Mathematics - III

Prof. Tanuja Srivastava
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture - 10
Strum Liouville Boundary Value Problems

Welcome to the lecture series on differential equations for under graduate students, today's lecture is in continuation of the Strum Liouville Boundary Value Problems. In this lecture we will discuss the properties of Strum Liouville Boundary Value Problems.

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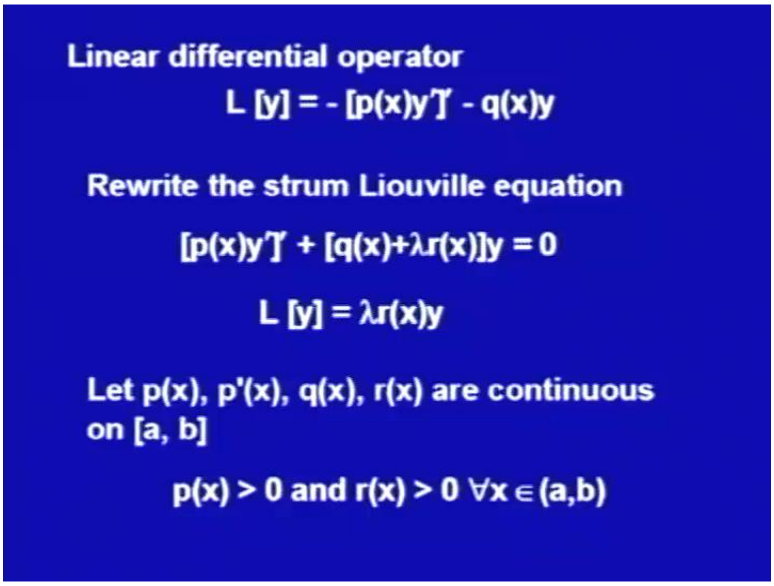

For properties let us, for convenience define a linear differential operator as L y is equal to minus p x y dash whole derivative minus q x into y. Here this L is actually an operator which is when operate it on the function y gives me the derivative of p x y dash with minus sign minus q x times y. Now, we can rewrite the strum Liouville differential equation are p x y dash whole derivative plus q x lambda r x y is equal to 0 in the terms of this linear differential operator as L y is equal to lambda r x times y.

Now, let us assume that these coefficients p x q x r x they are all continuous as well as p dash x is also continuous on the whole interval a b on this differential equation is being defined. Moreover, we would be assuming that the p x and r x to be positive on the whole interval that is for all x belonging to a and b.

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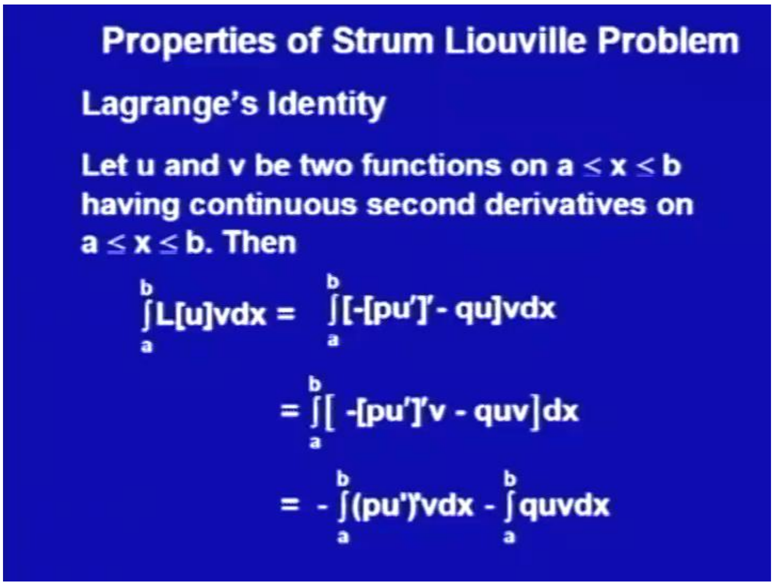

Then, an important property of the strum Liouville problem is known as Lagrange's identity. Let us first find it out that how we are obtaining it and what it is. Let u and v be two functions on interval a b having continuous second derivatives on the whole interval a b. Then we can write this integral L u into v integrated over x for though interval a b, now since L u is that is L is an operator which is operated on v, so just by the definition of that operator, we can write it as integral a to b minus p u dash whole dash minus q u v d x. Now, rewrite it as minus p u dash its whole derivative v minus q u v d x. Now, separating both the integrals we get minus a to b the derivative of p u dash into v integrated with respect to x minus integral of q u v with respect to x on the limits a to b.

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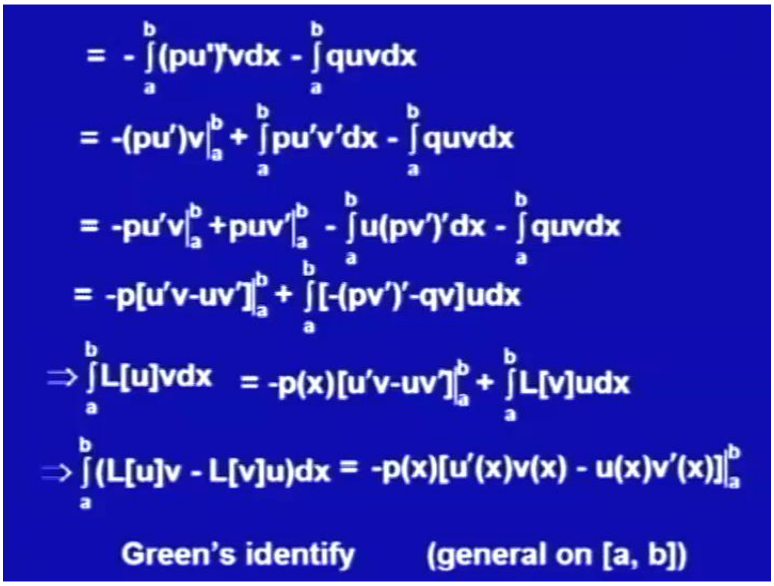

Now, if I integrate this first integral by part, we do get p u dash, because I have taken this p u dash and its whole derivative as the integrable function and into v evaluated from a to b plus integral a to b p u dash v dash d x and the second integral is as such integral q u v d x from with respect to x on a to b.

Now, the first integral here again we are going to do by with the integral with parts and the taking first function as u dash. So, what we will get the first thing as such p u dash v from evaluate from a to b plus p u v dash evaluate a to b minus integral u p v dash and its whole derivative integrated with respect to x on a to b minus the last integral as such q v with respect to x on the interval a to b.

Now, again what we are getting is sub making this rewriting the first two lines p u dash v and p u v dash we can write p v as common and we are getting it as u dash v minus u v dash evaluated from a to b plus this integral we are writing as one integral again. So, what we would get the derivative of p v dash with minus sign minus q v and u has been taken common integrated with respect to x on the interval a to b.

Now, you see this is what we are getting is as the operator L operated on the function v. Lets write it in this next line, so what we are getting is we have started with the integral L u v with respect to x on the interval a to b and we are getting it as minus p x u dash v minus u v evaluated from a to b plus a to b L of v u integrated with respect to x. This says if I am taking this both operator one and integrals on the one side.

This simply says is integral a to b L u v minus L v u d x is same as minus p x u dash v x minus u x v dash x evaluated from a to b. Now, this in general is also known as green's identity and this we are evaluating on the general interval a to b.

Now, we are talking about the properties of strum Liouville problem, so not only strum Liouville differential equation, but also the boundary conditions. So, we have assumed that u and v are such that they are being operated on this strum Liouville or this linear operator. Now, let us assume that they are satisfying the boundary value conditions of strum Liouville problem.

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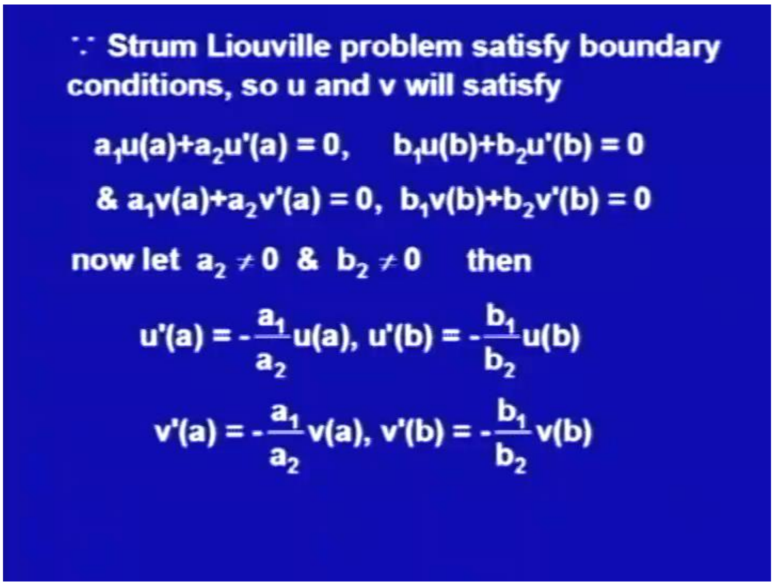

So, what we do say, that says is a 1 u a plus a 2 u dash a should be 0, b 1 u b plus b 2 u dash b should be 0. And similarly, for v a 1 v a plus a 2 v dash a should be 0 and b 1 v b plus b 2 v dash at b should be 0. Now here, we will assume that at least actually one of these a 1 a 2 b 1 b 2 must not be 0. So, we are assuming that a 2 is not 0 and b 2 is not 0 that says is we can write u dash u and v dash in the terms of u and v. Let us, see how u dash a, from the first equation can be written as minus a 1 upon a 2 u a and u dash b as minus b 1 upon b 2 u b. Similarly, v dash a as minus a 1 upon a 2 v a and v dash b as minus b 1 upon b 2 v b.

