Mathematics - III
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Module - 2
Lecture - 9
Boundary Value Problems

Welcome to the lecture series on differential equations. Today's lecture is on Boundary Value Problems. Till now we had learn about initial value problems, where we used to have a differential equation and some initial conditions on the unknown function and it is derivatives, we call them initial conditions, because we were having them at most of the time at the initial point. A similar, kind of problem is also existing in the practical's that is called the boundary value problems.

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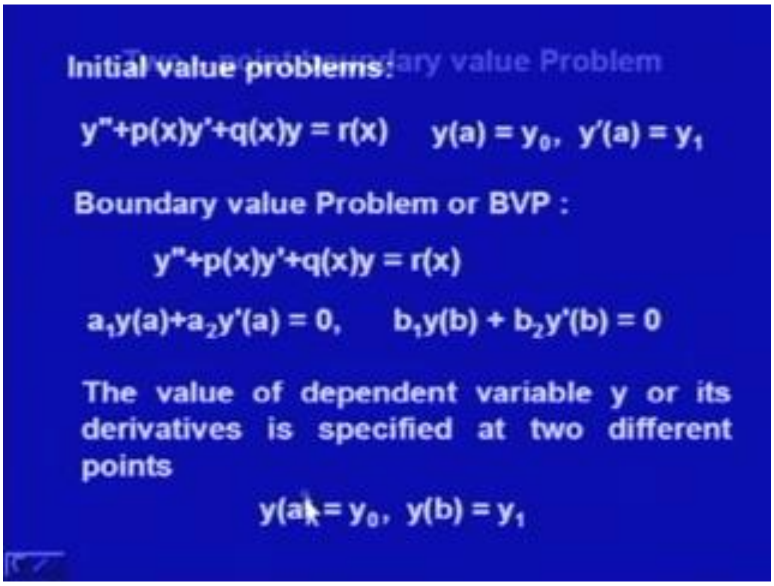

So, let us see a typical example of a initial value problem, here I had used one linear differential equation of the second order and the initial conditions as y the function at a is y naught and functions derivative at a is y 1, here we were having both the function and its derivative at the same point is been defined. In boundary value problem what we used to have a differential equation, so again I had used the same differential equation y double dash plus p x y dash plus q x y is equal to r x that is linear differential equation of the second order.

And, we may have the boundary conditions, which are such as a 1 y a at a plus a 2 y dash at a is equal to 0 and b 1 y at b plus b 2 y dash at b is equal to 0. This certainly this 1 a 2 and b 1 b 2 they should not be all 0, that is in this also both a 1 a 2should not to be 0, here also both b 1 b 2 should not be 0. We see here the difference between these 2 the differential equation is the same, here we were having the condition only at a point a and here we are having the condition at point a as well as at a point b and both involving our function as well as derivatives.

Now, these points a and b are typically the points on interval, where this equation is defined, so the end points a and b that is what we are calling them boundary value problems, that is the conditions are given at the boundary points. Thus what we are saying that the value of dependent variable y or its derivative is specified at two different points.

We can simplify our boundary conditions more say for example, here I have 1 simplified boundary condition, where the unknown function is a is specified at two different points only you see that y at a is y naught and y at b is y 1. Here we are not having any function derivative as they were in this original boundary conditions. Now, if I see the two problems the initial value problem this one, where the conditions are specified at a single point of the function and derivative of the function and this boundary value problem, where the differential equation is same, but the condition is defined at two different points they are looking similar.

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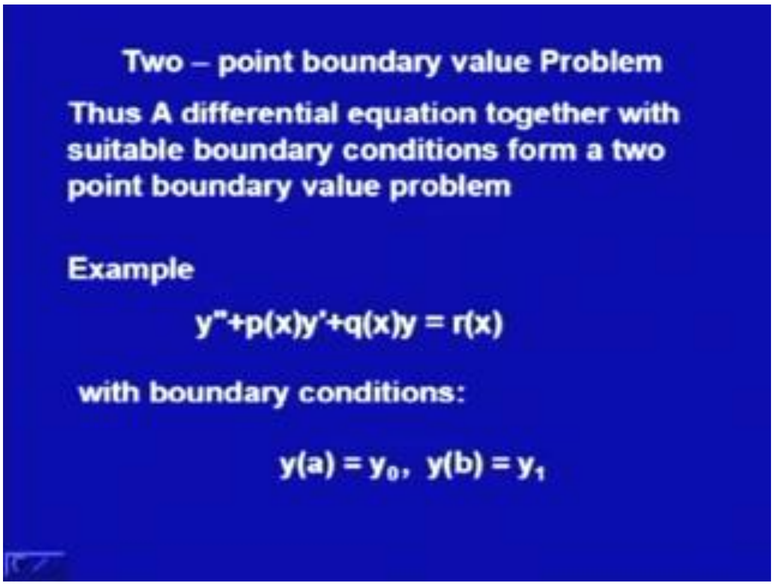

Now, what do we have read about the initial value problem that and with the we are just getting a formal definition of two point boundary value problem. A differential equation together with suitable boundary conditions form a two point boundary value problem, that we are just getting is that why the name is boundary value problem.

On the typical example again I am repeating the same example here, that y double dash plus p x y dash plus q x y is equal to r x, with boundary conditions at two specified points a and b y at a is y naught and y at b is y 1, so this is one simple example. Now, let us move with this typical example of ours and try to learn some more terms in this problem.

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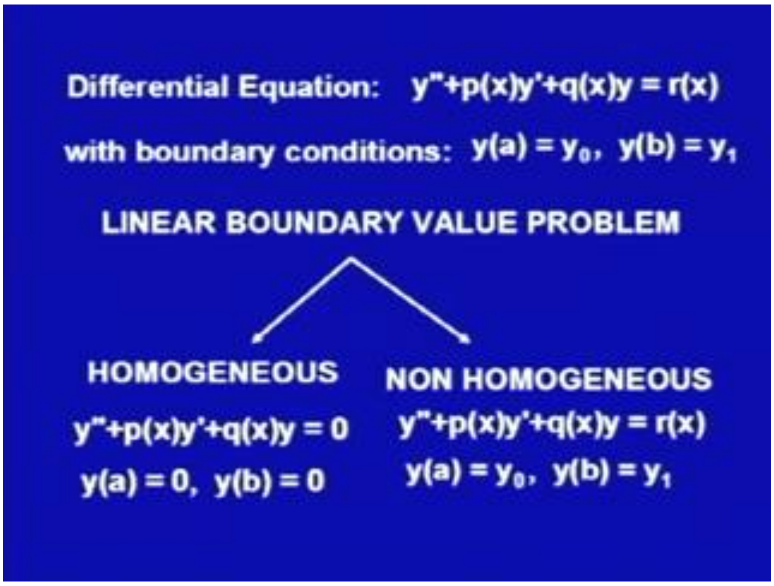

The differential equation, if we see in this boundary value problem, we do I certain we recognise that this is second order linear differential equation. Because, we are having is our s unknown function and its derivative they are occurring separately and this p x and q x are the function of x and r x is also a function of x. So, this is second order linear differential equation, where r x is not 0 its non homogeneous; otherwise, if 0 then it is homogeneous.

This is linear, so we would call whether is these two boundary conditions y at a is y naught and y at b is equal to y 1, so now this differential equation with these two boundary conditions, will be termed as linear boundary value problem. This is been namely classified into two forms one is homogeneous in homogeneous, what we will have as the differential equation would be homogeneous that is my r x would be 0 that says my differential equation would be y double dash plus p x y dash plus q x y is equal to 0.

Moreover you see, here that our boundary values are also that is boundary conditions are also changing, that is y at a is 0 and y at b is 0 this is homogeneous one. And; otherwise, it is called non homogeneous, so we will take this typical example, here even if r x is 0, but if y naught and y 1 are not 0 we will call them non homogeneous or if r x is not 0 and y not and y 1 are 0 is still it would be called non homogeneous one. So, this is 1 classification of linear boundary value problems.

