Mathematics - III
Prof. Tanuja Srivastava
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture - 11
Fourier series – Part -1

Welcome to the lecture series on differential equations for under graduate students. Today's lecture is on Fourier Series. In series solution of differential equations, we have learnt many series such as the Taylor series and Taylor polynomials.

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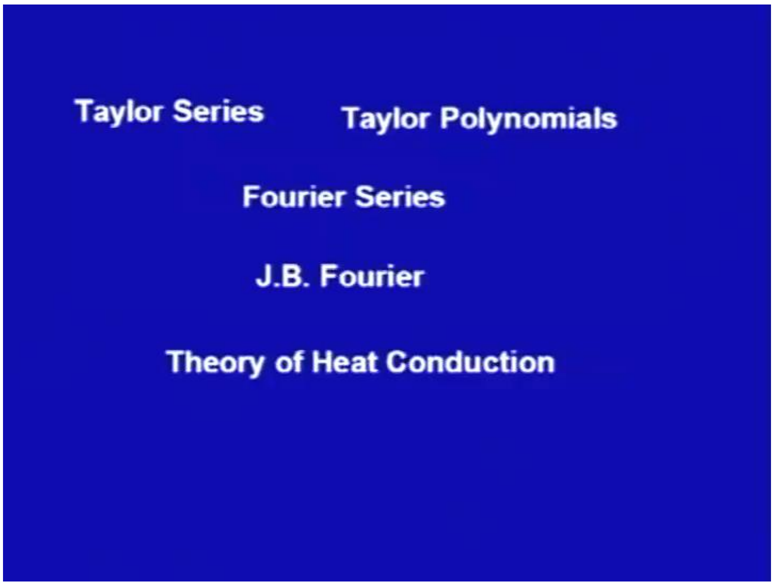

There we had learnt, that if we have to approximate the solution at a one point, we can use the Taylor series or Taylor polynomials or in other words, we had learnt that Taylor polynomials can approximate the function locally, but not globally that is if we have to find approximate the function at a particular point say for example, if we have to find out the limit we can use the Taylor series.

But if suppose we have to approximate the integral of a function over an interval, then of course, this Taylor polynomials or Taylor series they cannot give the answer, then what to do? The answer comes in the form of Fourier series found by J.B. Fourier in the development of the theory of heat conduction. Later audit has been find out that the series, which he has find out that is a very important mathematical tool in solving many other engineering and applications problems. Thus the series have been named after his name and they have been called the Fourier series.

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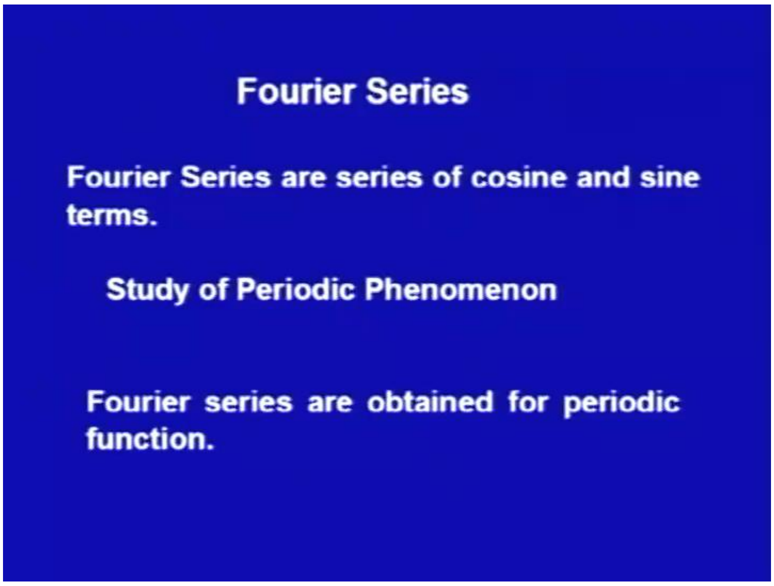

So, now let us try to find out, what is the Fourier series? Fourier series is a series of cos and sine terms this arises, basically in a study of periodic phenomenon or in other words we can say the Fourier series are obtained for the periodic functions, what are periodic functions?

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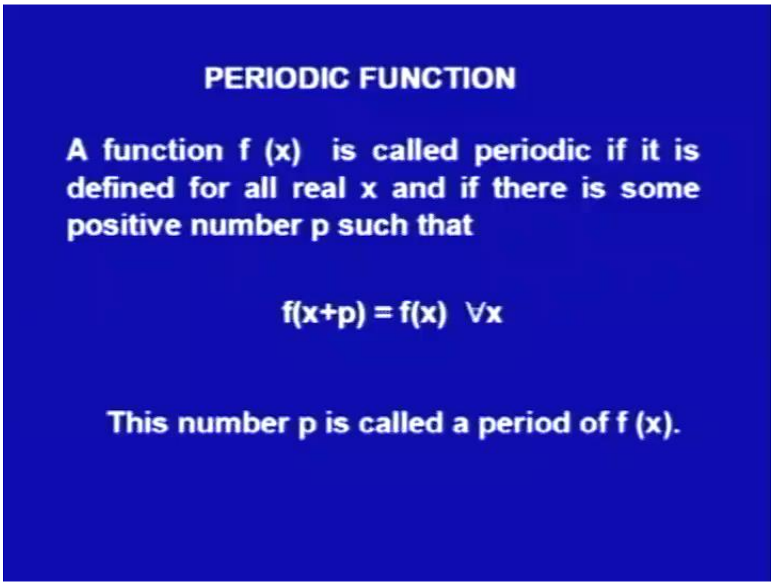

A function f x is called periodic, if it is defined for all real x and if there is some positive number p such that f x plus p is f x for all x that is it is repeating itself after some time and this number p is called the period of f x.

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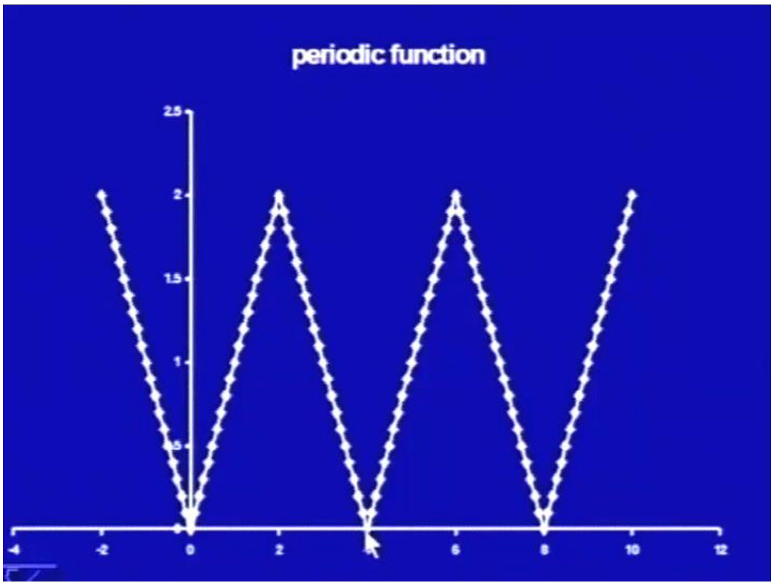

Say for example, in this particular function we see that the function is from 0 to 4 it is a triangle and then again it is repeating in the same one or if we just go like this one this is what is the periodic function, with period as 4.

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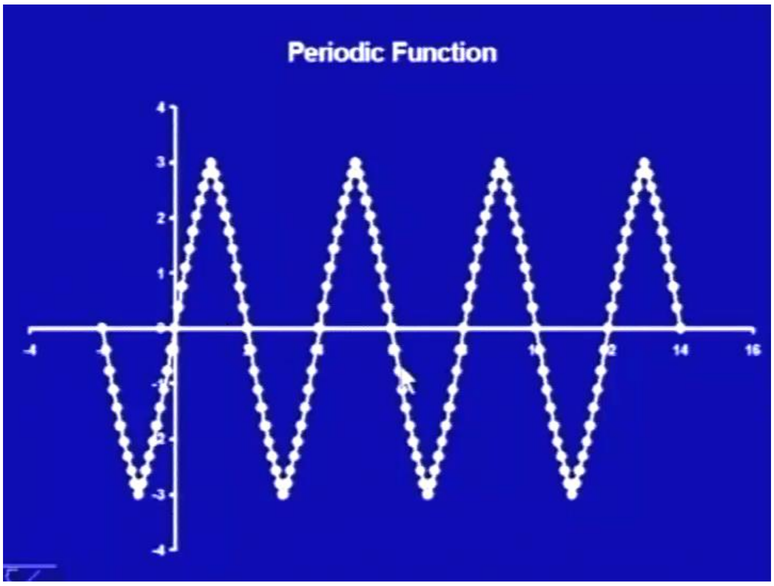

Similarly, let us see this another function here if we try to see from 0 to 2 and then from 2 to 4, again we are having is that function is moving like this one and after that it starts repeating itself that is if I take value at 0, that is same as value at 4, if I take value at 1 this is same as value at 5, so again it is a periodic function.

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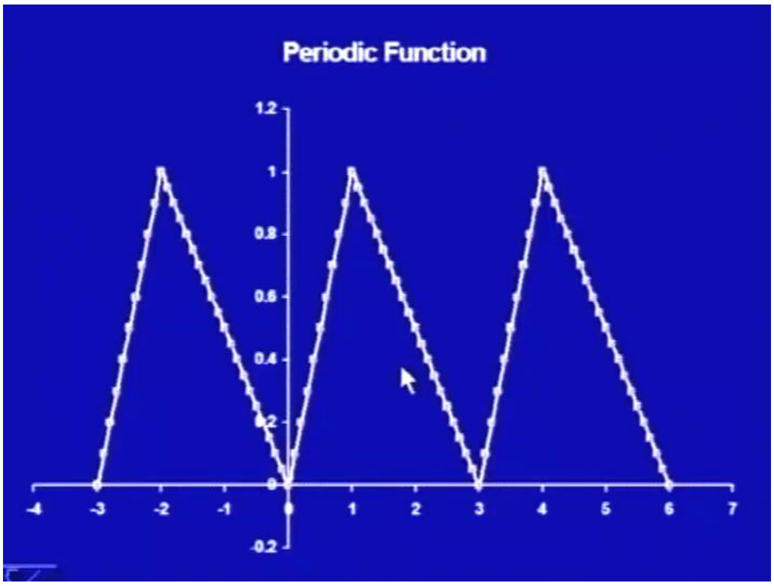

This is also one other example of a periodic function, here is one more example of periodic function.

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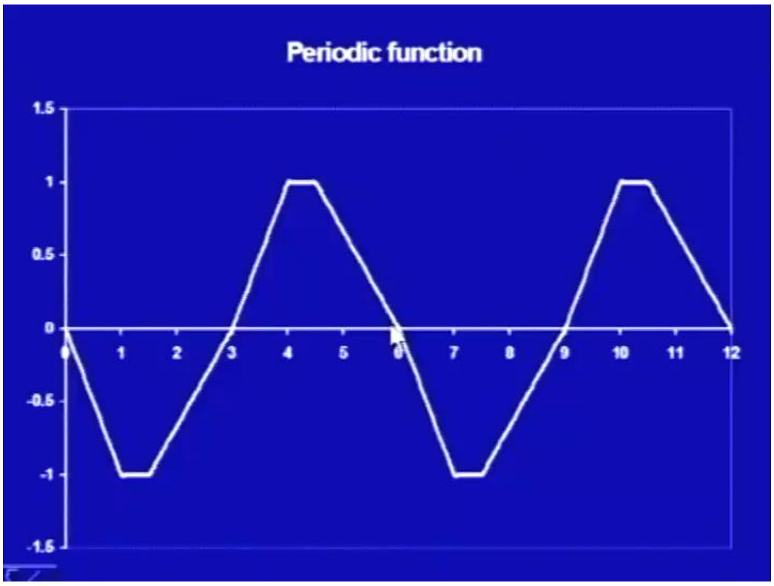

You see here, is that the period you do find out that the function is started from 0 it has gone up to 6 and then again it is started repeating, so now we find out that the period is 6.

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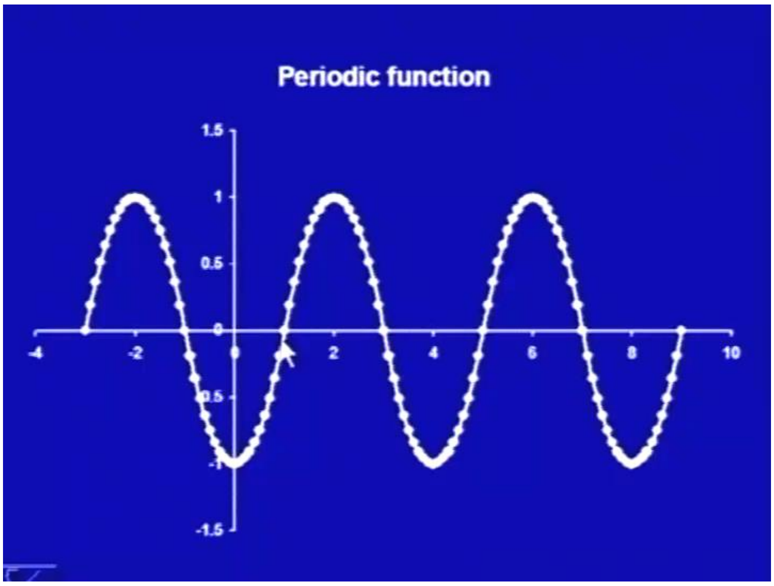

This is again one more periodic function.

