Mathematics - III
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Lecture - 2
Linear Differential Equations of the First Order and Orthogonal Trajectories

Dear viewers, in our last lecture we had discussed some standard techniques to obtain the exact solutions of certain categories of ordinary differential equations of first order and first degree. The categories of differential equations that we had considered were the differential equations in which the variables x and y can be separated or the ones in which the variables, the differential equation can be brought to the variable separable form after a certain substitution.

The second category of differential equation that we had considered where the homogeneous differential equations are the differential equations that are reducible to homogeneous form. And the third category of differential equation that we had considered where the exact differential equations are the differential equations, that can be brought to the exact form after a multiplication by a suitable function of x and y called the integrating factor. Now, in our today's lecture we will discuss Linear Differential Equation of the First Order and Orthogonal Trajectory.

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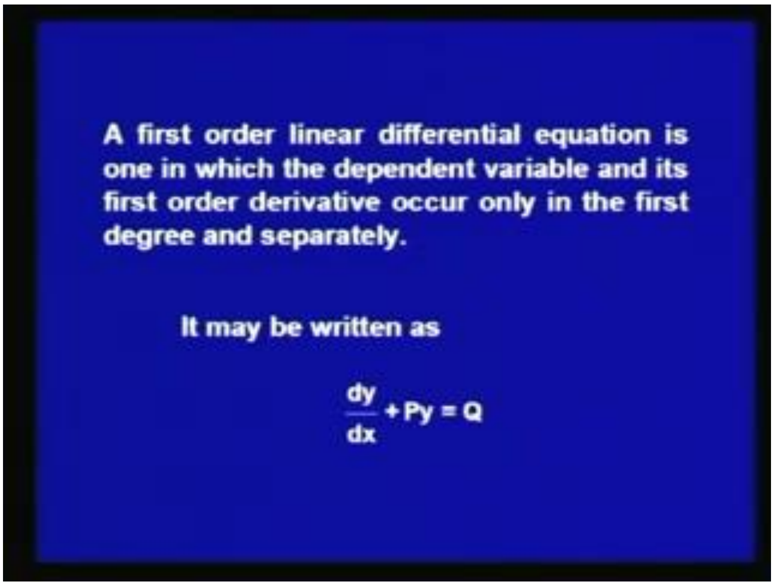

A first order linear differential equation is one in which the dependent variable and it is first order derivative occur only in the first degree and separately. We may write the differential equation as d y over d x plus P y equal to Q where y is the dependant variable and x is the independent variable.

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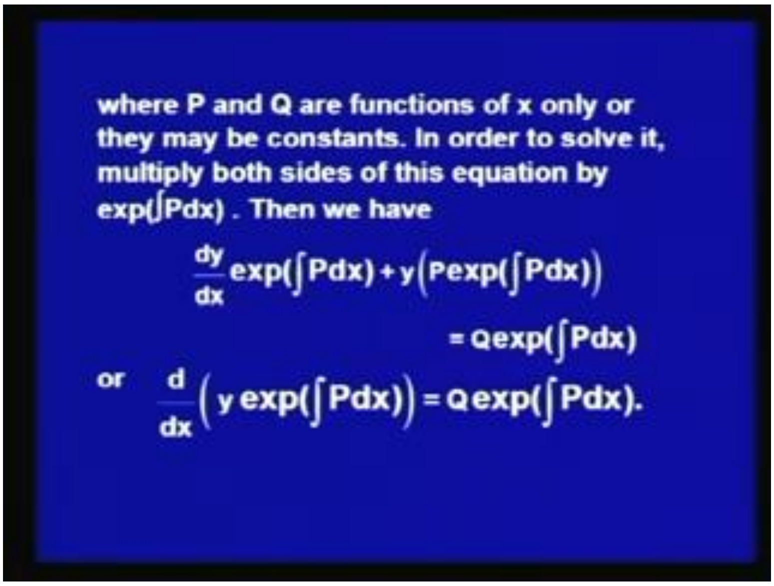

The P and Q are both functions of x only or constants, we can note that in the differential equation d y by d x plus P y equal to Q, the coefficient of d y by d x is unity. In order to solve this differential equation, let us multiply both sides of this differential equation by the function exponential of integral P d x. When we do so we get, d y by d x into exponential of integral P d x plus y times p into exponential of integral P d x equal to Q into exponential of integral P d x. Now, let us note that the left hand side of this equation is the derivative of y in to e to the power integral P d x and therefore, we have d over d x of y in to exponential of integral P d x equal to Q into exponential of integral P d x.

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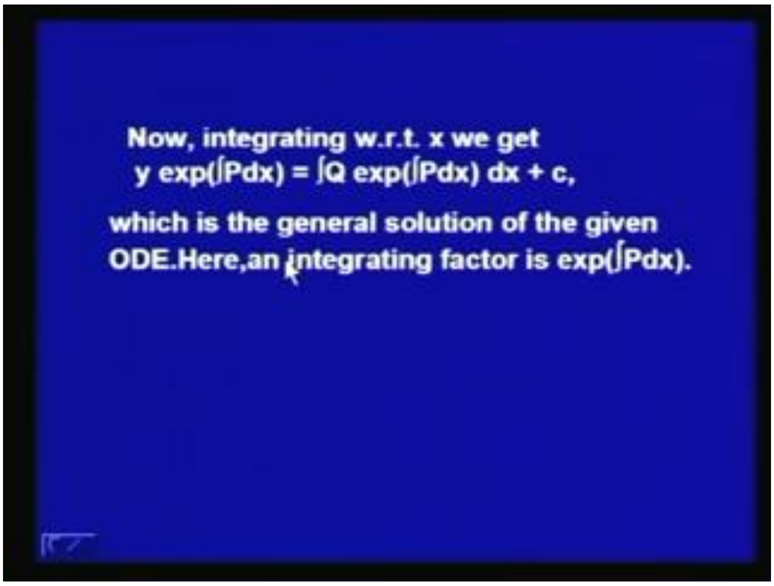

Now, let us integrate both sides of this equation with respect to x, we shall have y into exponential of integral P d x equal to integral of Q into exponential of integral P d x d x plus c, where c is an arbitrary constant. And thus this equation gives us the general solution of the given ordinary differential equation of first order, which is linear. Now, since on multiplying the linear differential equation of first order by the function exponential of integral P d x. We note that the left hand side becomes an exact differential of y in to e to the power integral P d x that is why we call the function exponential of integral P d x as an integrating factor. So, when we solve the linear differential equation of first order, the function e to the power integral P d x will be called an integrating factor.

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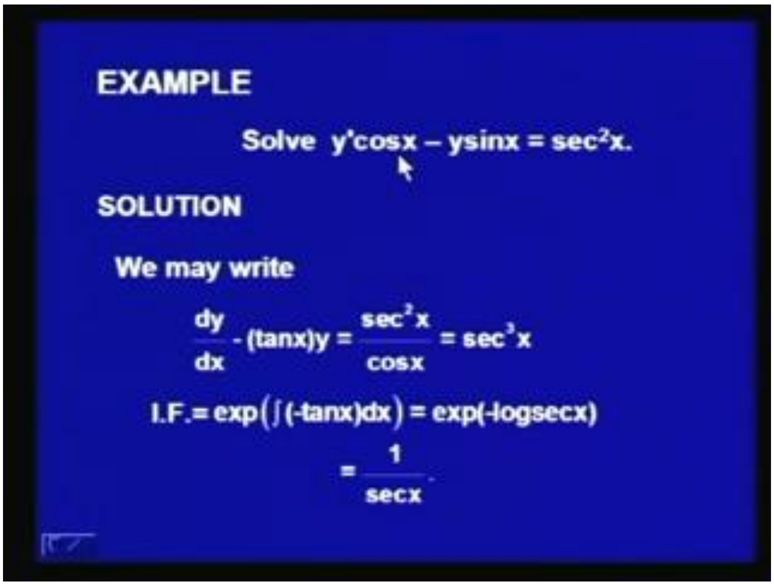

Now, let us try to solve a differential equation which can be brought to the linear form, suppose we consider the differential equation as y dash into cos x minus y sin x equal to sec square x. Then we can see that we may write this differential equation as d y by d x minus tan x into y equal to sec square x divided by cos x equal to sec cube x, we have to divide this differential equation by cos x to make the coefficient of d y by d x unity.

