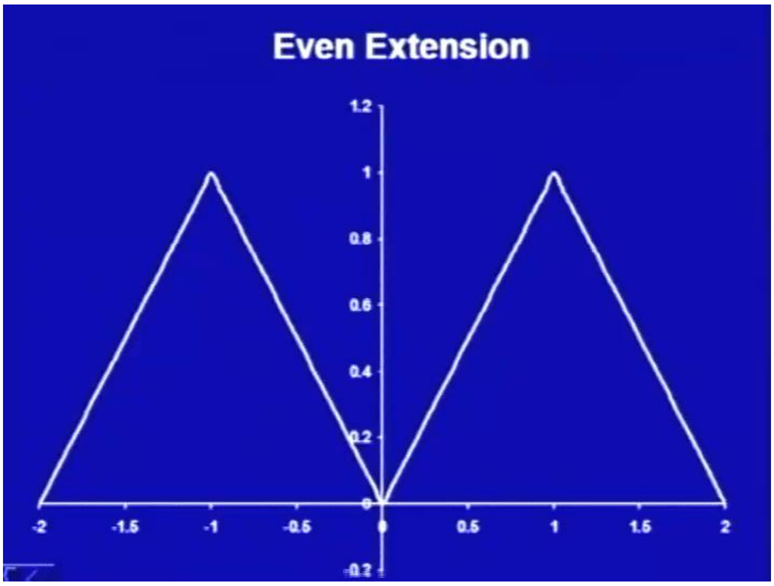

The function is defined 0 to 1 is a straight line, if I just go for the even extension I have to have the same kind of line on the negative side, that is when x is 0.5 my the function value should be plus 0.5. So, when x is minus 0.5 the value of the function at minus 0.5 should also be plus 0.5 value at 1 is 1, so value at minus 1 should also be 1. So, we are getting is that this line is that original 0 to 1 f x here I have made this even extension. So, this is the similar kind of thing.

Then we are making it periodic, periodic means is that is now from minus 1 to plus 1 with the period 2. So, from 1 if it is with peak into 2 it should be repeating this function that is from minus 1 to 0, so we are getting this one, similarly here this reputation has to be from 0 to 1. So, we are getting is this is looking like a triangle, but actually this is extension is only this 1 and then the periodicity is changing or that repeating the values.

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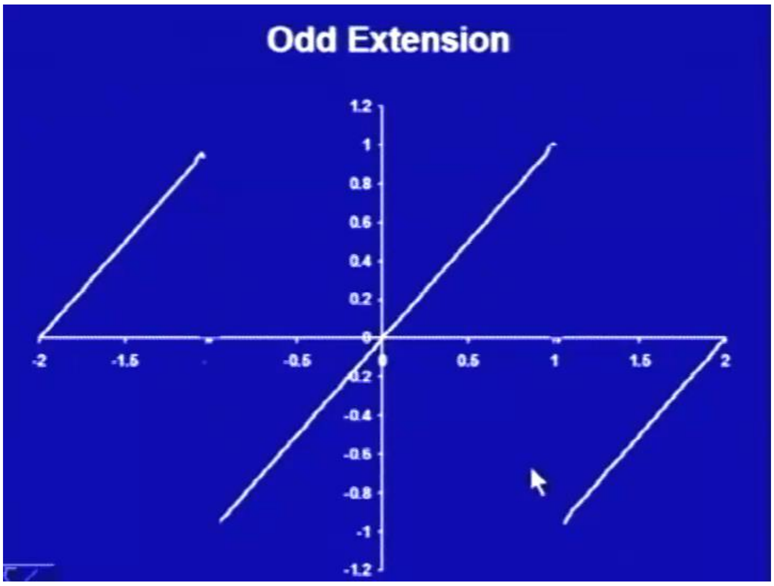

Odd extension, the function was 0 to 1 then I have to make it odd 1 that is whenever f of x is positive, then f of minus x has to be minus of f of x that is negative. So, you see the value at 0.5 we are getting is plus 0.5, so at minus 0.5 I should have the value as minus 0.5 that is this side. Similarly, it any points, so here we are having this point save at 1 the value is plus 1, so at minus 1 the value should be minus 1.

So, we are getting is that this is the original function 0 to 1 and this is it is odd extension again for periodicity. If I am repeating the function from minus 1 to plus 1 again from 1 to 2 if I am repeating it should have minus 1 to 0, that is this same function, similarly if it is repeating from minus 1 to 2 it should have been same as 0 to 1. So, minus 2 to 1 this is again repeating the same function, so this is the periodicity these ones, this is the extension in the interval minus 1 to plus 1, let see its half range expansions.