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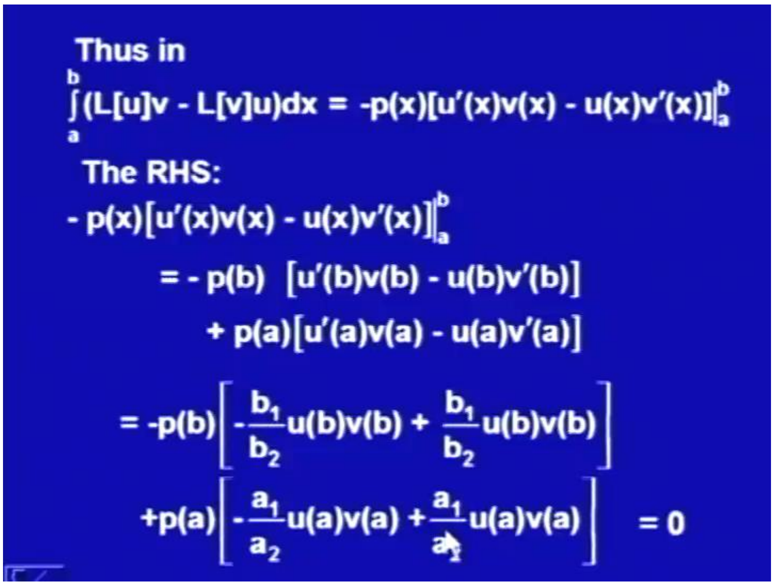

Now let us, come back to our u h green's property which says is that integral a to b L u v minus L v u d x is minus p x u dash x v x minus u x v dash x a to b. Now, the right hand side we are just write putting the values that is evaluation. So, it would be minus p b u dash b v b minus u b v dash b plus pa u dash a v a minus u a v dash a. Now, for u dash v dash at a and at b both places we will replace by the just now we had used that boundary conditions and from there the values that in the terms of u and v.

So, we can write as minus b 1 upon b 2 u b v b plus b 1 upon b 2 u b v b with multiplication as minus v b. And, the second term also p a minus a 1 upon a 2 u a v a plus a 1 upon a 2 u a v a. Now, we see the terms in the bracket both the terms in the bracket here, this is 0 as well as this is 0 this says is this whole right hand side will come out to be 0.

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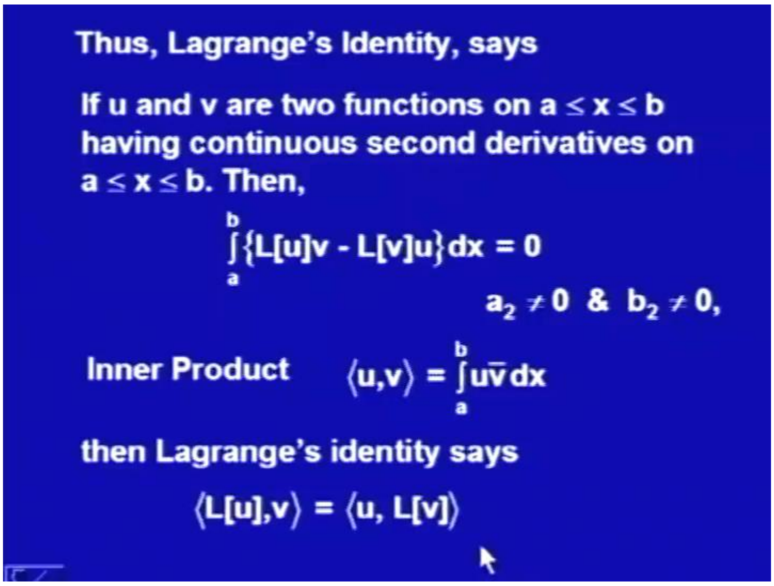

Now what we have got, thus we summarise the LaGrange's identity says if u and v are two functions defined on the interval a to b having continuous second derivatives on the same interval a to b. Then, integral of L u v minus L v u with respect to x on the whole interval a to b would be equal to 0 when in the boundary condition a 2 and b 2 are not 0. So, we are talking about strum Liouville boundary value problem that is the differential equation as well as the boundary conditions and in the boundary condition we are assuming that when a 2 and b 2 is not 0.

This thing we can also say in the terms of inner products, What are the inner products? Inner products we are defining as inner product u and v is defined as integral of u v bar with respect to x on the interval a to b. You see here in this one, we are having this v bar, v bar is we are assuming that u and v actually the complex functions and v bar is nothing but the complex conjugate. So of course, when u and v are not complex they are real v bar would be same as v, but notion of the inner product is defined in the terms of complex functions.

Now, let us rewrite this LaGrange's identity in the terms of this inner product, then what we are saying is inner product of L u comma v is same as inner product of u comma L v that is we can interchange this operator L and this is what LaGrange's identity is saying. Let us, see the implication of this LaGrange's identity on our strum Liouville boundary value problems.

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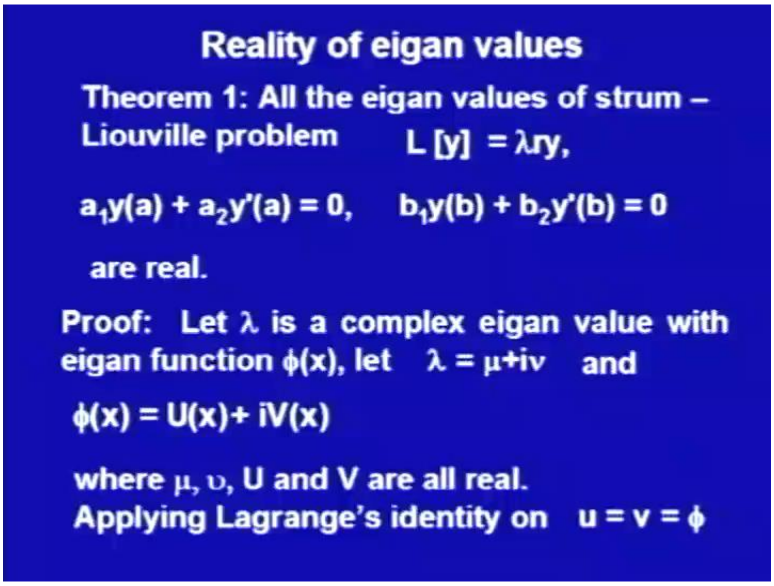

So, implication of LaGrange's identity the first implication is that reality of Eigen values what it is see, this is defined here as theorem all the Eigen values of strum Liouville problem L y is equal to lambda r y with boundary conditions a 1 y a plus a 2 y dash a is equal to 0, b 1 y b plus b 2 y dash b is equal to 0 has are real.

So, how to show it we would be showing it using a LaGrange's identity see the proof, let lambda is a complex Eigen value with Eigen function phi x, here what we are assuming we are starting with that lambda and phi both are complex that says is let lambda is of the form mu plus i nu. Now, what we would show we would show using the LaGrange's identity that this n u term that is nu is 0, so that we get lambda is equal to mu that is only real one since we had assumed that lambda and phi to be complex. So, phi also we are assuming of the form U plus i V x.

Now, with mu nu U and V all we are assuming as real now applying this LaGrange's identity on U and V that in LaGrange's identity we are having inner product of L u comma v is same as inner product of v comma L u there u and v if I assume same as phi.

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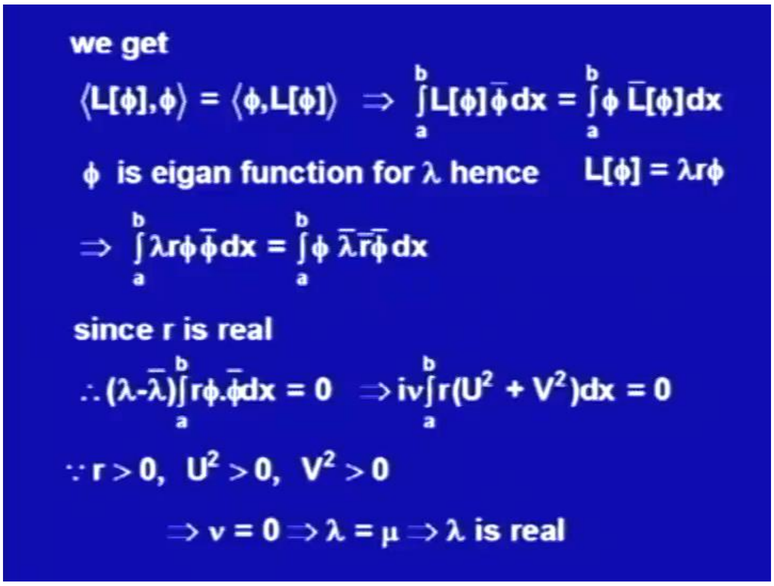

Then, what we would get? We get the inner product of L phi comma phi is same as inner product of phi comma L phi, let us expand it this says is integral a to b L phi phi bar d x should be same as integral a to b phi L bar phi d x where bar is denoting the complex conjugate. Now, since this phi is Eigen function for lambda, so L phi would be lambda r phi and L. So, let us substitute it in this identity, what we get, lambda r phi phi bar d x is equal to integral a to b phi lambda bar r bar phi bar d x. Because, L phi is lambda r phi its complex conjugate we would get lambda bar r bar phi bar.