Now, here if you compare this equation d y by d x minus tan x into y equal to sec cube x with this standard form of a linear differential equation of first order, we note that P is equal to minus tan x, while Q is equal to sec cube x. And so an integrating factor is e to the power integral of minus tan x d x, after integration of minus tan x we get minus log sec x. So, we get the integrating factor as exponential of minus log sec x, which may be written as exponential of log 1 over sec x and e to the power log 1 by sec x will be 1 by sec x, so integrating factor is 1 upon sec x or we may also call it cos x.

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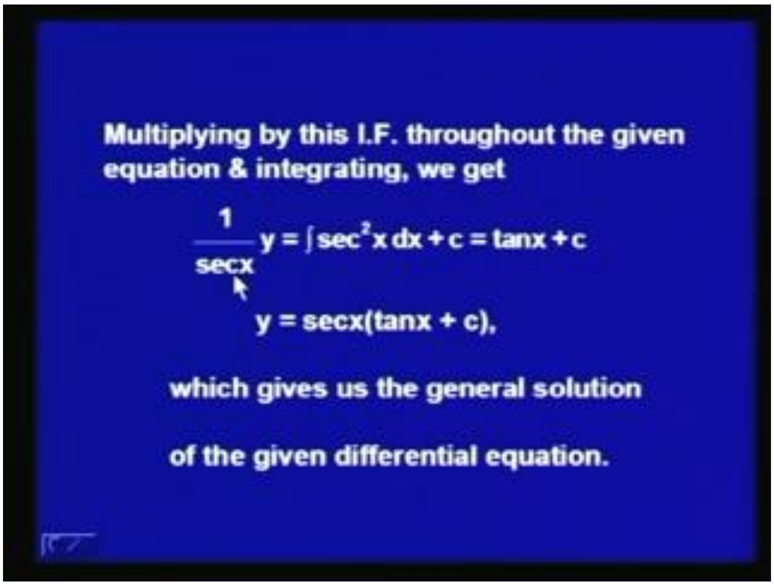

Now, let us multiply this integrating factor throughout the given linear differential equation ((Refer Time: 05:36)) d y by d x minus tan x into y equal to sec cube x, we shall have after integration with respect to x, 1 over sec x into y equal to integral of sec square x d x plus c. Now, integral of sec square x we know it is tan x, so we have the right hand side as tan x plus c after multiplying by sec x this equation becomes y equal to sec x into tan x plus c, which gives us the general solution of the given differential equation.

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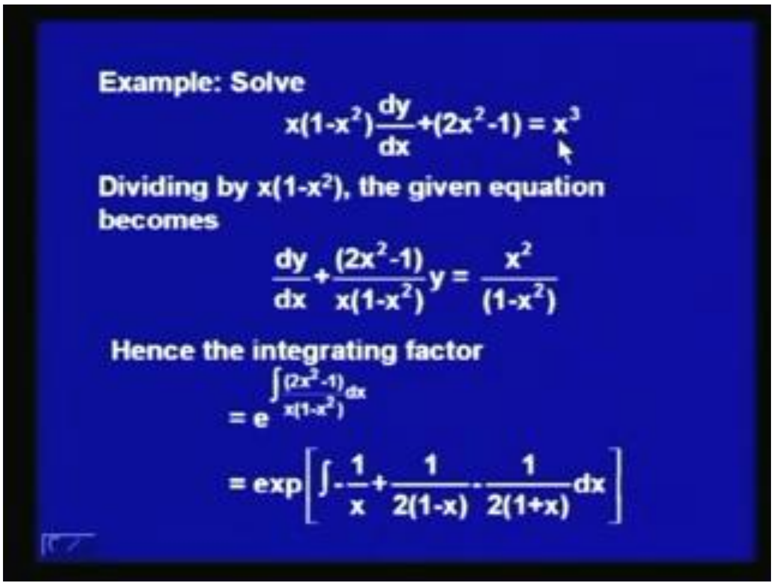

Now, let us take one more example on a linear differential equation of first order, the equation is x into 1 minus x square into d y by d x plus 2 x square minus 1 equal to x cube. So, here again we shall divide this differential equation by x into 1 minus x square in order to bring it to the standard form. So, divide by x in to 1 minus x square, we find that d y by d x plus 2 x square minus 1 upon x in to 1 minus x square into y is equal to x square upon 1 minus x square.

Now, when we compare with the standard form we note that P is equal to 2 x square minus 1 over x into 1 minus x square and Q is x square over 1 minus x square. So, the integrating factor here will be e to the power integral of 2 x square minus 1 over x into 1 minus x square d x, now let us break 2 x square minus 1 over x into 1 minus x square into partial fractions, the denominator has factors x into 1 minus x into 1 plus x. So, we will have the partial fractions of the function 2 x square minus 1 over x into 1 minus x square as minus 1 by x plus 2 over 1 minus x minus 2 over 1 plus x, we then integrate those functions.

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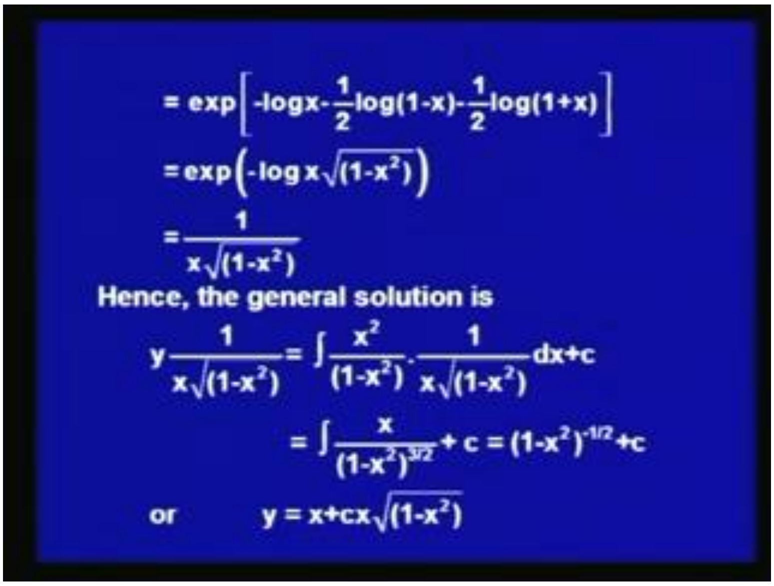

And find the integrating factor as e to the power minus log x minus half log 1 minus x minus half log 1 plus x, which is equal to e to the power minus log x into under root 1 minus x square which is also equal to 1 over x into under root 1 minus x square. And hence, the general solution is the dependant variable y into the integrating factor 1 over x under root 1 minus x square equal to integral of Q, that is x square over 1 minus x square into the integrating factor 1 over x into under root 1 minus x square d x plus c, which is equal to integral of x over 1 minus x square raised to the power 3 by 2 plus c.

Now, here when you put 1 minus x square as t and then change the variable from x to t we can easily show that the integral of this is 1 minus x square raise to the power minus half. And so we get the right hand side as 1 minus x square to the power minus half plus c, now we multiply this equation by x into under root 1 minus x square and then obtain y equal to x plus c into x into under root 1 minus x square, which is the general solution of the given differential equation.