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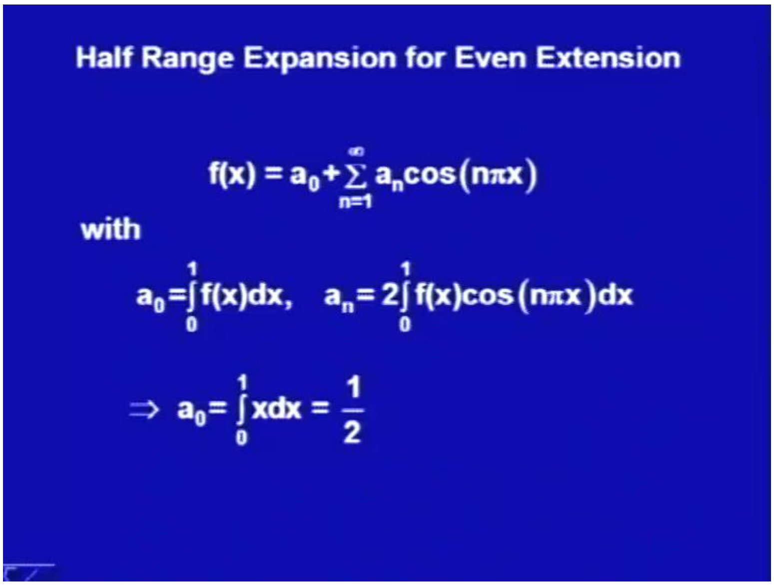

So, for even extension I want this kind of Fourier series, that is half range expansion Fourier cosine series only a naught plus summation n is running from 1 to infinity a n cos n pi x. Since L is here 1, so will not be getting n pi x over L, L is 1, so I would be getting it n pi x only find out what are these coefficients a naught and a 1, a naught should be integral 0 to 1 f x d x and a n should be 2 times integral 0 to 1 f x cos n pi x d x. So, with the given function f x as x with the interval 0 to 1 evaluate it a naught would be 0 to 1 x d x integral that is half.

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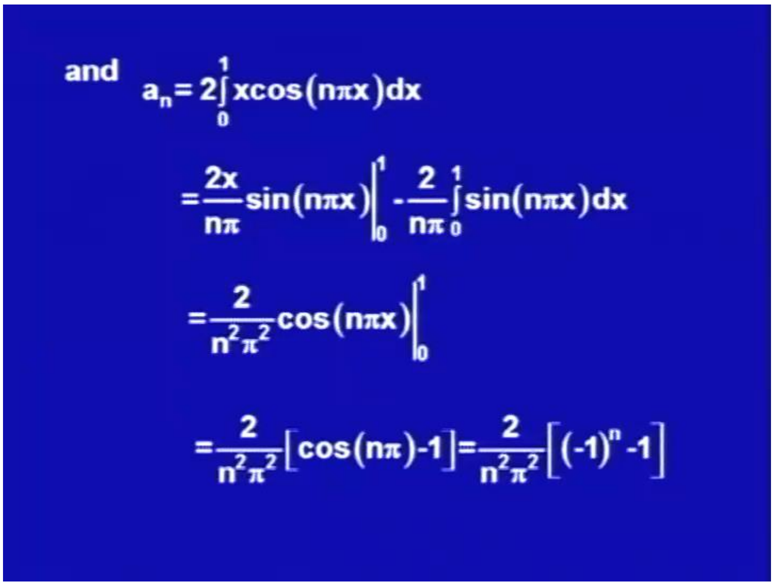

And a n would be 2 times 0 to 1 x cos n pi x d x evaluate this integral by parts, you get 2 x by n pi sin n pi x evaluated at from 0 to 1 minus 2 upon n pi 0 to 1 sin n pi x d x because, this is the first function. And it is derivative is 1 and this is the second function its integral is sin n pi x upon n pi, evaluate it we will get it 2 upon n square pi square since at x is equal to 1 sin n pi is 0 at x is equal to 0 sin 0 is 0, so this term is not giving me anything, here again this integral. So, minus sin n pi x it is integral is cos n pi over n pi.

So, we are getting is 2 upon n square pi square cos n pi x evaluated from 0 to 1 at x is equal to 1 it will be cos n pi at x is equal to 0 it will be cos 0. So, what we would be getting is 2 upon n square pi square cos n pi minus 1, the value of cos n pi is minus 1 to the power n, so we are getting is 2 upon n square pi square minus 1 to the power n minus 1.