Now, we are knowing is that strum Liouville problem r is real function. So, r bar would be same as r. Thus rewriting this and taking both the things on the one side we would get lambda minus lambda bar integral a to b r times phi dot phi bar d x is equal to 0.

Now, phi we had assume say of the form of u plus iv, so phi bar would be u minus iv, if you multiply both the I will get it as r times U square plus V square d x in the integral part and lambda minus lambda bar what it would be lambda we had assumed as mu plus i nu. So, lambda bar would be mu minus i nu when we are taking this subtraction we would be getting as minus 2 i nu.

And, that we would be plus and that two part we have taken as the 0 side. So, we would be getting this one. Now, you see here u and v were real function and r we had already assumed that it is a positive on the whole interval a to b. So, what we are having, r is positive U square is positive and V square is positive, we are integrating a positive function on interval a to b finite interval.

Of course, this integral cannot be 0, so what would be 0 in this identity, that nu should be 0. If nu is 0 my lambda would be nu plus 0 that is mu only that is lambda is real, so we had got using this LaGrange’s identity that my Eigen value of strum liouville problems are real this is the first property. The second property is again related with this Eigen values and Eigen function that requires one more definition, that property is belonging to orthoganlity.

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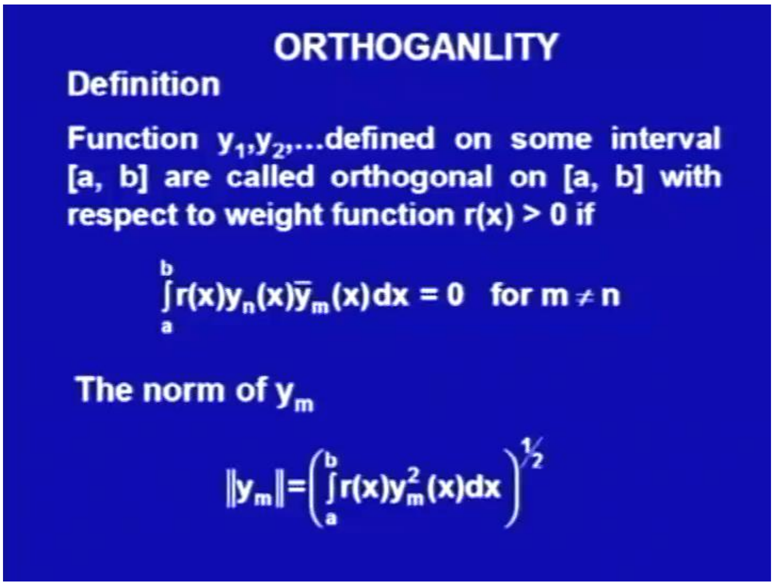

Let us, first see the definition what we call orthoganlity, functions y 1 y 2 y n would be defined which are defined on some interval a to b are called orthogonal on a to b with respect to a weight function r x that weight function has to be positive. If the integral r x y n y m bar x d x and a to b is 0 for all m not equal to n that is whenever we do have two different y m and y n. Then, y m into y m bar multiplied with r x this if I integrate over the whole range a to b it should give me 0, then we are calling that the function y n and y m are orthogonal with respect to r on the interval a to b.

Also, we are defining one more term here that is called the norm of the function y m, this we are denoting by the this notation y m. This is defined as r x y square y m square integrated with respect to x on the whole interval a to b and its square root. So, this term is defined as the norm of y m, one more from here if we take that norm of y m is to be 1.

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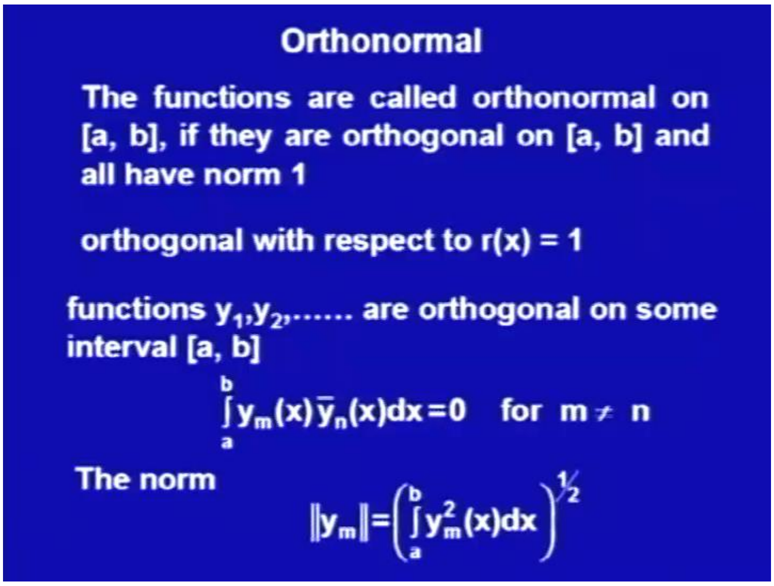

Then, we say the functions are called orthonormal on a b if they are orthogonal on a b again we are talking with respect to x and all have norm 1. So, if they are having norm one we are calling the functions to be normal and if they are normal and orthogonal we are calling them to be orthonormal.

Now, if r x that is the weight function is one uniformly all over there, then what we say is that we are not saying orthogonal with respect to our orthonormal with respect to we simply say orthogonal and the definition is also revise, so see the revised definition. The functions y 1 y 2 are orthogonal on some interval a b if integral y m x y n bar x d x that is with respect to x on the a b is 0 for m not equal to n.

And, the norm now has been we defined as integral y m square with respect to x on a to b and its whole square root. So, we have taken r to be 1 and here we are not using that with respect to r x as one that term we are deleting. So, now, before moving to the properties of strum liouville problem, let us first do one example to understand this orthoganlity.

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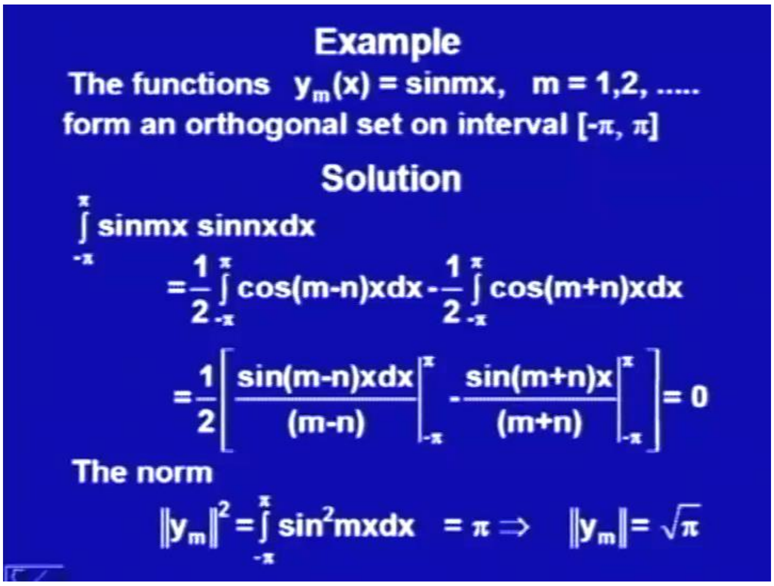

The functions sine m x for m as integer 1 2 and so on form an orthogonal set on the interval minus pi to plus pi. We see that says is that we have to show that this y m x for m is equal to 1 2 3 and so on is orthogonal. So, integral minus pi to pi sin m x sin n x with respect to x, now we do know sin m x sin n x we can write as half cos m minus n x minus half cos m plus n x, so we are substituting it and putting this in the integral and breaking it into two integrals. Now, integrating the both the integrals from minus pi to plus pi, we do get it that half sin m minus n x d x upon m minus n evaluated from minus pi to plus pi minus sin m plus n x upon m plus n minus pi to plus pi again evaluated.

Now, we see that m and n both are integers, so what we would be getting is that sin x with the multiple integer multiple here also sin x with n multiple as an integer and we do know that sin n pi plus or minus is always 0, that we are getting is that is integral is 0. So, we have been through that sin m x sin n x its integral from minus pi to plus pi d x is 0, so they are orthogonal. And of course, we are taking that m and n are not equal.