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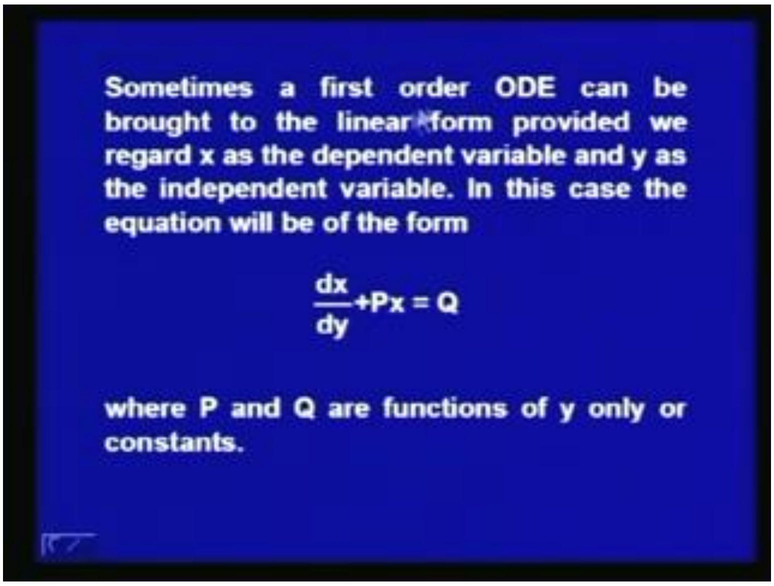

Now, sometimes what happens is that, a first order ordinary differential equation is not in the form which we have discussed earlier that is d y by d x plus P y equal to Q. But, it can be brought to the linear form provided we regard x as the dependant variable and y as the independent variable that is we interchange the rows of x and y. So, in this case the equation will be of the form d x over d y plus P x equal to Q, where P and Q now will be functions of y only or they will be constants.

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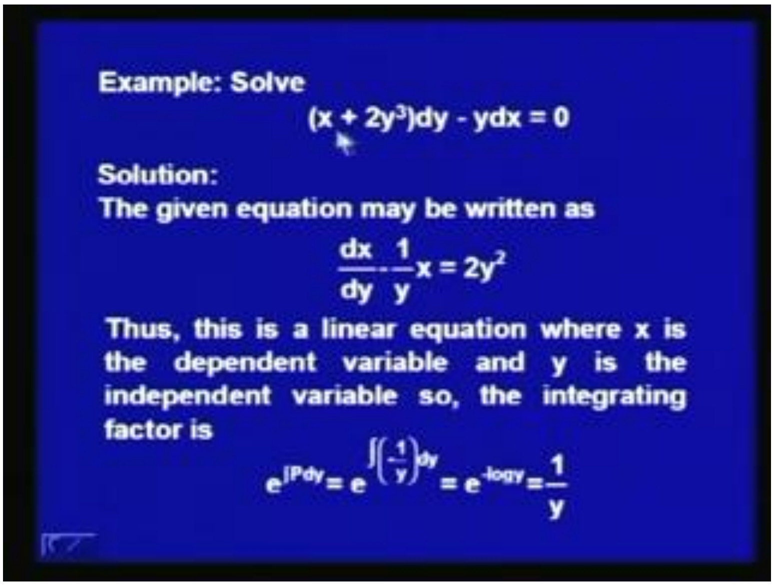

Let us take an example of this differential equation of this type, x plus 2 y cube into d y minus y d x equal to 0, then we can write this differential equation as d x over d y minus 1 over y into x equal to 2 y square. So, if you compare it with the form d x over d y plus P x equal to Q, we note that P is minus 1 over y and Q is 2 y square. And therefore, it is a linear differential equation, where x is taken as the dependent variable and y as the independent variable.

And so the integrating factor will be e to the power integral P d y, which is equal to e to the power minus 1 over y integral of minus 1 over y d y, an integral of minus 1 over y with respect to y is minus log y, so we have e to the power minus log y, which is the equal to 1 over y. First the integrating factor is 1 over y and so let us multiply the differential equation d x over d y minus 1 over y into x equal to 2 y square by the integrating factor 1 over y.

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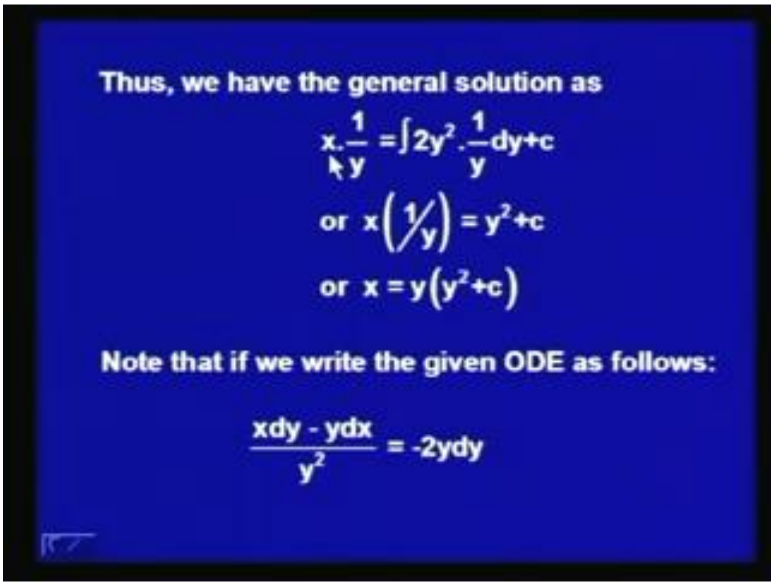

An integrate with respect to y, we find that the general solution is the dependent variable x into the integrating factor 1 over y equal to integral of Q, which is 2 y square into the integrating factor 1 over y d y plus c. After the integration on the right side, we get the solution of the differential equation as x into 1 over y equal to y square plus c or we may write it as x equal to y into y square plus c, which is the general solution of the given differential equation. Now, let us note the following that the given ODE can also be expressed as x d y minus y d x over y square equal to minus 2 y d y.

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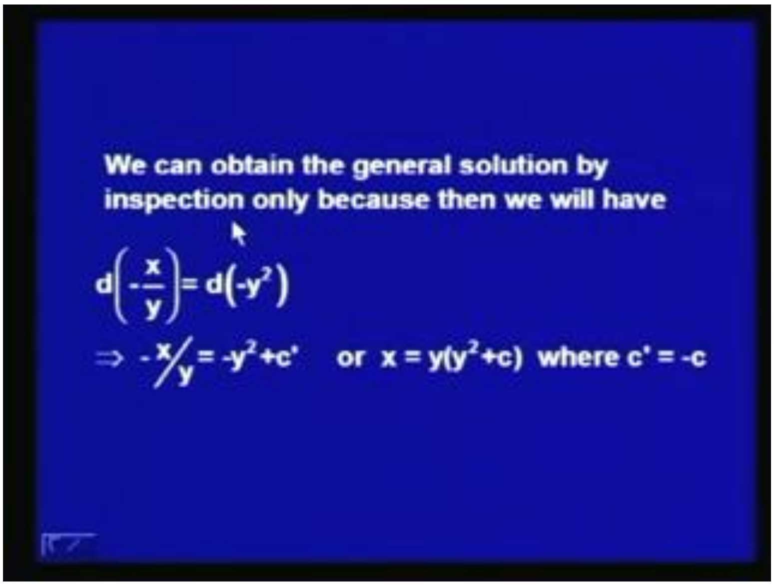

And so we can obtain the general solution of this ODE by inspection also, because the left hand side is the differential of minus x over y and the right hand side is the differential of minus y square. And so when we integrate on both sides, we find the general solution as minus x over y equal to minus y square plus c dash or x square equal to y into y square plus c, where c dash is equal to minus c.

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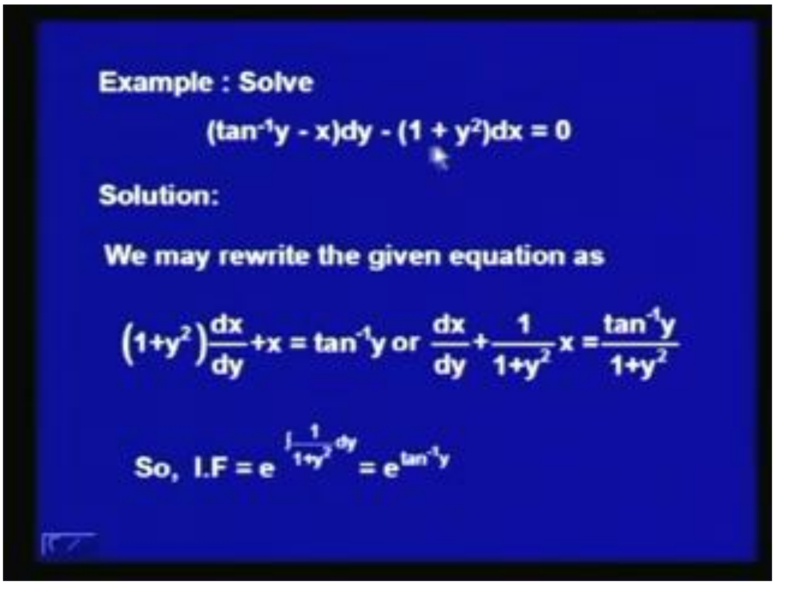

Let us take an example, one more example of differential equation of this type, say tan inverse y minus x into d y minus 1 plus y square into d x equal to 0. Now, we can write this differential equation in the form 1 plus y square into d x over d y plus x equal to tan inverse y or we may also write it as d x over d y plus 1 over 1 plus y square into x equal to tan inverse y over 1 plus y square.