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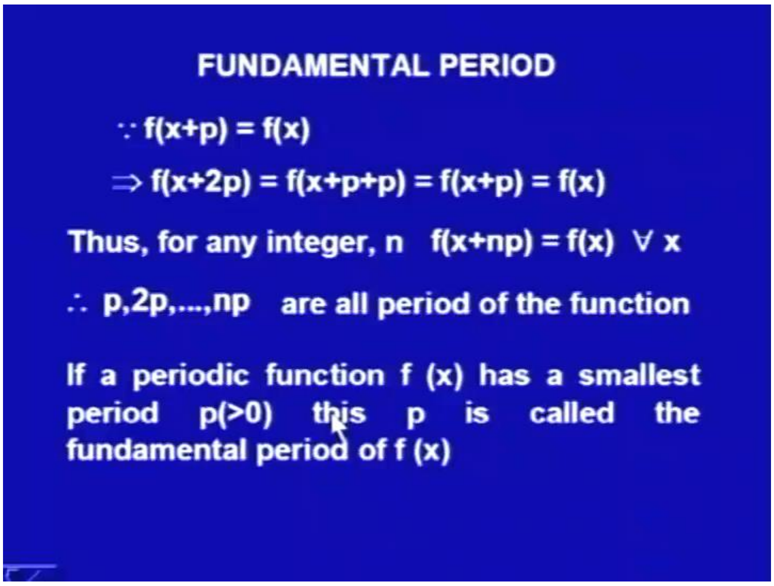

Now, we had seen that f x plus p is equal to f x this is what is we have defining as a periodic function. This says is if I take f x plus 2 p this would be f x of f of x plus p and plus p that we can write it as f x plus p by the first equation, which is same as f x, what it says is that, f x plus 2 p is also a same as f x moreover if we just go on like this one, then for any integer n we can get that f x plus n p would be f x for all x.

So, what it says is now I do have a function which is defined for all real number x such that f x plus p is f x, f x plus 2 p is f x and f x plus n p is same as f x for any integer n that says is my p 2 p or n p all are periods. So, we can say this function to be periodic with period p or 2 p or n p, so all these are period of the function.

If a periodic function f x has a smallest period p, then this p is called the fundamental period of f x. So, we had seen that if it is a period p, then 2 p n p all those would be the periods, so what we would call we just call this p that is smallest integer as the fundamental period and normally we just mean is that this period rather than using the fundamental period, which are simply say period.

So, let us see in our examples this was one periodic function if we see the function is repeating after 4. So, we have started from minus 3 to minus 1 to plus 1, so minus 3 to 1 you see is the function is same, then it is started repeating again it has come if I try to see this whole lot that is from here to here and again if I try to see again it is or rather, you if we just go with up to here and then if we see is again my function is repeating, so either I could say the period is from minus 3 to 1 that is the 4 or from minus 3 to 5, where it should be at till here 5.

So, what we are getting is that they either we could say that the period is this much or the period is this much, but the fundamental period where this is smallest one, when it starts repeating is this 1 that is from minus 3 to 1. Similarly, let us see in this one here also we find it out that it is starts repeating after this time, so this is the smallest value that is 6 after, which its starts repeating, so we just get that the 6 is the fundamental period for this function.

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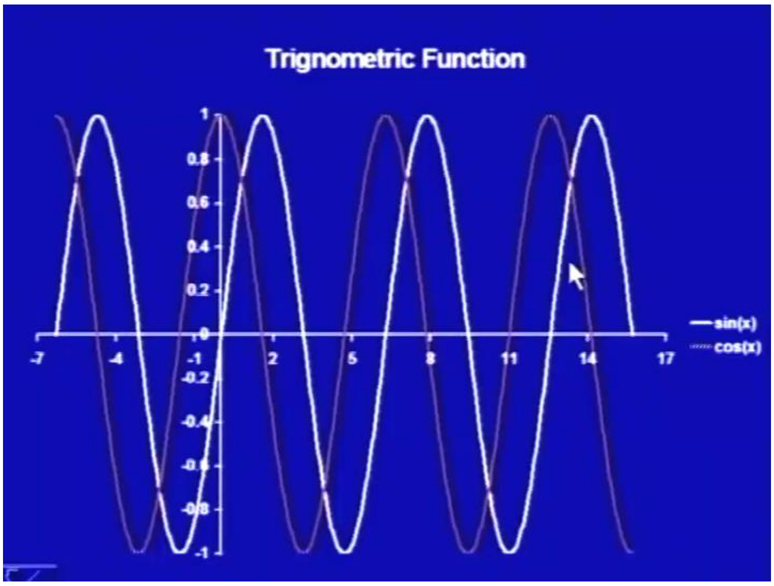

We have this trigonometric functions as we said is that sine and cosine functions we have defined in the Fourier series. So, we see that the sine function this has a period 2 pi, because 0 to pi it is a positive curve and then it is coming is negative, so 0 to 2 pi after that its starts repeating itself. Similarly, for the cos pi if I see again we could say either from minus pi to plus pi or then minus pi to plus pi and then pi to 2 pi. So, we do just find it out that is the function is repeating from here to here and, so that we just simply say that the period of the sine x and cosine x they are 2 pi.

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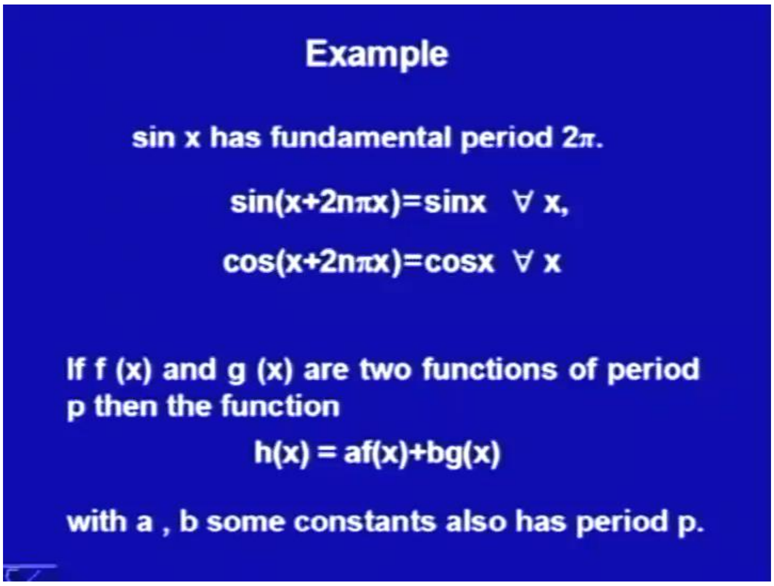

So, sine x has the fundamental period 2 pi that says simply that sine x plus 2 and pi x is sine x for all x this we do know already from trigonometry. And similarly for the cosine function also that is cosine of x plus 2 n pi x is equal to cos x for all x. One more result for this periodic function that is if f x and g x are 2 functions of period p, then if I take a function h x defined as a linear combination of f x and g x that is a of f x plus b of g x this will also be a periodic function and will have the period same as p, where this a and b are some constants.

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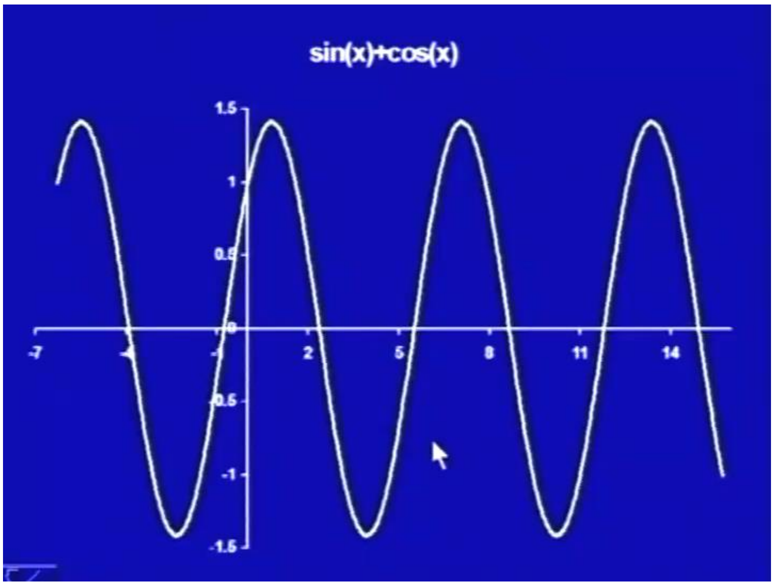

Say for example, if we could see here if I have taken this sine x and cosine x both are periodic function with fundamental period p. So, if I take sine x plus cosine x we find it out that this is again your function with period 2 pi, so this is the graph of sine x plus cosine x.

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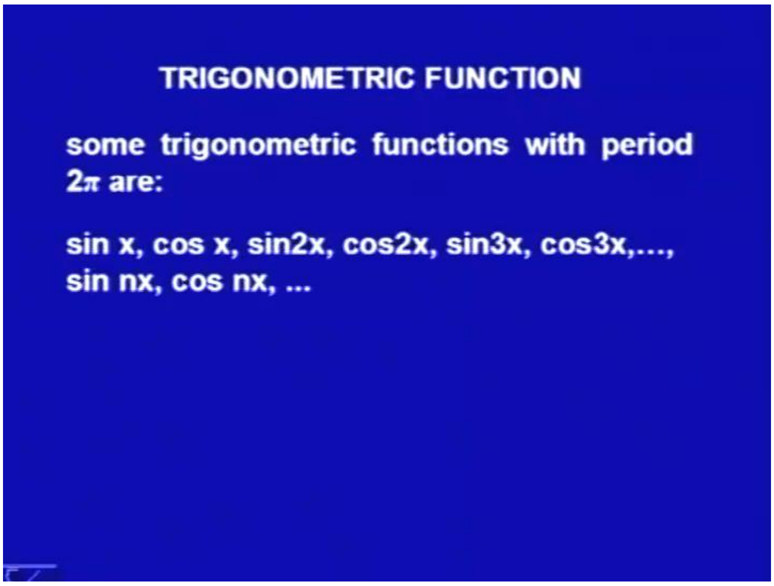

Now, trigonometric functions we will just come up some trigonometric functions with period 2 pi they are sine x, cosine x, sine 2 x, cosine 2 x, sine 3 x, cosine 3 x and so on.

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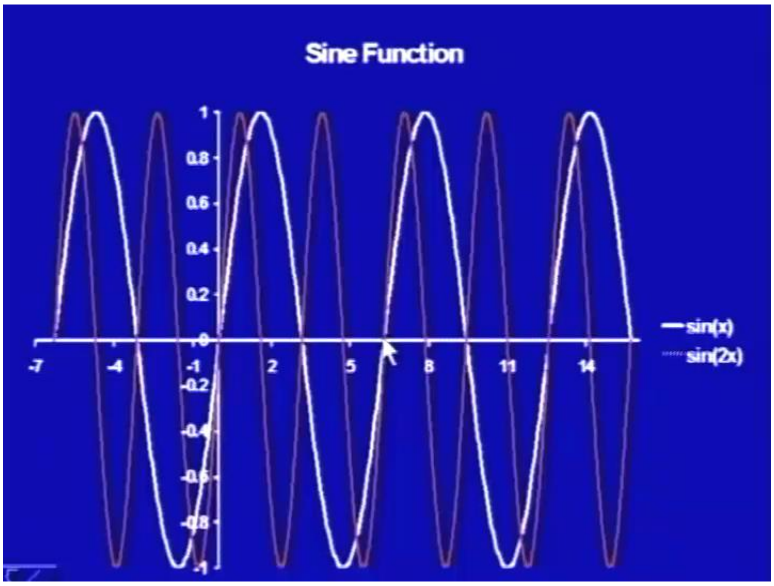

Let us see their graphs, here is the function sin x and sin 2 x this white line this is sin x function its period is 2 pi, so let say 0 to 2 pi from here to here, then this coloured line this is sin 2 x. We see is again it is also repeating after 2 pi, but we see here the fundamental period for sin 2 pi would be actually sin 2 x would be actually pi, you see it is repeating after here itself, but of course, as we say is that if p is the period 2 p and n p they all of the them called can be called as period. So, here we say is that sin x has the period 2 pi sin 2 x also has the period 2 pi.

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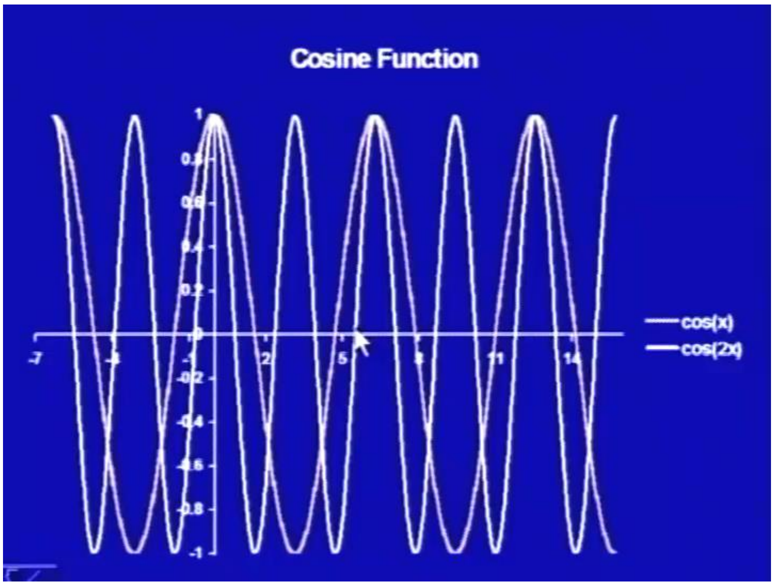

Similarly, for the cosine x we do have is that cosine x you do have the period from 0 to 2 pi and then again cosine 2 x this is cosine 2 x is this white line. So, you are finding it out again it is repeating after pi, but we could just go ahead with till pi and we are finding it out that this is also of 2 pi.