The norm more over we can find it out we see the norm of this function y m x that is minus pi to plus pi sin square m d x. If you evaluate with the simple ones we do get is equal to pi, so the norm of this y m that is sin m x is square root pi.

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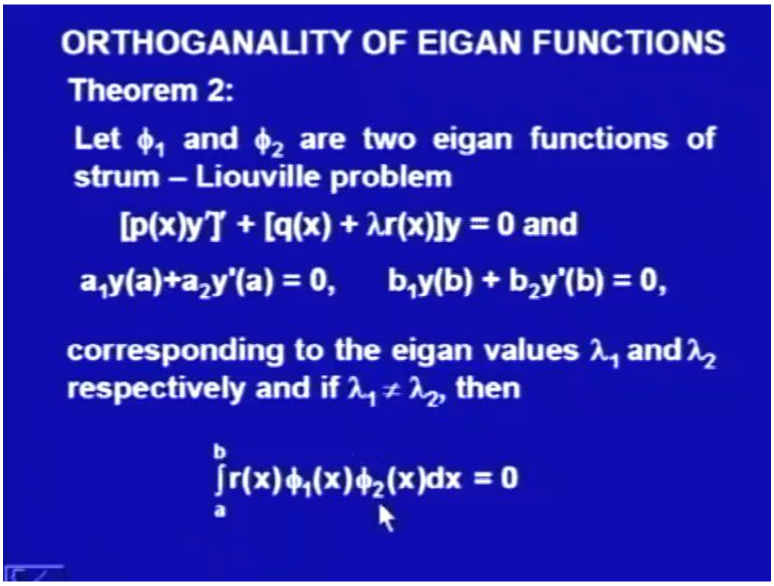

Now let us, come to the property of strum Liouville boundary value problem, it says is orthoganality of Eigen functions, what is this property again, we are writing in the terms of theorem. Let phi 1 and phi 2 are 2 Eigen functions of strum Liouville problem p x y dash whole derivative plus q x plus lambda r x y is equal to 0. With boundary conditions a 1 y at a plus a 2 y dash at a is equal to 0 b 1 y at b plus b 2 y dash at b is equal to 0 with respect corresponding to the Eigen values lambda 1 and lambda 2 respectively.

That is, phi 1 is Eigen function for lambda 1 and phi 2 is Eigen function for lambda 2. And, if lambda 1 and lambda 2 are not same then phi 1 and phi 2 are orthogonal or in other words a to b with respect to r x, this r x is the same which is coming in the strum Liouville differential equation. That is, integral a to b r x phi 1 x phi 2 x d x would be 0 , when your lambda 1 is not equal to lambda 2 and phi 1 and phi 2 are corresponding Eigen functions.

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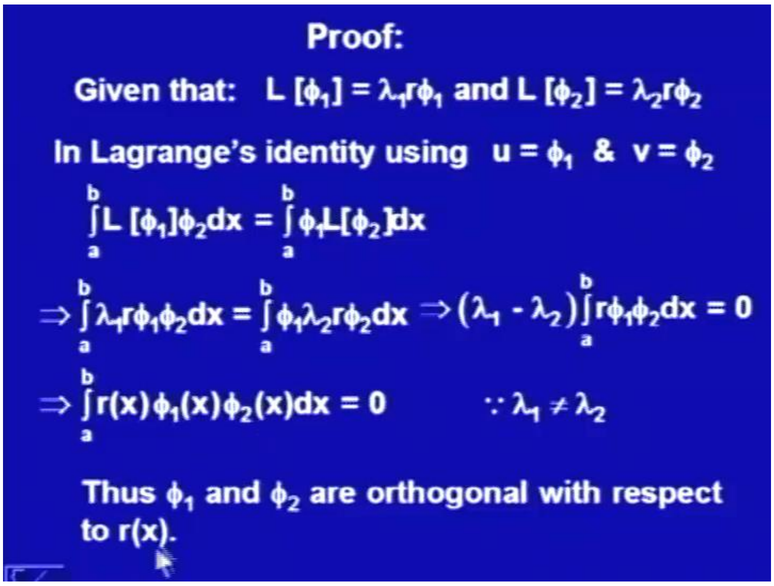

Proof of this theorem, let us see again we would be using this LaGrange's identity you see. We are been given that lambda 1 and lambda 2 are Eigen values and phi 1 and phi 2 are Eigen functions that is Lphi 1 is equal to lambda 1 r phi 1 and L phi 2 is equal to lambda 2 r phi 2 that is lambda 1 phi 1 lambda 2 phi 2 will satisfy our strum liouville equation.

Now, again we are going to use the LaGrange's identity what we will use in LaGrange's identity the two functions u and v now we will take them as phi 1 and phi 2. Then, LaGrange's identity says that inner product of L phi 1 with phi 2 is same as inner product of phi 1 L phi 2. Now, substitute this L phi 1 as lambda 1 r phi 1 and L phi 2 as lambda 2 r phi 2 we are getting lambda 1 r phi 1 phi 2 d x is equal to phi 1 lambda 2 r phi 2 d x.

Now, rewriting this we get lambda 1 minus lambda 2 times integral r phi 1 phi 2 d x is equal to 0. Now, we are knowing is that we have taken lambda 1 and lambda 2 distinct that is lambda 1 minus lambda 2 is not 0 that says is that integral r phi 1 phi 2 with respect to x on the integral on the interval a to b would be 0 that says phi 1 and phi 2 are orthogonal. So, they are orthogonal with respect to r x.

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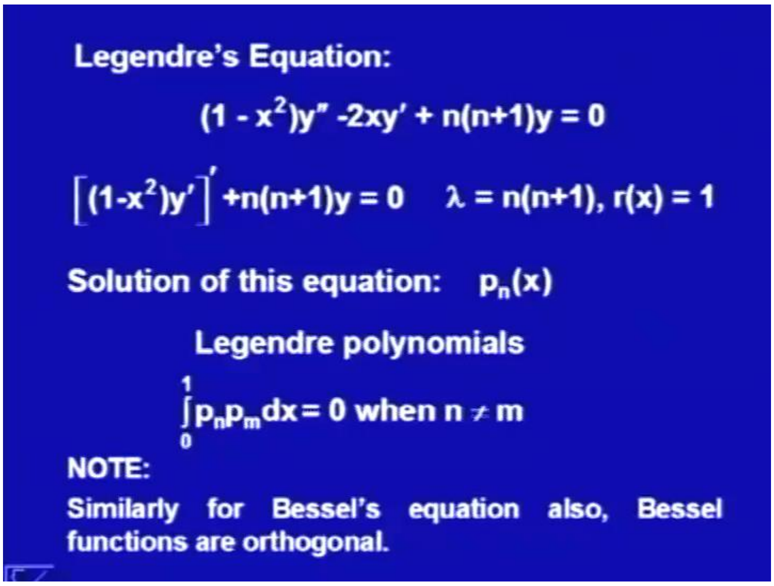

Now, let us see one more property or rather the use of this property you had already seen that Legendre equation, you had already done in the example this is actually strum Liouville differential equation.

Now, if we use the boundary conditions on 0 and 1, you had known that the solution of this equation one solution you had denoted it by as p n x and you called it Legendre polynomials. You had already proved that Legendre polynomials are orthogonal on the interval 0 1. Here, what I am not going to prove it here I am just telling you that you can see that because it is a strum Liouville differential equation using suitable boundary conditions, because, you are given they are the initial values.

You can show that this property can be find out from here also. Similarly you can, so this what you had already proved 0 to 1 integral p n pm d x is 0, whenever n is not equal to m. Moreover, you had already done that is Bessel's equation is also a special form of the strum Liouville equation and its solution you call the Bessel's functions you had already proved that the Bessel's functions are orthogonal, again you can show them using this LaGrange's identity.

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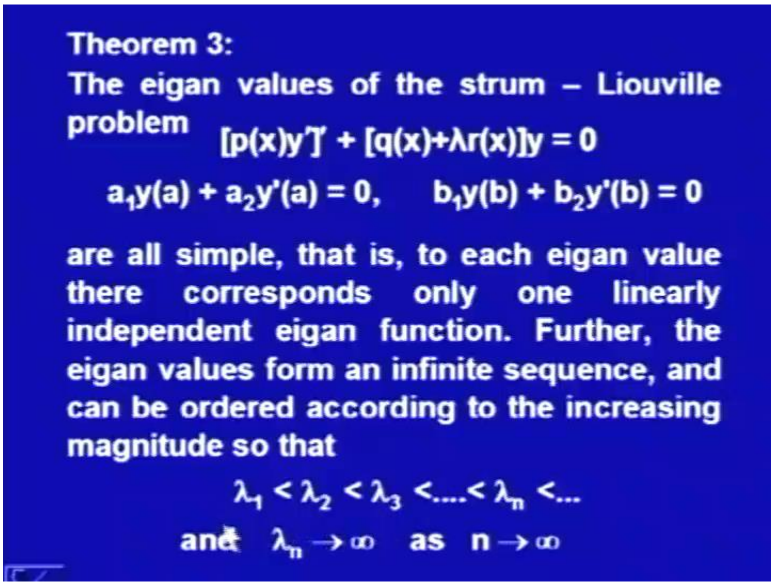

Now let us come to one more property, the Eigen values of the strum-Liouville problem p x y dash its derivative plus q x y lambda r x y is equal to 0 with boundary conditions as a 1 y a plus a 2 y dash a is 0 and b 1 y b plus b 2 y dash b is equal to 0. And, certainly all of a 1 a 2 and b 1 b 2 should not be 0. They are simple that is to each Eigen value there as corresponds only one linearly independent Eigen function. Further, Eigen values form an infinite sequence and can be ordered according to the increasing magnitude. So, that we can write as lambda one is less than lambda 2 is less than lambda 3 and so on.