So, it is a linear differential equation of first order in the variable x, the dependent variable is x, the independent variable is y and when we compare with this standard form P is 1 over 1 plus y square, while Q is tan inverse y over 1 plus y square. So, integrating factor is equal to e to the power integral of 1 over 1 plus y square d y, which is equal to e to the power tan inverse y.

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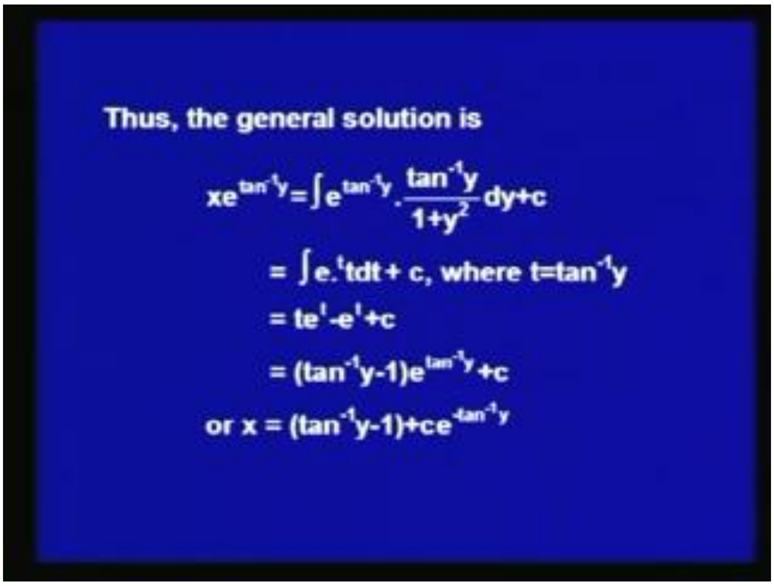

And hence, the general solution is x into e to the power tan inverse y equal to e to the power tan inverse y into tan inverse y over 1 plus y square d y plus c. Now, let us take tan inverse y equal to t here, then we will note that 1 over 1 plus y square d y is equal to d t and so the integral on the right hand side will reduce to integral of e to the power t into t d t plus c.

And the integral of t into e to the power t integration by parts will come out to be t into e to the power t minus e to the power t plus c. And let us now put the value of t that is tan inverse y, we will have the right hand side equal to tan inverse y minus 1 into e to the power tan inverse y plus c. And so we multiply the given the solution x into e to the power tan inverse y equal to tan inverse y minus 1 into e to the power tan inverse y plus c by the reciprocal of e to the power tan inverse y. And get the general solution of the given differential equation as x equal to tan inverse y minus 1 plus c into e to the power minus tan inverse y.

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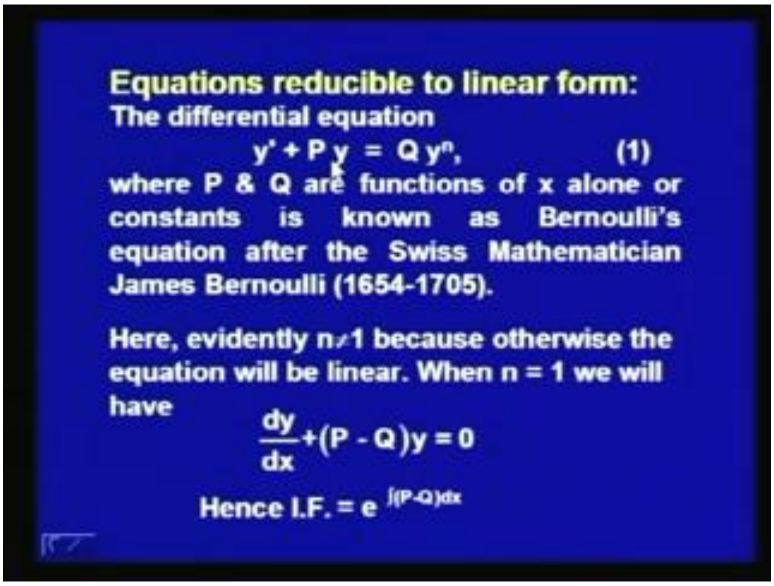

Next we consider the differential equations that are reducible to linear form, let us consider the differential equation y dash plus P y equal to Q into y to the power n, where P and Q are functions of x only or they are constants. This equation is known as Bernoulli's equation, after the Swiss mathematician James Bernoulli 1654 to 1705, now let us note here that here n is not equal to 1.

Because, if n is equal to 1 then y dash plus P y equal to Q y will be giving us d y over d x plus P minus Q into y equal to 0, which will be a linear differential equation of first order in the dependant variable y. And so integrating factor will be e to the power integral P minus Q into d x.

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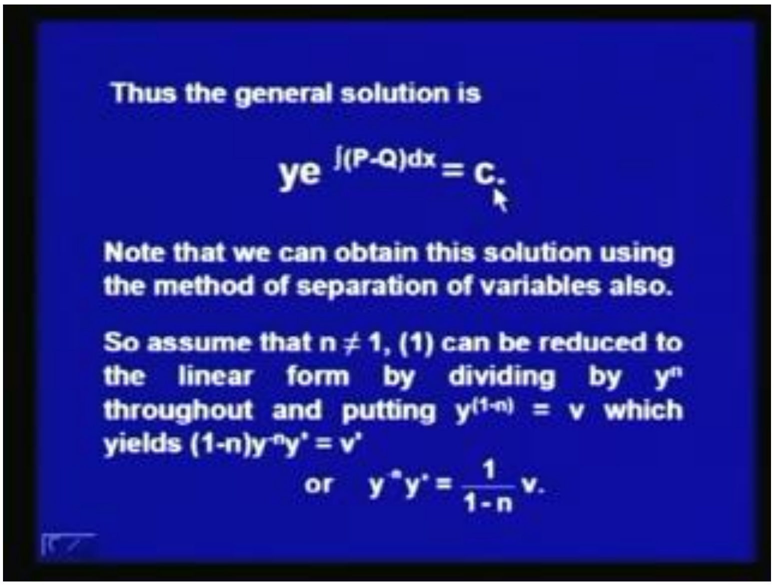

And so that is we can find the general solution as y into e to the power integral P minus Q d x equal to c, where c is an arbitrary constant, note that we can obtain this solution using the method of separation of variables. We can write ((Refer Time: 15:47)) this equation d y over d x plus P minus Q into y equal to 0 as d y upon y equal to Q minus P into d x.

So, we are able to separate the variables x and y and therefore, we can obtain the general solution of that differential equation, which will again be the same as y into e to the power integral P minus Q d x equal to c. Now, so we assume that n is not equal to 1 and then the differential equation y dash plus P y equal to Q y to the power n, can be reduced to the linear form by dividing by y to the power n.

((Refer Time: 16:35)) When you divide by y to the power n, what do get is y to the power minus n into y dash plus P into y to the power 1 minus n equal to Q and then what we will do is we will put y to the power 1 minus n equal to v. Now, here we differentiate with respect to x, we will get 1 minus n y to the power minus n into d y by d x which is y dash equal to d v by d x which we have denoted by v dash. And so since n is not equal to 1, we can divide this equation by 1 minus n giving us y to the power minus n into y dash equal to 1 over 1 minus n in to v.