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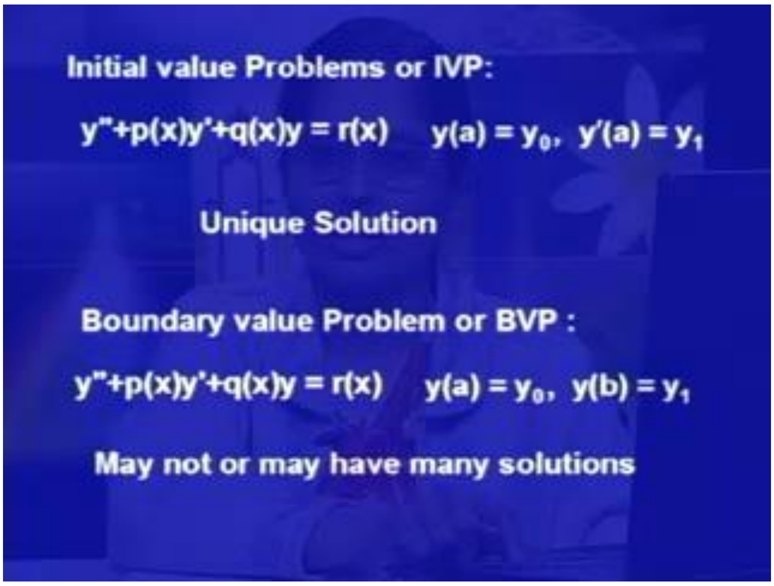

We had seen that in our initial value problem, if I was having this kind of equation of course, we are talking about linear equations and if we had this initial conditions at a of the function y and its derivative at a. We do know that under the assumption of this functions coefficients p x and q x and r x being continuous we had that this initial value problem was having a unique solution. Now, see our boundary value problem is having the same differential equation, but the boundary conditions that is y at a is y naught and y at b is equal to y 1.

Now, if I make the same assumption that is my coefficient p x q x and r x they are continuous, we cannot guarantee whether the solution may exist or solution will exist for this problem or not and if the solution does exist is still we cannot guarantee about the unique solution. These things that says is for under similar assumptions of p x q x and r x being continuous on the interval a b, we this boundary value problem may or may not have solutions and if it has solutions it may have one or many solutions. We well understand this or see with help of certain examples first and then we will go for the concepts.

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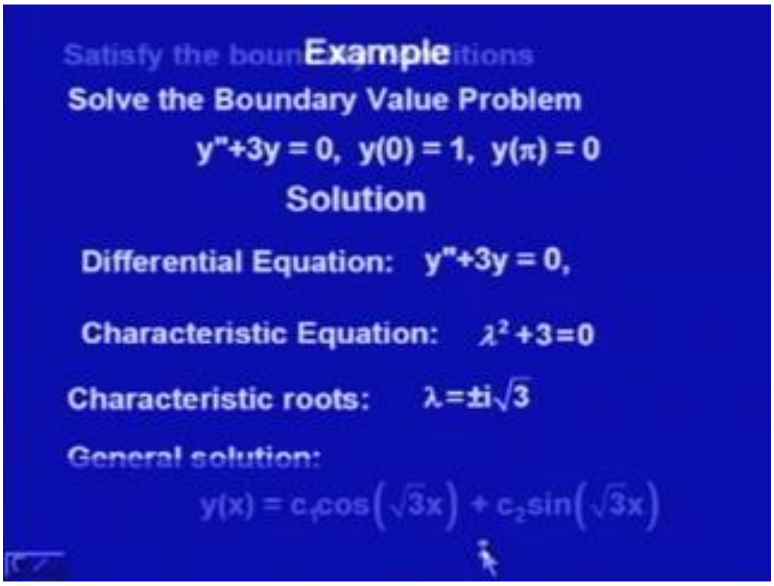

So, let us see some example first example, solve the boundary value problem y double dash plus 3 y is equal to 0, conditions y at 0 is 1 and y at pi is equal to 0, we see this is non homogeneous boundary value problem, since my boundary conditions are not 0. Solution, we see that our differential equation y double dash plus 3 y is equal to 0 this is homogeneous linear differential equation of second order, we do remember how to solve it.

So, we get the characteristic equation as lambda square plus 3 is equal to 0 it is roots is of course, plus minus square root 3 i that is the complex conjugate roots. So, in that case we do know that our general solution will be of the form c 1 cosine square root 3 x plus c 2 sin square root 3 x this is the general solution of this differential equation. Now, we have to get the solution of this boundary value problem that is we have to get a solution y such that these conditions are also satisfied. So, let us see in this general solution, if I use this condition, what we would get.

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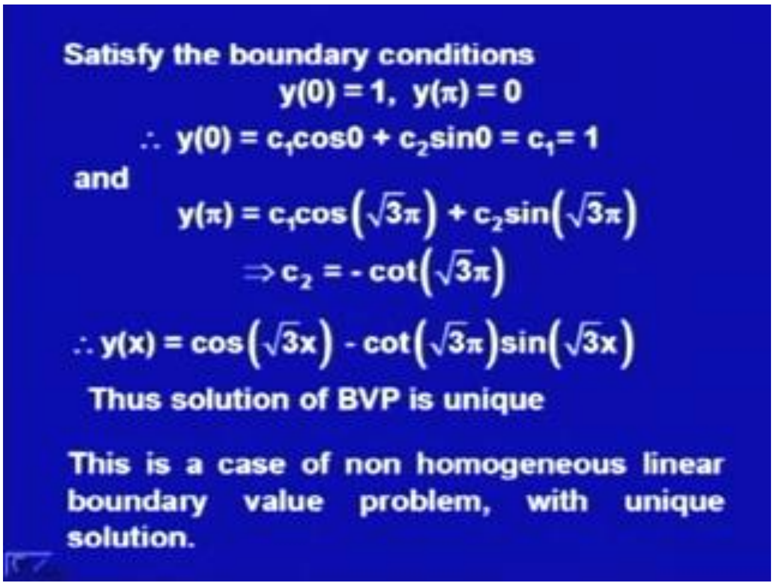

So, satisfy the boundary conditions that y at 0 is 1 and y at pi is 0, so if i put in our solution x is equal to 0 we would get c 1 cosine 0 plus c 2sin 0 this implies that c 1 is 1. And, if now I put x as pi I would get as c 1 cos square root 3 pi plus c 2sin square root 3 pi this should be 0, now since we already have in the first equation that c 1 is equal to 1 this will give me that c 2is equal to minus cot square root 3 pi.

Thus, we have got a solution of these that boundary value conditions have given me two values of c 1 and c 2c 1 as 1 and c 2as minus cot square root 3 pi. So, we have got the solution y x as cos square root 3 x minus cot square root 3 pi sin root 3 x, since we have got a single solution, we have got that this boundary value problem has a unique solution. This is of course, a case of non homogeneous now linear boundary value problem, with unique solution.

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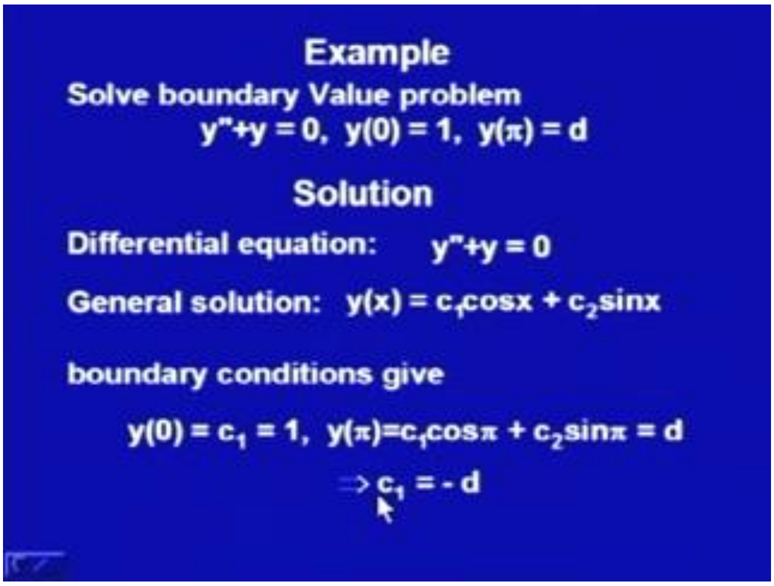

Let us have one more example solve the boundary value problem y double dash plus y is equal to 0, y at 0 is 1 and y at pi is equal to d. We see that from the previous example we have differed only in the differential equation as the coefficient of y has been changed from 3 to 1. But, we have changed our boundary condition only the second boundary condition there, we had used y at pi is equal to 0 here I am using y at pi is equal to d, now let us see what is the solution of this.

The given differential equation is y double dash plus y is equal to 0, we see again that this is linear differential equation. So, it is characteristic equation would be lambda square plus 1 is equal to 0 and we will get again the complex conjugate characteristic roots that is plus minus i, so the general solution would be of the form c 1 cos x plus c 2sin x.

