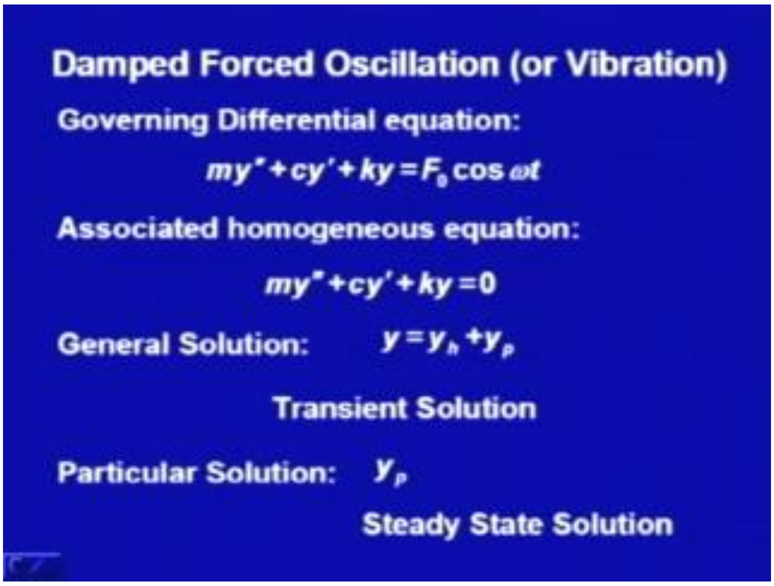
Mathematics - III
Prof. Tanuja Srivastava
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture - 7
Higher Order Linear Differential Equations

Welcome to the lecture series on differential equations for under graduate students. Today's topic is Higher Order Linear Differential Equations, but before going to the today's topic, we will first do the example for the last lecture, which we had left in the last lecture, just recall what we have done in the last lecture the damped forced oscillation or vibration.

(Refer Slide Time: 00:50)

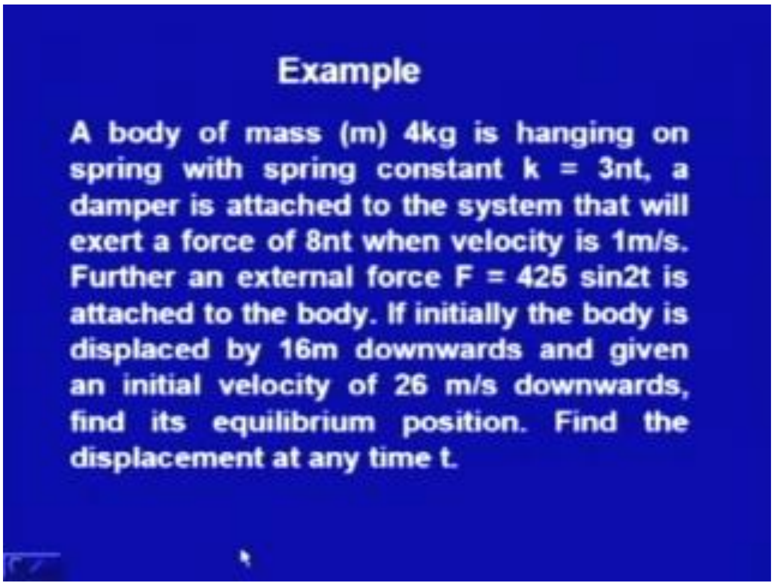


Here we were having that the system of mass spring system, were the damping was present and the vibration of the oscillation was governed by a force. That was that the differential equation which we were dealing was the second order differential equation and for this system the governing equation was m y double dash plus c y dash plus k y is equal to F naught times cos omega t.

You remember that we have taken this force as an example, that is we are talking about this kind of force where the force is F naught cos omega t that is the deriving force. And we want the systems response corresponding to this force, this is a non homogeneous equation. So, associated homogeneous equation would be m y double dash plus c y dash plus k y is equal to 0, the solution of this non homogeneous system we do know the general solution is of the form y h plus y p, where y h is the general solution of this corresponding homogeneous equation and y p is the particular solution of this non homogeneous equation.

This general solution is called the transient solution and this particular solution y p this is called the steady state solution. So, we are saying is that this steady state what the term is steady state means that is it would remain as such and this term is also saying is the vibration or oscillation will go on, so let us see all these things with the help of the examples, so do the example.

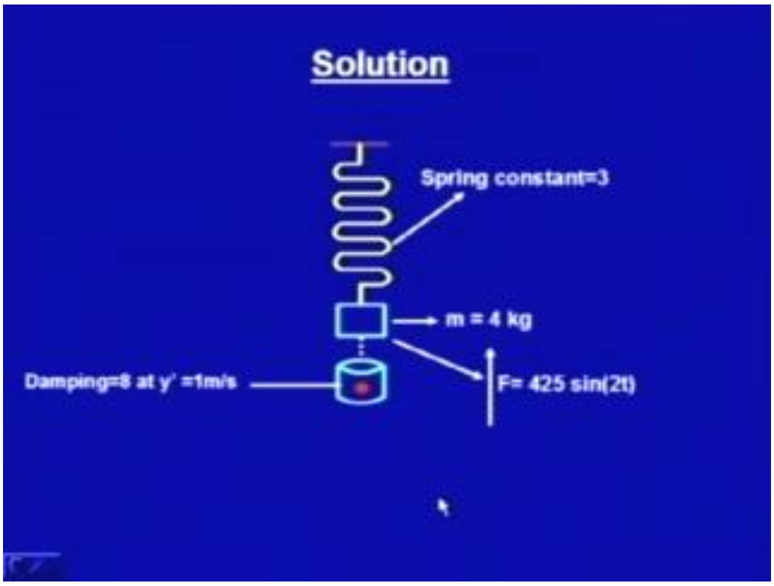
(Refer Slide Time: 02:31)



We do have a body of mass 4 Kg is hanging on the spring with spring constant as 3 Newton a damper is attached to the system that will exert a force of 8 Newton when velocity is 1 metre per second. Further an external force of 425 sin 2 t is attached to the body, if initially the body is displaced by 16 metre down wards and given that this initial velocity at that time is 26 metre per second downwards, then we have to find the equilibrium position and find the displacement at any time t.

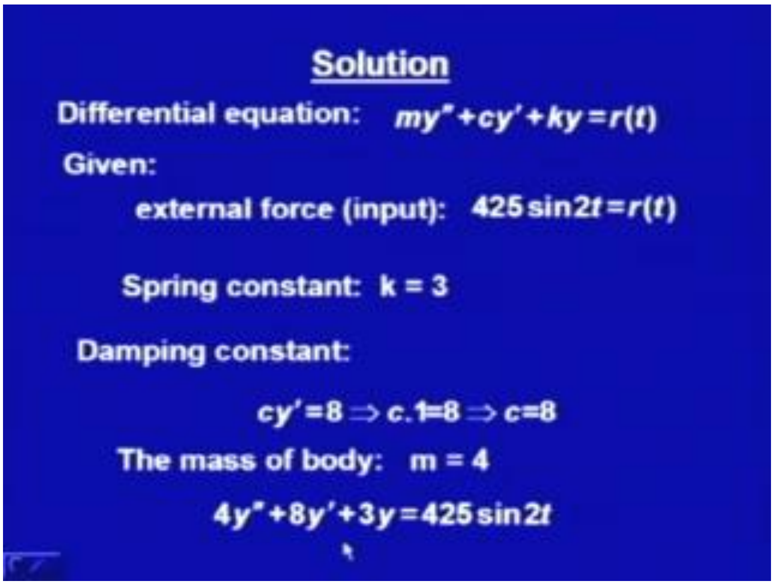
So, this question is basically of a mass spring system where, this we are having is a spring and which is attached with the mass. And the damper there is 1 external force is also there and initially it has been displaced by 16 metres downwards and that has generated a velocity of 26 metre per second downwards.

(Refer Slide Time: 03:36)



So, let us see is that is what is the system is, we are having this spring which has a spring constant as 3 there is a mass of 4 Kg which is attached to with this one, moreover a damper is also attached with this one, what this damper is doing this is changing the motion that is it is exerting a force of 8 Newton, when the velocity is 1 metre per second, we have to develop the equation differential equation for this system. And moreover in this system there is an external force is also governing this mass, that is external force of 425 sin 2 t this force is upwards.

(Refer Slide Time: 04:18)

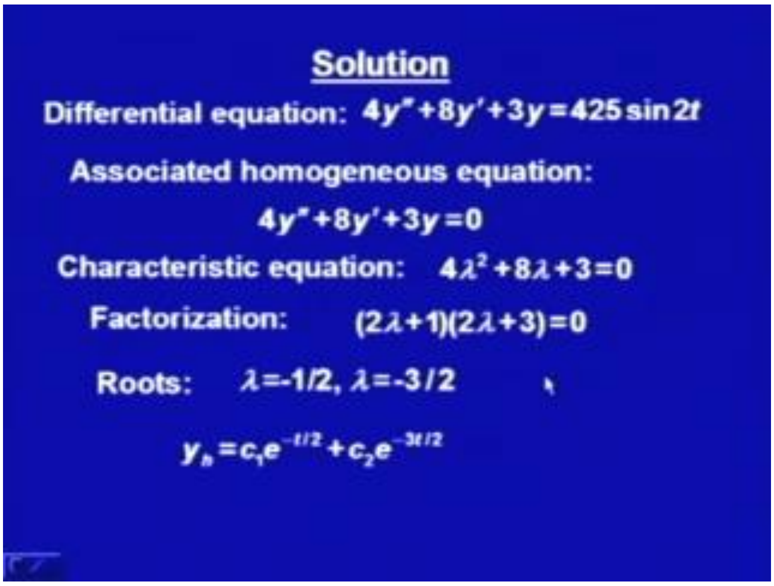


So, we do have that what will be the solution because, differential equation we are having is that damping is also there and the spring constant is also there. So, the equation would be in m y double dash plus c y dash plus k y is equal to r t since, an external force is also there. Now, what we are being given, we are given that this external force or this input is 425 sin 2 t that is r t is 425 sin 2 t, then we have been given this constant k spring constant that is 3.

Now, what is this damping constant c we are been given that the damping force is 8 Newton, when the velocity that is y dash is 1 metre per second that is we do have c y dash as 8. So, at y dash as 1 which gives me that the damping constant c to be 8 and moreover this mass we have been given is as 4 Kg, so what will be our equation now our equation would be 4 y double dash plus 8 y dash plus 3 y is equal to 425 sin 2 t. Now, we have got the governing equation for this given system, what we have to find it out here this y is actually the displacement at any time t.

So, we have to find out the displacement any time t that is the solution of this equation, which is y. Moreover, we have to find out when this system reaches to the equilibrium that is when it is reaching to the equilibrium position, that is just giving the response to the initial force that is this force.

(Refer Slide Time: 05:54)

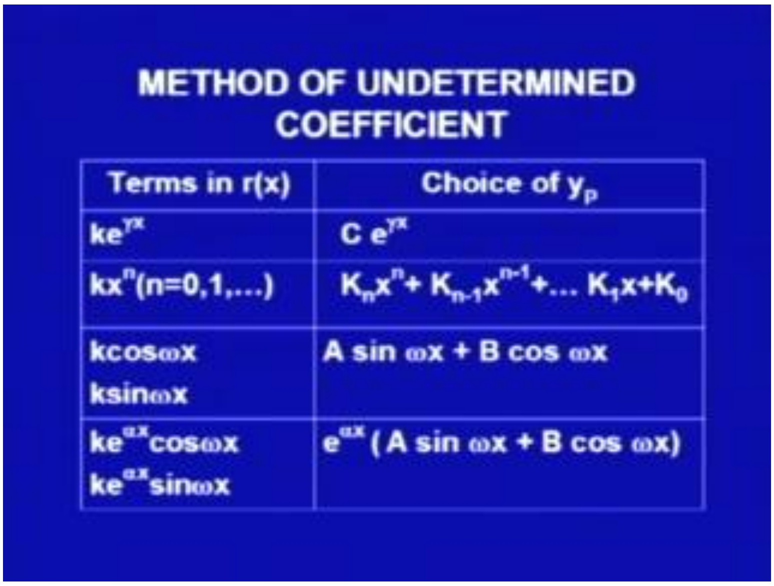


So, let us see that is how we are going to solve, so we are going to solve this differential equation. Now, this is a second order linear differential equation with the constant coefficients and the right hand side is not 0 that is it is non homogeneous, we do know that for solving this equation we require first to solve the associated homogeneous equation. What will be associated homogeneous equation that is the right hand side is 0 equation is same for y double dash plus 8 y dash plus3 y is equal to 0.