Moreover as, n is increasing lambda n is also increasing infinitely, let us see, what this theorem is saying? This theorem is saying that phi strum Liouville problem, it is all Eigen values are simple. You had understood the terms that simple you have got in the terms of roots and the corresponding solutions. So, multiple roots means is you are able to get two linearly independent solutions, so here what they are saying is all Eigen values are simple that is with respect to every Eigen value we will get only one linearly independent Eigen function. And moreover, one more thing we are saying that they are infinite many Eigen values for strum Liouville problem and they can be ordered in the increasing magnitude.

That is we can rename that is the smallest as lambda 1 then the lambda 2 and so on and they can increase still infinite. The proof of this theorem we are not going to do because it requires little bit mathematics which is beyond the course of this particular one. We would see that is so what we have learn the properties of strum Liouville problem.

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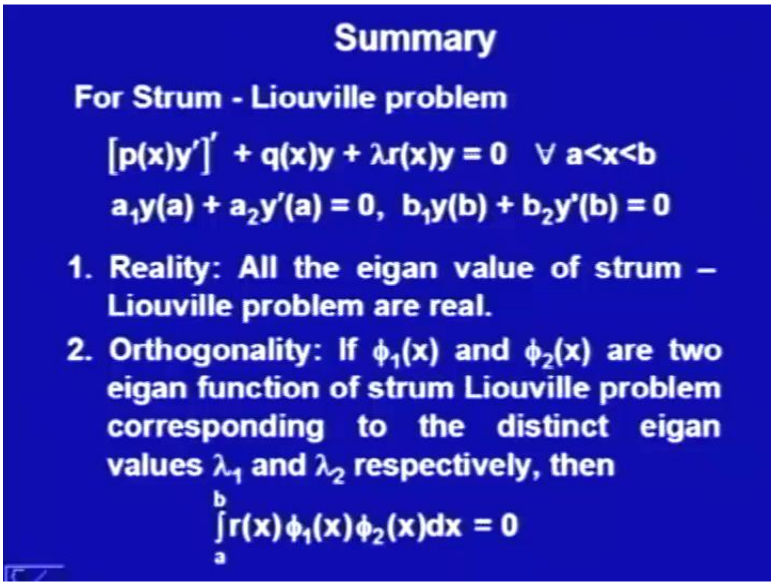

Let me, summarise it again we had learn that is for strum-Liouville problem, the problem I am writing again on the interval a b with the boundary conditions at a and at b. First property we have done is reality that all the Eigen values of strum-Liouville problem are real, second property we had learn about orthogonality. That, if phi 1 and phi 2 are two Eigen functions corresponding to the distinct Eigen values then they are orthogonal with respect to r, the r is the function which we are getting in this strum-Liouville equation.

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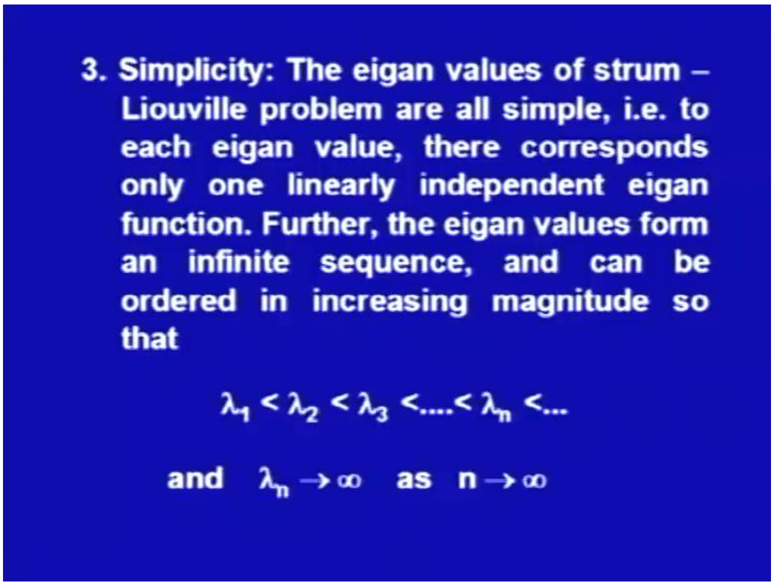

The third property just now we have done, that is called the simplicity, the Eigen values of strum-Liouville problem are simple that is each Eigen value there corresponds only one linearly independent Eigen function. And, this Eigen values form an infinite sequence and can be ordered in increasing magnitude.

As this one moreover as n approaches to infinity lambda n also approaches to infinity that says they can go as large as possible. Because, we are saying here, so we have said here that is all Eigen values are simple and corresponds only one linearly independent solution. But, we have seen in our previous examples that we are getting thus linearly independent solution that we can always write general solution is c times that solution. So, this c that is multiplicative constant we are talking about little bit that one.

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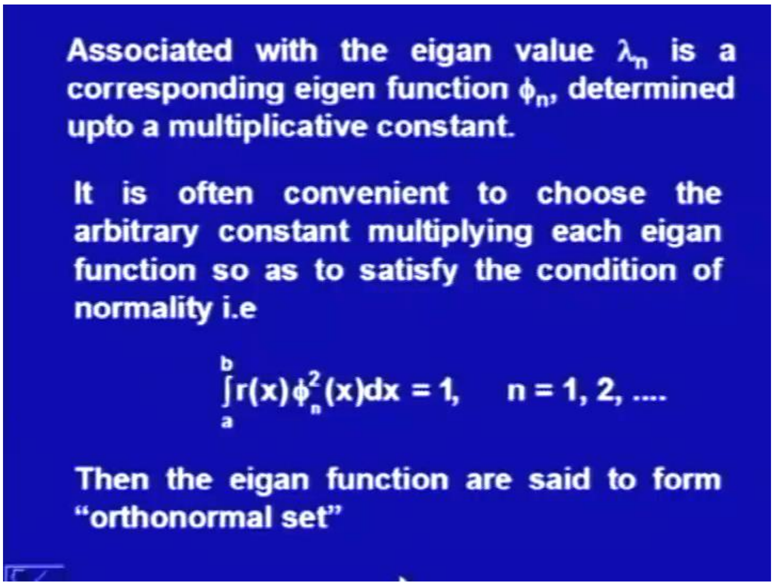

So, associated with each Eigen value lambda n, there is corresponding Eigen function phi n determined up to a multiplicative constant. It is often convenient to choose this constant such that it satisfy the condition of normality that is its norm is 1. So, that multiplicative constant many times we are choosing a particular value that is we are not talking about general solution, we have to talk about that particular solution and for that the constant we choose such that the function becomes normal.

Because, we already know the, for distinct Eigen values the Eigen functions are orthogonal, so we come as orthonormal set. So, normality means simply a to b r x phi n square x d x should be 1. Let us, try one of our examples, so what we are saying is that we just want them to form a orthonormal set. So, let us repeat our examples which we have done in the boundary value problem and strum Liouville problem and see that is how we are obtaining them.

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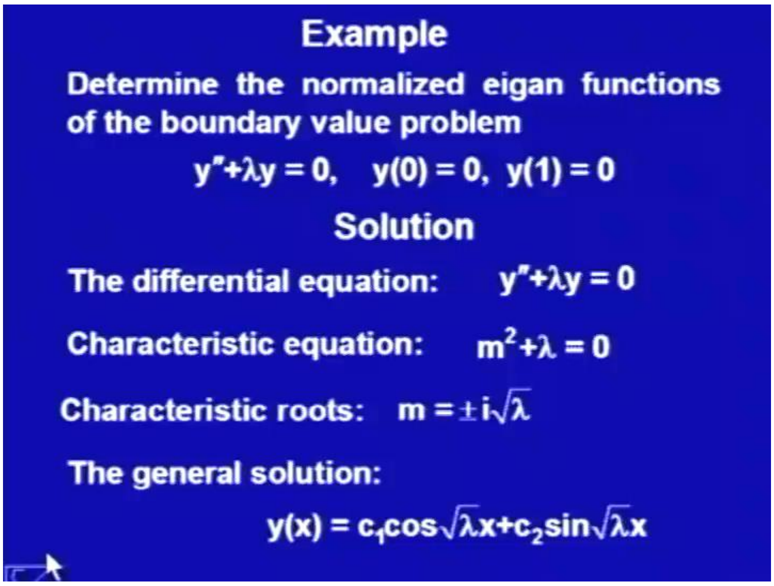

Determine the normalized Eigen functions of the boundary value problem y double dash plus lambda y is equal to 0, y at 0 is 0 and y at 1 is 0. You remember that this example we have already done and we had already shown that here, this can be written as y dash and its derivative plus lambda y is equal to 0. So, we can write it as the p as 1 and q as 0 and lambda is Eigen value and r as 1. So, this is again strum Liouville problem we want normalized Eigen functions.