Now, see with this boundary condition this is the general solution, now to try to satisfy the boundary conditions. So, put x is equal to 0 in this general solution, we would get that y at 0 we would get as c 1 and that is given as 1. So, we have got from the first condition c 1 as 1, when i put x is equal to pi, here in this solution we get y at pi is equal to c 1 cos pi plus c 2 sin pi and the given condition says is it should be equal to d, what its gives me that is it gives me that is, since c 1 is already 1 cosine pi is minus 1, I would be getting it and sin pi is 0, I would be getting it c 1 is equal to minus d.

Thus, what we have got that both the boundary conditions are giving me the value of c 1 only. The first condition is giving me c 1 is equal to 1 the second condition is giving me c 1 is equal to minus d now if this d is equal to minus 1, then certainly I do get the solution as c 1 is equal to 1.

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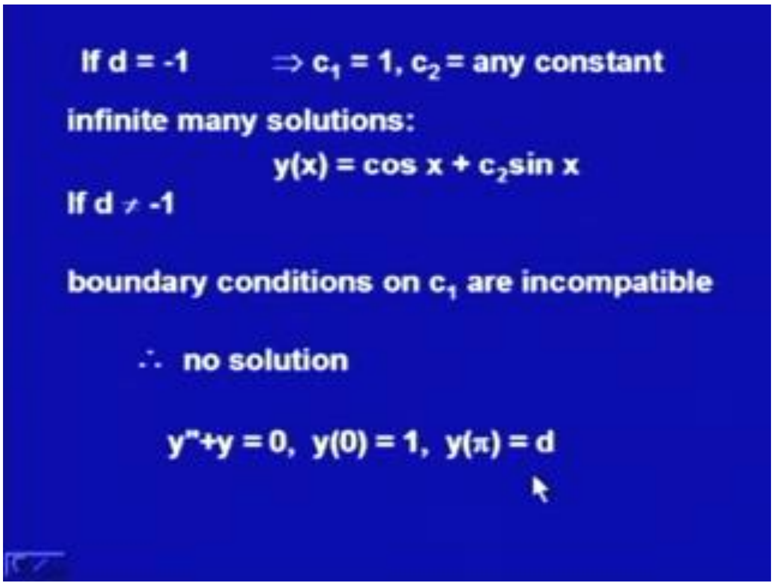

So, let us see if d is equal to minus 1, then I get c 1 is equal to 1 and c 2any constant. So, what will be the solution of my boundary value problem I would get infinite many solutions as cos x plus c 2sin x, where c 2is the arbitrary value. Now, if d is not minus 1, what I would be getting c 1 is equal to 1 and c 2is equal to minus d and says d is any value other than minus 1, I am getting two different values for c 1 that says the boundary conditions are not compatible on c 1, what it says is there is no solution for this boundary value problem.

So, now we had say in this example that if we do have the boundary condition, where I had actually I would taken a example a general 1 the condition, where the general constant d. And, we have got that if I could a particular value to that d as minus 1, we are getting infinite many solutions and if it is not minus when we are getting no solution.

So, now we have got the example that is under the same condition, if you see my differential equation the coefficients are continuous functions the boundary conditions are simple one. But what we are having is that is this may have many solutions or may not have solution depending upon d.

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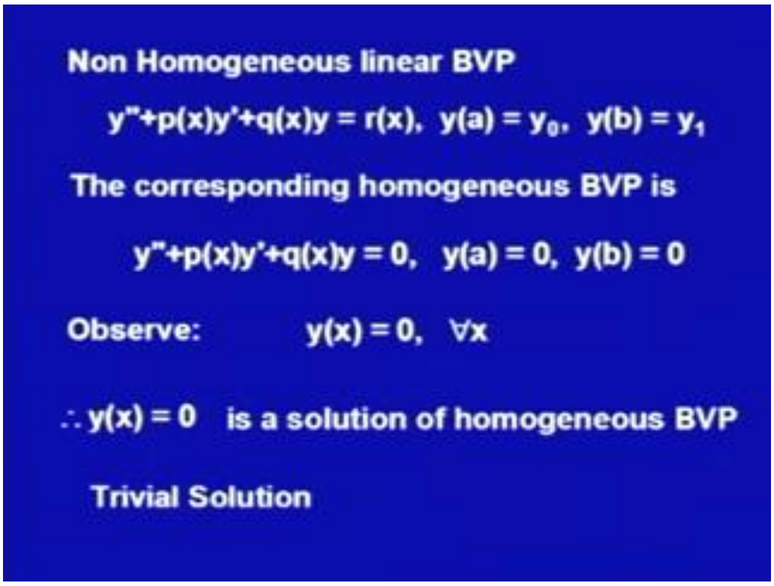

Now, let us see what we had learn about this non homogeneous both the examples we have taken of non homogeneous linear boundary value problems. Now, let us go back to that is before actually solving this boundary value problem, we will see is that is how we have done this initial value problems, because our differential equation is same only the conditions have changed.

We remember that, if we are having this non homogeneous linear differential equation corresponding to that we always had a homogeneous linear differential equation. And, the solution of non homogeneous linear differential equation had come through the solution of homogeneous linear differential equation and the same thing was true for the initial value problem as well. And, there we have got that if we are using initial value problem we are getting a unique solution.

Here we are getting is that is in non homogeneous linear differential boundary value problems, we have got that is either we are not having a solution or we are having a unique solution or we may have many solutions. Let us see what corresponding to that can we have a homogeneous boundary value problem, certainly we can have. So, if my non homogeneous boundary value problem again I am taking that simple example of, where this boundary conditions are only at two points.

So, if I do have this boundary value problem we to have simple linear differential equation of the second order non homogeneous and the conditions were y naught and y 1 are not necessary, you see or rather we can take we are taking them non 0. Then, the corresponding homogeneous boundary value problem would be when r x is 0, that is it is same as that linear differential equation that is I am having is homogeneous linear differential equation.

Moreover I do require that the conditions also must be 0, if you do remember in initial value problem, we had never discussed about the conditions initial conditions. We have just discussed about the differential equation that is non homogeneous correspondingly that we had homogeneous linear differential equation we got the general solution from, here then we got general solution of non homogeneous 1 and then we satisfied the initial value.

Now, here what we are saying is that is homogeneous boundary value problem says is we are homogenizing or we are making this y naught and y 1 that is the conditions for boundary value, then also as 0, so this is the corresponding homogeneous boundary value problem. In initial value problems, we had got that the solution we are getting through homogeneous one.

Let us see is there any inside that is, if I try to solve this homogeneous boundary value problem can I get something or can I assured of any solution and if that is happening can I relate it with this non homogeneous, let us try to say it we do not know exactly at this moment. What we are observing here 1 thing is that if I put y is equal to 0, then this will satisfy this equation as well the initial this boundary condition.

So, this y is equal to 0 for all x this will satisfy this boundary value problem and certainly this would be a solution this solution what we will call trivial solution. And of course, we are not interested in this trivial solution, because it is not giving me anything or it is not giving us any insight about this differential equation, so we are certainly interested in non trivial solution that is solution other than 0 value.

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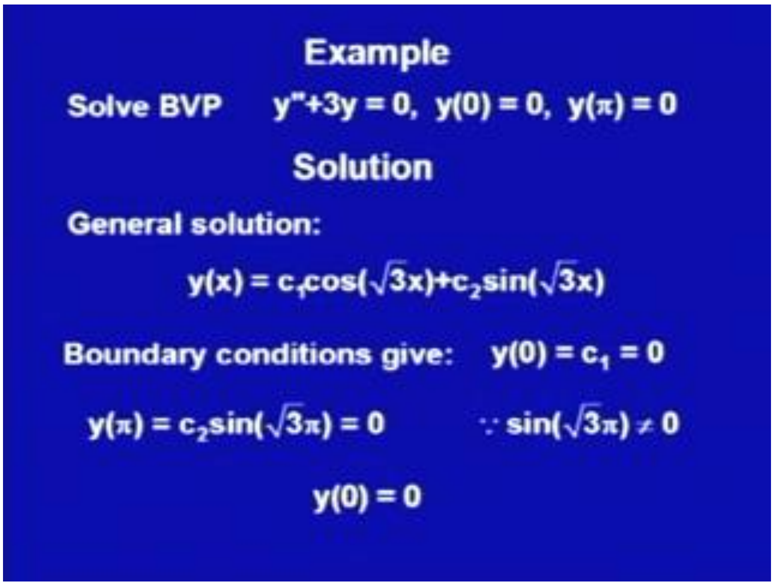

Let us see some non some homogeneous boundary value problems and their solution with the help of example. So, again i am having this example y double dash plus 3 y is equal to 0 y at 0 is 0 and y at y pi is equal to 0, if we see this is corresponding to the our first example of homogeneous boundary value problem. In the first example, we had this same equation, but the conditions were here different that is they were not 0.