Now, for solving this homogeneous equation we do know we require the characteristic equation of this one, what will be the characteristic equation from here it should be 4 lambda square plus 8 lambda plus 3 is equal to 0 for finding the roots of this equation, we will first factorize it. So, factorization gives me 2 lambda plus 1 into 2 lambda plus 3 now equate it to 0 we will get the two roots, one is minus half and another is minus 3 by 2 from here, so we are getting two real roots which have distinct.

So, what will be the linear independent solutions, two linear independent solution corresponding to these two roots, as we do know they should be e to the power minus t by 2 and e to the power minus 3 by 2. So, the general solution for this homogeneous equation would be c 1 e to the power minus t by 2 plus c 2 e to the power minus 3 t by 2. Now, for finding out the particular solution for this non homogeneous one, since we have this linear differential equation with the constant coefficient and this right hand side is of a special form we can use the method of undetermined coefficient.

(Refer Slide Time: 07:38)



So, based on this kind of thing we just go to the table, which says is that if my right hand side is of the form k sin omega x then I should choose my y p as A sin omega x plus B cos omega x, so let us move to the solution.

(Refer Slide Time: 07:57)

We do have this right hand side as 425 sin 2 t, so according to the table my r t is since 425 sin 2 t. So, I should choose y p as a cos 2 t plus b sin 2 t, now here this a and b are the constant which we have to determine for this equation which gives that is, so that this y p becomes a particular solution of this equation. For this what I have to do, I have to find out y p dash and y p double dash and substitute it over in this equation.

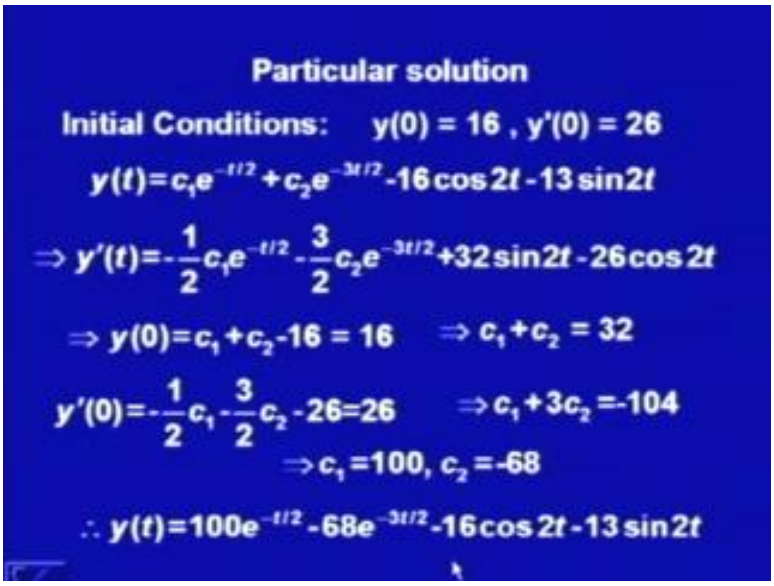
So, y p dash would be minus 2 a sin 2 t plus 2 b cos 2 t, what will be y p double dash again differentiate minus 4 a cos 2 t minus 4 b sin 2 t, now substitute this y p y p dash and y p double dash in this given equation. So, this substitution will give me after all these clarifications, we get it that is 16 b minus 13 a cos 2 t minus 16 a plus 13 b sin 2 t is equal to 425 sin 2 t. Now, to find out this a and b we will equate the coefficient of cos 2 t and sin 2 t on both the sides.

So, on the left side we see the coefficient of cos 2 t is 16 b minus 13 a on the right hand side there is no term involving cos 2 t that is the coefficient is 0. So, we get the first equation as 16 b minus 13 a plus is equal to 0, now the coefficient of sin 2 t and the left side is minus of 16 a plus 13 b and the right side is 425. So, we get the second equation as 16 a plus 13 b is equal to minus 425, now we are getting two equations in two unknowns, these are simple algebraic equations we can get the unique solution and that solution is actually a as minus 16 and b as minus 13.

So, what will be our particular solution y p would be minus 16 cos 2 t minus 13 sin 2 t, so what will be the general solution for this non homogeneous equation, that would be as y h plus y p, now y h we do remember that it we have got c 1 e to the power minus t by 2 plus c 2 e to the power minus 3 t by 2 and plus this y p that is minus 16 cos 2 t minus 13 sin 2 t, so this is the general solution of this non homogeneous system.

Now, we are been given the conditions initial one's that the system has been put to the movement by displacing initially the object by 16 metre downwards and having generating velocity of 26 metre per second downwards.

(Refer Slide Time: 10:51)



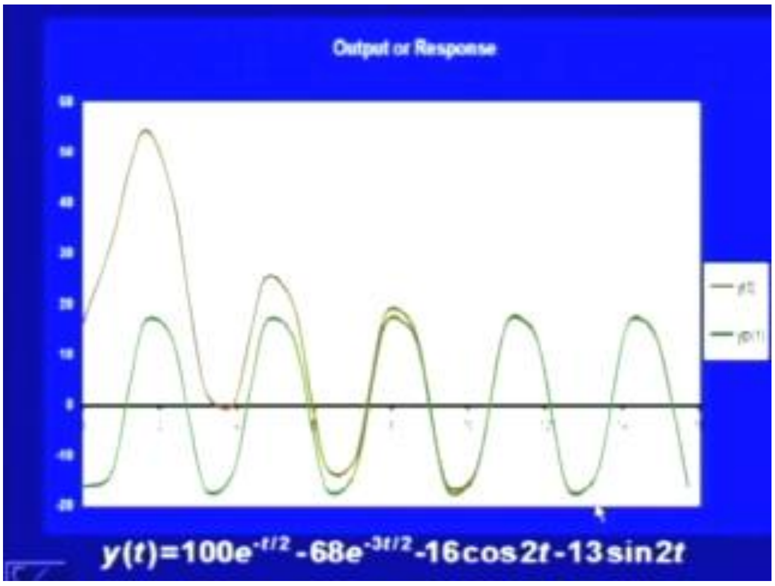
So, we have to get the particular solution for that, the initial conditions y 0 is 16 because, it is downwards, so the sign is plus. Similarly, the velocity that is y dash at 0 that is 26 metre per second downwards that is again the sign is plus, so we have got y 0 at 16 and y dash at 0 as 26. Now, what was our general solution, our general solution was this one, so we find out first what will be y dash t, y dash t would be minus half c 1 e to the power minus t by 2 minus 3 by 2 c 2 e to the power minus 3 by 2 plus 32 sin 2 t minus 26 cos 2 t.

Now, initial condition means that is y at 0, when I put t is equal to 0 here I will get c 1, here I will get c 2 and here I will get minus 16 this part would be 0. So, we would get y at 0 is c 1 plus c 2 minus 16, this is given as equal to 16, so the first equation we have got c 1 plus c 2 is equal to 32. Now, second initial condition is about y dash at 0, so when I put t is equal to 0 in this y dash t what I will get minus half c 1 minus 3 by 2 c 2 this term would be 0 and minus 26, that is minus half c 1 minus 3 by 2 c 2 minus 26 this is given to be equal to 26, so we will get the second equation as c 1 plus 3 c 2 is equal to minus 104.

Now, we have got two equations in two unknowns we will solve it and we get from here that c 1 has 100 and c 2 as minus 68. So, what is our particular solution for the given system, that is y t is equal to 100 e to the power minus t by 2 minus 68 e to the power minus 3 t by 2 minus 16 cos 2 t minus 13 sin 2 t. Now, we are been asked two questions, the question one was that is what is the when it reaches to equilibrium position and second was that is what is the displacement at any time t displacement at any time t that is y t.

So, we are getting is that by this function we are giving the displacement at any time t, now when it is reaching to the steady state solution that is the equilibrium, equilibrium means is it is reaching to the steady state solution, this solution and this part is 0, let us see these things by the graphs.

(Refer Slide Time: 13:27)



So, we see that this graph we do have this red line is actually the general solution, that is 100 e to the power minus t by 2 minus 68 e to the power minus 3 t by 2 minus 16 cos 2 t minus 13 sin 2 t and while this green line this representing the particular solution, that we had obtained as y h plus y p that is minus 16 cos 2 t minus 13 sin 2 t. Now, we see this solution that is solution of this problem, we do have that initially it is displaced by 16 metre downwards that is the positive side it is generating an velocity of 26 metre per second, that is the slope here is 26 metre or 26 with positive direction.

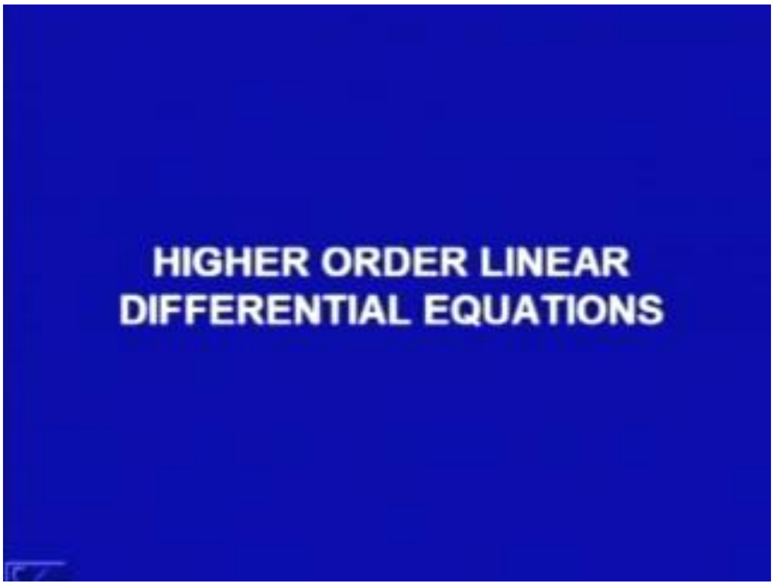
So, this movement is that is the displacement is in the positive side more higher, then it is coming down. Because, it is a spring it is coming down to something is near 0 or other going little bit negative, when it is again going up and then coming down and so on it goes on. We see after some time this red line and this green line they are matching ones, this we called if you do remember the general solution as the transient solution and this y p that is the particular solution as the steady state solution.

So, this is the graph where we are saying is that this is the displacement or this will be the motion of that system, which after sometimes you are seeing is that it is matching towards the green line, if you see is that is exactly it is started matching after this at say 10 seconds, after this 10 second it is exactly matching with the green and red, that is we are not able to see finally, that is where is the green and where is the red.

So, we find out that is after that the system has stabilized now the movement is not because of that initial pull or initial velocity. Now, it is only the response of the function deriving force that is 425 sin 2 t, so we are reaching to the steady state solution after say 10 seconds. So, this is what the solution of we could say that displacement at any time t that is been given by this red graph, when it is reaching to the steady state solution or equilibrium that is at 10 seconds after 10 seconds it is reaching.