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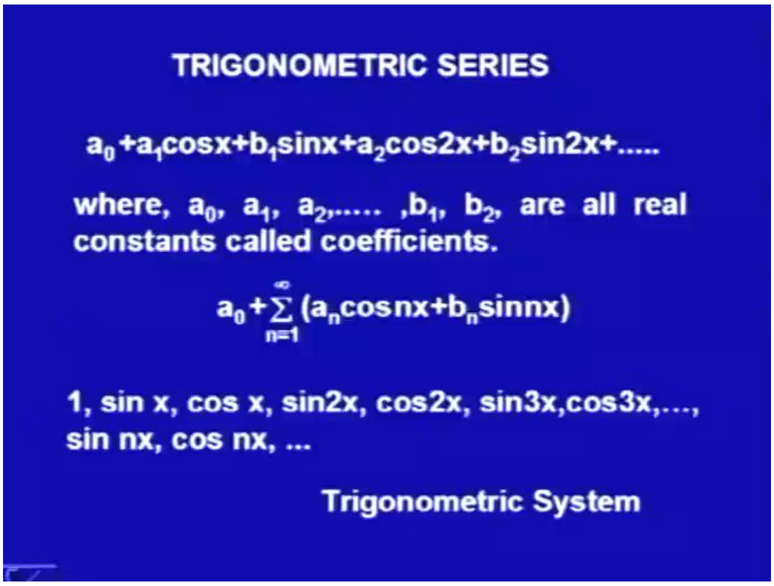

So, we can say that is sine x cosine x sine 2 x cosine 2 x and so on. They would be all of the period 2 pi, so if I take this series a naught plus a 1 cosine x plus b 1 sin x plus a 2 cosine 2 x plus b 2 sin 2 x and so on. Here cosine x sin x cosine 2 x sin 2 x all these have period 2 pi a naught is a constant, constant function we can have with any period, so we can say as that period 2 pi again.

So, we do have this is a linear combination of the functions with the period 2 pi, so this series would be actually a function with period 2 pi. So, this trigonometric series we could say is a function with period 2 pi, where all these a naught a 1 b 1 b 2 they are real constants and they are called the coefficients.

We can rewrite this series as a naught plus summation n is running from 1 to infinity a n cos n x plus b n sin n x. Now, what we are getting is 1 sin x cosine x sin 2 x cosine 2 x and so on, these we would be calling the trigonometric system, so now we are just formulated the terms before going for the Fourier series, Fourier series we said is a series of containing the terms of sin x and cosine x.

Here, we have taken a trigonometric series, which is containing cosine x and sin x terms and moreover, we had also learnt one thing is about that the periodic functions and we had find out that the sin x cosine x all of them are periodic functions. So, this series we could treat as a function with period 2 pi this basic functions 1 sin x cosine x and so on, this system we would call trigonometric system, now come to the Fourier series.

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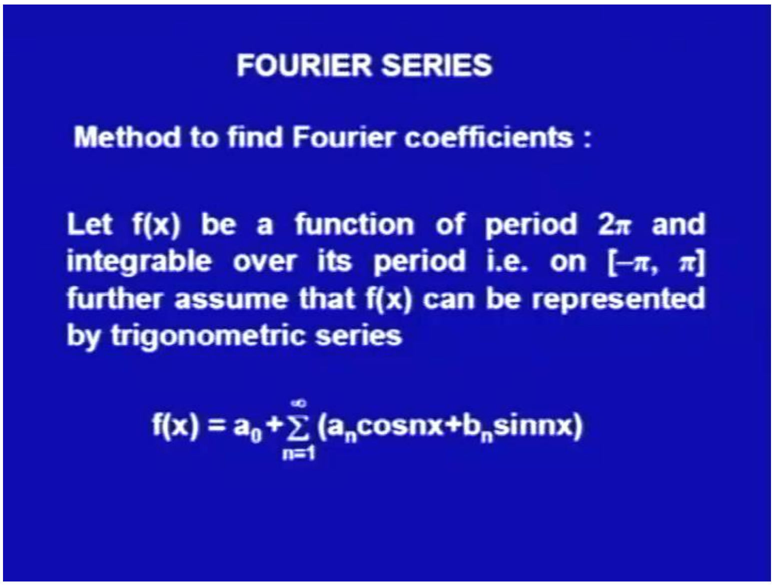

How do we find out the Fourier coefficients, what we are saying is that is the trigonometric series, which we had find out we said is that is it is containing the functions, so this complete series can be taken as a periodic function with period 2 pi. So, the question comes only that is if any function, which is of period p or period 2 pi that can be approximated by that trigonometric series only thing is we have to find out what are those coefficients a naught a 1 a 2 b 1 b 2 and so on, how to find it out, let us see.

First, we are assuming that f x be a function of period 2 pi and this is integral over its period, whenever it is having a period 2 pi normally we write it out that is it is from minus pi to plus pi rather than writing it from 0 to pi. This convention we can write anything, but normally we write minus pi to plus pi further assume that f x can be represented by trigonometric series as a naught plus summation n is running from 1 to infinity a n cos n x plus b n sin n x.

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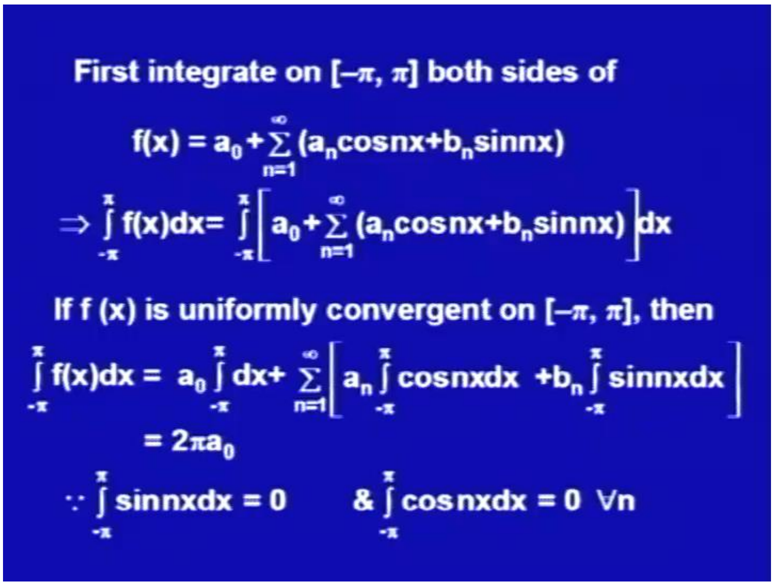

Now, we integrate this function and the series on both the sides from minus pi to plus pi, so we integrate it this says is that I would get integral minus pi to plus pi f x d x is same as integral minus pi to plus pi, a naught plus summation n is running from 1 to infinity a n cos n x plus b n sin n x d x.

Now, integrate it out the right hand side for integration we are assuming that f x is uniformly convergent on minus pi to plus pi, then this integral can be written the right hand side we can write as the integral of some functions I can write as sum of integral function these things, you might have learn in some other courses that is when we can integrate or the series and integration sign that summation integration sign can be interchanged.

So, like that if I am writing that it should be a naught minus pi to plus pi d x plus summation n is equal to 1 to infinity a n integral minus pi to plus pi cos n x d n x plus b n times minus pi to plus pi integral sin n x d x, that should be equal to 2 pi a naught why, because integral minus pi to plus pi sin n x d x is 0 and integral minus 2 pi to plus pi cos n x d x is 0 for all n. So, what is been left is only the first term a naught minus pi to plus pi d x its integral is x.

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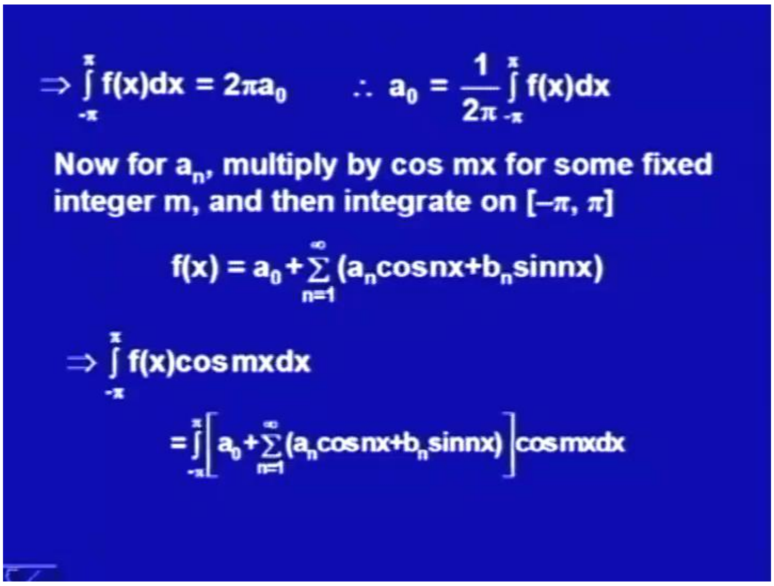

So, from plus pi to minus pi this is 2 pi, what it give me it give me that integral minus pi to plus pi f x d x is 2 pi a naught or in other words a naught is equal to 1 upon 2 pi minus pi to plus pi f x d x, so the first coefficient a naught we had find that, now for the other coefficients.

So for a n, what we will do we will multiply with cos n x for some fixed integer n and integrate from minus pi to plus pi this whole series that is I multiply both the sides with the cos n x, here and here this whole series and then integrate over minus pi to plus pi. This says is left hand side minus pi to plus pi f x cos n x d x would be same as minus pi to plus pi integral a naught cos summation of n running from 1 to infinity a n cos n x plus b n sin n x into cos n x d x.

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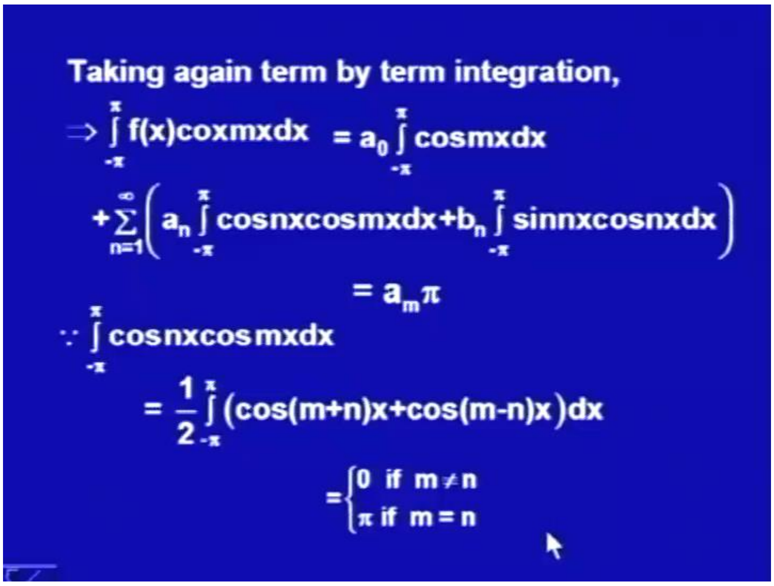

Now, again we will assume that uniform continuity and convergence and that this series summation and this integral sign, they can been interchanged that will give me this minus pi to plus pi f x cos m x d x as a naught integral minus pi to plus pi cos m x d x plus summation n is running from 1 to infinity a n integral minus pi to plus pi cos n x cos m x d x plus b n integral minus pi to plus pi sin n x cos n x d x, now integrate it.

The first term just, now we had seen that in the last integral that integral minus to pi to plus pi cos m x d x is 0 for all integer n, so this will be 0 and then we have integral cos n x, cos m x sin n x and cos n x. This integral will come as a m times pi y, let us see we have to integrate one by one these integral.

So, let us see first this integral cos n x, cos m x this integral this function cos n x, cos m x we can write as half of cos m plus n x plus cos of m minus n x d x, and now this m plus n and m minus n both are integers m plus n is an integer m minus n would be an integer, but it would be 0 if m is same as n and then what we are getting is that if m is not equal to n, I would get this integral as some cos p as a k x and some cos l x both are integers and we already know that integral minus pi to plus pi cos n x is 0 for all integer n.

If m is equal to n in that case it would be this part would be 0 of course, but here what we would get we would get cos 0, cos 0 is 1. So, what we will get integral minus pi to plus pi 1 d x and that would be 2 pi half into 2 pi that would give me pi, so we are getting is that this first integral would be pi if m is same as n moreover this integral this integral, you could see that it is it would be 0.