Now, let us see the solution of this we had seen the general solution of this differential equation was c 1 cos square root 3 x plus c 2sin square root 3 x, now try to satisfy these boundary values, so if I put x is equal to 0, I would get c 1 as 0. Now, if I put x is equal to pi I would get y pi is equal c 1 cos square root 3 pi plus c 2sin square root 3 pi, but since c 1 is 0, so I would get only c 2sin square root 3 pi is equal to 0. Now, since sin square root 3 pi is not 0, this simply gives me c 2is equal to 0, so what we are getting is both c 1 and c 2 0 that says I am getting the trivial solution only 0.

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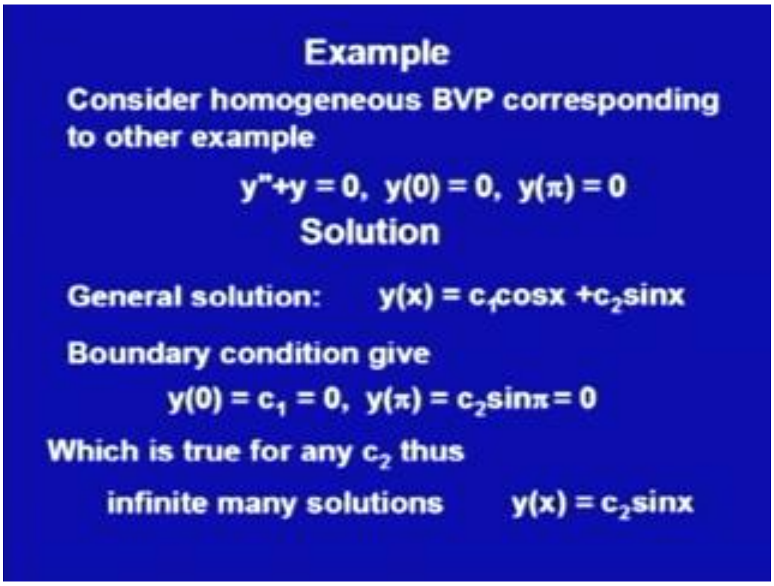

So, now let us see the another example, this homogeneous boundary value problem corresponding to the other example do you remember that there example, we had y double dash plus y is equal to 0, here I had put the boundary conditions also at 0. So, we are having y at 0 is 0 and y at pi is equal to 0, solution we do remember that the corresponding to this differential equation the general solution was c 1 cos x plus c 2sin x.

Now, try to satisfy this boundary conditions, so at x is equal to 0 I would get c 1 is equal to 0 and at x is equal to pi, I would get c 2that is c 1 sin pi cosine pi and since c 1 is 0, I would get only c 2sin pi and since sin pi is 0, I would get c 2in a constant. So, what we are getting the solution we are getting the solution that this boundary conditions are giving me c 1 as 0 and c 2any arbitrary value. Thus we get the solution of this boundary value problem as infinite many solutions c 2sin x infinite many, because c 2is any arbitrary constant.

Now, in these examples we had seen that our differential equation that, because we have coming with the homogeneous boundary value problems. So, our and both the times our boundary values remain 0 and pi so our boundary conditions remains same, now only thing what we have changed was our differential equation. In the first example I had y double dash plus 3 y, here and here I am having is y double dash plus y that is the coefficient of y has changed.

And, in the first example we had got that that boundary value problem was having a unique solution and this boundary value problem is the in the first example we got that the boundary value problem was not having any non trivial solution and in this 1 we are having a non trivial solution rather infinite many solutions. Now, let us see this same problem that is, where I am changing only, because boundary conditions have not changing my boundary values are not changing, so only thing which is changing is in this equation and in this equation also the thing, which is changing is only the coefficient of y.

So, let us try to see this particular problems or rather we could say is that, now for solving boundary value problems, we are first trying with very special kind of problems and those special kind of problems, now we would call them as Eigen value problems also.

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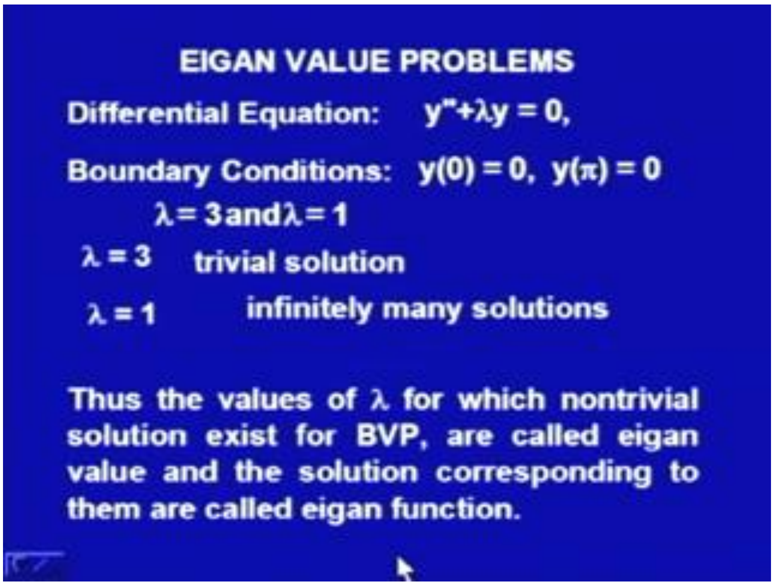

Let us see what these we are saying we do have a differential equation of the form y double dash plus lambda y is equal to 0 and the boundary conditions y at 0 is 0 and y at pi is equal to 0. Now, we had already seen that in our examples, we had the different values of lambda, when I had lambda is equal to and 1, when I had this lambda is equal to 3 we have got trivial solution only, when lambda is equal to 1, we have got infinite many solutions.

So, what we are gaining is that, because of this lambda that is this value of lambda is changing and we are getting either solution or you could say only trivial solution, let us see here no solution will not be a question, because we will always get the trivial solution trivial solution we will keep in the category of no solution.

So, what we are getting is, because of this value of lambda we may get trivial solution or non trivial solution. So, now we are going to define the terms the values of lambda for which nontrivial solution exist for boundary value problem are called Eigen value and the solution corresponding to them are called Eigen function.

So, what we have got we have seen that lambda is equal to 3 for this will not be an Eigen value, because we are getting only trivial solution on what we call Eigen value if it is nontrivial solution is existing. So, lambda is equal to 1, since nontrivial solution is existing that would be called an Eigen value and corresponding solution, what was the solution if I do remember that was c 2sin x, so that is the corresponding Eigen function.

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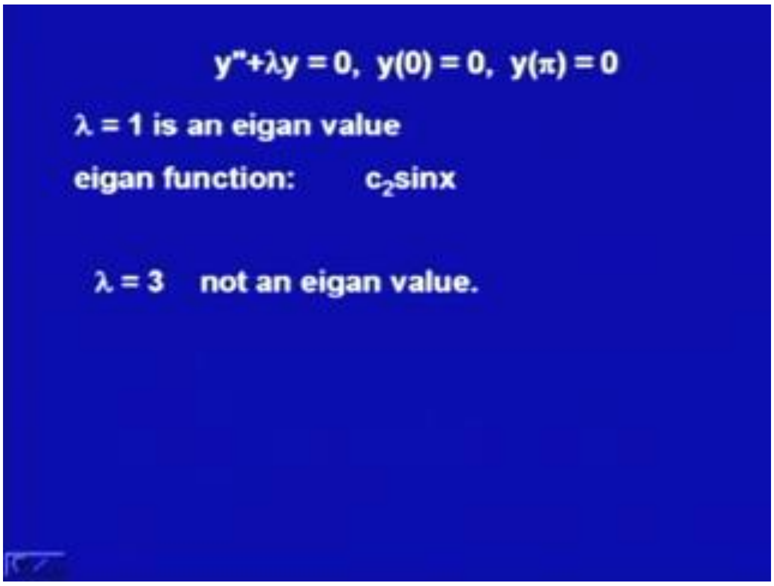

So, now, this problem we would call Eigen value problem, what we are having is again I am summarising the things y double dash plus lambda y is equal to 0, y at 0 is 0 and y at pi is equal to 0, lambda is equal to 1 is an Eigen value and the corresponding Eigen function is c 2sin x, lambda is equal to 3 is not an Eigen value. Now, let us have something that is more general, here we have just seen two typical examples, then we find out all the lambda values for which the nontrivial solution does exist, let us see if we could do it.