Again, we will just try to solve it the complete here, because we had done it in the last lecture. So, the differential equation given is y double dash plus lambda y is equal to 0, the characteristic equation would be m square plus lambda is equal to 0 giving me the characteristic roots as plus minus i root lambda. So, the general solution would be c 1 cos root lambda x plus c 2 sin root lambda x.

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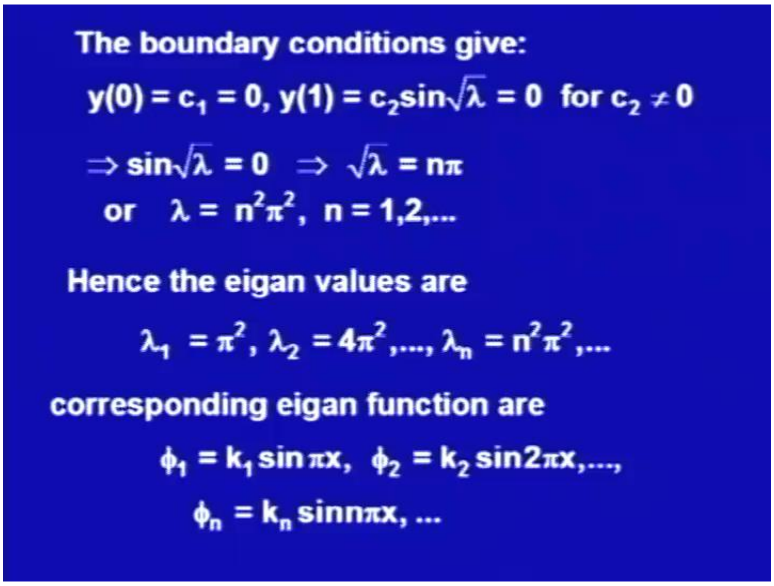

Now, satisfying the.boundary conditions would give me at 0 c 1 is 0 at 1 c 2 sin root lambda is 0, because c 1 is already 0. So, for c 2 to be non 0, so that we get nontrivial solution we require sin root lambda should be 0. This says root lambda should be of the form n pi or lambda should be of the form n square pi square for n integer 1 2 and so on.

So, what we have got our Eigen values at pi square 4 pi square n square pi square and so on and what is our Eigen function you see, c 2 sin root lambda. So, corresponding Eigen functions would be phi 1 as k 1 sin pi x phi 2 as k 2 sin 2 pi x. So, this c 2 I have changed as according to this lambda 1 and lambda 2 we are writing k 1 and k 2 and so on. Thus phi n would be k n sin n pi x, now we have to choose these k n such that these functions phi n are normal that is...

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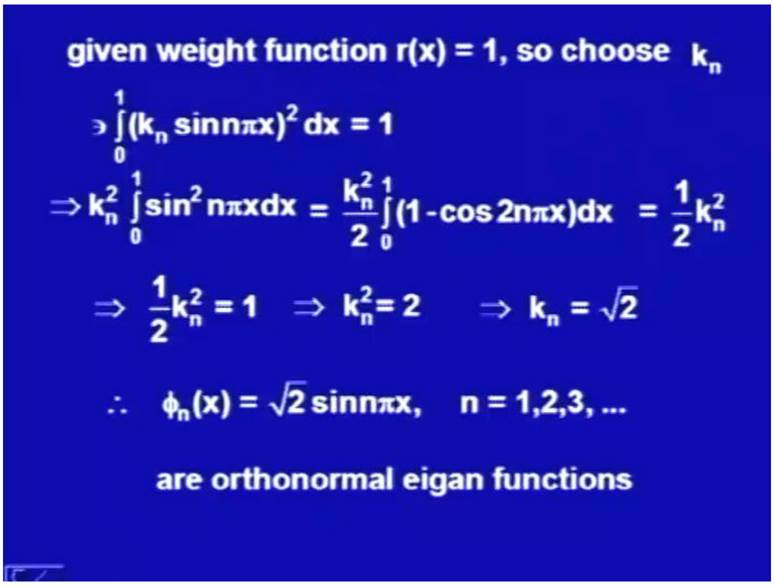

Now, given the weight function r x as 1 we want to choose k n, such that 0 to 1 k n sin n pi x whole square with respect integral with respect to x should be 1. Now, integrate this left hand side that is find out this integral, this says is this integral would be k n square 0 to 1 sin square n pi x d x which we could write as 1 minus cos 2 n pi x that is sin square n pi x is written as 1 minus cos 2 n pi x. Now, integrated with respect to x on the range 0 to 1, we do get it to be half k n square this integral you can do by yourself.

Thus, we get that half k n square should be 1 or k n square should be 2 or k n should be square root 2. This value of square root 2 this will give us the normal values or normal functions, so our phi n x should be of the form square root 2 sin n pi x for n is equal to 1 2 3 and so on. We already know that they are orthogonal now we have substituted for this k n as square root 2. So, they are normal as well. So, these phi and x are forming the orthonormal Eigen functions or the set of orthonormal Eigen functions for this given boundary value problem which is our strum Liouville problem as well.

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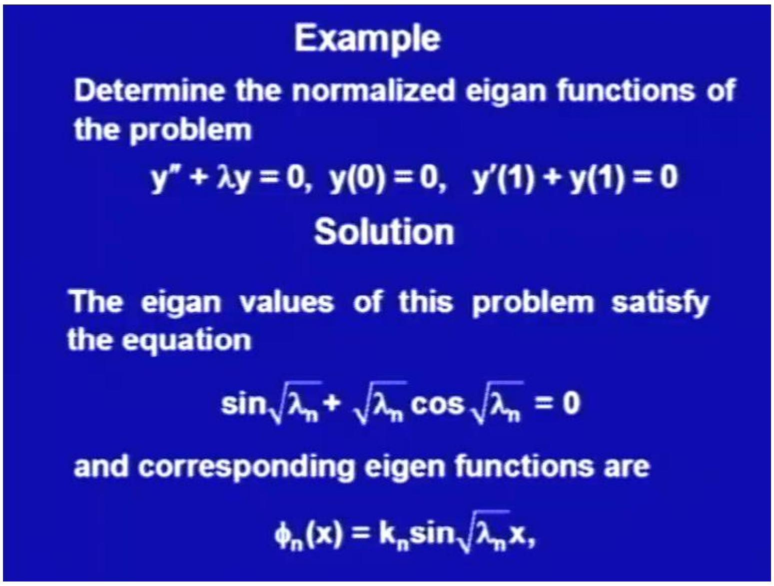

Now let us, do one more example, determine the normalized Eigen function of the problem y double dash plus lambda y is equal to 0 y at 0 is 0 and y dash at 1 plus y at 1 is 0. So, we are getting again this strum Liouville problem we had already seen this example and we had found out that the Eigen function or Eigen values of this problem are those which are satisfying the equation sin root lambda n plus root lambda n cos root lambda n is equal to 0, those lambda n are forming the Eigen values of this problem. And the corresponding Eigen functions also we had seen that were of the form k n sin root lambda n x.

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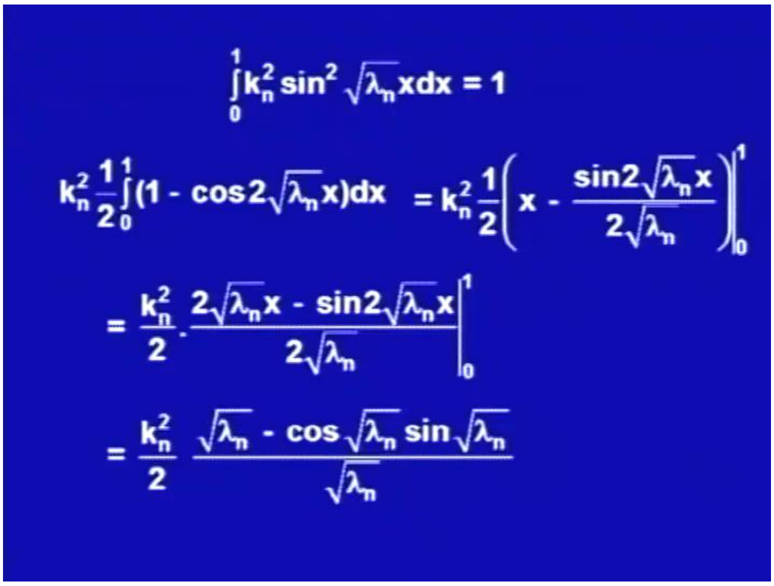

Now, we have to determine the normalized Eigen functions, that is we have to find out this k n such that, integral k n square sin square root lambda n x d x is 1. Now, this integrate this integral it says is k n square half 1 minus cos 2 root lambda n x d x, which is we can write as k n square into half x minus sin 2 root lambda n x upon 2 root lambda n that is the integral of this evaluated from 0 to 1. Now, if I rewrite this one we are getting k n square by 2 into 2 root lambda n x minus sin 2 root lambda n x upon 2 root lambda n. Now, two we are taking common and this 1.