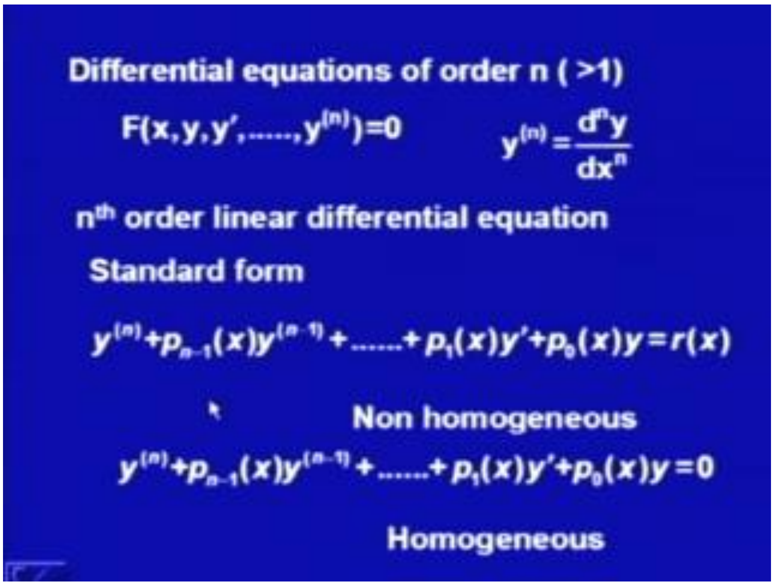
So, this is what we had concluded with this vibration one, that is it will move on like this, this oscillation will on go that is why we are calling it vibration it is going with equal ones and like that one it will go on because of that, input deriving force 425 sin 2 t.

(Refer Slide Time: 16:22)



Now, we will move to towards today’s topic that is higher order linear differential equations. Now, we will move on from the second order to any arbitrary order in equations.

(Refer Slide Time: 16:39)

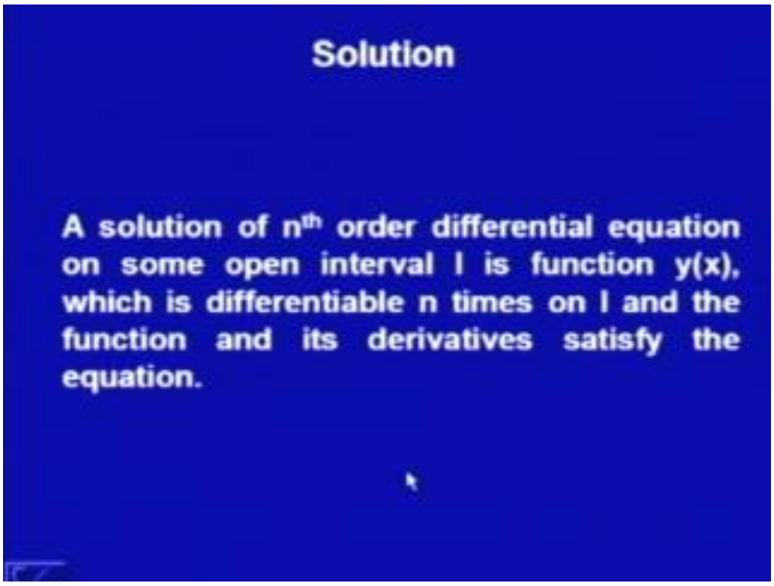


So, what will be a differential equation of any order in that should be a function of x y y dash y n is equal to 0, where this y n we are saying is representing the derivative of y with respect to x. This equation would be called linear, if this is of the form y n plus p n minus 1 x y to the power n minus 1'th derivative of y y n minus 1 plus and so on p 1 x y dash plus p naught x y is equal to r x. This is equation is of order n, we see the highest derivative which is occurring in this equation that is of n'th order, so the order of this equation is n.

The coefficients of y n y n minus 1 y dash y all these are functions of x only that is p naught x p 1 x p n minus 1 x and r x all these are the function of x only. So, we this equation is linear since none of the terms y or y dash or it is derivatives are coming in the same terms simultaneously, so it is a linear one. Now, if this right hand side is identically 0 the equation will be of the form y n plus p n minus 1 x y n minus 1 plus, so on p 1 x y dash plus p naught x y is equal to 0 this would be called the homogeneous linear equation of order n.

When this is not 0, then we call it non homogeneous linear equation of order n, this we are calling as the standard form since the coefficient of the n'th derivative y n is 1, this could be any p n x, but that to make it as of order n we require that p n x must not be 0 function. So, we do can have this p n x as 1, so this we are calling a standard form now let us do revisit some definitions in the differential equations.

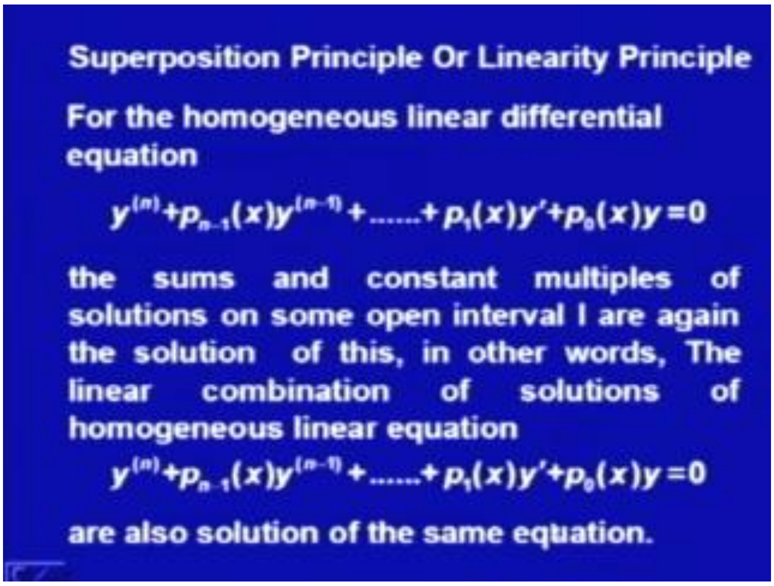
(Refer Slide Time: 18:45)



So, first is solution, what will be the solution of in a n'th order differential equation, as usual we have done that the solution should be any function, which is satisfying the equation. So, what is special about this one, a solution of in a n'th order differential equation on some open interval I is the function y x, which is differentiable n times on I and the function and it is derivatives satisfy the equation.

So, the function and it is derivatives satisfy the equation that fulfil the condition of solution, what I require that this function would be solution only if it is n times differentiable. So, that is the difference between the whatever the definitions we have done earlier, now we will concentrate first on the homogeneous equations, we have seen in the second order homogeneous equations that we have got one fundamental theorem, which said is that is linear combination of solution is also the solution.

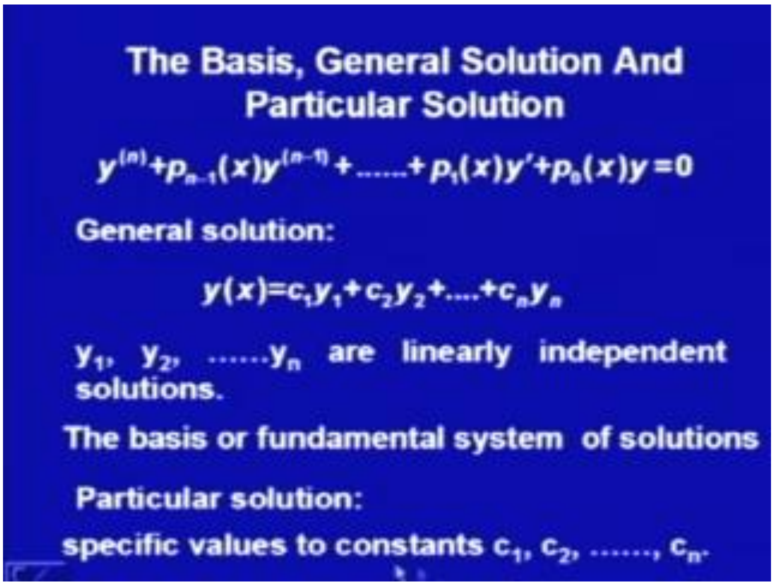
(Refer Slide Time: 19:48)



Similarly, here also we do have superposition principle or the linearity principle what this says, this says for homogeneous linear differential equation y n plus p n minus 1 x y n minus 1 and so on plus p 1 x y dash plus p naught x y is equal to 0. The sum and the constant multiples of the solutions on some open interval I are again the solution of this equation.

In other words the linear combination of the solution of the homogeneous linear equation are also solution of the same equation, that is if suppose I do have y 1 and y 2 as the two solution for this homogeneous linear equation. Then c 1 y 1 plus c 2 y 2 will again be a solution of this homogeneous linear equation of order n, let us just see some more terms over here.

(Refer Slide Time: 20:43)

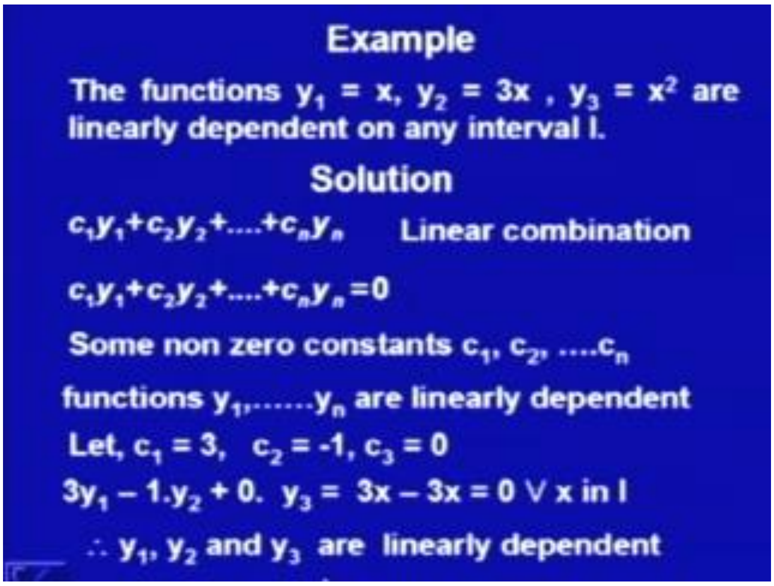


That basis, general solution and particular solution, first the general solution for in a third order differential equation, which is homogeneous of this form that is standard form y n plus p n minus 1 x y n minus 1 plus, so on plus p 1 x y dash plus p naught x y is equal to 0. The general solution will be of the form c 1 y 1 plus c 2 y 2 plus, plus c n y n where this y 1, y 2, y n they are linearly independent solution of this equation and this c 1, c 2, c n are the arbitrary constants.

Now, this linearly independent solutions of this equation they are called basis or fundamental system of solutions. Moreover, what we do have here is since this is n'th order equation, this will actually have n linearly independent solutions and those n linearly independent solutions will form the basis or the fundamental system of the solution. And the general solution of this equation will consist of the linear combination of the all these linearly independent solutions, where this c 1, c 2, c n would be arbitrary constants.

And this general solution will include all the solutions, moreover a particular solution would be what when I do give this specific values to these constants c 1, c 2, c n that solution would be called particular solution that can be obtained by giving certain conditions, we say is that it satisfying certain conditions, so that I could find out the values of this c 1, c 2, c n.