So, to find Eigen values and Eigen function of this boundary value problem y double dash plus lambda y is equal to 0, y at 0 is 0 and y at pi is equal to 0. Since in our examples we had seen the different values of lambda are giving me that is either solution is existing or there is no solution that says is this is depending on the value of lambda, now lambda is any real number.

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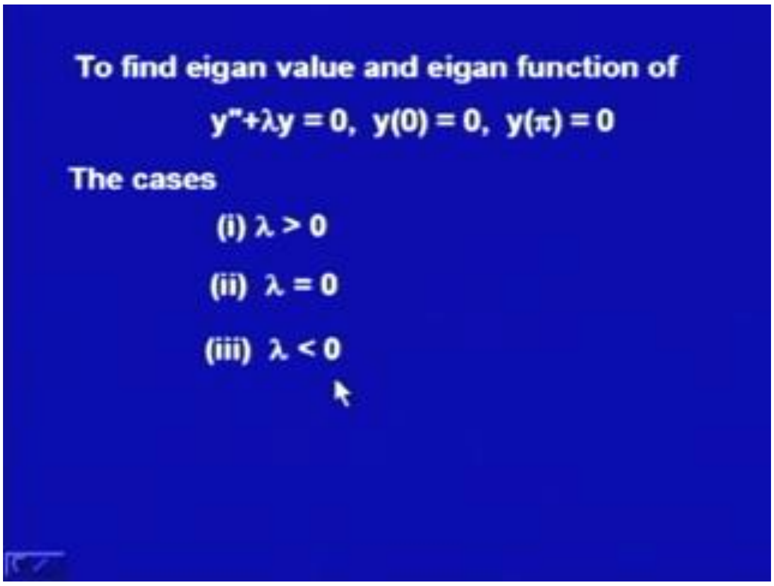

Let us divide into 3 cases it what the 3 cases positive lambda, lambda 0 and negative lambda, because that would give me actually my general solution of this differential equation as different, let us discuss these cases one by one.

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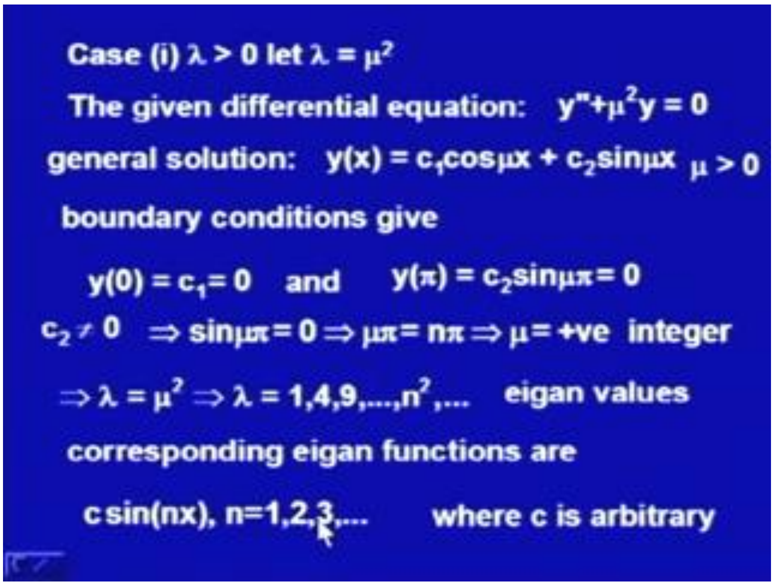

Let us take the first case, case one lambda positive value, so suppose lambda is mu square, then the given differential equation y double dash plus lambda y is equal to 0, will become the y double dash plus mu square y is equal to 0. Now, what will be its characteristic equation that should be n square plus mu square is equal to 0 and will give me the two complex conjugate characteristic roots that is plus minus mu i.

Thus we would get the general solution of the form c 1 cos mu x plus c 2sin mu x, Now you see is that is why I have taken lambda is equal to mu square otherwise I have to write here square root lambda. And we will assume here that mu is positive, because it does not matter whether mu is negative or positive lambda will always be mu square that is a positive real number, so we will take mu as positive.

Now, the boundary conditions that y at 0 is equal to 0 and y at pi is 0 would give me that c 1 is 0 the first condition when x is equal to 0 when x is equal to pi, you will get y pi as c 2sin mu pi is equal to 0, because c 1 is already 0. Now, for getting nontrivial solution I must get that c 2should not be 0, that says is that I must get sin mu pi is e equal to 0, we do know that it will be 0 if mu pi is equal to m pi that says is mu must be a positive integer that is why you do not understand that, why I have taken mu to be positive.

Now, what we have got from here that I will get nontrivial solution for this differential equation or this boundary value problem. If my the value lambda is equal to mu square and mu is a positive integer that says is the Eigen value is or that lambda should be 1, 4, 9 and so on like n square, because n is a positive integer.

So, these would be, so my Eigen values for the problem should be 1 4 9 and n square in general n square and the corresponding Eigen functions would be c times sin n x, thus we would be getting is c 2is that is non 0 value that is arbitrary values. So, we would have c sin n x for n is equal to 1 2 3, so I would be getting is my Eigen value as n square and corresponding Eigen function as c sin n x. So, lambda positive 1 4 9 and n square like that they are Eigen values, now you see is that lambda is equal to 3 is not coming over here and, so we have got that it was not an Eigen value.

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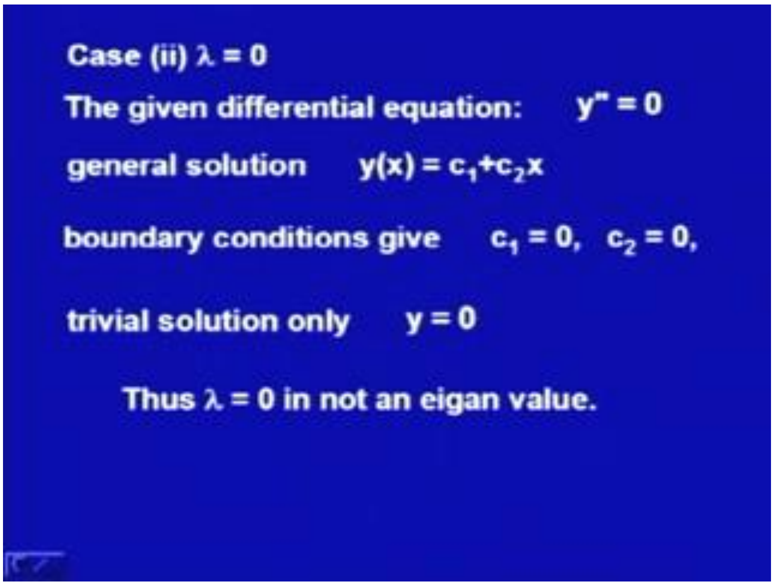

Now, let us see the second case lambda is equal to 0, in this I would given differential equation what it will become y double dash is equal to 0, that is we would be having two characteristic roots 0 and 0 that is equal curve. And, in that case we do know our general solution will be of the form c 1 plus c 2x.

Now, if I try this boundary conditions for this at x is equal to 0 I will get C 1 is equal to 0 at x is equal to pi I would get c 1 plus c 2pi is equal to 0, c 1 is already 0 pi is not 0 gives me c 2is equal to 0 that is at lambda is equal to 0 the boundary conditions are providing me only the trivial solution, where both c 1 and c 2 are 0. So, we get only trivial solution thus lambda is equal to 0 is not an Eigen value.

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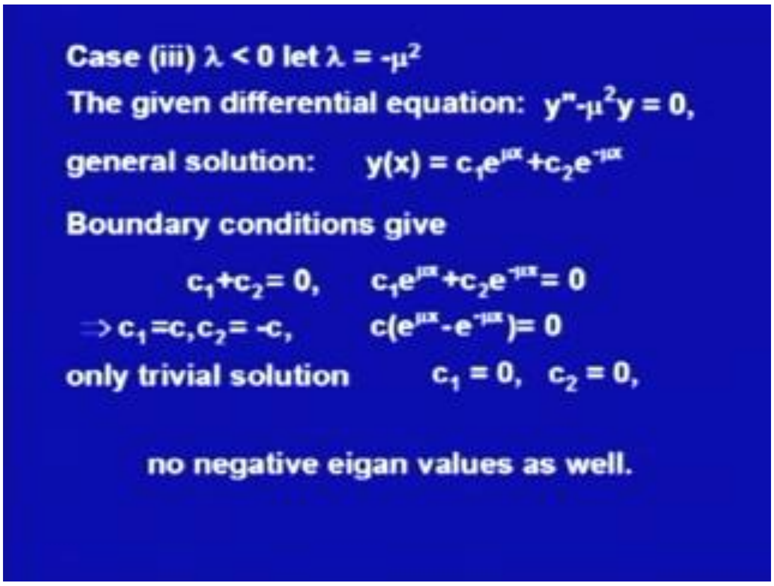

Now, let us come to the third case, n lambda is negative, so let us assume lambda is equal to minus mu square, where mu square of course would be positive, because we are taking mu to be a positive real number.