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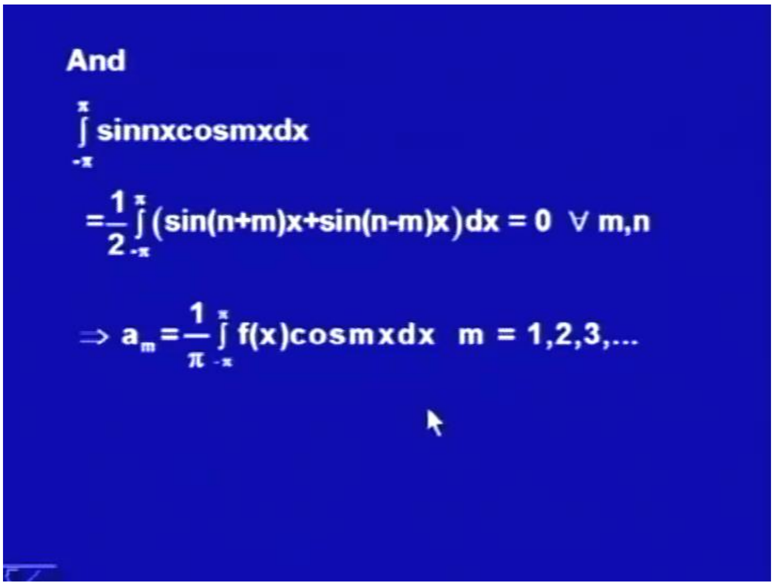

So, now come up to this second integral sin n x, cos m x, d x this we could write as sin n plus m x plus sin n minus m x d x. Now, this is again sin k x and this is sin l x, when n is not equal to m and we already know that integral minus pi to plus pi sin n x d x is 0 for all integer n, when n is equal to m this is some integer, but this would be 0 and sin 0 is 0.

So, again this whole function would be 0 for all m and n that is why the second part of the integral would not be contributing anything and what we have got the result, we have got that a m is equal to 1 upon pi minus pi to plus pi f x cos m x d x this is true for all m 1 2 3 and so on.

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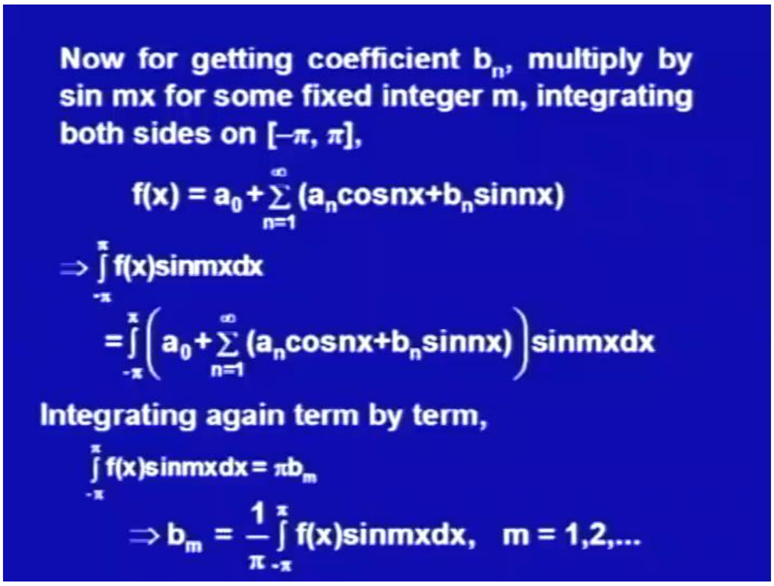

Now, move to find out b n for finding out b n, what we will do we will multiply with sin m x this series f x a naught plus summation running from n is equal to 1 to infinity a n cos n x plus b n sin n x and integrate it on complete interval minus pi to plus pi. So, what we would get that left hand side integral minus pi to plus pi f x sin m x d x and right hand side as minus pi to plus pi a naught plus summation running from 1 to infinity a n cos n x plus b n sin n x multiplied with sin m x d x.

Again we will assume that the uniform convergence and its summation sign and integral sign can be interchange, so I would get it first term a naught sin m x d x that is 0, integral minus pi to plus pi. Second term second we would be getting is a n times integral minus pi to plus pi cos n x into sin m x d x. And that in the previous integral we have just find it out that there it was cos sin n x and cos m x that is n and m both interchanged that is all, but that does not make any difference that was 0 for all n and m.

So, the first integral that is the coefficients a n is that is whatever, we are getting that integral would be 0. Another integral would be b n as sin n x into sin m x and that is pi b m y, because we could see is that that the second integral sin n x into sin m x that can be written as cos m plus n x minus cos m minus n x like that And that again would be pi only when m is equal to n and that is, what we are getting here it is pi b m thus we are getting it that b m is equal to 1 upon pi integral minus pi to plus pi f x sin m x d x for all m 1 2 and so on.

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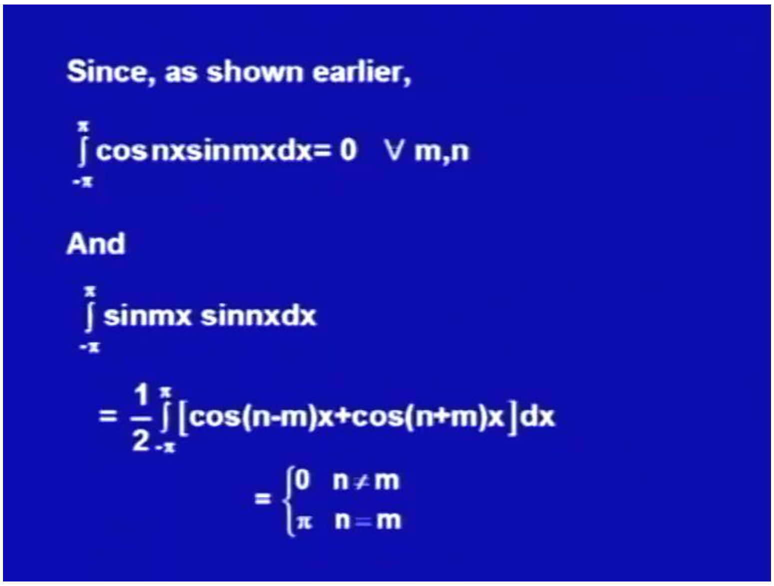

So, we see is that is as earlier we have shown that cos n x sin m x d x is 0 for all m and n and this sin m x sine n x that can written as half times minus pi to plus pi integral cos n minus m x plus cos n plus m x d x and that would be 0 if n is not equal to m and if n is equal to m that would be pi.

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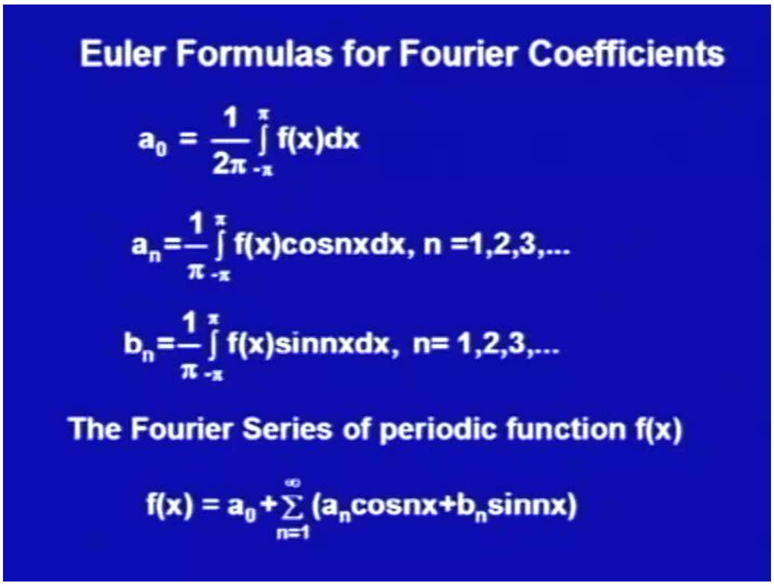

So, we have got the coefficients these coefficients we call that Euler formula for Fourier coefficients, what we had obtained? We had obtained that a naught is equal to 1 upon 2 pi minus pi to plus pi f x, d x a n is 1 upon pi minus pi to plus pi f x cos n x, d x and b n is 1 upon pi minus pi to plus pi f x, sin n x, d x and this is true for all n 1 2 3 and so on.

These are called the Euler formula for Fourier coefficients, what are the Fourier coefficients, that is if we do have any periodic function with period 2 pi, we can represent it as a trigonometric series containing a naught plus summation n is running from 1 to infinity a n cos n x plus b n sin n x kind of thing and, where this a naught a n and b n, we can obtain using this Euler's formula for the coefficients. So, the Fourier series of the periodic function f x is a naught plus summation n is running from 1 to infinity a n cos n x plus b n sin n x with Fourier coefficients a naught a n and b n as obtained above.

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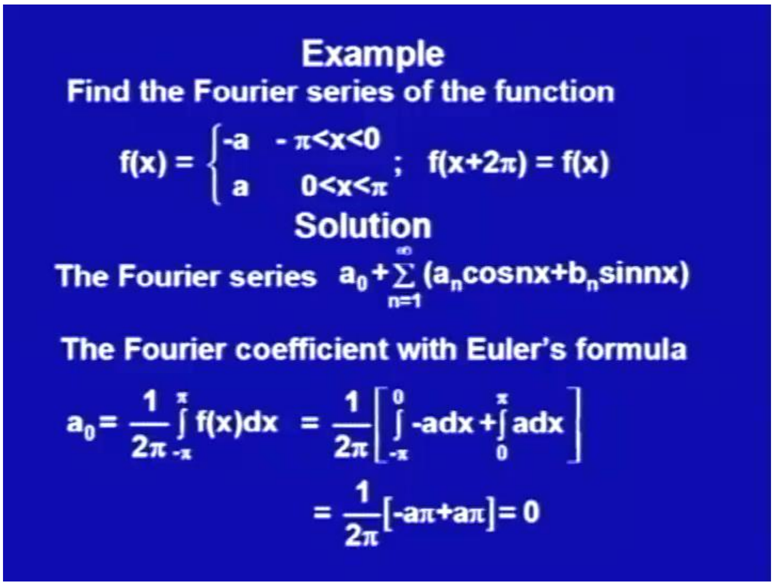

Now, let us try with example find the Fourier series of the function f x, which is minus a in the interval minus pi to 0 and a in the interval 0 to pi and its periodic that is f x plus 2 pi is equal to f x, you have to find out the Fourier series of this function. Let us see, how we are going to do, the Fourier series is we have to obtain a trigonometric series of the form a naught plus summation n is equal to 1 to infinity a n cos n x plus b n sin n x.

And, the Fourier coefficients will obtain by the Euler's formula, so first for a naught 1 upon 2 pi minus pi to plus pi f x d x, f x is the above function, so what we are finding it out minus pi to 0 it is minus a and 0 to pi it is a d x. So, if I integrate over here, what I do get is, minus a pi plus a pi that is 0, so a naught is 0.

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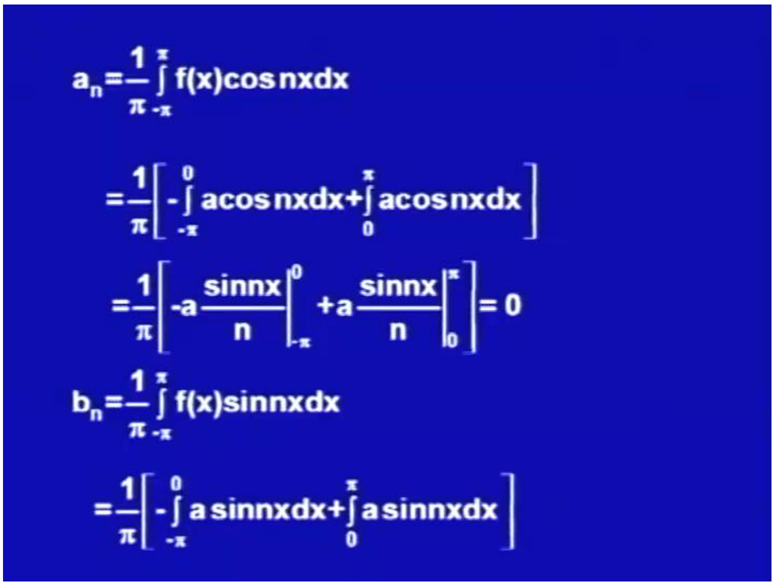

Now, let us come to the a n a n is 1 upon pi minus pi to plus pi f x cos n x d x my f x is minus a n minus pi to 0 and from 0 to pi it is a. So, we just write this integral as 1 upon pi integral minus pi to 0 as minus a times cos n x d x and 0 to pi as plus a times cos n x d x. Now, integrate it we get minus a times sin n x upon n 0 to minus pi to 0 plus a times sin n x upon n from 0 to pi.

And, we do know that as x is equal to 0 sin 0 is 0 and at x is equal to pi, whether plus or minus sin n pi is always 0, so we are getting this is 0. So, we have got a naught is 0 as well as a n is 0 for all n, now let us come to the b n, b n is 1 upon pi minus pi to plus pi f x sin n x d x. Again we will put the value of f x that is from minus pi to 0 it is minus a times sin n x d x plus integral 0 to pi a times sin n x d x, now if I integrate it.