(Refer Slide Time: 22:32)



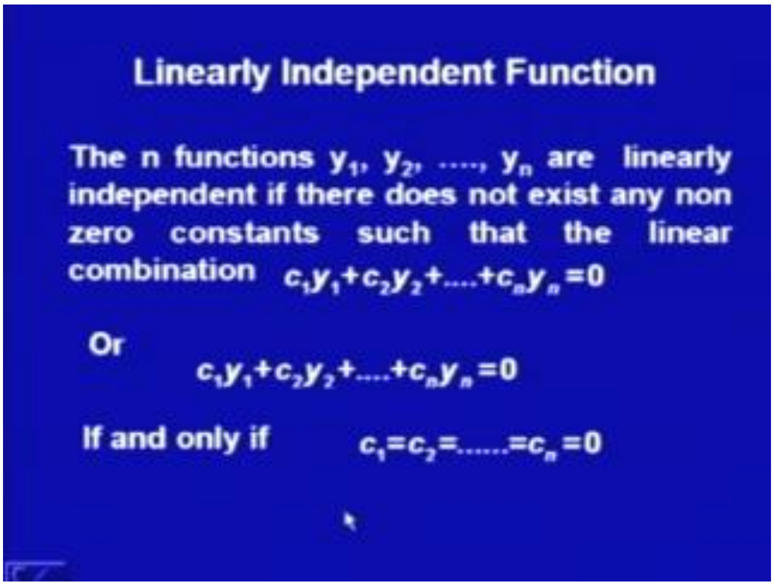
Let us do some examples to understand the concepts just now we had learnt of course, we have done it in the second order and the first order. But, we will again redo for the n'th order just, first example is the function x 3 x x square or linearly dependent on interval I that is first we are trying to understand the linear dependence and independence. We have already done the linear dependence and independence in the terms of two solutions, now we are moving to the higher one.

So, first thing is that is what we would call linear dependence or independence c 1 y 1 plus c 2 y 2 plus c n y n this is called a linear combination where, y 1, y 2, y n are the functions and this c 1, c 2, c n are constant this is called a linear combination. Now, if this linear combination is identically equal to 0 for some constants c 1, c 2, c n where all this c 1, c 2, c n are non 0 that is for some non 0 constants, then we call that y 1, y 2, y n are linearly dependent.

So, what we have learnt that from linear combination if it is identically 0 for some non 0 constant, then the functions y 1, y 2, y n would be linearly dependent. Now, in this question what is been asked, that is show that they are linearly dependent on any interval. Let us choose here c 1 as, 3 c 2 as minus 1 and c3 as 0, so now, I have chosen two constants which are non 0 and 1 0 constants, since they are three functions I have chosen this three constants.

Now, let us see what would be c 1 y 1 plus c 2 y 2 plus c3 y3 this would be actually 3 times x minus 3 times x which is equal to 0 for all x, whatever be the interval I that says is that linear combination is 0 with non zero constants, the non zero constants we had 3 and minus 1. Thus what we had conclude, that y 1, y 2, y 3 are linearly dependent, now let us revisit the definition of independence also as we said is that is n general solution, we require n linearly independent solution, so what we would call linear independence.

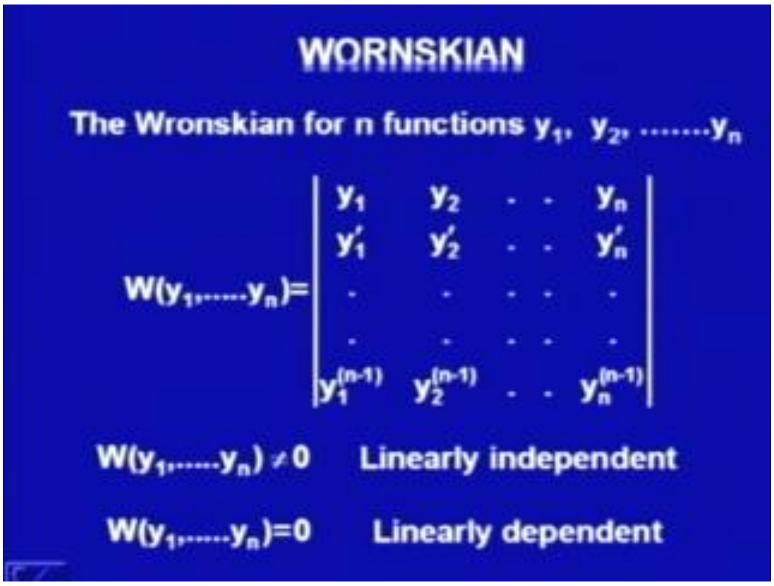
(Refer Slide Time: 24:57)



So, for the n function second we will use with the linear combination the n functions y 1, y 2, y n are linearly independent if there does not exist any non zero constant such that the linear combination c 1 y 1 plus c 2 y 2 plus c n y n is 0. That says is that, this linear combination would be 0 if only if this c 1, c 2, c n are all 0 that is in other words we say, the linear combination is 0 if and only if all c 1, c 2, c n are 0, then we call that y 1, y 2, y n are linearly independent.

We had also learned this linear dependence and independence if you do remember, in second order equation using something called the wronski determinant or wronskian that is again we can see the wronskian in the form of n functions or n solution. So, let us do the definition of dependence and independence in the form of wronskian.

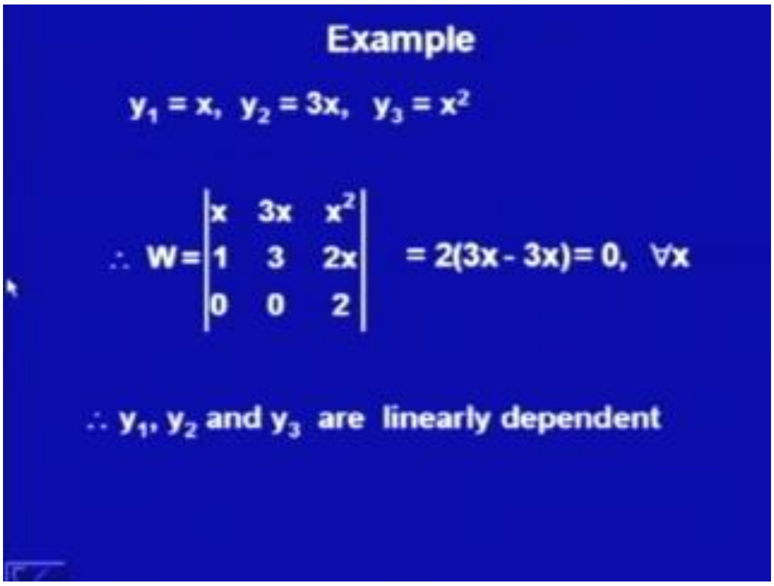
(Refer Slide Time: 26:12)



What will be the wronskian of n functions that we define as for n functions y 1, y 2, y n the wronskian W y 1, y 2, y n is being defined as the determinant where the first row is containing all the functions y 1, y 2 and y n. The second row will contain the derivative of y 1 dash that is y 2 dash and y n dash and so on, the n'th row will contain the n minus first derivative of the function y 1, y 2, y n. So, the wronskian for the n function would be a determinant of order n containing these functions and their derivatives.

We would say that when the functions will be linearly dependent and independent, if this wronskian that is this determinant is not 0 for any x, we call that y 1, y 2, y n are linearly independent, if this determinant is 0 for some x, we call that functions are linearly dependent.

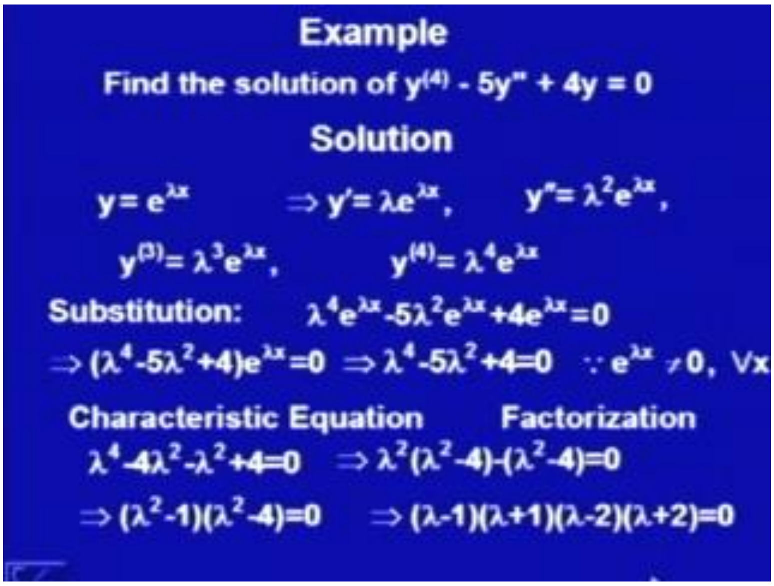
(Refer Slide Time: 27:20)



So, now, let us just use this definition in our own example we had the example that we had the functions x 3 x and x square. What is the wronskian of these functions of course, that should be first row will contain the functions x 3 x x square, second row the first derivative it is derivative is 1, it is derivative is 3 and it is derivative is 2 x. Now, the second derivative of the function or the again the derivative of the last row, so the derivative of 1 is 0 3 is 0 and 2 x is 2.

Now, if I find out the value of this determinant I would just like to expand it with respect to the third row. So, with respect to this two we would be getting is the determinant of order second x and 3 x and 1 and 3, so what we will get 3 x minus 3 x that is 2 times 3 x minus 3 x which is 0 for all x that again gives me that is my wronskian has come out to be 0 for all x, so y 1, y 2, y 3; they are linearly dependent now let us do one more example to learn the about the solution.

(Refer Slide Time: 28:26)



Find the solution of y 4 minus 5 y dash plus 4 y is equal to 0, y 4 means the 4'th derivative of the function y. Now, we see this is a linear differential equation, where the coefficients are 4, 5 and 1 that is they are constants, now we just try with the similar conditions, which we have done in the first order and second order when we had the constant coefficients, we tried with the solutions of the kind e to the power lambda x.

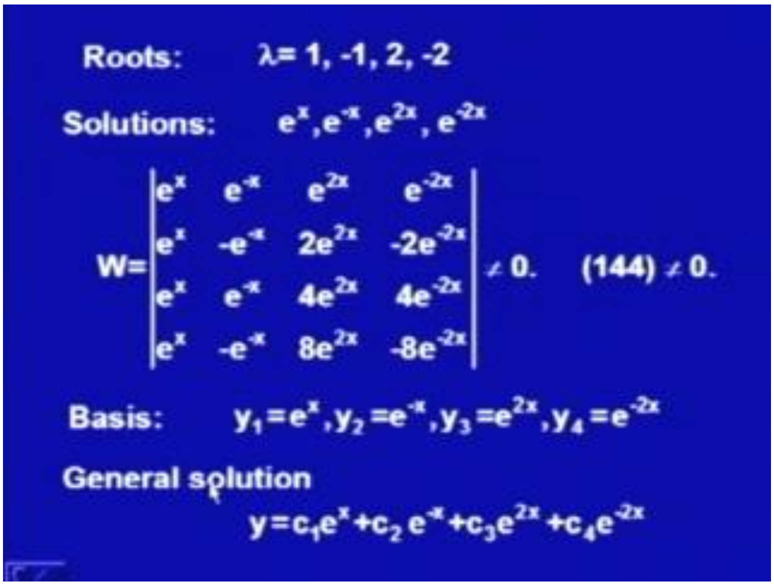
So, here again we will move to the form of solution that is let say e to the power lambda x is a solution. If this is a solution, then this function and it is derivatives y dash y double dash and so on they must satisfy this equation. So, let us try what will be y dash lambda times e to the power lambda x, y double dash lambda square times e to the power lambda x, the third derivative lambda cube times e to the power lambda x and the 4'th derivative would be lambda to the power 4 e to the power lambda x.

Now, I substitute if this is a solution I would substitute this and we will try to satisfy the equation. So, if I am substituting in this equation I would get lambda to the power 4 e to the power lambda x minus 5 lambda square e to the power lambda x plus 4 e to the power lambda x is equal to 0. Now, rewriting it we get lambda to the power 4 minus 5 lambda square plus 4 into e to the power lambda x is equal to 0.