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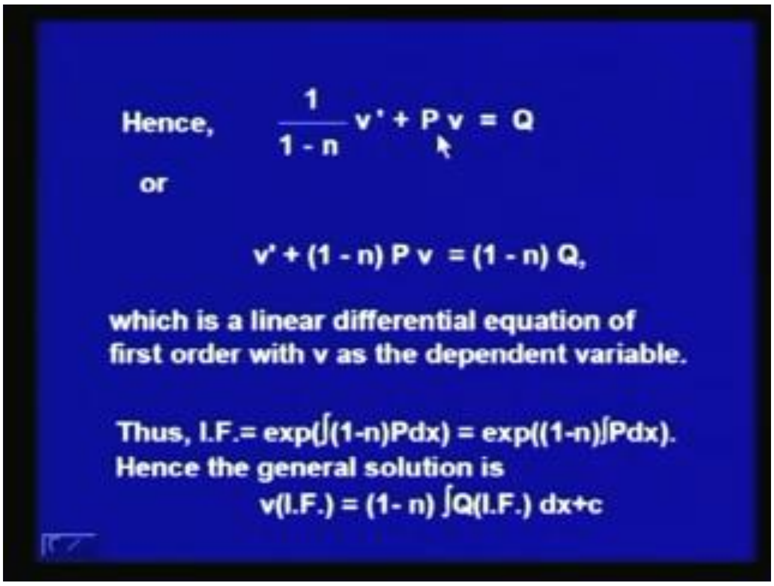

And hence, the equation becomes 1 over 1 minus n into d v by d x plus P v equal to Q or after multiplying by 1 minus n we get v dash plus 1 minus n into P into v equal to 1 minus n into Q. So, this is a linear differential equation of first order in the dependant variable v, where the coefficient of v is 1 minus n into P, which is our new P and the right hand side is 1 minus into Q, which is the new Q.

And so we can find the integrating factor here as e to the power integral of 1 minus n into P d x, which is equal to e to the power 1 minus n into integral P d x. And so the general solution of this equation we may write the dependant variable v into the integrating factor equal to 1 minus n times integral of Q into integrating factor d x plus c.

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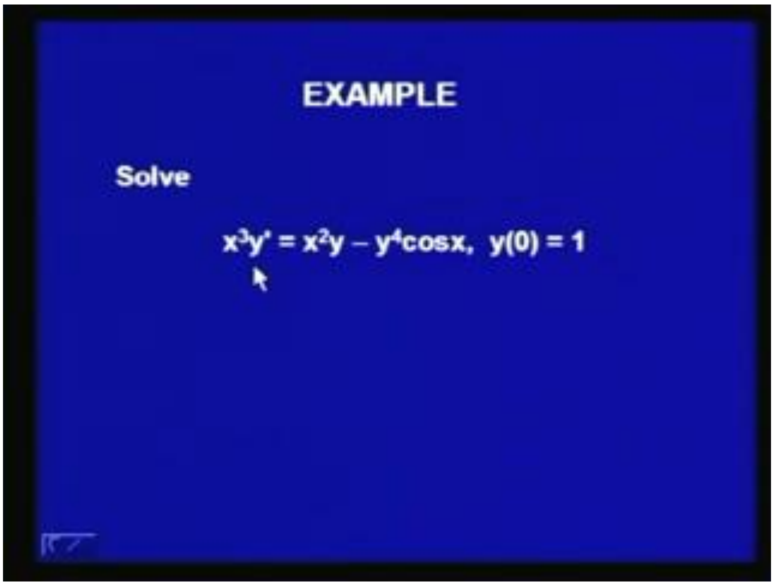

Let us take an example of a differential equation x cube y dash equal to x square y minus y 4 into cos x, where we are given the initial condition that at x equal to 0, y is equal to 1. So, in order to solve this differential equation first what will we do is, we will divide by x cube.

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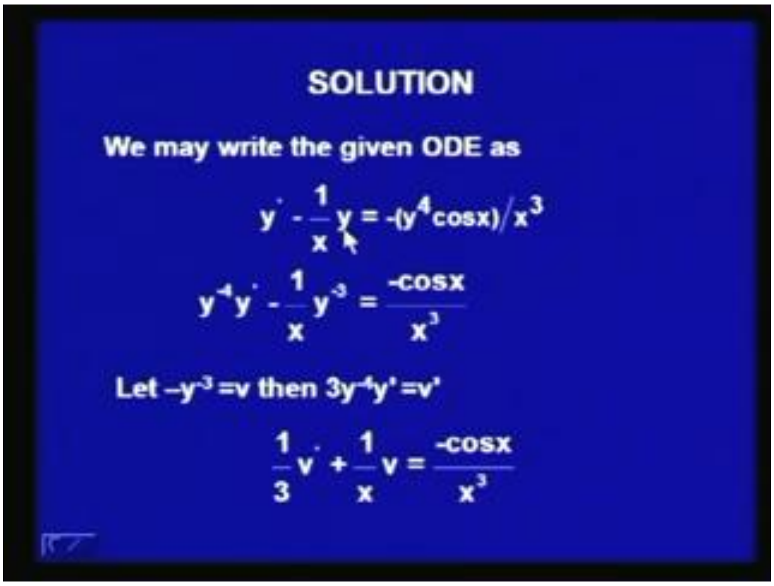

When we divide by x cube, we can write the differential equation as y dash minus 1 over x into y equal to minus y 4 cos x over x cube. Now, this is not a linear differential equation in y, but it can be made linear if we divide it by y to the power 4 and then put minus y to the power minus 3 equal to v. So, when we divide by minus with y to the power 4 we get y to the power minus 4 into y dash minus 1 over x into y to the power minus 3 equal to minus cos x over x cube.

Now, let us put minus y to the power minus 3 equal to v, then when we differentiate with respect to x, we will have 3 into y to the power minus 4 into y dash equal to v dash. So, making use of this the equation reduces to 1 by 3 into v dash plus 1 over x into v equal to minus cos x over x cube.

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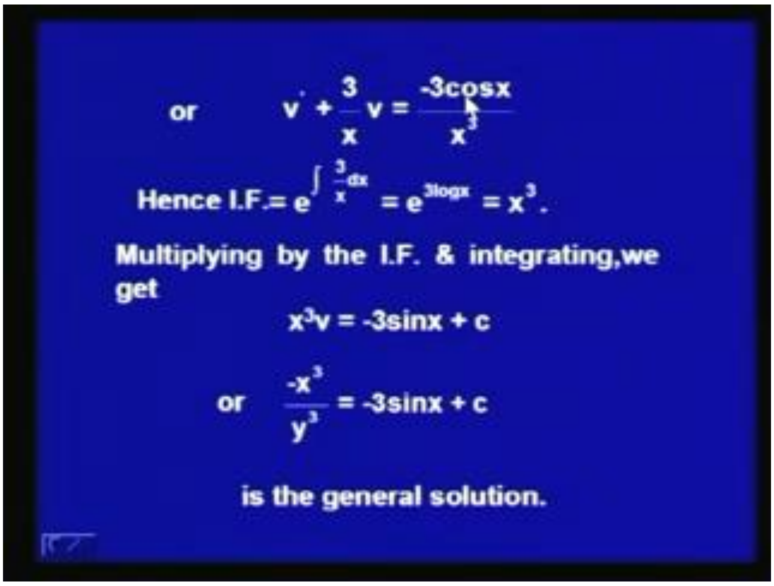

Or we may write it as v dash plus 3 over x into v equal to minus 3 cos x over x cube, now let us find, so this is a linear differential equation of first order in the dependant variable v. So, let us find the integrating factor here it is e to the power integral 3 over x d x or e to the power 3 log x which is equal to x cube; let us multiply the equation v dash plus 3 over x v equal to minus 3 cos x over x cube by the integrating factor x cube.

And integrate with respect to x, we shall have x cube which is the integrating factor into the dependant variable v equal to minus 3 sin x plus c after integration with respect to x or we will have minus x cube upon y cube equal to minus 3 sin x plus c, where we have put the value of v as minus 1 upon y cube. So, minus x cube on y cube equal to x plus c gives us the general solution of the given differential equation.