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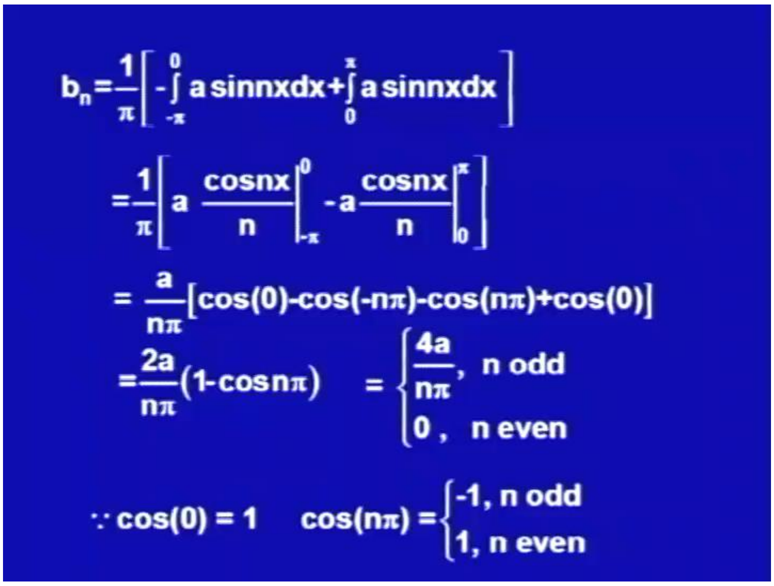

We do get it as 1 upon pi a times integral of sin n x is minus cosine n x upon n and minus a times cosine n x upon n 0 to pi. So, when x is 0 cos 0 is 1 from x is minus pi it is minus cos n pi and, when cos is plus pi x is plus, plus pi put, because pi, so we would be getting it as a upon n pi cosine 0 minus cosine minus n pi minus cosine n pi plus cosine 0.

Since, cosine of minus x is same as cosine of x, we would be getting it as 1 minus cosine n pi and 2 as common cos 0 is 1. So, what we are getting is for n odd; that means, n is 1 3 and 5 and so on, I would be getting it as cos pi by 2 that is cos n pi by 2 is 0, so I would be getting it as 4 a upon n pi and it will be 0 for n even. So, we would be getting it since cos 0 is 1 and cos n pi is minus 1 for n odd and plus 1 for n even.

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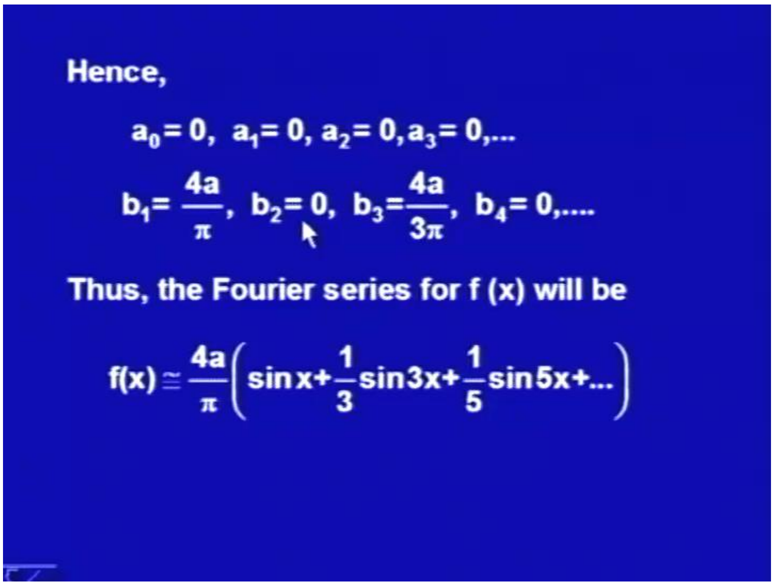

So, this is what we have got b n that says is b n we have got as a naught 0 1 a 2 is 0 b n is 4 a b 1 is 4 a upon pi b 2 is 0 b 3 is 4 a upon 3 pi and so on. Thus the Fourier series we would obtain is 4 a upon pi sin x plus 1 by 3 sin 3 x plus 1 by 5 sin 5 x, you see is that coefficients a naught a 1 they are 0, we are getting just only the b n is and that summation n is running from 1 to infinity b n sin n x that is n would be making it b 1 is 4 a upon pi b 2 is 0. So, again we would be getting is 3 times, so this is 4 a upon 3, so 4 a upon pi we have taken common sin x plus 1 upon 3 sin 3 x plus 1 upon 5 sin x sin 5 x and so on.

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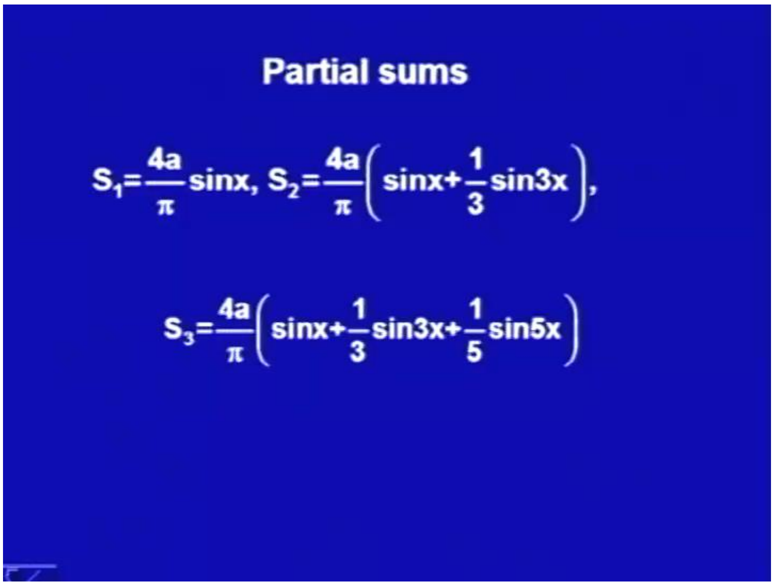

Now, let us see is that is, what we have obtained for this function if I take only the first term S 1, we call it 4 a upon pi sin x, this is the first partial sum we are saying S2 as 4 a upon pi sin x plus 1 by 3 sin 3 x and so on say, let us say S 3 till 3 terms we are just writing it over here, and let us try to see if I try to approximate the function by this Fourier series, let us see that is, how it is approximating.

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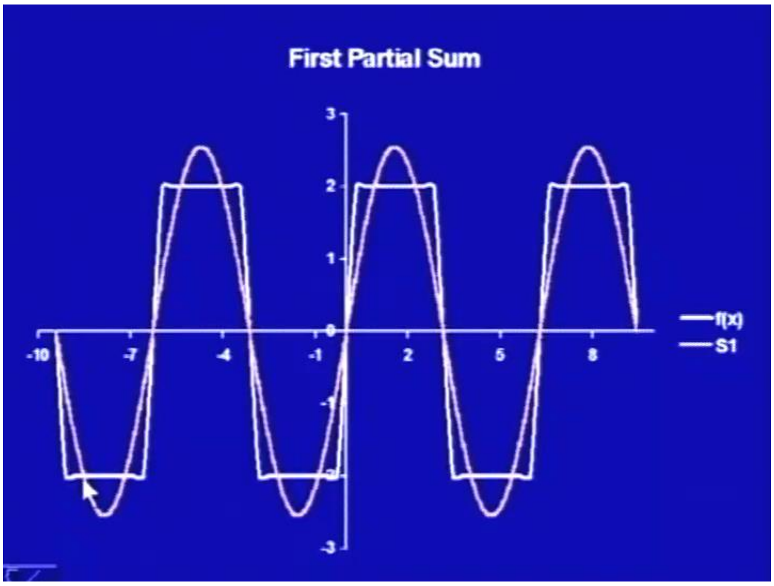

Let us see by the graph, if I take only S 1 this is the function minus a we could say is 0 to minus pi it is minus a and 0 to pi it is plus, so this is the periodic function you are getting is just this is square kind of is square waves kind of function it is coming up. And, if I approximate it by first partial sum that is my only 4 a upon pi sin x we are getting this function, you see is that is we are approximating it. So, we are getting is that it is matching from this end points and all those things, but as we are approaching towards this upper ones we are reaching above ones.

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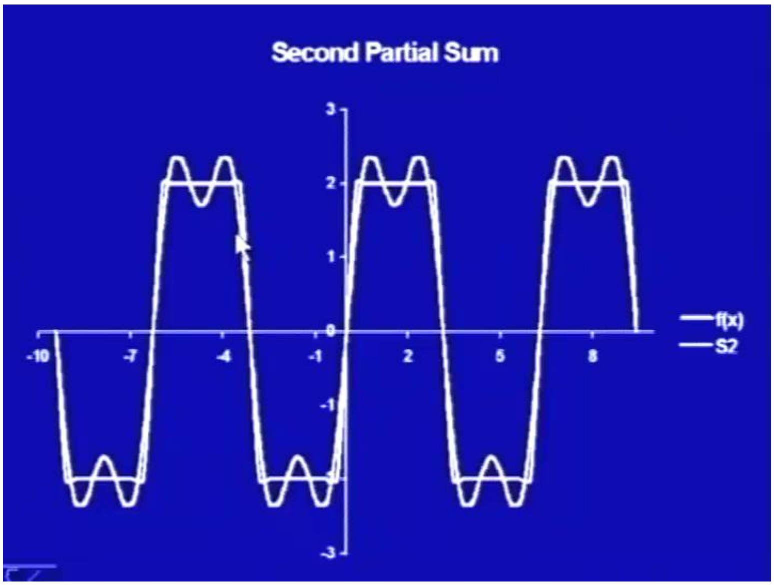

Let us see, if I use the first two functions that is S 2, now you see is that is we are first we are moving varying for our, now we are coming little bit inside here, let us see if i am just using the third function that is S3.

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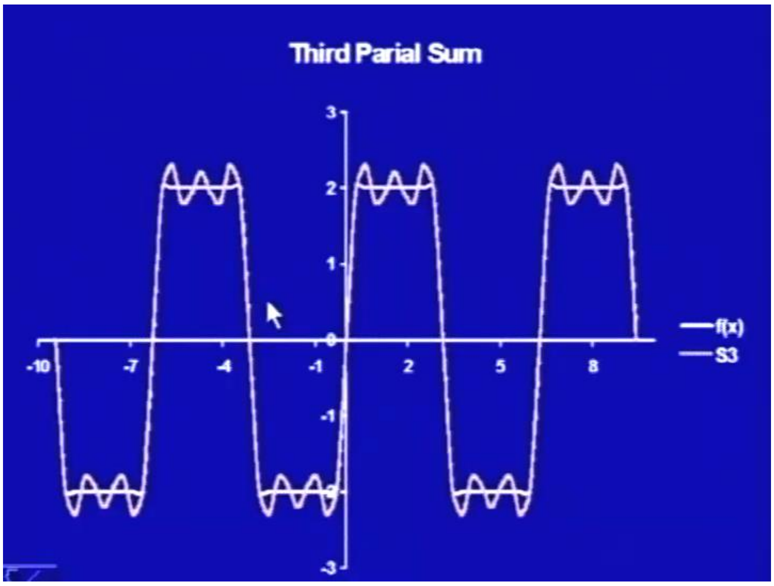

That is 3 terms we had used, now we are approximate, so now if I am going on increasing my n that is if I am approximating my function by this Fourier series as i am increasing by the number of terms in the approximation.

I would be getting is that this function would be more near to this whatever the function actual function, we are getting is you see is that is here also, we are matching too much, you see in the previous one also.

First one, you see we are deviating from here, then we have started matching it till here, so little bit deviation over here, let us see if this points and then we have gone this. Third one, we find out that is we are not deviating, we are deviating only at this point and then as we are going to increase the number of terms in this one, we would be more stabilising or we would be more correctly approximating this function over this whole interval or that is on the whole real line.

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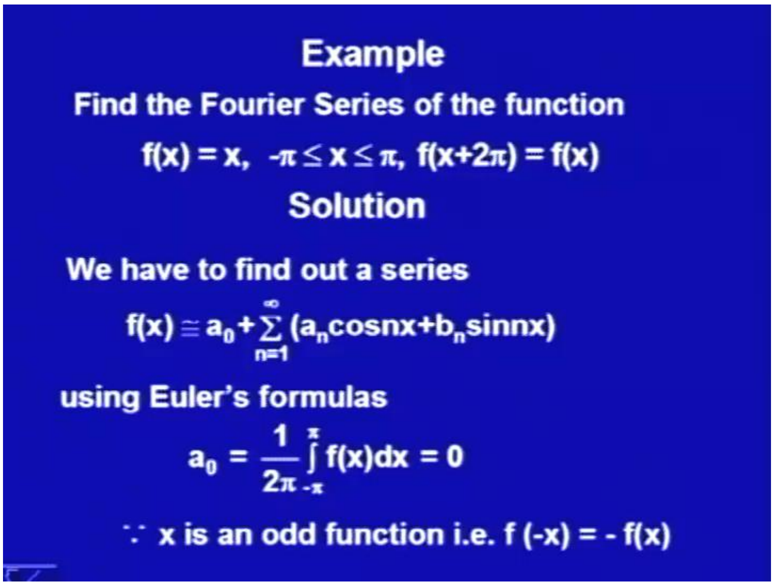

Let us see one more example, find the Fourier series of this function f x is equal to x from minus pi to plus pi and its periodic that is f x plus 2 pi is equal to f x. Since, we have just obtained only to find out the Fourier series for the functions, which has period 2 pi, so we are just taking those examples.

So, let us see now, what we have to do that simply says is we have to find out a series of this form a naught plus summation n is running from 1 to infinity a n cos n x plus b n sin n x and this coefficients a naught a n and b n we would be using by finding by using this Euler's formula, so a naught is 1 upon 2 pi minus pi to plus pi f x d x. Since, my function f x is x from minus pi to plus pi we do find it out that f of minus x is same as f of x, so we are minus of f of x.