Now, if e to the power this y has a solution this must be true, this will be true only if either e to the power lambda x is 0 or this coefficient that lambda at power 4 minus 5 lambda square plus 4 is 0. Now, this e to the power lambda x this cannot be 0 because, we are taking this as a solution, so it should not be trivial solution that it should not be 0 for all x, that says is that my lambda to the power 4 minus 5 lambda square plus 4 should be 0, this if you do see is that is this similar to that what we are getting is characteristic equation in second order.

So, this also we call the characteristic equation, now this is a 4'th order one let us factorize this equation. What this factorization would give me lambda to the power 4 minus 4 lambda square minus lambda square plus 4, now you see if I take common from here lambda square and here from minus 1. We would get lambda square into lambda square minus 4 minus lambda square minus 4 is equal to 0, which gives me lambda square minus 1 into lambda square minus 4 is equal to 0 that says is further factorization lambda minus 1 into lambda plus 1 lambda minus 2 into lambda plus 2 is equal to 0, so we have got the roots of this equation as 1 minus 1 2 and minus 2.

(Refer Slide Time: 31:21)



So, the roots are 1 minus 1 2 and minus 2 now what we have seen, we have got the 4'th order equation the characteristic equation was of the order 4. So, we got the degree 4, so we have got the 4 roots, now all these roots are now we are seeing is they are distinct ((Refer Time: 31:41)). So, let us say that is we will get the 4 solution, say the solutions we would say corresponding to the 1 e to the power x corresponding to the minus 1 e to the power minus x corresponding to 2 e to the power 2 x and corresponding to the minus 2 e to the power minus 2 x.

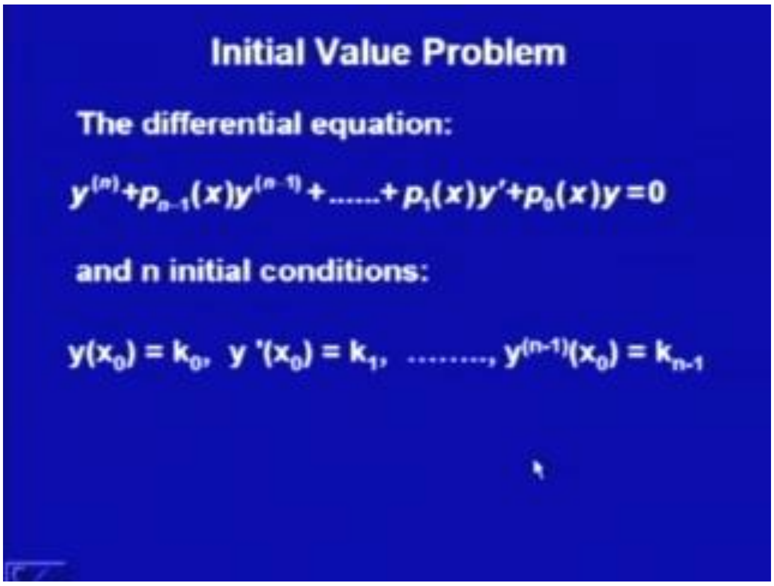
Now, this solutions will form a basis only if they are linearly independent, so for checking linear independence let us use the wronskian. So, wronskian we are having it is the four solutions, so we will have the determinant of order four, where the first row of course, as usual we would have all the solutions e to the power x e to the power minus x e to the power 2 x e to the power minus 2 x.

Second row the derivatives, so e to the power x minus e to the power minus x here 2 times e to the power 2 x this one would be minus 2 times e to the power minus 2 x, third row again the second derivative or the derivative of the previous row. So, like this one and the last row would be the derivative of the previous row, so again we are having e to the power x minus e to the power minus x 8 times e to the power 2 x and minus 8 times e to the power minus 2 x.

Now, to solve this determinant we can take common from the first column e to the power x, in the second column we will get e to the power minus x, the third column we would get e to the power 2 x and the 4'th column we would get e to the power minus 2 x. So, this multiplication of all these we would get as 1 and what will be remaining here is the first column 1, second column 1 minus 1 1 minus 1, the third column we will get 1, 2, 4, 8 and in the 4'th column we will get 1 minus 2 4 minus 8.

Now, if we just solve this determinant we will get the value of this as 144 which is not 0, that says is that all my these solutions, which we had form out they are linearly independent. So, what will be the basis for the system that is basis or the fundamental system for of this solutions that would be y 1 is e to the power x, y 2 as e to the power minus x, y 3 as e to the power 2 x and y 4 as e to the power minus 2 x. So, the general solution for the given differential equation would be c 1 e to the power x plus c 2 e to the power minus x plus c 3 e to the power 2 x plus c 4 e to the power minus 2 x, so we have got the general solution for this differential equations.

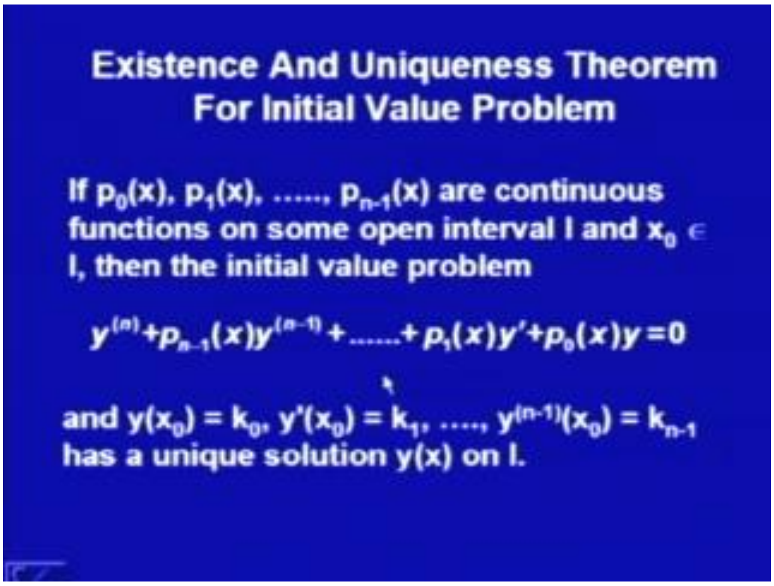
(Refer Slide Time: 34:16)



Now, we will move to the next term initial value problem, what is the initial value problem, the initial value problem will contain the differential equation that is n'th order differential equation, homogeneous one and n initial conditions. Since, this is n'th order differential equation, we will have now n initial condition, we had seen the first order equation we had got only single initial condition, in the second order equation we have got two initial conditions.

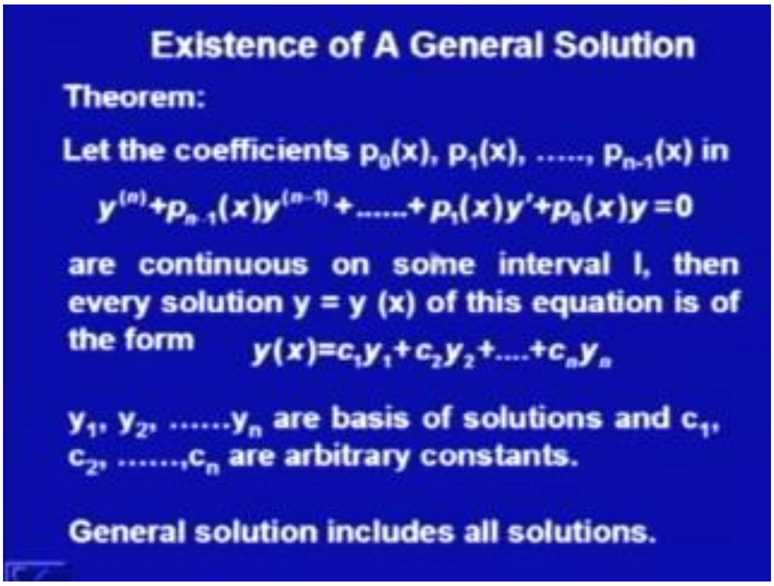
Now, we will have n initial conditions what are those n initial conditions, they would be y at some point x naught is k naught y dash at some point x naught is k 1 and so on the n minus 1 derivative at x naught is k n minus 1. So, this would be called initial value problem they would give neither actually the particular values for the constant c 1, c 2, c n that is arbitrary constant in the general solution.

(Refer Slide Time: 35:11)



Now, existence and uniqueness theorem for initial value problem, what we say is that if p naught x, p 1 x and p n minus 1 x are continuous functions on some open interval I and x naught belongs to I, then the initial value problem, which is containing this n'th order equation y n p n minus 1 x y n minus 1 plus p 1 x y dash plus p naught x y is equal to 0. And n initial conditions y at x naught is equal to k naught y dash at x naught is equal to k 1 and y n minus 1 x at x naught is k n minus 1, this will have a unique solution on the interval I that is i would get a single values the unique values for c 1, c 2, c n in the general solution.

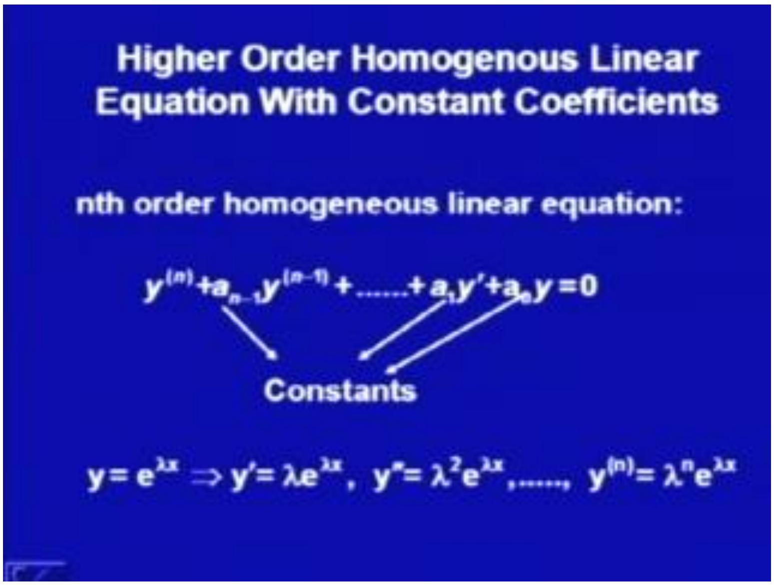
(Refer Slide Time: 36:07)



Existence of general solution, results says if the coefficients p naught x, p 1 x and p n minus 1 x in this linear differential equation, which is homogeneous y n p n minus 1 x y n minus 1, and so on, they are continuous on some interval I. Then every solution y x of this equation will be of the form c 1 y 1 plus c 2 y 2 plus c n y n, where this y 1, y 2, y n are the basis of the solutions and c 1, c 2, c n are arbitrary constants.

That is this n'th order equation will contain or will have n linearly independent solutions and the general solution would be a linear combination of those linearly independent solutions. Moreover, all this solution of this equation would be included in the general solution, that is there would be no singular solution for this differential equation of linear differential equation of order n, which is homogeneous.