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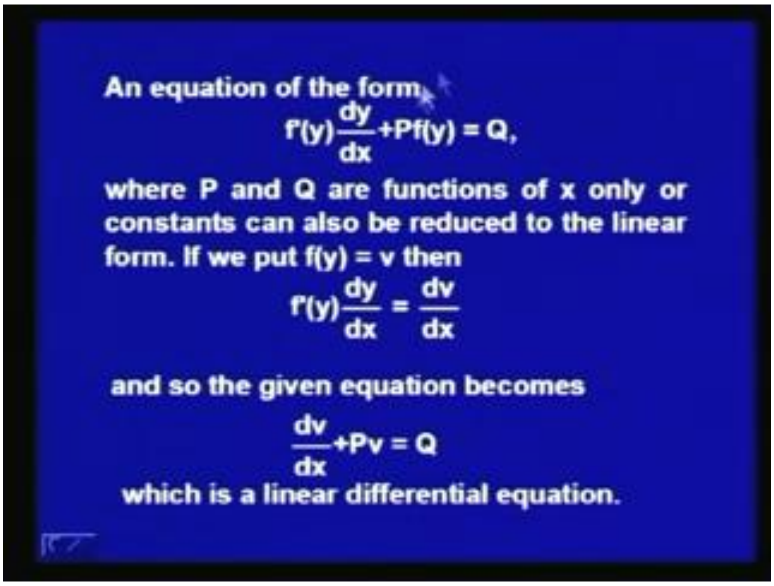

Now, next we consider an equation of the form f dash y into d y by d x plus P into f y equal to Q, where P and Q are functions of x only or they may be constants, now such a differential equation can also be reduced to the linear form. Let us if we put f y equal to a variable v, then when we differentiate with respect to x we shall have f dash y into d y by d x equal to d v by d x.

And so the given equation will become d v over d x plus P into v equal to Q, which we can see is a linear differential equation in the dependant variable v and so can be solved by the method of solving a linear differential equation. Now, let us discuss in alternative method for finding the solution of a linear differential equation of first order.

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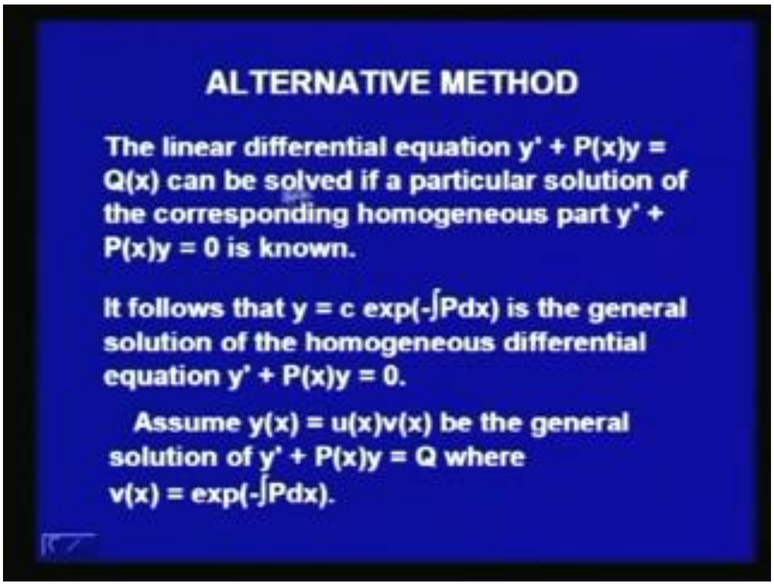

As we know the linear differential equation of first order in the dependent variable is of the form y dash plus P x into y equal to Q x. Now, it can be solved also if we know a particular solution of the corresponding homogeneous part that is the solution of y dash plus P x into y equal to 0. We can see here that y dash plus P x into y equal to 0 can be solved easily by separating the variables x and y.

We can write y dash plus P x into y equal to 0 as d y over y equal to minus P x d x, when we integrate both sides we shall find that y equal to c times e to the power minus integral P d x is the general solution of the differential equation y dash plus P x into y equal to 0. So, what we do is let us assume that y x is equal to u x into v x be the general solution of y dash plus P x in to y equal to Q x, where we take v x as e to the power minus integral P d x, which is the particular solution of the homogeneous part. Y dash plus P x into y equal to 0, we have taken c as 1 in the general solution to get v, v is equal to e to the power minus integral P d x.

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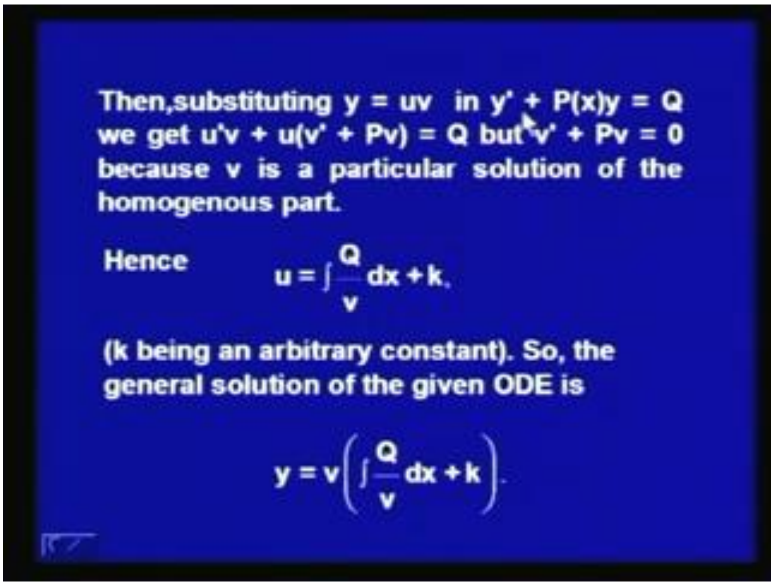

So, then substituting y equal to u v in the given differential equation y dash plus P x into y equal to Q, we will have y dash is u dash v plus u v dash y is u v, so we have P into u v and then it is equal to Q. But, since v is a particular solution of the homogeneous part of the given differential equation, therefore v dash plus P v will be equal to 0 and so this will reduce to u dash v equal to Q.

And we then divide u dash v equal to Q y the function v and integrate with respect to x we shall have u equal to integral of Q over v into d x plus k, where k is an arbitrary constant. And so we have found the function u and therefore, the general solution of the given ODE, e will be y equal to v into u which is integral of Q over v into d x plus k.

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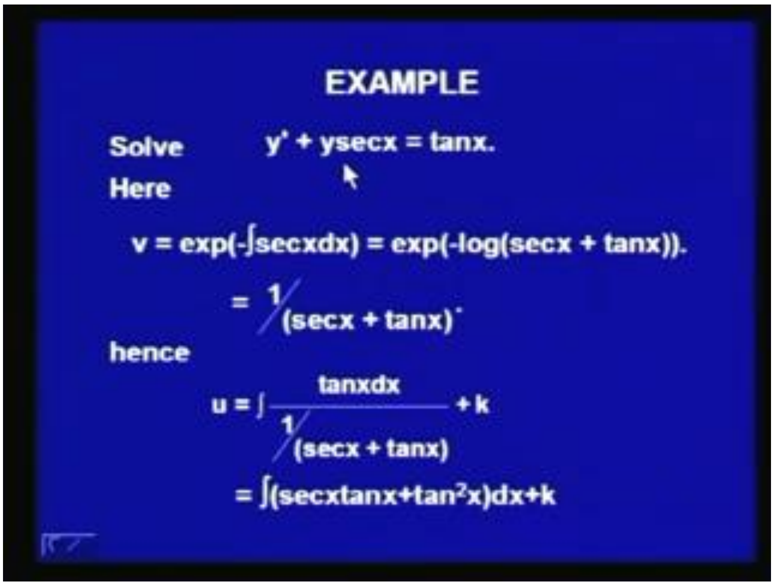

Now, let us illustrate this method by an example, so let us consider the differential equation y dash plus y into sec x equal to tan x, so here P x is equal to sec x and Q x is equal to tan x. And therefore, a particular solution of the homogeneous part y dash plus y sec x equal to 0 that is v is equal to e to the power minus integral sec x d x, now we know that integral of sec x with respect to x is log sec x plus tan x.