Now, then what will our differential equation become y double dash plus lambda y is equal to 0 it will become y double dash minus mu square y is equal to 0. So, what will be its characteristic equations n square minus mu square is equal to 0 that says I would get characteristic roots two different values mu and minus mu. Thus my general solution would be of the form c 1 e to the power mu x plus c 2e to the power minus mu x.

Now, here if I try to find out with the boundary conditions whether boundary conditions what they are giving, they would give me if at x is equal to 0 we would get c 1 plus c 2is equal to 0 at x is equal to pi we would get c 1 e to the power mu pi plus c 2e to the power minus mu pi is equal to 0. Now, this first condition that c 1 plus c 2is equal to 0 gives me that c 1 is equal to minus c 2or c 2is equal to minus c 1, so let us take it c 1 to be c, then c 2is minus c.

Now, substitute this in this second equation, what it gives me that c times e to the power mu pi minus e to the power minus mu pi is equal to 0. Now, we do know that e to the power mu pi minus e to the power minus mu pi this does not have or this will not be 0 for c real that says is, again we are getting only trivial solution that says that c 1 c 2should be 0 and, so negative Eigen values as well.

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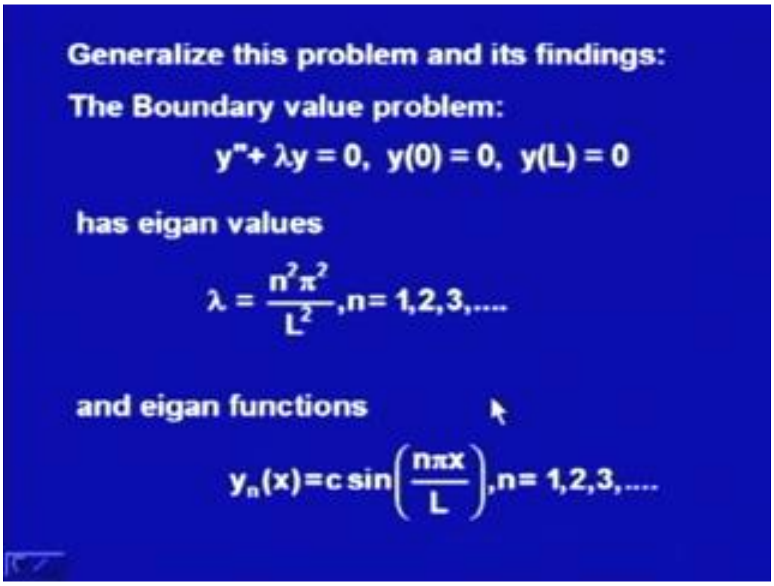

Now, what we had learnt that, now generalize this problem and its findings, what we had learnt the boundary value problem Y double dash plus lambda y is equal to 0, y at 0 is 0 and y at L is equal to 0, now what we had got there the my condition was y at pi.

Now, not working with the interval 0 pi only, we are generalizing it to 0 to l if I generalize it, what we would get that it will have Eigen values only as of n square pi square upon L square, what we are doing is that is we are, now you can do this as an exercise for yourself that is rather than putting the boundary value boundary condition y at pi is equal to 0 put y at L is equal to 0 and try to satisfy it we would that the lambda value for the lambda, you would get is n square upon pi n square into pi square upon L square for n being a positive integer 1 2 3 and so on.

And the corresponding Eigen functions would be c times sin n pi x over L for n is equal to 1 2 3 and so on that is if Eigen value is n square pi square upon L square its Eigen function is c sin n pi x over L, that is we would have infinite many solutions corresponding to this Eigen value, this is what we have got generally that is this kind of problem, what we are having is Eigen values and Eigen function of this kind of boundary value problem.

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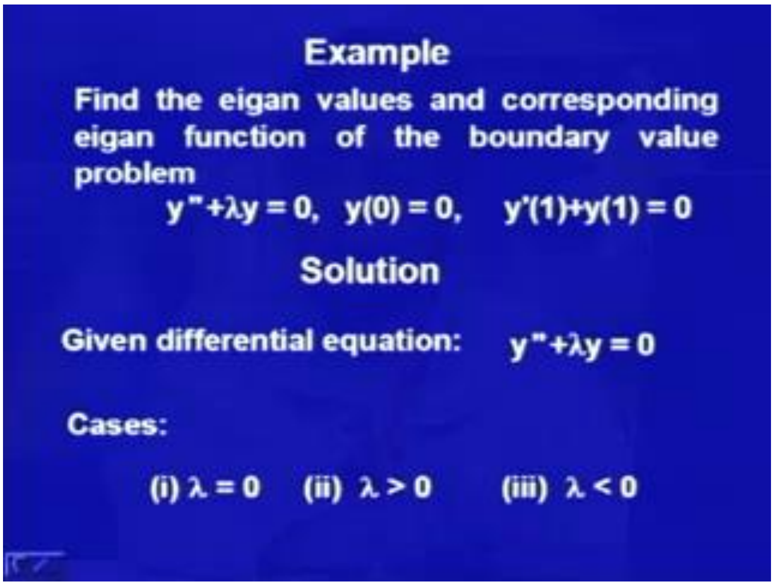

Let us see one example here find the Eigen values and corresponding Eigen function of the boundary value problem y double dash plus lambda y is equal to 0 the conditions are, now are you see y at 0 is 0 and the second condition is y dash at 1 plus y at 1 is equal to 0. Till now, what we had seen my differential equation is same, we are having homogeneous boundary value problem, but my boundary conditions are, now not that simple boundary condition that at two points the value of the function, now we are involving the derivative as well.

Let us see try to find out the Eigen values and Eigen function of this example as well or this particular. Again we see the differential equation is y double dash plus lambda y is equal to 0 we will again see with the cases, because this value of the lambda would be making our general solution differently. So, again we will take those three cases that lambda is 0 lambda positive and lambda negative discuss these three cases one by one.

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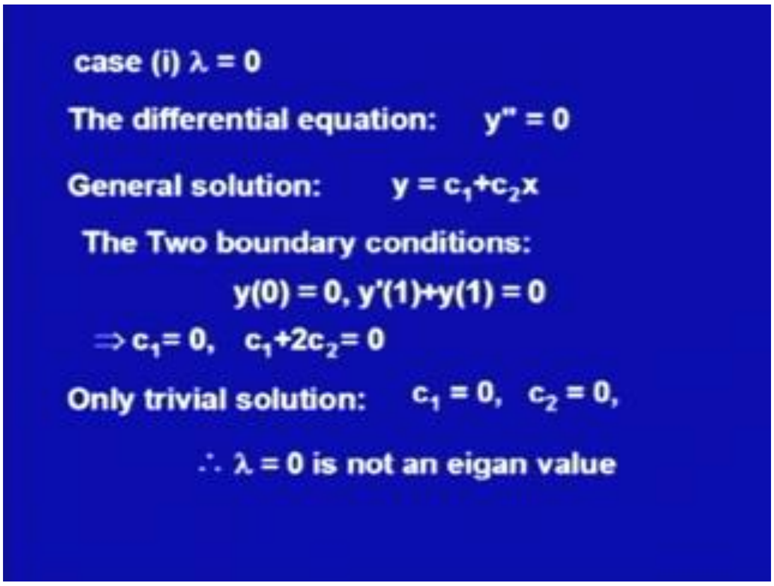

First the case lambda is equal to 0 in this case our differential equation will become Y double dash is equal to 0 as we had already seen the general solution in this case would be c 1 plus c 2x. Now, our boundary conditions involve y dash also, so let us see the two boundary condition y at 0 is 0 and y dash at 1 plus y at 1 is equal to 0, if I put x is equal to 0 I would get c 1 is equal to 0.

And, the second condition says is y dash at 1, what will be y dash x it would be simply c 2 that is c 2 and y at 1 would be c 1 plus c 2,so we are getting c 1 plus twice c 2is equal to 0, since c 1 is already 0 from the first condition the second condition would give me c 2is 0. That is again we are getting the trivial solution of this equations c 1 is 0 and c 2is 0 that says is I would get only trivial solution y as 0 so lambda is equal to 0 is not an Eigen value for this problem.