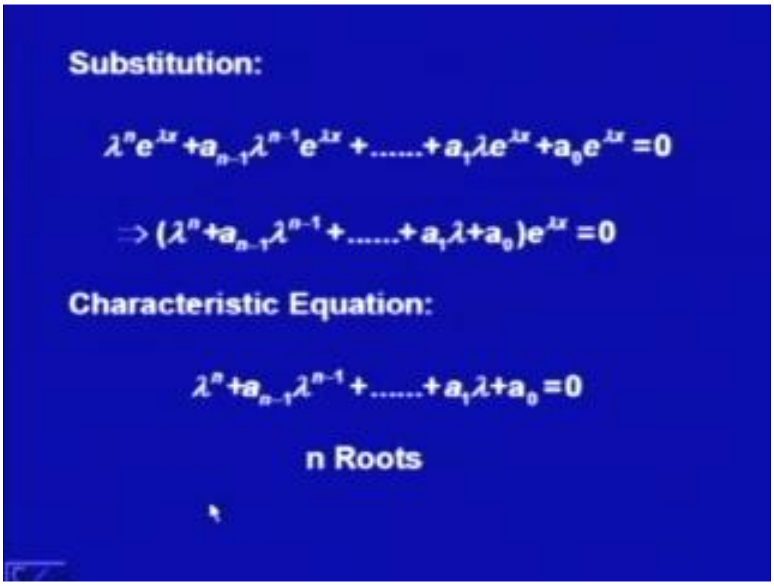
(Refer Slide Time: 37:10)



So, let us come first with the higher order homogeneous linear equation with constant coefficients, what n'th order homogeneous linear equation, which is of the form y n plus a n minus 1 y n minus 1 plus, so on a 1 y dash plus a naught y is equal to 0, where this coefficients a n minus 1, a n minus 2 a 1 a naught they are all constants. So, now, they are not the function of x, they are just the constants, then we call this n'th order homogeneous linear equation with constant coefficients.

Now, we do have done the solution of this kind of equations in the first order and second order, there we had used the function of the form e to the power lambda x as the solutions. So, here also we will find out the method to solve this equation, we will use again this kind function as a solution, that says is that we are having y dash as lambda to the times e to the power lambda x y double dash as lambda ((Refer Time: 38:16)) and so on. The n'th derivative would be lambda to the power n times e to the power lambda x, if this function is a solution of this one, then we would get that this function and these derivatives must satisfy this equation.

(Refer Slide Time: 38:34)

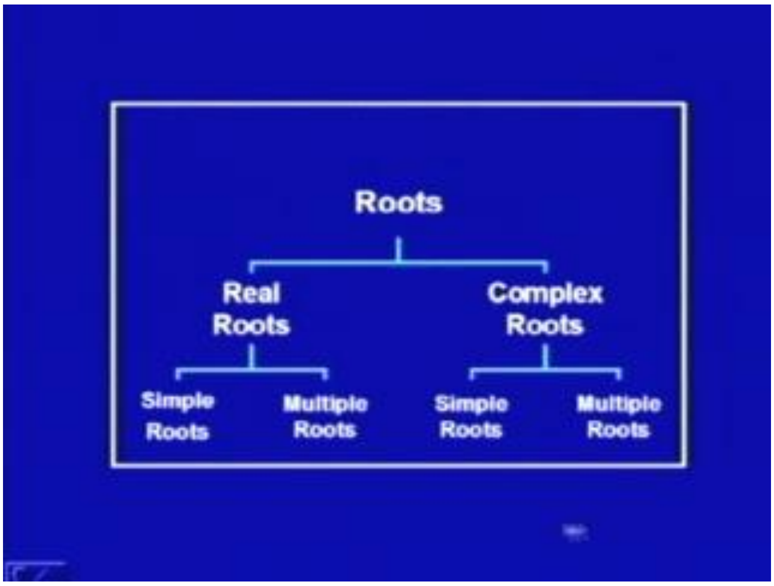


So, if I substitute this here in this equation we would get lambda n e to the power lambda x plus a n minus 1 times lambda to the power n minus 1 into e to the power lambda x and so on a naught e to the power lambda x is equal to 0. Rewriting this I would get 1 equation as lambda n to the power n plus a n minus 1 times lambda to the power n minus 1 and so on plus a 1 lambda plus a naught into e to the power lambda x is equal to 0.

Since, we have started that e to the power lambda x is a solution, so it cannot be 0, so if e to the power lambda x is a solution that must satisfy this equation and this equation will be satisfied if either this part is 0 or this part is 0, this part cannot be 0. So, we will get that this part is 0, which we are calling the characteristic equation, so the characteristic equation we are getting lambda to the power n plus a n minus 1 lambda to the power n minus 1 and so on plus a 1 lambda plus a naught is equal to 0.

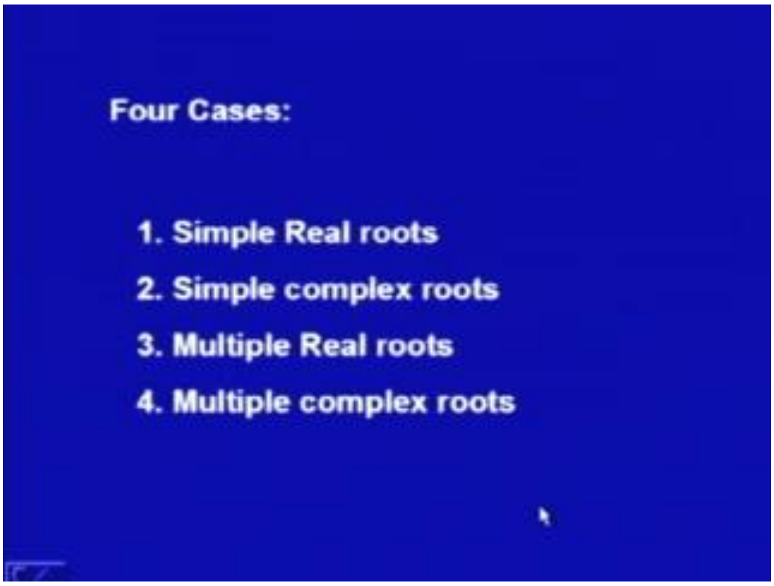
Now, we see this is equation of degree n till now we have done the second order equation where we have got the characteristic equation as having quadratic equation. Just now we had finished one example, where we had used a 4'th order equation and the characteristic equation we have got of degree 4 and that equation was having 4 roots. So, this equation we will have n roots that says is now, if you do remember in the second order equation we have got the three cases that is their real roots, double roots and then complex roots, now since they are n roots.

(Refer Slide Time: 40:16)



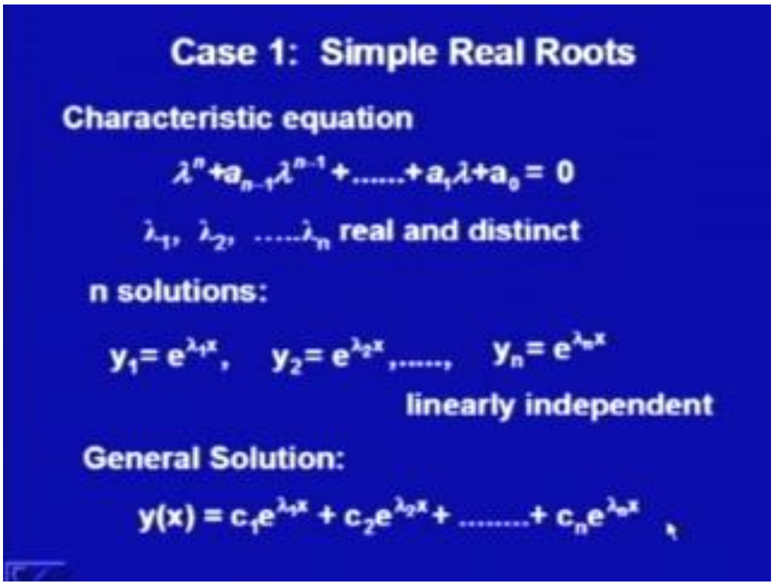
So, we will have many more cases what will be those kind of roots, let us see they would be either real roots or the complex roots. Then the real roots also could be either simple roots, that is they are not repeated one or the multiple roots that is they are repeated one. Similarly, in the complex roots also we can the simple roots that would be the a conjugate pair or that conjugate pair is been repeated many times that is the multiple roots.

(Refer Slide Time: 40:37)



That is in all we have got four cases, simple real roots, simple complex roots, multiple real roots and multiple complex roots. Let us see one by one all these four cases and what will be n linearly independent solutions in all these four cases.

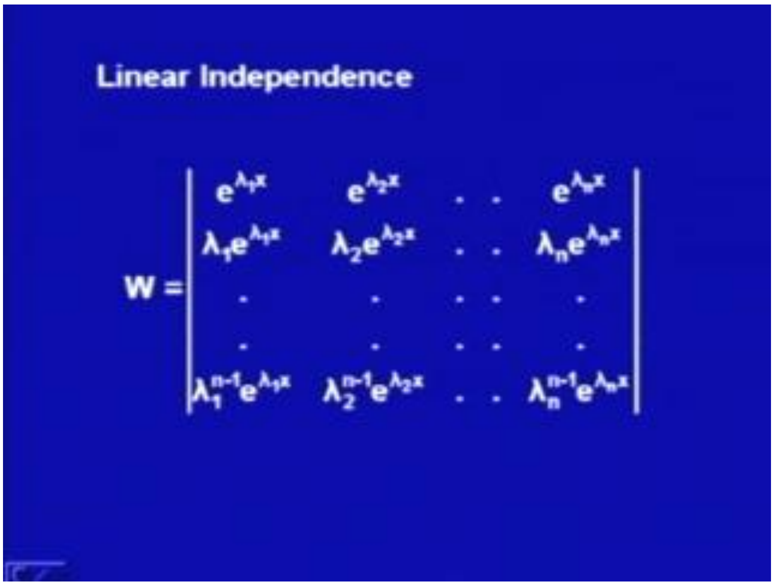
(Refer Slide Time: 40:52)



So, the case 1 simple real roots that says, if the characteristic equation lambda n plus a n minus 1 lambda to the power n minus 1 and so on plus a 1 lambda plus a naught is equal to 0. This is having n roots lambda 1, lambda 2, lambda n, they are all real and distinct, then we would say n linearly independent solution should be e to the power lambda 1 x e to the power lambda 2 x and so on e to the power lambda n x.

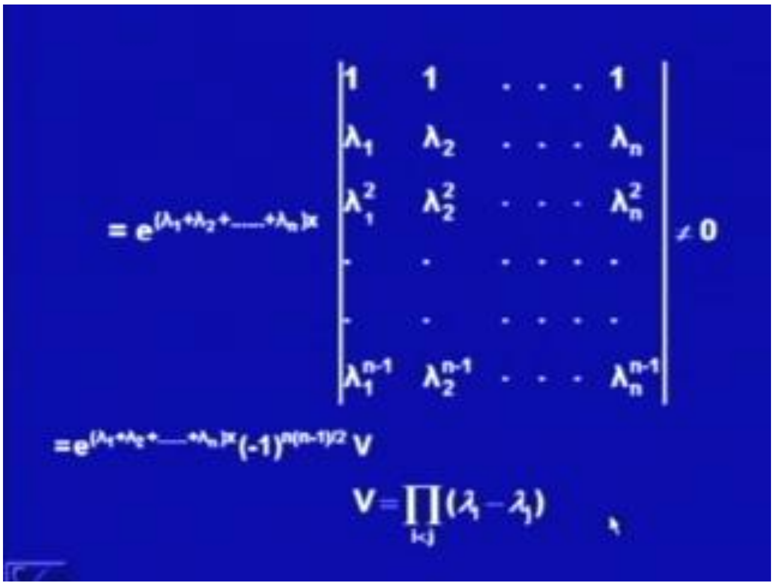
These solutions would be linearly independent, if they are linearly independent then we will get the general solution as c 1 e to the power lambda 1 x plus c 2 e to the power lambda 2 x plus c n e to the power lambda n x. Now, we said is if they are linearly independent, now let us just check whether they are linearly independent or not.

(Refer Slide Time: 41:54)



We will again use the wronskian, so wronskian for this would be n'th order determinant where the first row will contain all the solutions e to the power lambda 1 x e to the power lambda 2 x and so on e to the power lambda n x, second row will have the first derivative and so on, the n'th row will have n minus 1th derivative of these functions. So now, if I again go to solve this determinant we will find it out that from the first column I can take common e to the power lambda 1 x second column e to the power lambda 2 x, and so on.