So, v will be equal to e to the power minus log sec x plus tan x, which is also equal to 1 over sec x plus tan x and hence, u will be equal to integral of Q over v; Q is tan x v is 1 over sec x plus tan x into d x plus k, the right hand side is therefore integral of sec x tan x plus tan square x into d x plus k.

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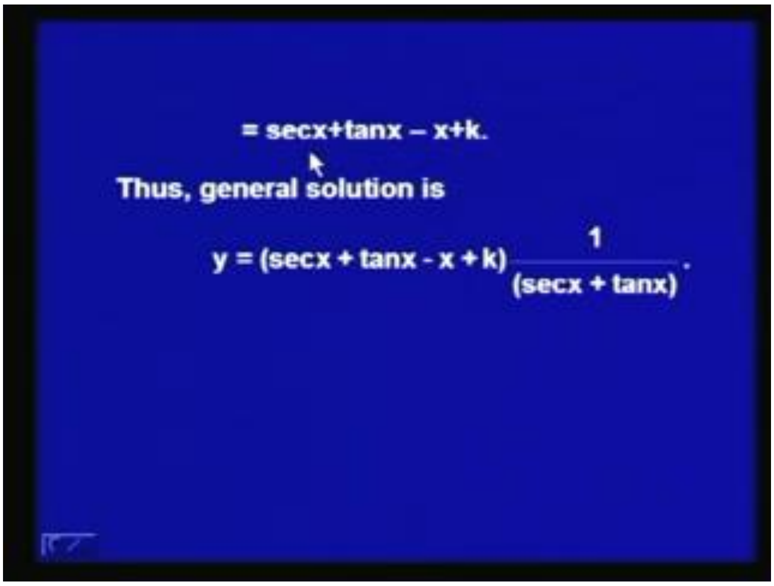

After integration we can see that we find sec x plus tan x minus x plus k, when you replace tan square x by sec square x minus 1, so the general solution is y equal to u which is sec x plus tan x minus x plus k into v, that is 1 over sec x plus tan x.

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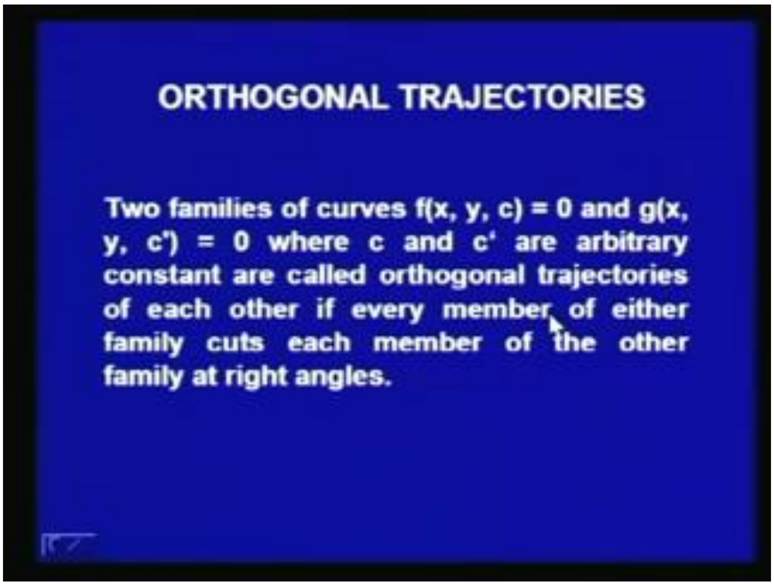

Now, we consider an important application of differential equations of first order that is we consider the orthogonal trajectories, we can use differential equations to find the curves which cut the given family of curves at right angles. Let us consider two families of curves f (x, y, c) equal to 0, and g (x, y, c dash) equal to 0, where c and c dash are arbitrary constants.

The two families of curves given by f (x, y, c) equal to 0 and g (x, y, c dash) equal to 0 are called orthogonal trajectories of each other, if every member of either family cuts each member of the other family at right angles. For example, let us consider the family of parabolas y equal to c x square, we shall see that the family of parabolas y equal to c x square has orthogonal trajectories given by the ellipses x square by 2 plus y square equal to c dash. The concept of orthogonal trajectories plays an important role in various fields of physics for example, let us consider an electric field between two concentric cylinders. Then the pass along which the current flows are orthogonal trajectories of the equipotential curves and vice versa.

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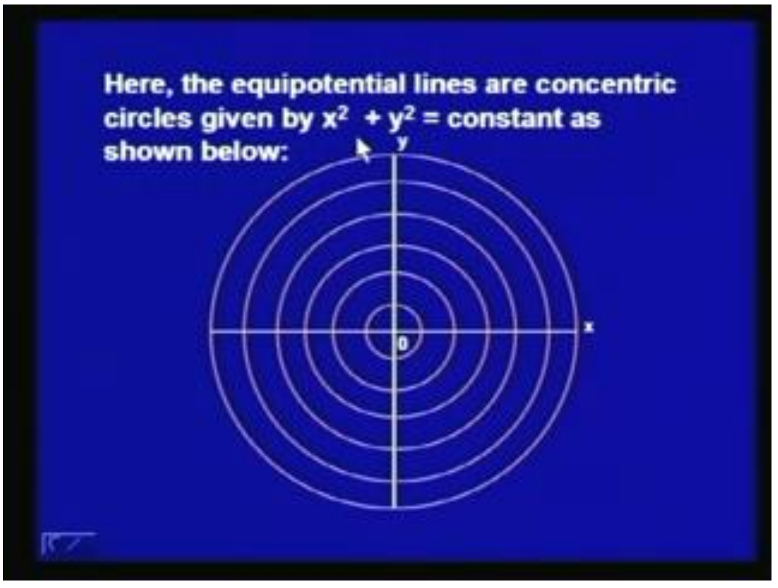

Here the equipotential lines are concentric circles given by x square plus y square equal to constant as shown in this figure, you can see that they are all concentric circles the center being at the origin.

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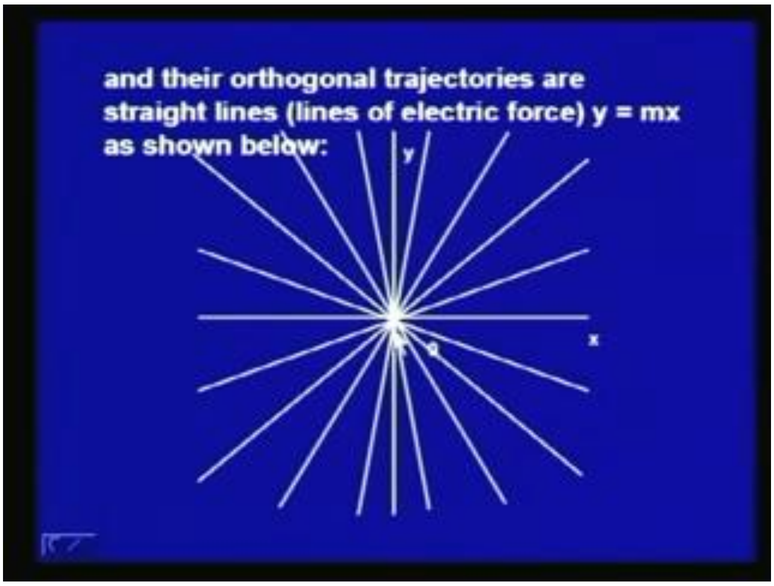

So, they are equipotential curves and their orthogonal trajectories are this straight lines that is the lines of electric force given by y equal to m x, where m is the parameter, you can see in this figure that the electric lines are lines of force, that is the straight lines given by y equal to m x are the lines which pass through the origin. So, they are the lines of electric force and they happen to be the orthogonal trajectories of the concentric circles, which are shown in this picture.