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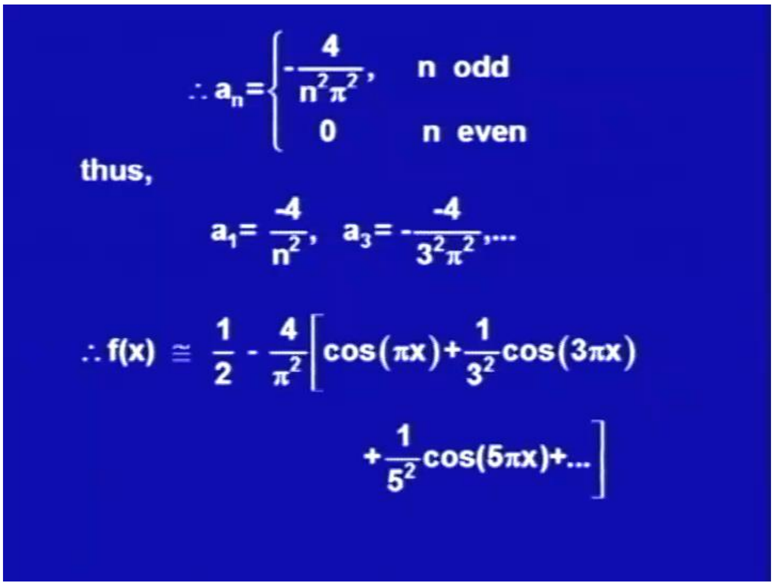

$$\therefore a_n = \begin{cases} -\dfrac{4}{n^2\pi^2}, & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

thus,

$$a_1 = \frac{-4}{n^2}, \quad a_3 = -\frac{-4}{3^2\pi^2}, \ldots$$

$$\therefore f(x) \cong \frac{1}{2} - \frac{4}{\pi^2}\left[\cos(\pi x) + \frac{1}{3^2}\cos(3\pi x) + \frac{1}{5^2}\cos(5\pi x) + \ldots\right]$$

Which says is an we would be getting is for odd n minus 4 upon n square pi square and for even n it would be 0. So, what will be our Fourier series a 1 would be minus 4 upon pi square a 3 would be minus 4 upon 3 square pi square and so on f x a naught we have got as 1 4 upon pi square is common. So, we are getting is 4 upon pi square cos pi x plus 1 by 3 square cos 3 pi x plus 1 upon 5 square cos 5 pi x and so on this is the Fourier series.

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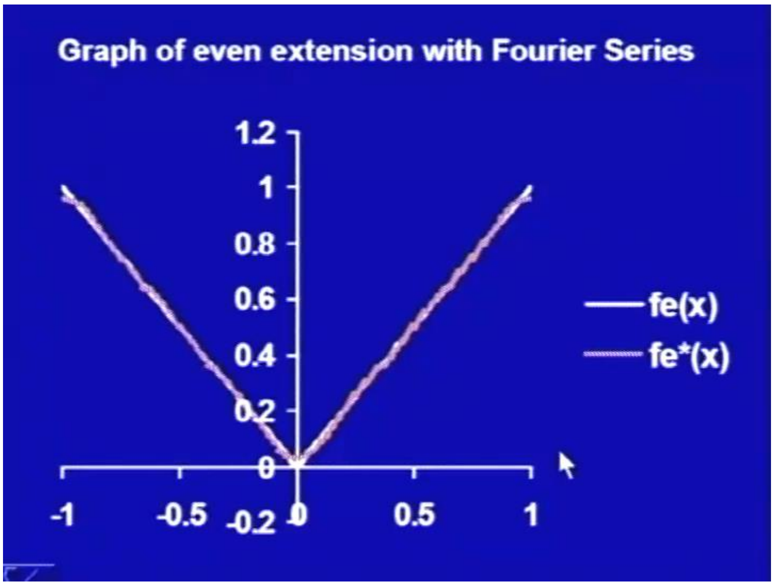

Let us see is how it is making the even extension, this is the even extension 0 to 1 and this even extension over here. When we are plotting the Fourier series over here, we are finding it out that it is at the points where it is changing the values that is at 0 at 1 and minus 1, we are getting is little bit distraction from that or this is not exactly matching over here, but here it is quite nicely overlapping one.