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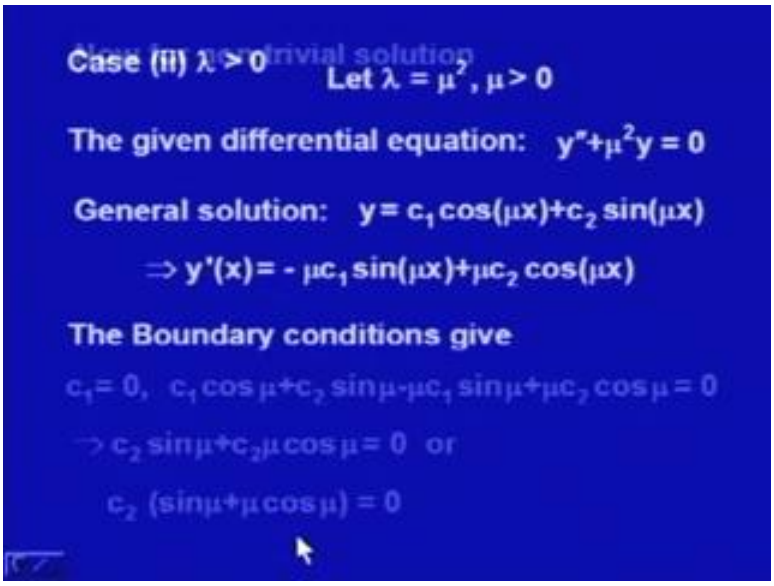

Now, come to the case two, lambda as positive value again, let lambda as mu square and we will again take mu to be positive real number. The given differential equation will become y double dash plus mu square y is equal to 0, we had already seen that the general solution in this case is of the form C 1 cos mu x plus c 2sin mu x.

Now, put the boundary conditions, so d for boundary conditions with the y dash also, so first calculate y dash x that would be minus mu c 1 sin mu x plus mu c 2sin cos mu x. Now, y at 0 is 0 that is if I put x is equal to 0 in the first general solution I would get c 1 is equal to 0, then when I am putting y at 1 here what I would get c 1 cosine mu plus c 2sin mu when we put y dash at 1 I would get minus mu c 1 sin mu plus mu c 2cos mu and this is given is equal to 0.

Now, simplify it what we are getting is c 2sin mu plus c 2mu cosine mu is 0 or more simplified form c 2times sin mu plus mu cosine mu is equal to 0, whether because this term c 1 is 0. So, the term containing c 1 we had eliminated over here.

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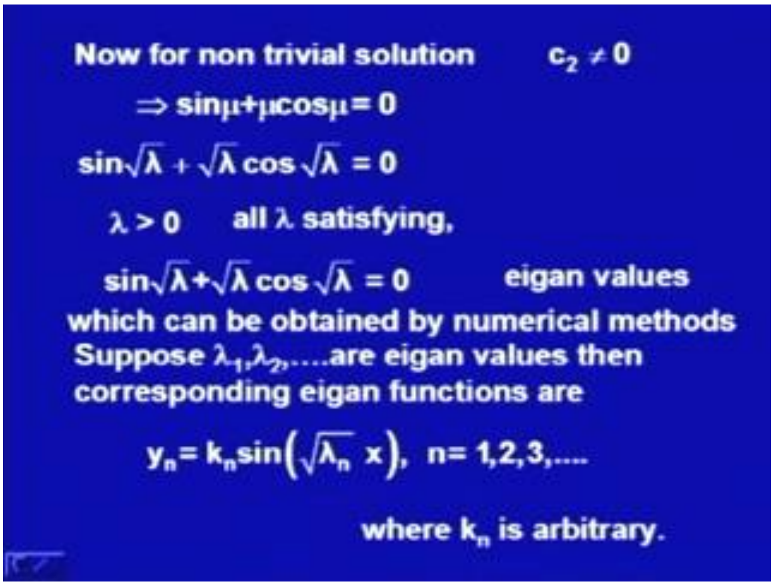

And from here, we will get nontrivial solution that is nontrivial solution; that means, I require c 2to be not 0 that says is my this coefficient of c 2that is sin mu plus mu cosine mu this should be 0 that is what is we will writing here that is for c 2not to be 0 implies that sin mu plus mu cosine mu should be 0.

So, what we are getting is now on replacing mu with square root lambda we are saying is lambda is an Eigen value only if it satisfies this equation, what is that equation sin root lambda plus root lambda times cosine root lambda is equal to 0. That is all lambda which are satisfying this equation they would be the Eigen values for our boundary value problem, where my differential equation was y double dash plus lambda y is equal to 0, but the boundary conditions have changed. So, you see that is boundary conditions are changing our Eigen values are changing and lambda has to be positive.

So, we do get lambda positive and satisfying these equation would form the Eigen values of lambda. Now, how to find out this 1 of course, this is not a simple equation, which we can solve directly yeah numerical methods that is put certain values for the lambda and then we get that is what are those particular values, so which can be obtained by numerical methods.

Of course, corresponding Eigen function would be, so suppose lambda 1 lambda 2 that is those are, which are satisfying this equation they would be of course, certainly more than 1 values they are Eigen values then the corresponding Eigen functions are k n sin square root lambda n x for n is equal to 1 23. So, now this n we are using as the different values of the lambda for, which this equation is satisfied we are naming them as lambda 1 lambda 2 lambda n, in this Eigen functions k n is arbitrary that is corresponding to each Eigen value we would get infinite many solutions and we are calling them Eigen functions.

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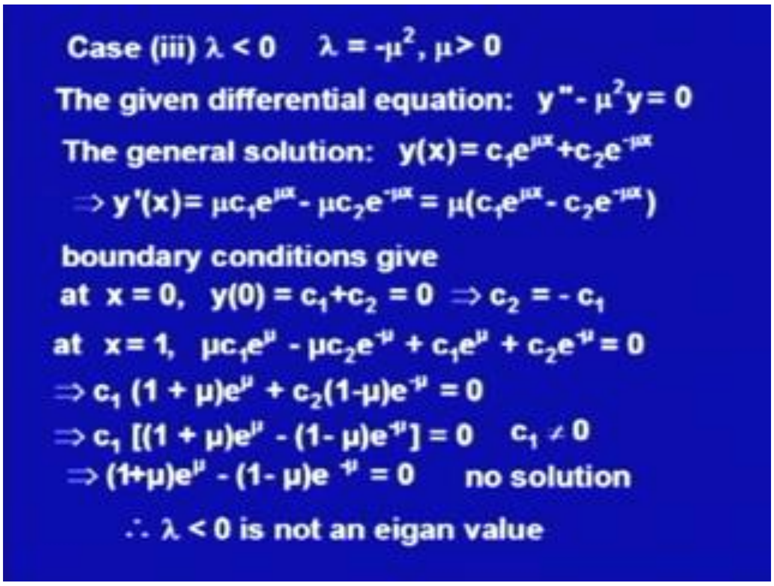

Now, discuss the third case when lambda is negative, so let us again suppose in the similar manner that lambda is equal to minus mu square and mu is a positive real number. The given differential equation will become y double dash minus mu square y is equal to 0, as we already know that in this case the general solution would be of the form c 1 e to the power mu x plus c 2 e to the power minus mu x.

Now, what will be this y dash mu times c 1 e to the power mu x minus mu times c 2 e to the power minus mu x or in more simplified manner mu times c 1 e to the power mu x minus c 2 e to the power minus mu x. Now, boundary conditions would give me y at 0 is 0 that is c 1 plus c 2is equal to 0 or c 2is equal to minus c 1 and y at 1 and y dash at 1 we would get mu c 1 e to the power mu minus mu c 2 e to the power minus mu plus c 1 e to the power mu plus c 2 e to the power minus mu is equal to 0, which we simplify in we would get c 1 times 1 plus mu e to the power mu plus c 2times 1 minus mu e to the power minus mu is equal to 0.

If we try to solve these equation that is c 2is equal to minus c 1 if I put I would get c 1 as 1 plus c 1 into 1 plus mu times e to the power minus mu minus 1 minus mu e to the power minus mu is equal to 0.

For getting nontrivial solution I must get c 1 to the nonzero and that says is I should get the coefficient of c 1 as 0 or rather we should get 1 plus mu e to the power mu minus 1 minus mu e to the power minus mu is equal to 0, but this equation does not have a solution, which are this equation does not give me any solution. So, what we are getting is we cannot get nontrivial solution in this case also.