(Refer Slide Time: 42:35)

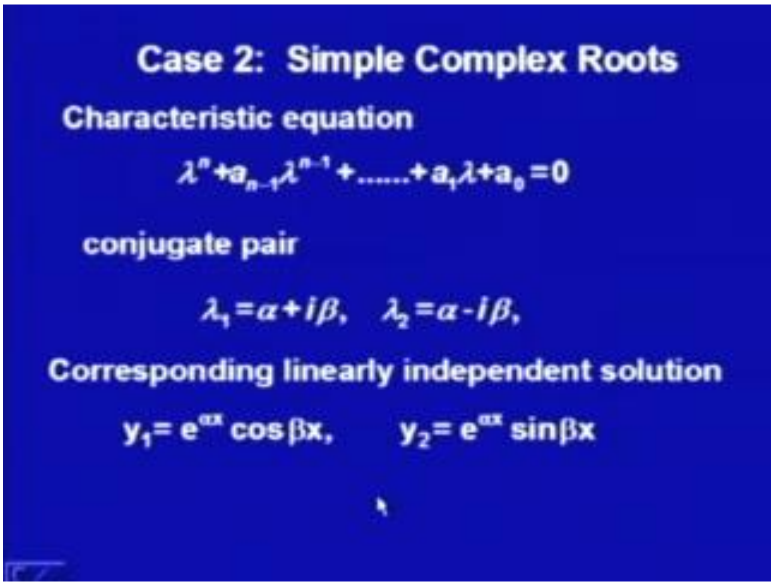


The last column e to the power lambda n x that E to the power lambda 1 plus lambda 2 plus lambda n plus lambda n x, and the determinant we will get 1 1 1 the first row second row lambda 1 lambda 2 lambda n, the third row lambda 1 square lambda 2 square lambda n square and so on the last row we would have lambda 1 to the power n minus 1 lambda 2 to the power n minus 1 and so on lambda n to the power n minus 1.

This is a special kind of determinant, this is not 0 we can check it by adjust that is by doing all those row operations and column, operations for finding it out after the value of this determinant it turns out to be all this term e to the power lambda 1 plus lambda 2 and so on. Here, minus 1 to the power n into n minus 1 by 2 into v, here v is actually the product of lambda i minus lambda j for all i less than j.

Now, since we are having this all my roots lambda 1 lambda 2 lambda n are distinct that is lambda i minus lambda j will not be 0 for all i less than j, when i is not equal to j they will not be 0 that says is this product of non 0 real numbers, this will never be 0 and that gives that this determinant will not be 0. So, wronskian is not 0 that says my solutions e to the power lambda 1 x e to the power lambda 2 x they are linearly independent.

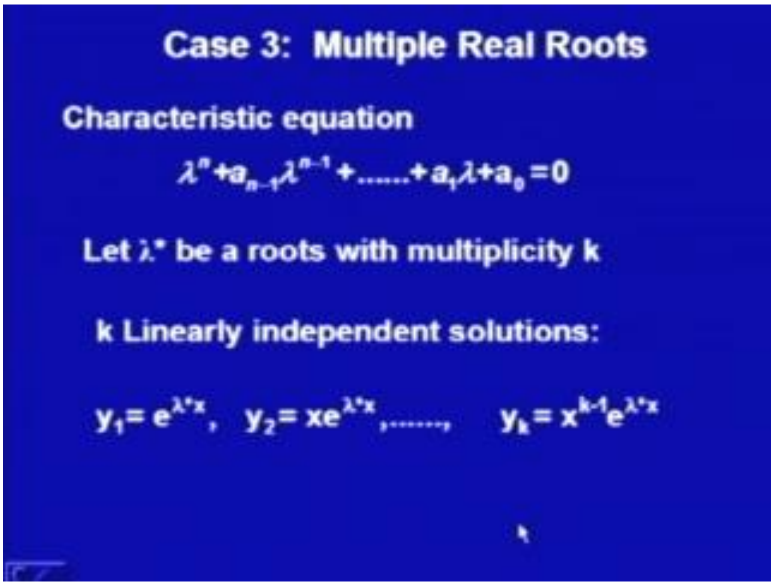
(Refer Slide Time: 44:00)



Now, let us come to the second case that is simple complex root, we do know that is whenever the complex roots are occurring they would occur in the pairs. So, if my characteristic equation does has a complex root, they will always occur in conjugate pair that is if the one root is of the form alpha plus i beta I must have the second root of the form alpha minus i beta, now suppose this is simple that it is occurring only once.

So, corresponding to this lambda 1 and lambda 2 these two roots we will have two linearly independent solutions as e to the power alpha x cos beta x and e to the power alpha x sin beta x, they are linearly independent of each other. Moreover, if I am having all other roots, they are simple real roots or they are different simple complex roots, they will also be linearly independent to them, these things we will learn in the examples.

(Refer Slide Time: 44:59)

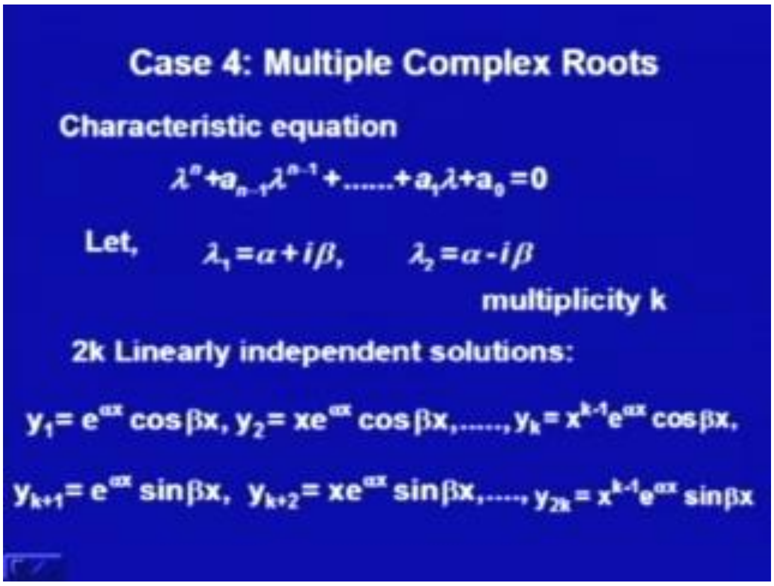


Now, come to the case 3 that is multiple real roots, what it says if my characteristic equation lambda to the power n plus a n minus 1 lambda to the power n minus 1 and so on plus a 1 lambda plus a naught is equal to 0. This has real roots all the roots are real, but all the roots are not distinct, say some root lambda star is repeated k times that is I do have some things lambda minus lambda star to the power k that we are calling multiplicity k.

Then, one solution would be of course, e to the power lambda star, but what will be about the other solution. So, we will require actually k linearly independent solution, those k linearly independent solution with respect to this lambda star would be actually this, you see first solution of course, as e to the power lambda star x. Then the second solution we will have x times e to the power lambda star x and so on, the k'th solution we will have x to the power k minus 1 e to the power lambda star x.

How we had obtained if you do remember we have done, the double real root in the case of second order equation. And there the second solution we had obtained by the method of variation of parameter are that by substituting the solution, so with the same method we are finding out these k linearly independent solution with respect to this root lambda star, which is been repeated k times.

(Refer Slide Time: 46:31)



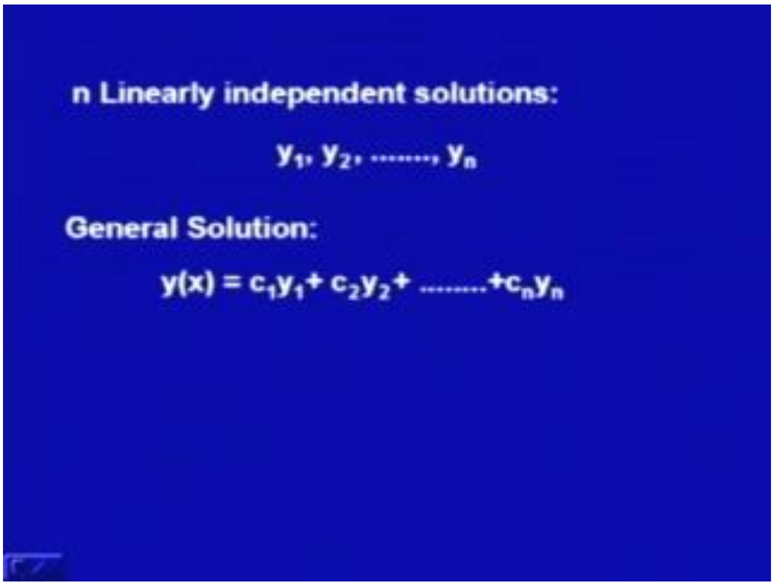
Now, let us come to this when this we do have case 4 multiple complex roots, multiple complex roots; that means, it is not only the single root the complete pair would have been repeated. So, if this characteristic equation does has a complex conjugate pair that is alpha plus i beta and alpha minus i beta and this is been repeated k times that is this complete pair is having multiplicity k, that says now we require 2 k linearly independent solutions.

So, what will be those 2 k linearly independent solution, let us see that is how we are obtaining this 2 k linearly independent solution one we will obtain with respective, if you do remember if they would have been simple one, we would have e to the power alpha x into cos beta and e to the power alpha x sin beta. So, we will have first e to the power alpha x cos beta that is the one solution, the other solution with the multiplicity k that is k we would have x times e to the power alpha x cos beta x and so on x to the power k minus 1 e to the power alpha x cosin beta x.

Similarly, the k plus first solution we will go with this e to the power alpha x sin beta x and again we will have the k solution, that is k plus 2 as x times e to the power alpha x sin beta x and so on 2 k'th solution would be x to the power k minus 1 e to the power alpha x sin beta x. So, we are having the roots that solutions one corresponding to the e to the power alpha x cos beta x they we would multiply a x x square and so on k minus 1.

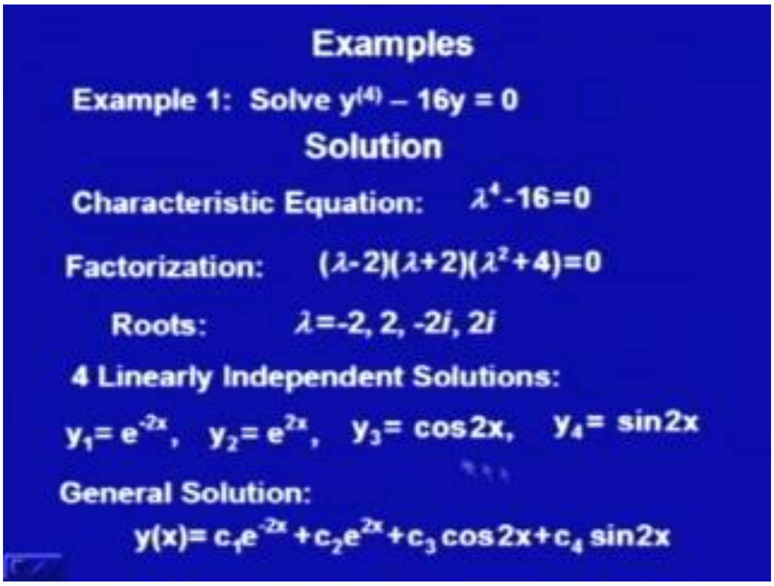
And in the second solutions we do have is that is e to the power alpha x sin beta x then multiply x x square and so on, till x to the power k minus 1.

(Refer Slide Time: 48:23)



That is what we had learnt in each case we will get n linearly independent solution, whatever be the situation and those solutions let say they are y 1, y 2, y n. So, the general solution will always be of the form c 1 y 1 plus c 2 y 2 plus c n y n, now let us do some example to understand these things.