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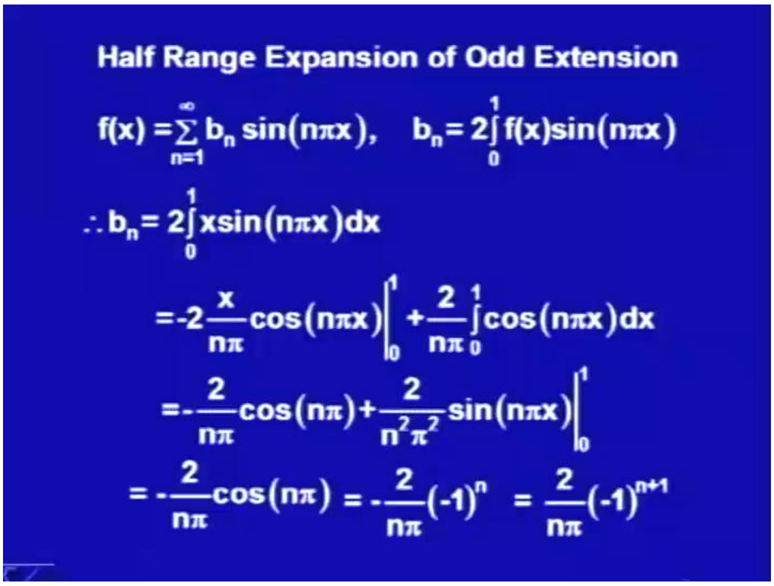

Let us move to the odd extension and it is half range expansion, we want f x as of the form b n sin n pi x where, b n would be 2 times 0 to 1 f x sin n pi x. Hence, evaluate this f x is x for the whole interval 0 to 1, so b n would be 2 times 0 to 1 x sin n pi x d x, again evaluate this integral with by parts. So, this integral would be minus 2 x upon n pi cos n pi x 0 to 1 plus 2 upon n pi 0 to 1 cos n pi x d x because, the derivative of x is 1.

Here, when I am putting x is equal to 1 I will get cos n pi and 1 and when x is equal to 0 this will give me 0. So, we would be getting is minus 2 upon n pi cos n pi plus this integral is sin n pi x over n pi, so we will get 2 upon n square pi square sin n pi x 0 to 1, when x is equal to 1 we will get sin n pi when x is equal is 0 we will get sin 0 both values are 0. So, we are not getting anything from here it will give only 0, we are getting it minus 2 upon n pi cos n pi, the value of cos n pi is minus 1 to the power n. So, minus 2 upon n pi minus 1 to the power n this minus sin we are just joining over here, so we get b n as 2 upon n pi minus 1 to the power n plus 1.

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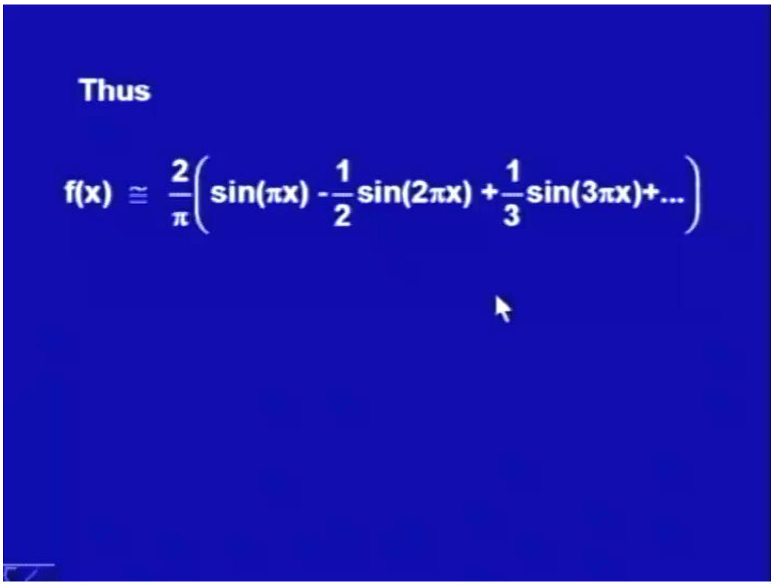

Thus the Fourier series we would get as 2 upon pi sin pi x minus half sin 2 pi x plus 1 by 3 sin 3 pi x and so on, let us see how it is in the graph how it is approximating the function.