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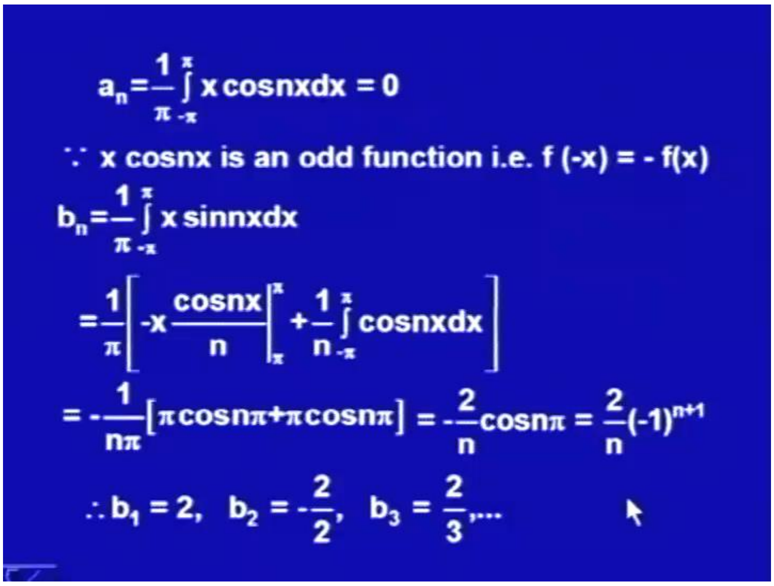

So, we are getting it as 0 it is an odd function moreover when we come to this a n 1 upon pi minus pi to plus pi x sin f x sin n x d x this is also 0, because minus pi to plus pi x sine n x d x is also 0, since x cos n x is an odd function that is f of minus x is minus of f of x. So, we have b n, now 1 upon pi minus pi to plus pi x times sin n x d x, now integrate it by term by parts.

So, it is 1 upon pi the integral of sin n x is minus cosine n x upon n integrate from minus pi to plus pi evaluate and then another integral plus 1 upon n minus pi to plus pi cosine n x d x, because the derivative of x is 1. Now, when I am keeping the values and evaluating it I would be getting it as minus 1 upon n pi pi cos n pi plus pi cosine n pi that is the second integral as well.

And, this is minus 2 upon n cos n pi, which is 2 upon n minus 1 to the power n plus 1, because cos n pi is minus 1 to the power n and 1 minus sin I have taken from here that is this minus sign have merged over. So, what this is, what we have got b n, which says is my b 1 would be 2 n is 1 b 2 would be 2 upon 2 that is and with the sign would be minus 1 to the power 3 that is minus b 3 would be 2 by 3 with plus sign and so on.

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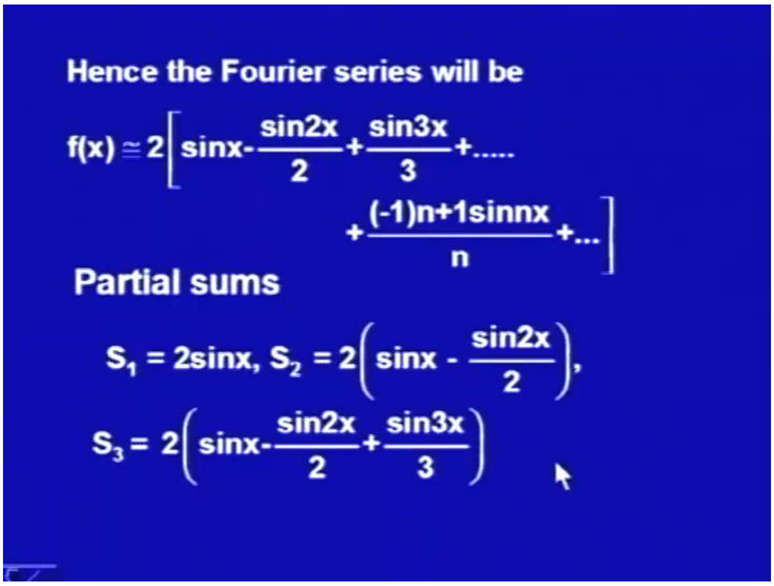

So, what we will get the finally, Fourier series a naught and a n is we have got 0 and this 2, I have taken as common first term sin n x minus second term with minus 1 by 2 sin 2 x plus sin 3 x with 1 by 3 and so on, will go on that is next sign would be minus actually, so this is, what is my Fourier series, now again try to approximate it with the function in the graph we would see.

So, for that lets first find out that the partial sums, partial sums we are calling it that is the first term, second term that the sum up to first term, second term and so on. So this is, what is my complete Fourier series, and the partial sums are S 1 is 2 sin x S 2 is 2 times sin x minus sin 2 x by 2 s 3 is 2 times sin x minus sin 2 x upon 2 plus sin 3 x upon 3 and so on like that.

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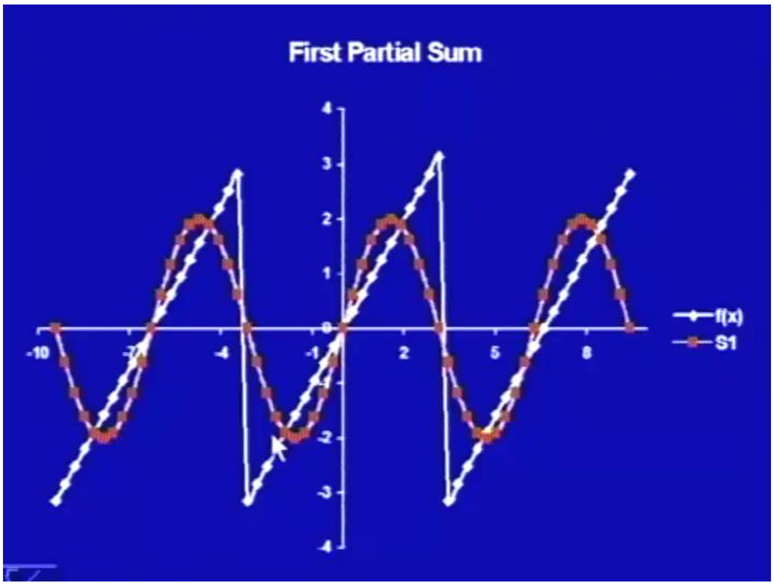

So, let us see in the graphs, you see here is in this graph actually this line, which is been shown here, is joining this one that line is not there this is just in the graph that this line is coming up here, we just have that this function is, where we have marked once this is not the line this is just showing it up.

And, if I approximate this function by the first partial sum you see our approximation is from here to here, it is coming up little bit that is at the points, where it is meeting with x axis at those points its matching, but we are deviating from here and, because of this that is what I have shown that is, the function if I just start S1 this function will be the straight lines.

But, if I try this function to be the continuous one, because it is a piece wise continuous here the function is discontinuous, so if I try to join it would be like this 1, but this is what it is this Fourier series is it is also making it one and that says is, we are deviating from the function little bit more.

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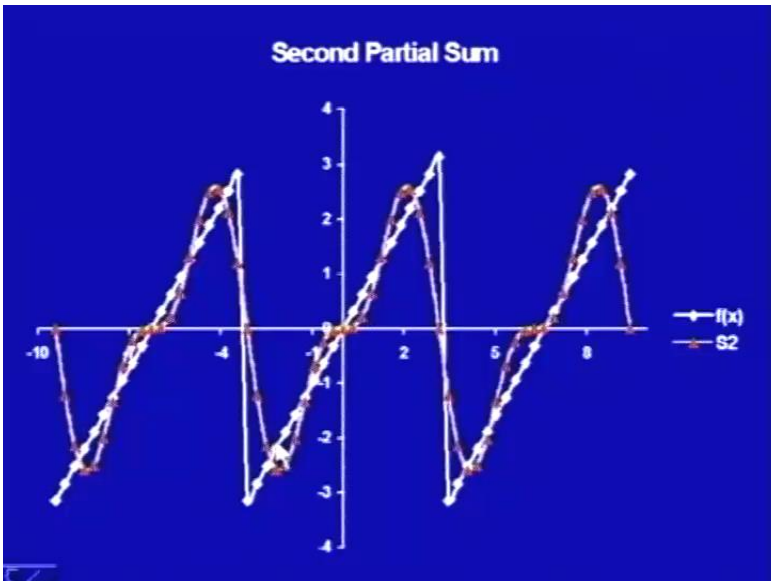

Now, let us move to the second partial sum this second partial sum, we are getting is that of course, because of this discontinuity we are deviating, but now we are little bit more near that is here, we are trying to be more near to this one.

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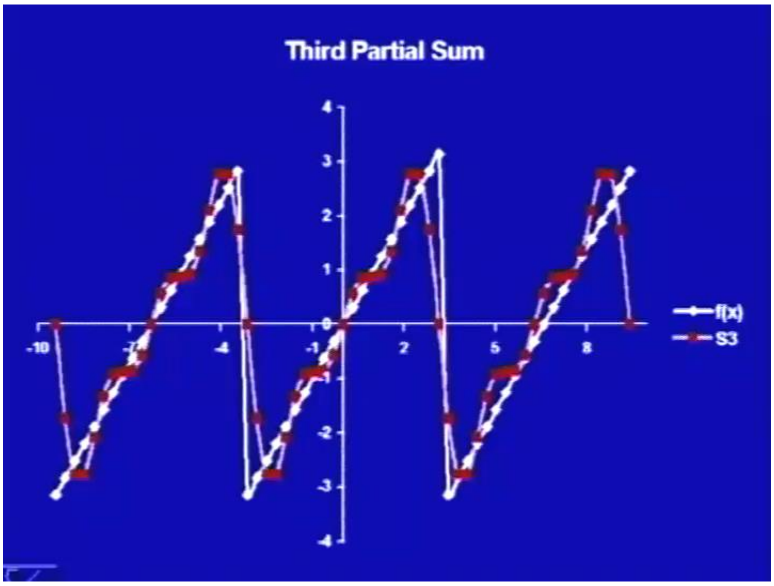

As we move to the third partial sum, you see we are more near to the approximate values or the points over there. So, like that if we just increase on that is if by whole Fourier series I am approximating it, you see is now here, on this line the points are decreasing, the line actually in the function this is the point and this is the point no other point in between, but if we see in the first partial sum, you see we are getting many points, so over here. But, if I just come to the second partial sum I am getting less points over here, when I am moving to third partial sum I am getting again more less points.

So, as we are increasing my aim that is the terms we will be we would be actually approximating this function more nicely, so you do find it out that the Fourier series in this one only this we require is that the function has to be continuous it may be discontinuous that is what we are calling piece wise continuous. We can approximate the function by the Fourier series the function has to be periodic and the period is 2 pi.

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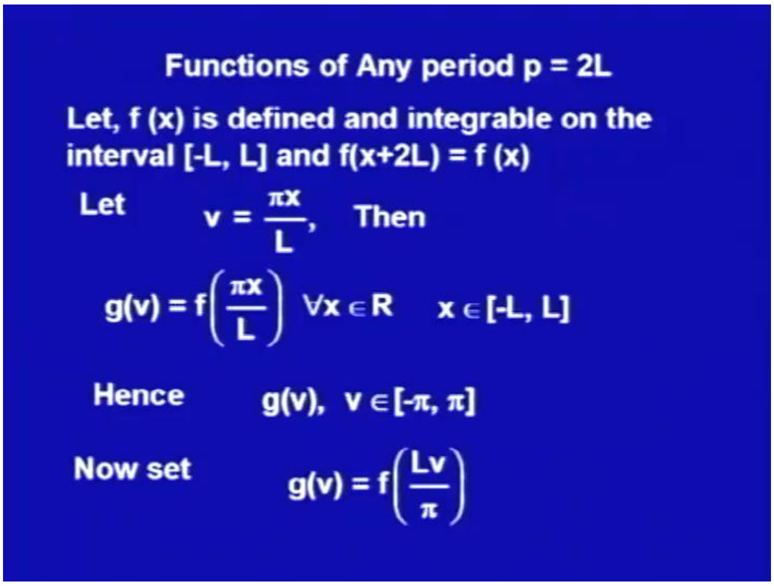

Now, it is not that is we do have only the functions of the period 2 pi only we can have any period, so can we find out the Fourier series for any period 2 L, now we are just using it 2 L. So, let say f x is defined and integrable on the interval minus L to plus L and f x plus 2 Lis f x that is my period p is 2 L, then we try to find it out can we find out the Fourier series for this.

We will just try to obtain it with the method as we are seeing is we are approximating it with the series containing sin x and cosine x sin x and cosine x they are periodic functions with period 2 pi, so we will just try to find out the answer from there itself. So, let us say defining new variable pi x upon L, let us call it v, now if I take the function f x, which is defined and minus L to plus L.

Now, we would take the new function g v as f of pi x upon L, so I am defining another function g for all x belonging to R. So, my original function my f x is defined from minus L to plus L. Now, I have changed my variable to v as pi x upon L, so what will happen since x is in minus L to plus L, what I would get that my new function g v this would be actually from minus pi to plus pi.

Since, when x is minus L, I would get the value for v as minus pi and when x is plus L I would get the value for v as plus pi. So, now if I change the variable x to v I would get the new function and the new function I defined like this one, I would get the new function from minus pi to plus pi and since I am defining it for all x and the original function f x is periodic with f x plus 2 L is equal to f x I would get g v plus 2 pi is equal to g v.