And, when we are putting the value of 1,we are getting is that is k n square by 2 root lambda n minus cos root lambda n root lambda n, because sin 2 root lambda n we are writing as 2 cos lambda n and sin lambda n at x is equal to 1 and at 0 because sin is 0 is 0 and x at 0 is 0 we are getting 0 there.

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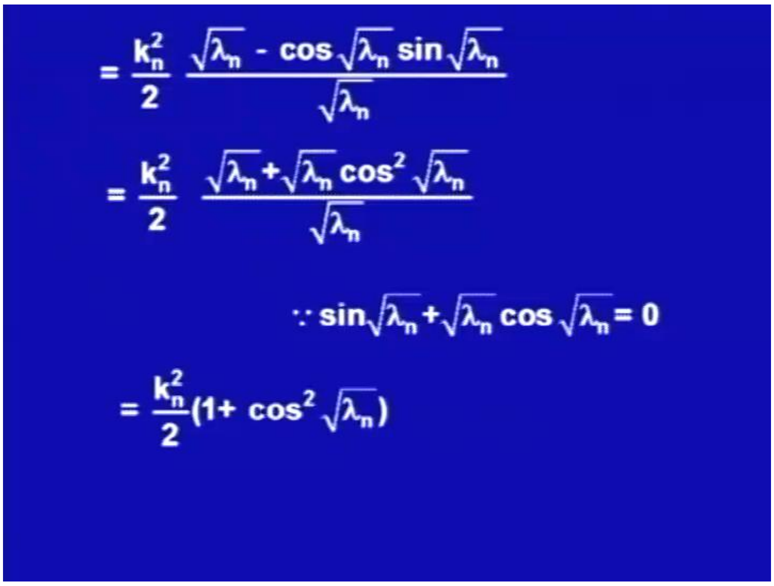

Now, this thing we can write as k n square by 2 lambda n plus root lambda n cos square root lambda n upon lambda n why since we do know that the lambda n are the Eigen functions which are satisfying this equation sin root lambda n plus root lambda n cos lambda n is equal to 0. So, substituting root sin root lambda n, as minus lambda root lambda n cos root lambda n we do get this one which says is that we are getting k n square by 2 into 1 plus cos square root lambda n.

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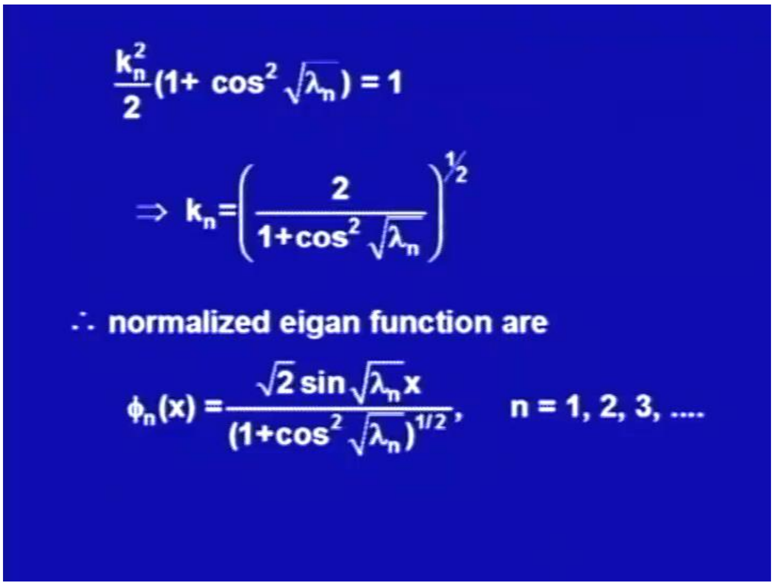

Now, we want that this has to be equal to 1, this says is my k n square must be equal to 2 upon 1 plus cos square root lambda n and whole square root. So, what would be our normalized Eigen function, normalized Eigen functions are phi n x is equal to root 2 sin square root 2 square root lambda n times x upon 1 plus cos square of square root lambda n whole square root for n is equal to 1 2 3. They are forming orthonormal set.

Because, we had already shown that these problems are nothing but the strum Liouville problems and they are satisfying this condition that is a 1 a 2 is a 2 and b 2 are not 0. So, they are having that real Eigen values as well as that Eigen functions are orthogonal. And here, we are choosing these multiplicative constants such that we do get the norm as 1. So, this is also forming our normalized this 1.

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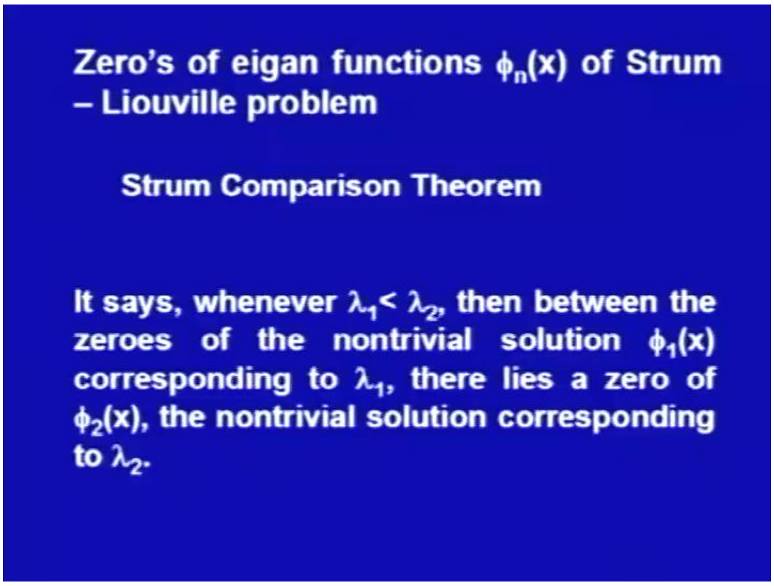

Now, let us see one more result about the zeros of Eigen functions phi n x of strum Liouville problem, what we mean by the zero? Zeros means is that where this phi n x the function is 0 for which values of x I have be wherever it is crossing the x axis if I draw the graph of phi n x.

This is very interesting result and this is known as the strum comparison theorem, rather than giving you exactly what is the result here I would just give you some point or some part of it that what is strum comparison theorem. It says is that whenever lambda 1 is less than lambda 2.

Then, between the zeros of nontrivial solution phi 1 corresponding to lambda 1, there lies a 0 of phi 2, the nontrivial solution corresponding to lambda 2, what we are saying is, we had already find it out that strum Liouville problem has real Eigen values, its Eigen functions corresponding to the distinct Eigen values are orthogonal and they are simple.

Now, what we are saying is one more result we have got that they can be ordered also. Now here, if i take any two Eigen values such that lambda 1 is less than lambda 2 and their corresponding Eigen functions if we plot. Then, corresponding to the smaller Eigen value the Eigen function would have zeros and the zeros of the second Eigen function what certainly lie between them lets have this more nice explanation over here.

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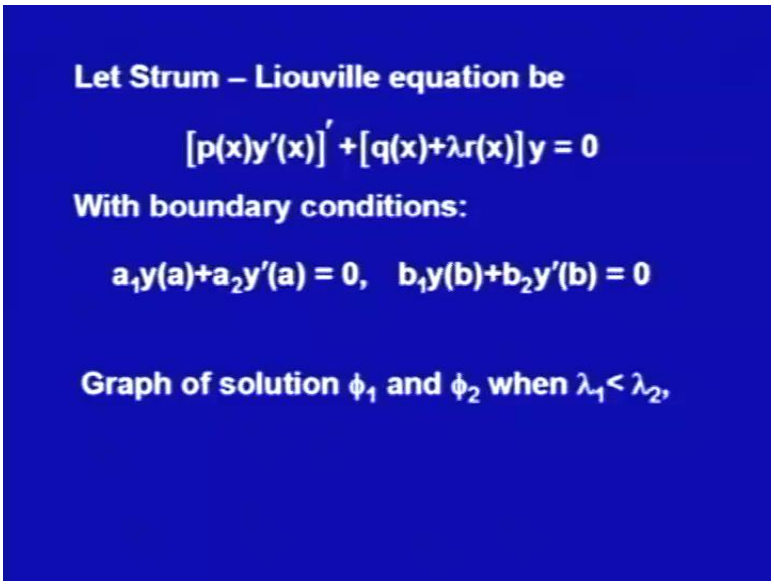

Let the strum-Liouville equation be this as usual one, with our boundary conditions a 1 y a plus a 2 y dash a is equal to 0 b 1 y b plus b 2 y dash b is equal to 0. And let us have that, some lambda 1 and lambda 2 are its two Eigen values and let us have corresponding Eigen functions as phi 1 and phi 2. Then this theorem is saying is that is phi 1 is the Eigen function corresponding to the smaller Eigen value and phi 2 is the Eigen function corresponding to the larger Eigen value.