(Refer Slide Time: 48:45)



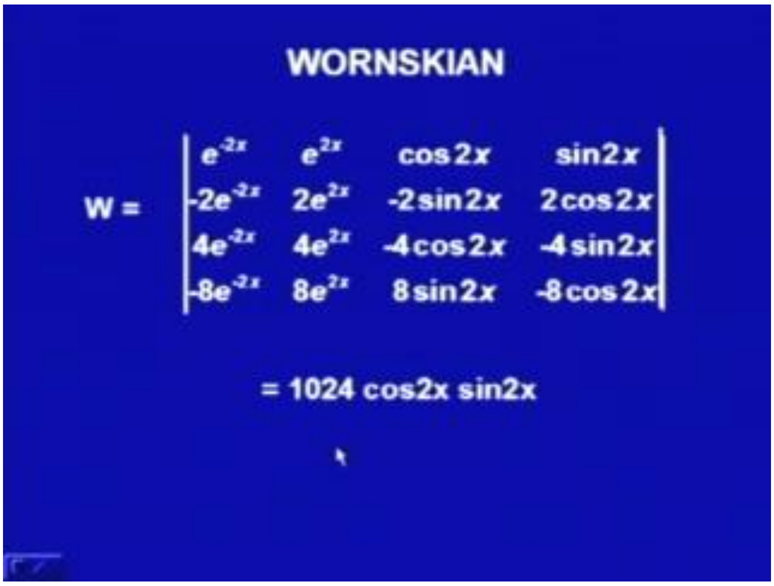
Let us do the first example, solve y 4 minus 16 y is equal to 0 what is the solution, so this is a linear differential equation of order 4 and it is homogeneous. So, first we would find out the characteristic equation, the characteristic equation if you do see we have got is lambda to the power n plus a n minus 1 lambda to the power n minus 1 and so on. So, here what we will get lambda to the power 4 minus 16 is equal to 0.

Now, factorize this one what we will get lambda minus 2 lambda plus 2 into lambda square plus 4 is equal to 0. Thus what we will get the roots we will get the roots as 2 minus 2 and minus 2 i and plus 2 i, so what we are getting actually we are getting 4 roots, where the 2 roots are real and they are distinct the 2 roots or the complex conjugate. And in the complex roots we are having is that the term alpha plus i beta and alpha minus i beta I am having is that alpha part is 0 and we are having the beta as 2.

So, what will be corresponding 4 linearly independent solution with respect to this real roots we would have because, they are distinct e to the power minus 2 x with 2 we would have e to the power 2 x. Now, with this complex conjugate ones, we would have cos 2 x and sin 2 x, now all these 4 roots are this 4 solutions are linearly independent.

As I said is that is we have find out these two linearly independent, these two linearly independent we had already claimed it. Now, we want to check that is all these 4 are linearly independent, let us again move to the method of wronskian that is first find out the wronskian of this one. So, the general solution would be of course, if they are linearly independent, this would be of the form c 1 e to the power minus 2 x plus c 2 e to the power 2 x plus c 3 cos 2 x plus c 4 sin 2 x, let us check about the linear independence.

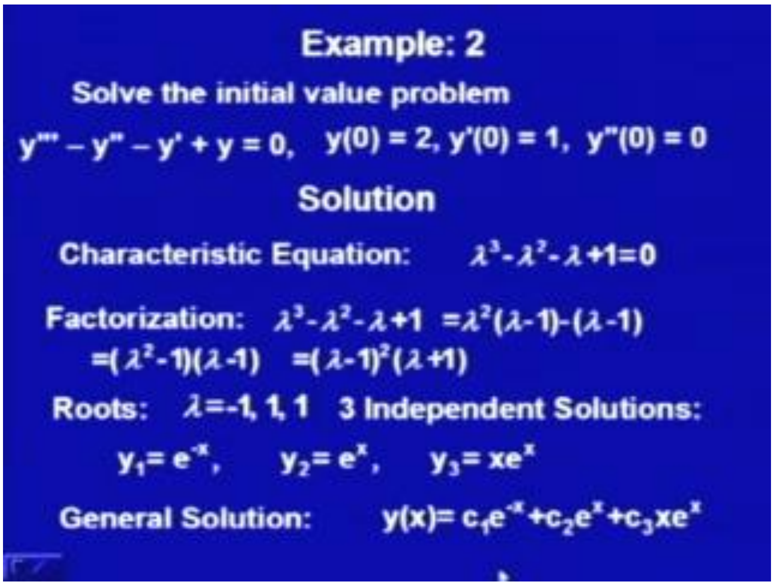
(Refer Slide Time: 50:51)



What will be the wronskian because, it is a 4 solutions, so it will be of order 4 determinant where the first row would have all the solutions e to the power minus 2 x e to the power 2 x cos 2 x and sin 2 x. Second row we will have the derivatives, so as usual the derivatives, then the 3'rd row derivative of the second row and the 4'th row the derivative of the 3'rd row, so we have got this determinant.

Now, again we will take e to the power minus 2 x common from the first column e to the power 2 x from the second column and from here we will get cos 2 x and sin 2 x. So, what we are getting is that is it would be actually 1024 times cos 2 x sin 2 x, this will be not exactly cos 2 x sin 2 x this would be a function of this one, so this will not be 0 for any x.

(Refer Slide Time: 51:50)



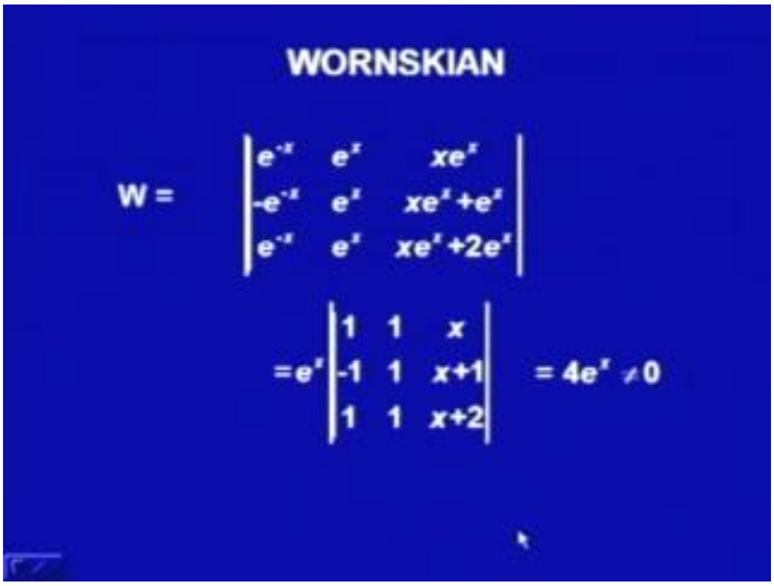
That, says is that we are getting the 4 linearly independent solutions, so come to the second example. Solve the initial value problem y triple dash minus y double dash minus y dash plus y is equal to 0 with initial conditions y at 0 is 2 y dash at 0 is 1 and y double dash at 0 is 0. Now, so the equation is homogeneous linear differential equation of order 3 with 3 initial conditions.

So, for solving it we first get the characteristic equation as usual lambda cube minus lambda square minus lambda plus 1 is equal to 0. We see that with the factorization that it is nothing but, lambda square if we write a common from here and here from minus 1 I would get lambda minus 1 and lambda minus 1, that is we would get lambda square minus 1 into lambda minus 1 which is same as lambda minus 1 into lambda plus 1 and lambda 1, so lambda minus 1 whole square into lambda plus 1.

If I equate it to 0 I will get the three roots where the root 1 is repeated 2 times because, I am having lambda minus 1 whole square and lambda plus 1 give me the root minus 1. So, I get the roots minus 1 1 and 1, so I have one root which is repeated 2 times, what will be the three linearly independent solution, the solution corresponding to the first root would have e to the power minus x, with the second root I would had e to the power x.

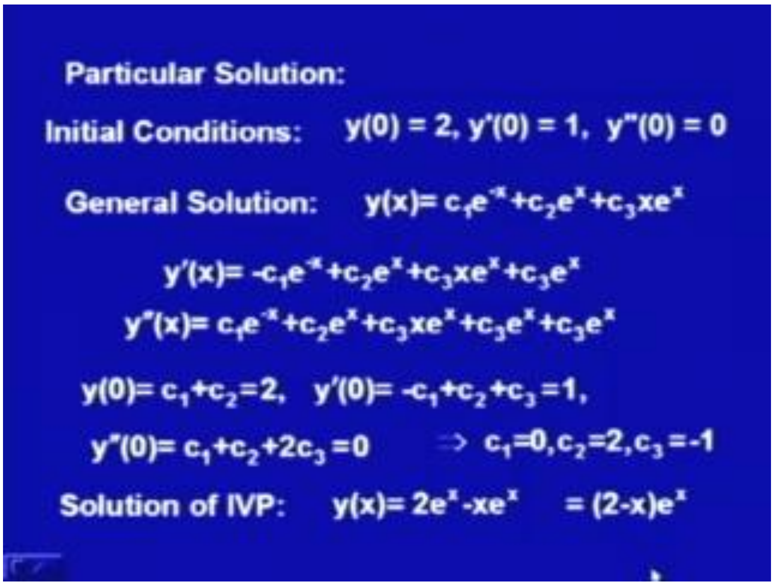
Now, the third root is repeated 1, so we will go with the method that is we multiply x with this one and the third solution would be x times e to the power x, these three are linearly independent solutions. So, the general solution will be of the form c 1 e to the power minus x plus c 2 e to the power x plus c 3 x times e to the power x, again we can check that these three are linearly independent with the help of wronskian.

(Refer Slide Time: 53:53)



Wronskian here would be e to the power minus x e to the power x x times e to the power x. So, if we take the derivatives on the second derivatives, now if I take the common e to the power minus x e to the power x and e to the power x I would left with e to the power x. And the determinant as of the form 1 1 x minus 1 1 x plus 1 and 1 1 x plus 2, if we solve it we do get it is equal to 4 times e to the power x, which is not 0 that says is that these three solutions are linearly independent. Now, we have to solve the initial value problem that is we have to find out the particular solution.

(Refer Slide Time: 54:30)



The given initial conditions are y at 0 is 2 y dash at 0 is 1 and y double dash at 0 is 0, our general solution was c 1 e to the power minus x plus c 2 e to the power x plus c 3 x times e to the power x, what will be y dash that would be minus c 1 e to the power minus x plus c 2 e to the power x plus c 3 x e to the power x plus c 3 e to the power x. And double dash that is second derivative would be c 1 e to the power minus x plus c 2 e to the power x plus c 3 x times e to the power x plus 2 twice c 3 e to the power x that is 1 c 3 e to the power x as here and here.

Now, put the initial condition that is put y x at 0 here, so y at 0 I will get c 1 plus c 2 which is given is equal to 2 from the second one, if I am putting x is equal to 0, this will give me minus c 1 plus c 2 plus c 3 this is equal to 1 this is given. And the third condition gives me c 1 plus c 2 plus 2 c 3 is equal to 0 from here, so I am getting three equations in three unknowns and we can solve it, we can get the unique solution that solution we are getting as 0 c 1 as 0 c 2 as 2 and c 3 as minus 1.

So, what will be the solution of our initial value problem that will be 2 times e to the power x minus x times e to the power x that is c 1 is 0 c 2 is 2 and c 1 is minus 3. So finally, what we are getting is 2 minus x times e to the power x, we can check that this is solution for the given differential equation, moreover this solution is satisfying all these initial conditions, so this is the solution of initial value problem.

So, today we had learnt about higher order homogeneous linear equations with constant coefficients, we had learn how to solve them, we had learn certain definitions in the terms of higher order one. And we had learnt homogeneous one that is higher or n'th order linear differential equations with constant coefficients with right hand side is 0.

Thank you.