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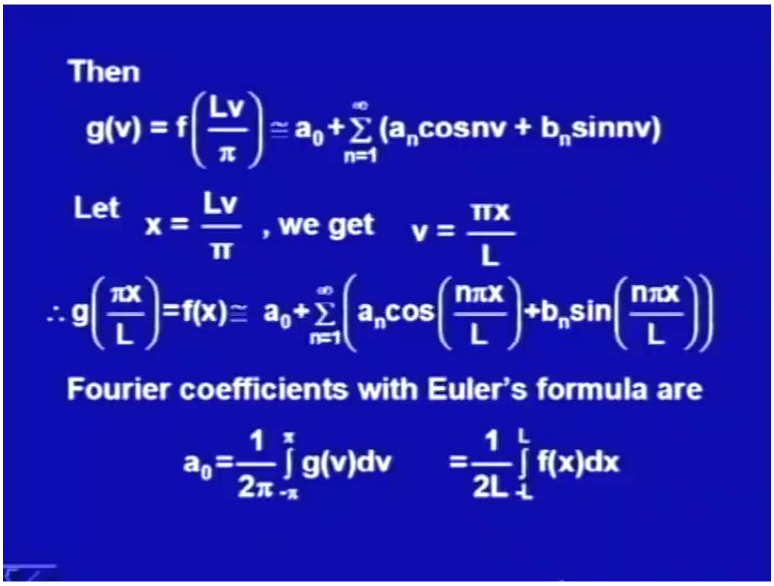

So, now set g v as f o f L v upon pi, now I am again rechanging it now, you see, what it gives me, that now this new function this is, now periodic with period 2 pi in the v. So, I can write it as a naught plus summation n is running from 1 to infinity a n cos n v plus b n sin n v, now rather than x I am using it v and what I would get this function coefficients using the Euler's formula. So, x is the L v upon pi we get v as pi x upon L and now, what I am getting is g of pi x upon L that is f x, which is, because finally I have to find out the Fourier series for f x.

So, I am again rewriting it on getting it is a naught plus summation n is running from 1 to infinity a n cos now n v, v I would write pi x upon L, so n pi x upon l plus b n sin n pi x upon L. Now, this is what we have got the Fourier series for function f x, where the period of function is 2 L. And, the Fourier coefficients with the Euler's formula again we will just go back a naught, now you see a naught from here it should be if g v is the function at that the period 2 pi and with any function with period 2 pi we do know that a naught is integral minus pi to plus pi 1 upon 2 pi g v d v that there it was f x d x.

So, because we are changing it to x to v this would be, now if I rewrite it, what I would get is, I would get it minus l to plus, because g v is same as f x, where g v is defined from minus pi to plus pi f x is defined from minus L to plus L. So, I would get it this is same as 1 upon 2 L minus L to plus L integral f x d x.

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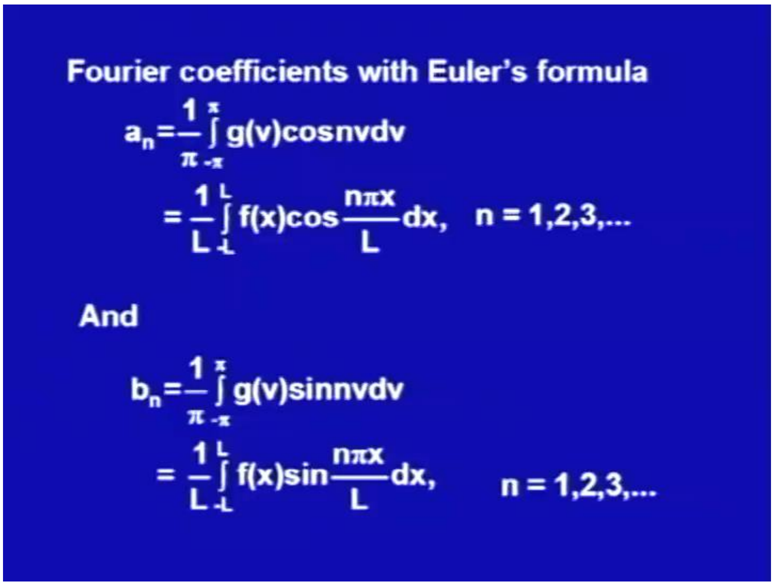

Similarly, we would go with a n, a n would be 1 upon pi minus pi to plus pi g v cos n v d v, now we would change to the x and we would get it as 1 upon L minus L to plus L integral f x cos n pi x over L d x and this would be for all n 1 2 3 and so on. And b n we would be getting is 1 upon pi minus pi to plus pi g v sin n v d v again changing it to the x we would get 1 upon L minus L to plus L f x sin n pi x over L d x thus what we have got well is true for again for 1 2 3 and so on.

So, now, we have got Fourier series for any periodic function, the period could be any period p, which we are saying is equal to 2 L that is for our convenience we are just writing it as 2 L that can be obtained using the Euler's formula this new Euler's formula are, now a naught is 1 upon 2 L minus L to plus L f x d x and a n is 1 upon L integral minus L to plus L f x cos n pi x over L d x and b n is 1 upon L integral minus L to plus L f x sin n pi x over L d x. And now, the Fourier series we would be having is a naught plus summation n is running from 1 to infinity a n cos n pi x over L plus b n sin n pi x over L.

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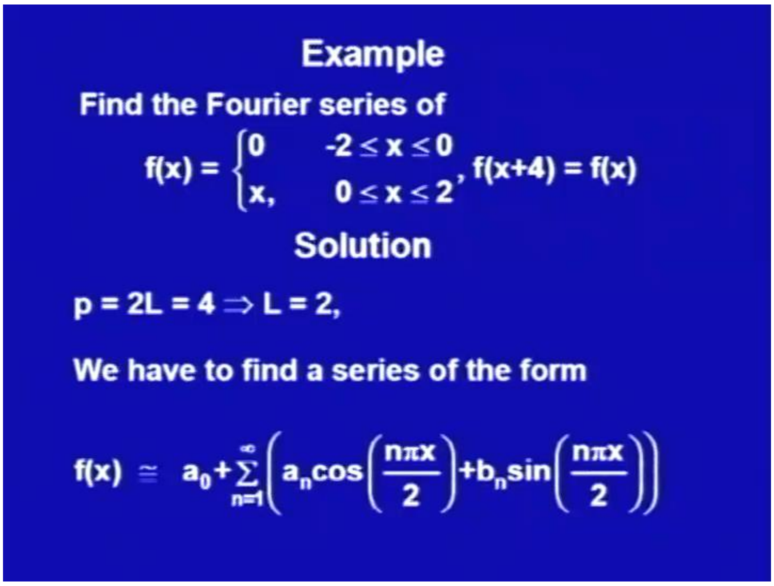

So, now let us see one example over this find the Fourier series of the function f x, which is 0 from minus 2 to 0 and x from 0 to 2 and the period is 4 that is f x plus 4 is equal to f x. So here, my period p is 4 that says is L is 2, because we are taking is p is equal to 2 L say now, what we would get L we have obtained as 2. Now, we will find out the Fourier series of this forma a naught plus summation n is running from 1 to infinity a n cos n pi x by 2 plus b n sin n pi x by 2, because my L is, now 2 and what a naught a n and b n, we will obtained using the Euler's formula.

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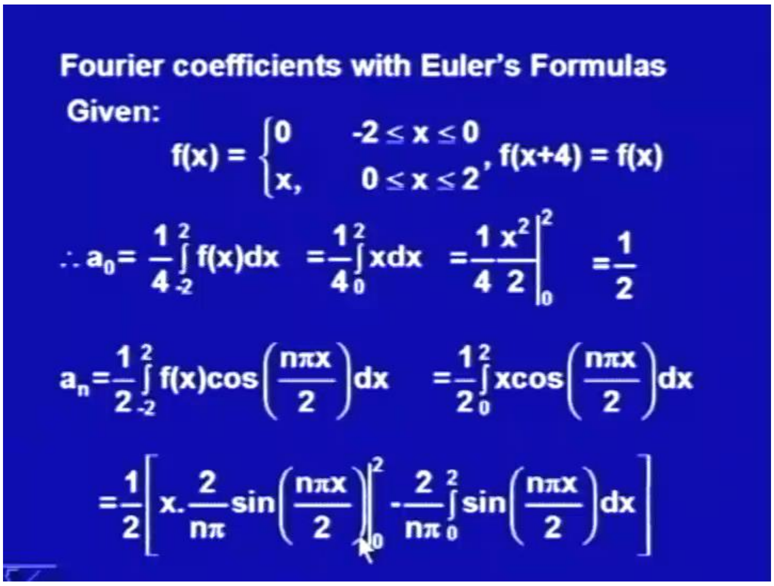

So, we have been given this is the function a naught is 1 upon 12 that is 1 upon 4 minus 2 to plus 2 f x d x, f x is 0 from minus 2 to 0 and x from 0 to 2. So, we would get it as 1 by 4 0 to 2 x d x and that would get x square by 2 and so on, integrated over this 1 and evaluated from 0 to 2 I would get it 4 by 2 that is 2 that is we would be getting it as 1 by 2.

Now, a n 1 by L that is 1 by 2 integral minus L to plus L that is minus 2 to plus 2 f x into cos n pi x over L that is L is 2 d x, f x is 0 from minus 2 to 0, so this integral would be half times integral 0 to 2 x times cos n pi x by 2 d x. Now, integrate it by parts we would get half as outside this first x is as such integral of cos n pi x by 2 is 2 upon n pi sin n pi by x sin n pi x by 2 evaluated from 0 to 2 minus 2 upon n pi this derivative of x is 1, 0 to 2 sin n pi x by 2 d x, which when I would be evaluating it at x is equal to 2 I would be getting it as sin n pi that is 0 and x is equal to 0 this is 0 as well as sin 0 is 0.

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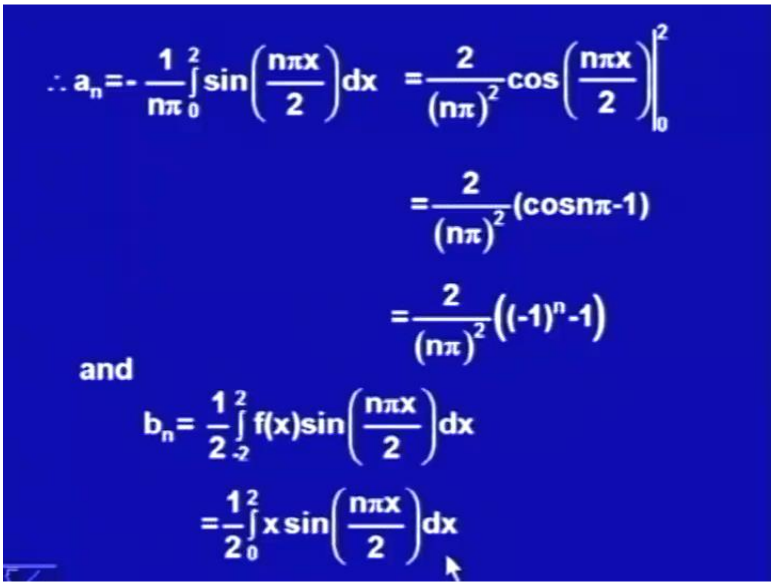

So, the first function would be first term would be 0 and we would be getting it as minus 1 upon n pi 0 to 2 sin n pi x by 2 d x, which when again we would be integrating it we would be getting is 2 upon n pi square cos n pi x by 2 0 to 2 at 2, I would be getting it as cos n pi and at 0 I would be getting it as cos 0. So, this is 2 upon n pi square cos n pi minus 1, because cos 0 is 1.

So, cos n pi is minus 1 to the power n, so we are getting this 2 upon n pi square minus 1 to the power n minus 1 and b n finally, we would be getting it as 1 upon 2 that is 1 upon L integral minus L to plus L that is minus 2 to plus 2 f x sin n pi x by L, L is 2 d x. Function is 0 from minus 2 to 0, so the integral would change to half times integral 0 to 2 x times sin n pi x by 2 d x.

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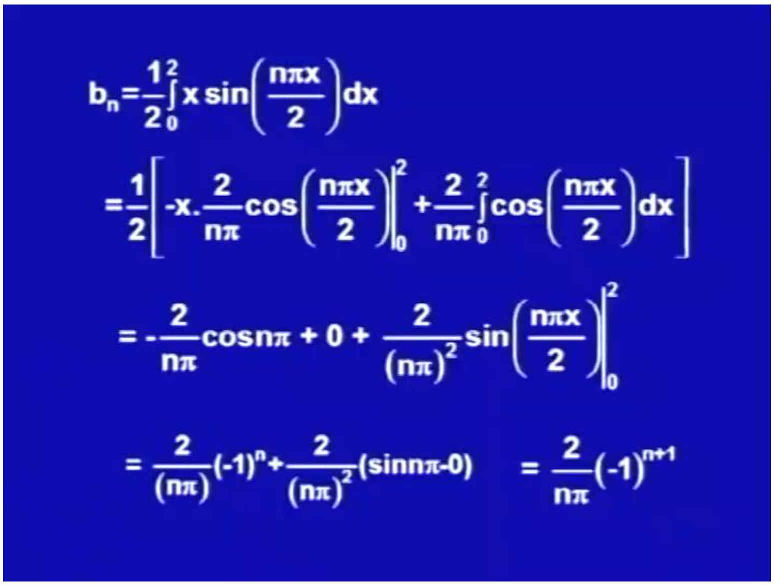

Again integrating it by parts we do get it as minus x 2 upon n pi cos n pi x by 2, 0 to 2 plus 2 upon n pi 0 to 2 integral cos n pi x by 2 d x. Now, here when we keep x is equal to 2 I would get cos n pi, when x is equal to 0 this 0 will give 0, so we would be getting it is the first term as minus 2 upon n pi cos n pi.