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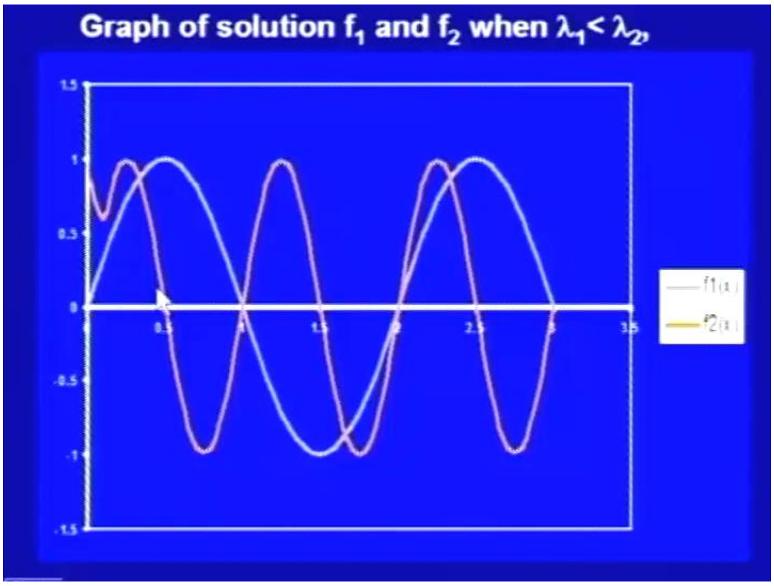

Then, you see that graph what it is saying is, it is simply telling us you see this lighter line this is corresponding to the larger or this smaller Eigen value we are having the 0 over here 0 over here 0 over here. And, you see this another coloured line this is for the graph of phi 2 that is corresponding to the larger Eigen value. You see its 0 is here that is between the zeros of these two similarly of course, the another 0 we are matching here.

Here, again we are getting the zeros of these function or these two and in between them there is a 0 of this, one that says is basically as we are increasing our Eigen values, because they can be ordered corresponding Eigen functions are giving as the zeros more nearly or we could get more kind of this kind of structure. Let us see with the help of one example of our this one also,

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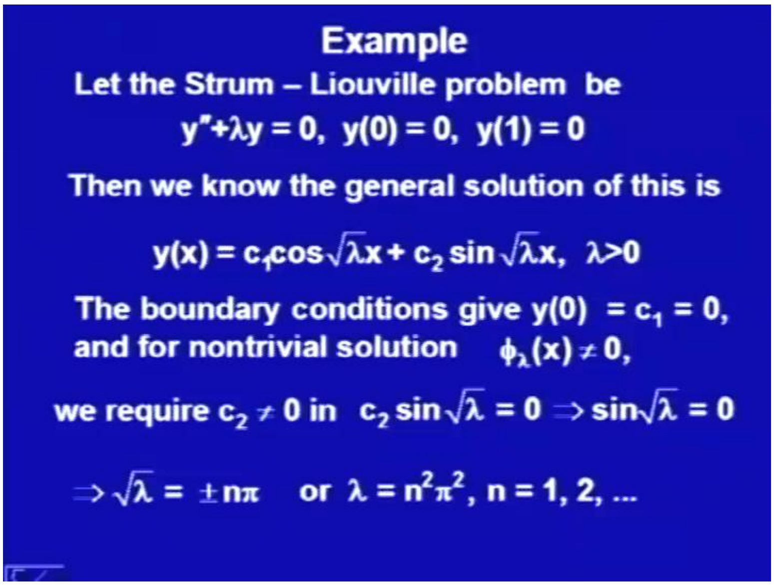

Let our strum-Liouville problem as first one we have already discussed y double dash plus lambda y is equal to 0 y at 0 is 0 and y at 1 is 0. Then, we do know that general solution of this we had already done as c 1 cos root lambda x plus c 2 sin root lambda x lambda is greater than 0, because for this we do know that for lambda positive only this has Eigen values. And, corresponding boundary conditions had given as nontrivial solution phi lambda not 0, as you see you do remember this just now we have done c 2 sin root lambda is equal to 0 or we do require sin root lambda is equal to 0. This says root lambda should be of the form plus minus n pi this is the Eigen value or lambda is n square pi square for n is equal to 1, 2, and so on.

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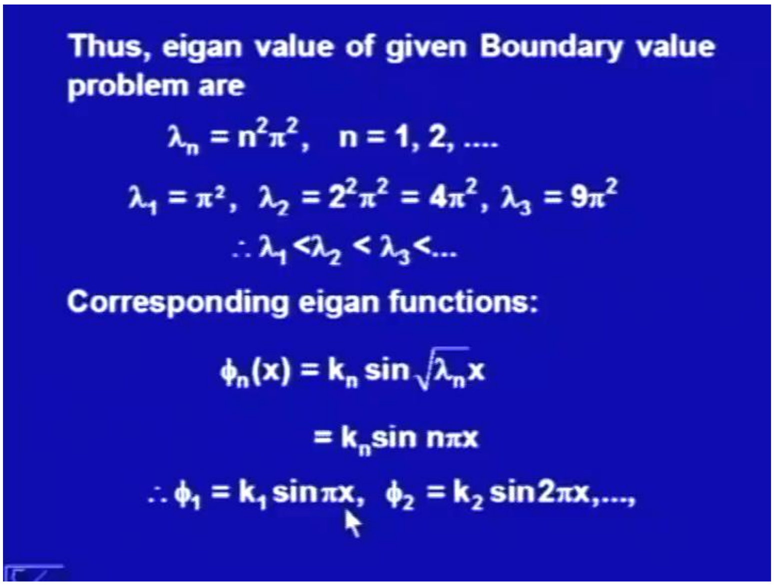

So, Eigen values of the given boundary value problems are n square pi square corresponding that is pi square 2 square pi square are 4 pi square and so on. And, let us take we see that is how we had ordered them, pi square is certainly smaller than 4 pi square is certainly smaller than 9 pi square and so on. So, we can order them. Corresponding Eigen functions, k n sin root lambda n x just now we had find it out, now let us take root lambda n should be n pi, so k n sin n pi x. Let us take the first two Eigen values lambda 1 and lambda 2, and the first two corresponding Eigen functions k 1 sin pi x and k 2 sin 2 pi x.

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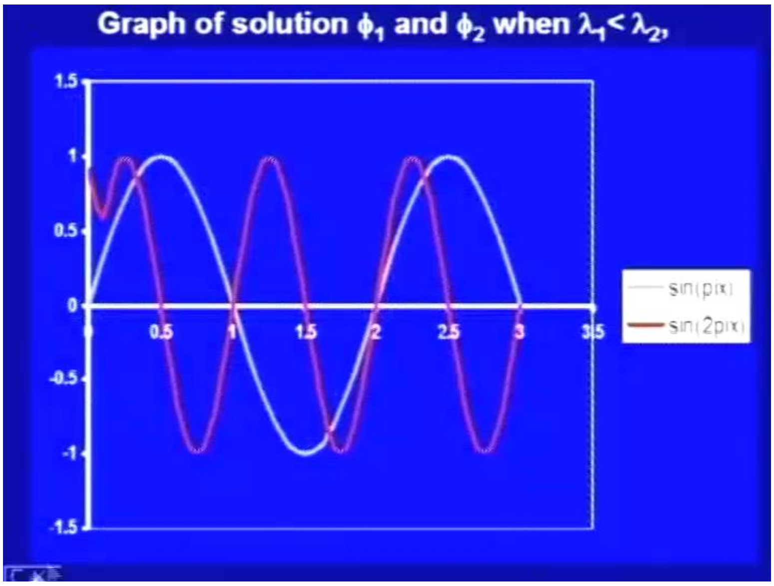

Now, let us see the zeros of these two function in the graph, this is the graph of sin pi x this is the graph of sin 2 pi x. Now, again you are seeing is sin pi x means is that is I would be having the at x is equal to 0 at x is equal to 1 at x is equal to 2 that is sin n pi is 0. Similarly, for sin 2 pi x we do know certainly that it should be n is equal to 0, n is equal to half, n is equal to 1, n is equal to 3 by 2, because it is 2 n. So, 2 n is integer or rather we do will get all those half values also we would be getting zeros. So, we would be getting of course, some points we are matching the zeros, but certainly between the 2 0es we are getting this one.

Basically, this result is saying is that as we are getting more higher Eigen values our corresponding Eigen functions will become more dense or the zeros would be more near on the real line we would have more alterations, this is all we do have for our strum Liouville boundary value problems.

So, today we had learnt properties of strum Liouville boundary value problem, the last lecture we had learnt boundary value problems. So, basically these two lectures of forming one unit for the boundary value, problems are more specifically we have done strum Liouville boundary value problems.

For this, we had learn that they are not as simple as the initial value problems we are not guaranteeing that is when they are going to have the solutions. So, we had generated from there that is they are having Eigen values and Eigen functions or rather when they are having a solution non trivial solution we call them Eigen values and Eigen functions.

So, first we try to find out the Eigen values corresponding Eigen functions those are the solutions of the boundary value problem. And, for this we have done the special kind of equations those we called strum Liouville equations and there we had used the simple linear boundary values and we called in complete strum Liouville boundary value problems. We have learnt certain properties of these problems they are remaining in the real line that is all the solutions are existing on the real numbers itself, the that is all for this one.

Thank you.