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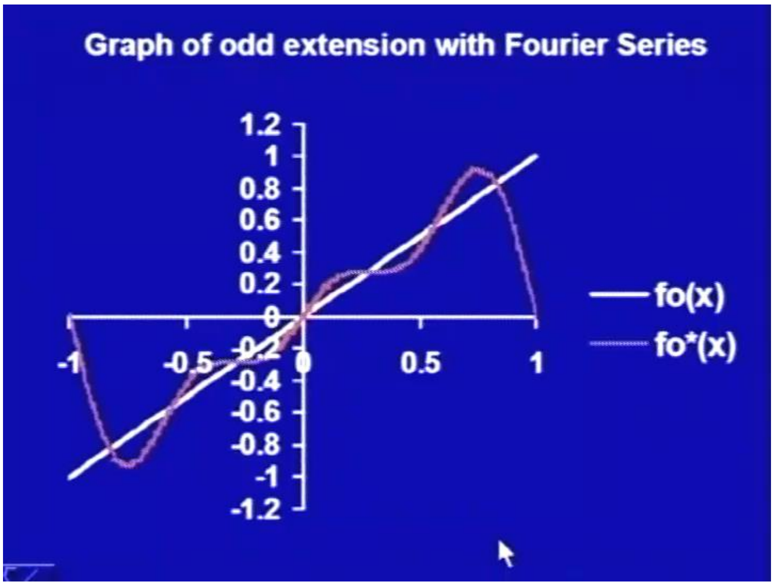

This is what is our odd extension and the Fourier series just this sin series, which we have find it out that is this one. Now, you see here also we are distracting one, but the points is that is when we are ending up both the sides the function as such the function was here and if we see the periodic function from here again we are starting in this manner, that is this has to go like this one.

What it says is at a point 1, if I take the periodic function might the function discontinuous that is if at 1 would be here from the left when we are reaching it was reaching to the plus 1. When from the right we are reaching it is reaching to the minus 1 that is it is discontinuous over here and that is what we are getting is our Fourier series is just going down, if you have seen in the examples where we had find out the Fourier series, where the function of this kind and we are getting is that the values are distracted one.

So, we are getting the Fourier series we are functions we are getting like that one of course, the points are less, but it was joining one. So, that is what we are seeing is it is a joining over here, so this is what is our odd extension, so this is what we had learn today if the function is I having certain properties that is even or odd, we get either Fourier cosine series or Fourier sin series.

Moreover, for finding out the coefficients we are evaluating the integrals only on the half of the period, that is if the period is from minus L to plus L, we are evaluating it only the all integrals from 0 to l. So, when it is even function we are evaluating the integrals for a naught and a n that is the coefficients of the cosine function and we get Fourier cosine series. When the function is odd we are getting only Fourier sin series and the coefficients b n again we are evaluating the integrals from 0 to L that is the half period only rather than minus L to plus L.

This property and one more property we had learn that is, if we are adding the functions or we are taking linear combination of any two functions for which we can find out the Fourier series. Then the Fourier series of that linear combination of those functions can be just given as the linear combination of the Fourier series itself, from these two properties and the property of this even and odd functions, that has given insight to us that we can use this Fourier series for the functions which are not periodic and they are defined only on some interval of the form 0 to l.

What we have done is, we had extended them either to the even extension or to the odd extension, that is the function which is defined from 0 to l only, we have made it from minus L to plus L. Either taking the function from minus L to 0 as the same function from 0 to L that is f of x as same as f of minus x or we have taken an odd extension that is taking f of minus x as minus of f of x and then we repeated it from minus L to plus L made it periodic with period 2 L.

Then we can find out the Fourier series for this extended functions and we called it half range expansions, our aim was actually to find out the Fourier series for the function which is defined only for 0 to L. But, since the method we are knowing is defined for the periodic functions, which has defined on the complete real line, so we made it periodic by extending it from minus L to plus L, either making it the function to be even or making with the function to the odd.

So, that we are getting the integrals or the evaluation of the coefficient the integrals we are using for 0 to l only, that is the given function only which where it is defined from 0 to L. And we had seen that we are in our two examples of course, both the times we had find it out that the even extension was much nicer looking in the graphs, while rather than the odd.

That is because of that particular examples, which we have taken they were discontinuous at the end points. And then Fourier sin series was just going down like that because of the graphs, which we have drawn like that one, but we have got that we can make it from 0 to L. Next, we will see is that is whenever we are finding it out the graph is matching with the function or graph is not matching with the function, what is this properties, what is the value of the function at that particular point. Value of, because we have started the Fourier series, where we want the integral over an interval to be approximated by a series. So, next lecture we will see something more about the Fourier series for today that is all.

Thank you.