Since, 2 is here and 2 is here, so it will be cancelling it out and this 2 and this 2 is again cancelling out, so I would be getting is this integral as 2 upon n pi whole square sin n pi x by 2, evaluate it at 0 to 2 pi, when x is equal to 2 I would get sin n pi and when x is equal to 0 I will get sin 0 both are 0. So, that is not going to put any term with there we would get it as 2 upon n pi minus cos n pi is minus 1 to the power n plus 2 upon n pi square sin n pi minus 0 both the terms will give me 0. So, finally, we are getting is 2 upon n pi minus 1 to the power n plus 1 that is minus sign here it is been deleted little bit, so we are getting it this minus 1 sign with here that is minus 1 to the power n plus1.

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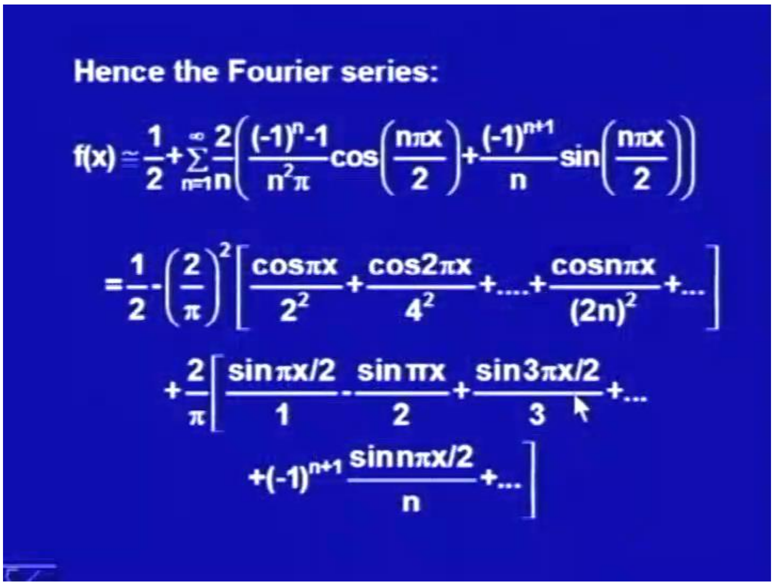

So, the Fourier series would be f x as half plus summation n is running from 1 to infinity 2 upon n minus 1 to the power n minus 1 upon n square pi cos n pi x over 2 plus minus 1 to the power n plus 1 upon n sin n pi x over 2.

Thus we would be writing it out that is expanding it half minus 2 upon pi whole square that is we are taking common and we would be getting the terms, here cos pi x upon 2 square plus cos 2 pi x upon 4 square plus and so on, cos n pi x over upon 2 n whole square and so on, and then we would be getting the sin terms as 2 upon pi sin pi x by 2 upon 1 minus sin pi x by 2 minus sin 3 pi x by 2 upon 3 and so on.

So finally, we would get the here the general term as minus 1 to the power n plus 1 sin n pi x by 2 upon n and plus 1. So, we are getting this is the Fourier series of the function f x, which was having the period of period 4 it was not of the form 2 pi, but we could get it as here, we are getting the sin terms as pi x and like this 1.

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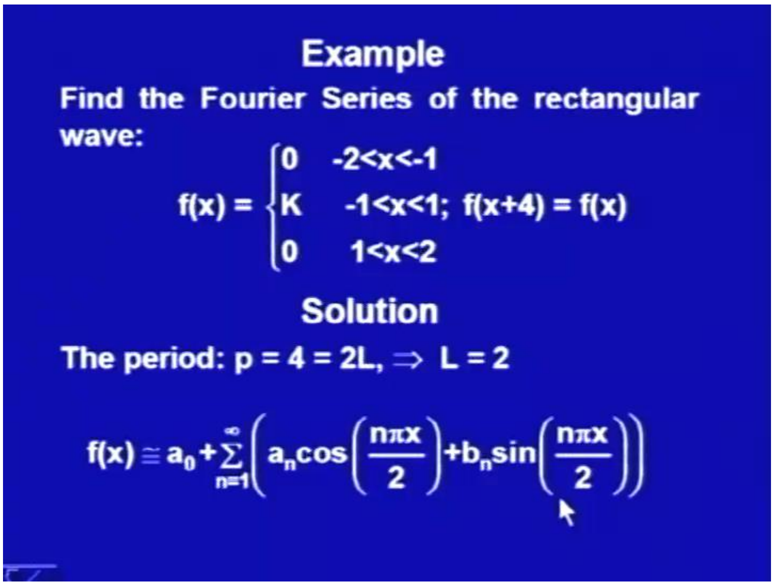

Let see another example, find the Fourier series of the rectangular wave, which is 0 from minus 2 to minus 1 and k from minus 1 to plus 1 and 0 from 1 to 2 and the period is again 4 f x plus 4 is f x. So, we do have again a function of period 4, so my p is 2 l is 4, so l would be 2 again and my I want the Fourier series of this form, so again here l is 2, so I would be getting is n pi x by 2. So, we would be getting the we want a series of the form a naught plus summation n is running from 1 to infinity a n cos n pi x by 2 plus b n sin n pi x by 2.

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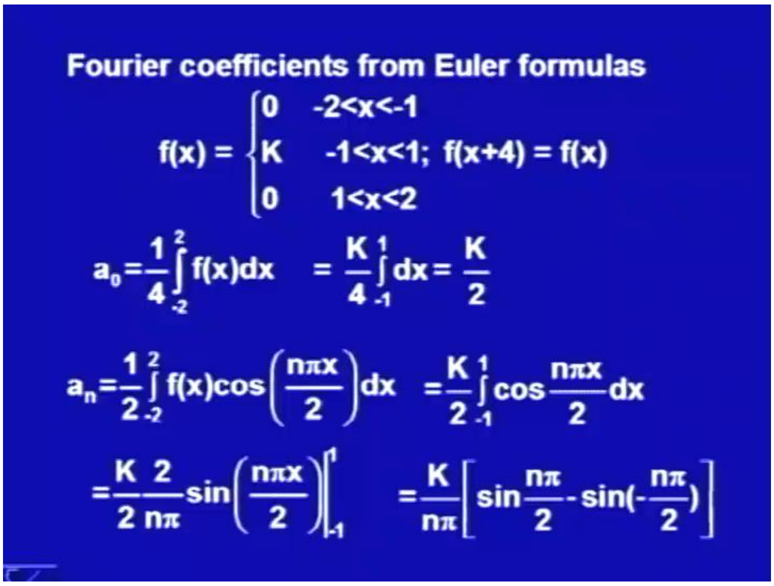

This a naught a 1 a n is and b n is will find out using the Euler's formula this is my f x, which is given to us, so a naught 1 by 4 minus 2 to plus 2 f x d x. This is 0 function is 0 from minus 2 to minus 1 and 0 from 1 to 2, so it is a function is only k from minus 1 to plus 1, so we would be getting it as k by 2 this integral.

Next a n 1 by l that is 1 by 2 minus 2 to plus 2 f x cos n pi x by 2 d x again the function is k only in minus 1 to plus 1. So, this integral would be k by 2 minus 1 to plus 1 cos n pi x by 2 d x, now integrate this function cos n pi x by 2 we do get is sin n pi x by 2 into 2 upon n pi. So, k by 2 into 2 upon n pi sin n pi x by 2 from minus 1 to plus 1,when i keep x is equal to plus 1 I would get sin n pi by 2 and, when I keep x is equal to minus 1 I would get minus n pi by 2.

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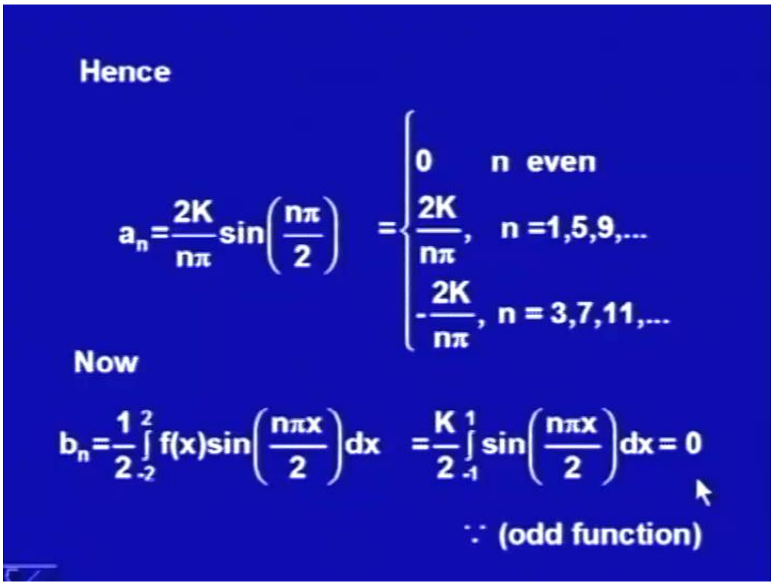

So, the value for sin n pi by 2 is 1 and value for sin minus n pi by 2 is minus 1 that will give me a n as 2 k upon n pi into sin n pi by 2. Now, sin n pi by 2 that is when n is even I would get it is of the forms sin 2 n pi upon 2 that is sin n pi that will always be 0, when it is odd with 1 5 9and so on, I would be getting it as a plus 1. So, this will be 2 k upon n pi, when it is 3 7 11, I would be getting it as minus 1, so it is minus 2 k upon n pi.

Because, we do know that sin pi by 2 is plus 1 sin 3 pi by 2 is minus 1 and so on, so we just find out that this is, what a n we had find it out. Now, b n 1 upon 2 minus 2 to plus 2 f x sin n pi x by 2 d x, since the function is k only in the interval minus k to minus 1 to plus 1, I would get this integral as k by 2 minus 1 to plus 1 sin n pi x by 2 d x. Now, function sine x is odd function, so minus 1 to plus 1 it will always be 0.

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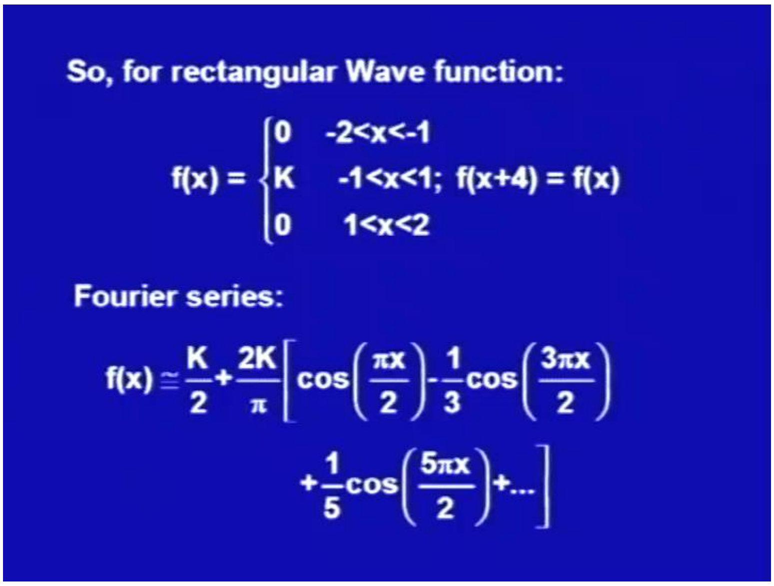

So, what we have got we have got that for rectangular wave of function f x 0 to 0 between minus 2 to minus 1 and 1 to 2 and k from minus 1 to plus 1. The Fourier theory is k by 2 this is a naught, which we had find out plus 2 k upon pi cos pi x by 2 minus 1 by 3 cos 3 pi x by 2 and so on, plus 1 upon 5 cos 5 pi x by 2 and so on this is, what is the Fourier series for this function we have got.

So, today we had learn what is Fourier series we had learnt that this is a series of sin and cosine functions and it could be given as that a naught plus a 1 cosine x plus a 2 cosine 2 x and so on, plus b 1 sine x plus b 2 sine 2 x and so on. Then, the function was of the period 2 pi, we had also learnt how to obtain for any function f x this coefficients a naught a 1 a 2 b 1 b 2 so on, which we are calling Fourier coefficients, we had learnt Euler's formula for that.

This we have done for a periodic function with period 2 pi, then we have move to the any periodic function that is the period is of any p or rather we call it any 2 L rather than 2 pi. And we had learnt how to obtain the Fourier series for a periodic function with period 2 L, again we had modified our Euler's formula for finding out the Fourier coefficients and we had find it out.

We had seen in the examples, we have done two examples for Fourier series for periodic functions with period 2 pi there we had seen also in the graphs that is how those functions, when we are approximating by the Fourier series as we are moving towards many terms we are obtaining that the functions are being approximated by those Fourier series. And we have learnt for the other functions periodic functions with period 2 pi, next some more properties of this Fourier series for this periodic functions we will learn in the next lecture, so today is lecture is over here.

Thank you.