So, with the differential equation, now we have got that lambda is equal to 0 is not an Eigen value lambda negative is not an Eigen value the only thing lambda positive and satisfying 1 particular condition or 1 particular equation, they are Eigen values and the corresponding Eigen functions also we had seen.

Now, we had seen this Eigen value and Eigen problem this was very particular kind of differential equation, we have discussed y double dash plus lambda y is equal to 0 and we have discussed the particular boundary conditions also we had little bit variated our boundary condition. Now, we will discuss one more practical problem or that is the problem, which is up arising in many applications the name is associated with strum and Liouville and thus we would be calling them strum Liouville boundary value problems.

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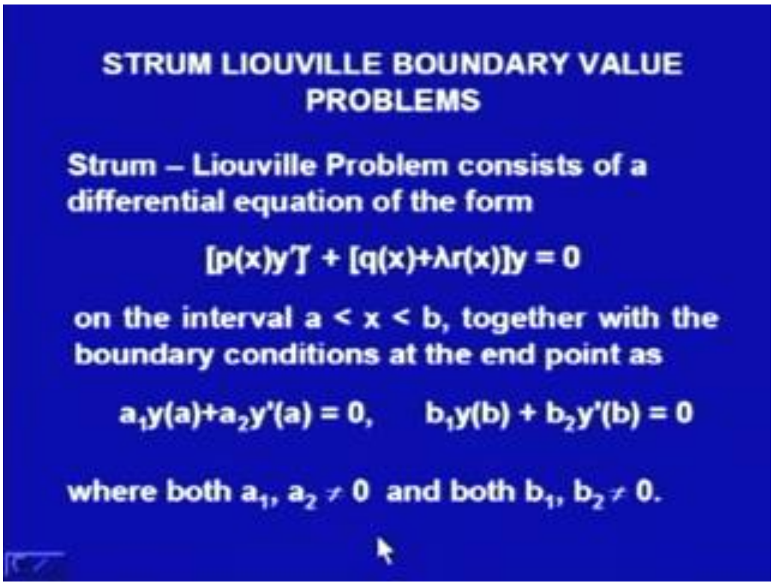

Let us first define, what this problem is a Strum- Liouville problem consist of a differential equation this is of a special form p x y dash its whole derivative plus q x plus lambda r x times y is equal to 0, that is the equation of this form. If any differential equation, we are getting this is second order differential equation certainly it is ready to be linear and if it can be written in this form and the boundary condition this equation must be defined on an interval a to b and together with the boundary conditions at the end points a and b as a 1 y a plus a 2y dash a is equal to 0, b 1 y b plus b 2y dash b is equal to 0.

Now, we see the Strum-liouville problem our differential equation is linear homogeneous the boundary conditions they are linear and homogeneous. So, this is actually a linear homogeneous boundary value problem, where the equation is of a special form in this Strum-liouville problem we do require for this boundary conditions that both a 1 a 2and b 1 b 2simultaneously should not be 0.

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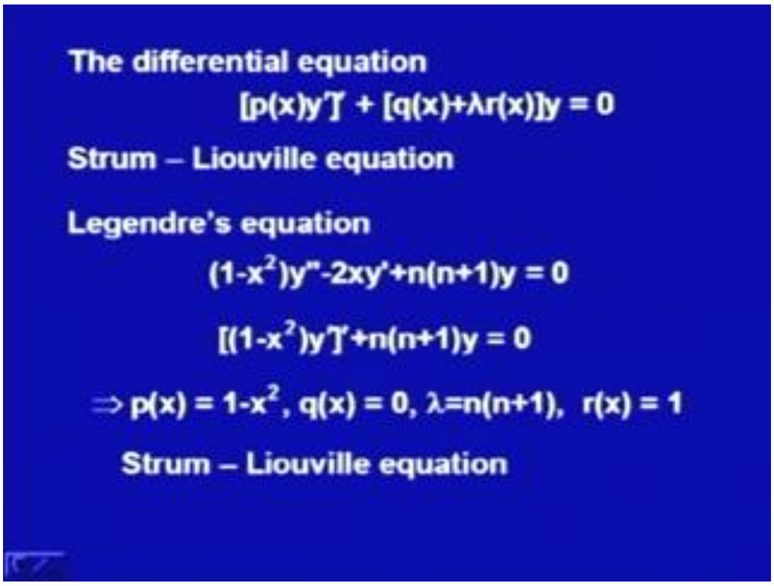

This differential equation i dash its wholes derivative plus q x plus lambda times r x y is equal to 0 this is known as Strum-liouville equation. So, the Strum-liouville boundary value problem is basically about this equation and the general boundary linear boundary conditions. If do remember the Legendre equation 1 minus x square y double dash minus 2x y dash plus n into n plus 1 y is equal to 0, you had already solved this equation. You, see that we can rewrite this equation as 1 minus x square y dash its whole derivative plus n into n plus 1 y is equal to 0.

Now, you see here if I compare it with this Strum-liouville equation I get p x as 1 minus x square q x as 0 and lambda as n into n plus 1 and r x as 1, then we do know that Legendre equation is also if Strum-liouville equation. Similarly, actually if you see that is boundary this Bessel's equation, which you had already done in the series solution that is also Strum-liouville equation, but that requires little bit calculations.

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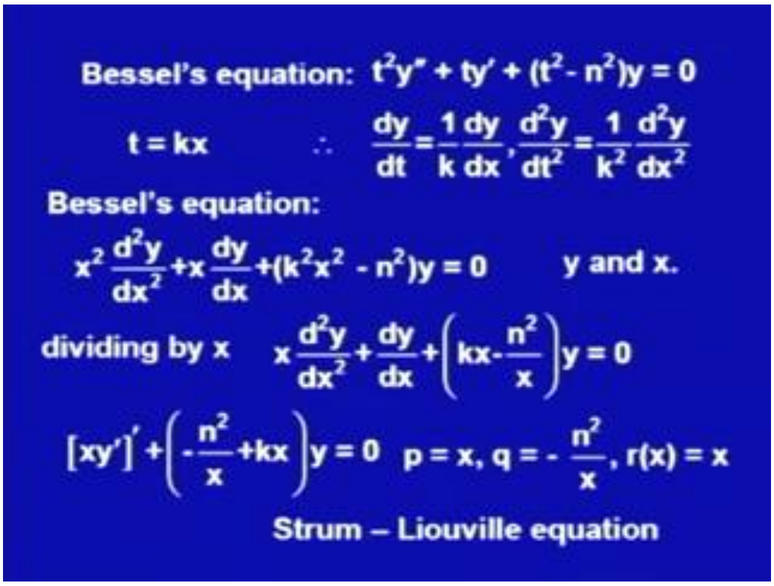

So, let us see I am writing this Bessel's equation in y and t, so t square y double dash plus t y dash plus t square minus n square times y is equal to 0 this is the Bessel's equation. Let us change the variable t with k x that says is my y dash and y double dash would also that it changed, so d y over d t I would be that is here y dash and y double dash with respect to the function with the variable t. So, d y over d t would be 1 upon k d y over d x d 2y over d t 2would be 1 upon k square d 2y over d x 2.

If I substitute this in this given Bessel equation, I get x square d 2y over d x 2plus x times d y over d x plus k square x square minus n square times y is equal to 0, we see that this is an equation in y and x. Now, divide this equation by x, so what we get x times d 2y over d x 2plus d y over d x plus k times x minus n square upon x into y is equal to 0, now rewrite this in the form of Strum-liouville equation. So, I do get x times y dash its derivative plus minus n square upon x plus k x into y is equal to 0.

Now, if I compare with this p x, q x and lambda we would get p is equal to x q x as minus n square upon x and r x as x and of course, lambda is here k then this is also Strum-liouville equation.

Now, if you do remember we have done in this whole lecture one, special equation y double dash plus lambda y is equal to 0. If you see, that is also we could write as Strum-liouville equation and that you can check with as an exercise with this today I am finishing up our today is lecture of boundary value problem, where I have defined you 1 special problem Strum-liouville problem.

Now, in the next lecture actually, we will discuss little bit about the properties of this Strum-liouville problem and how to get the solution of this particular problem. So, today we had learnt boundary value problems and we had learnt that they are very different from the initial value problems it is not necessary that we do get we have not done the general case, we have done very special kind of equations and we had done that is that special kind of equation was stunning out to be one of very practical appearing problem that Strum-liouville problem and with that I would be ending today is lecture.

Thank you.