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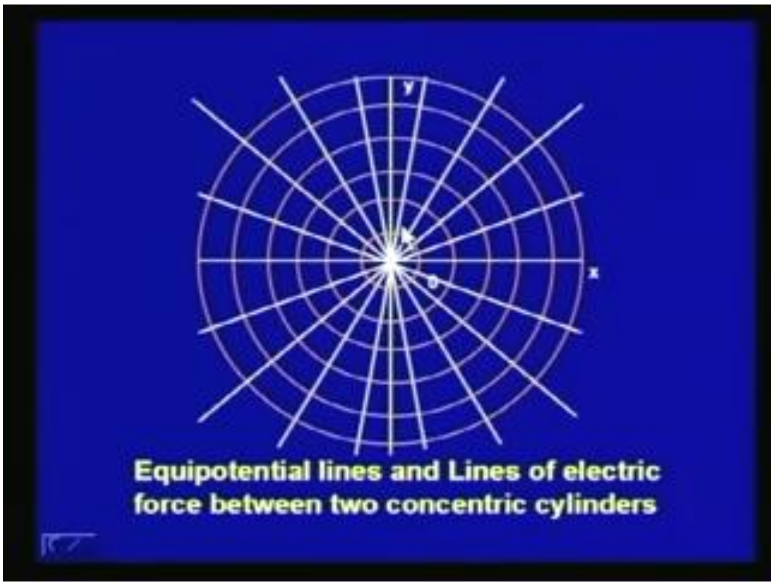

In this figure you can see that the concentric circles the familiar concentric circles are orthogonal to the family of straight lines, these the lines of force electric force. So, we have equipotential lines and lines of electric force between two concentric cylinders, the two families intersect each other orthogonally, so they are orthogonal trajectories of each other.

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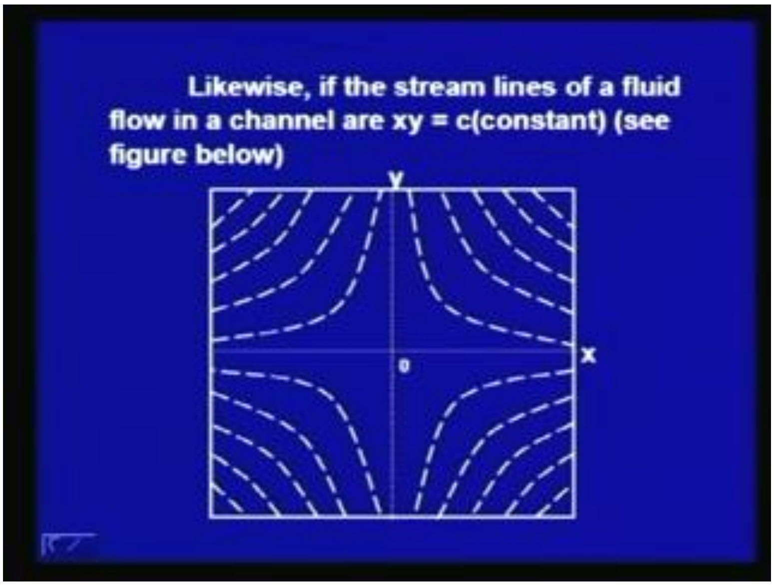

Let us consider another example, let us consider the stream lines of a fluid flow in a channel suppose they are given by x y equal to c, a constant we know that x y equal to c give us a family of rectangular hyperbolas. So, in this figure we have drawn a this family of rectangular hyperbolas, they are the stream lines of a fluid flow in a channel.

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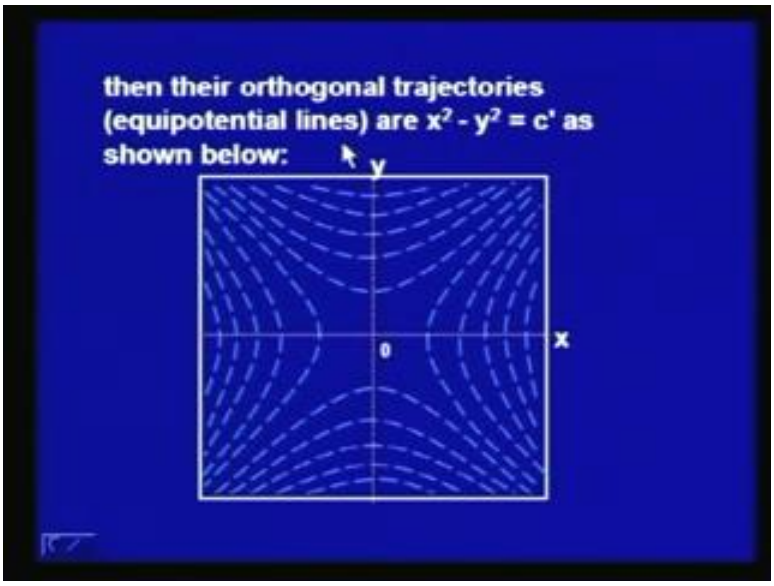

Then, their orthogonal trajectories that is the equipotential lines turn out to be x square minus y square equal to c dash, they again are a family of hyperbolas as shown in this picture.

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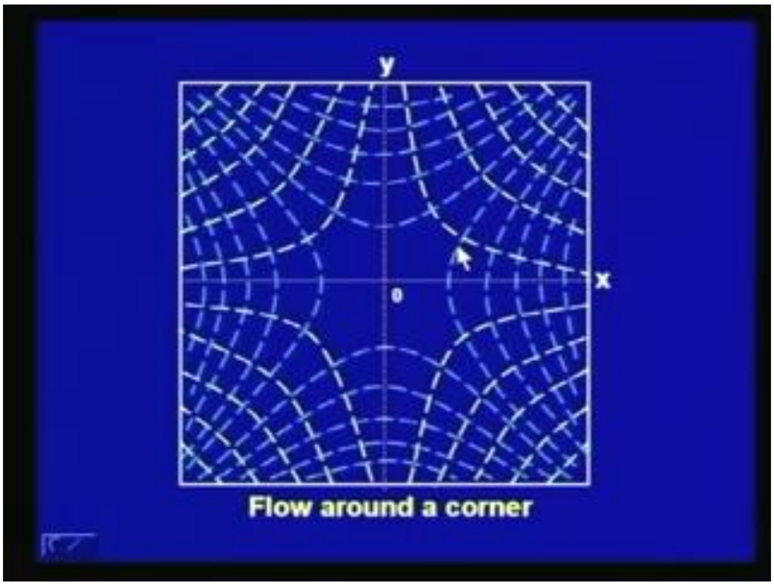

And in this picture we show that the two families of hyperbolas cut each other at right angles, so they are orthogonal trajectories of each other; every member of one family cuts every member of the other family at right angles.

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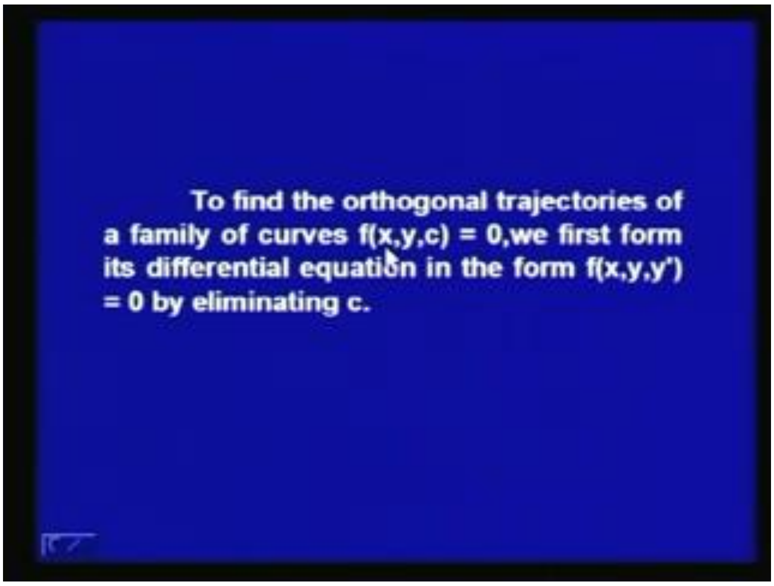

Now, let us see how to find the orthogonal trajectories for a given family of curves, suppose the family of curves is given by the equation f (x, y, c) equal to 0. Then first we form its differential equation in the form f (x, y, y dash) equal to 0 by eliminating c that is we arrive at a differential equation a first order d y by d x equal to g (x, y).

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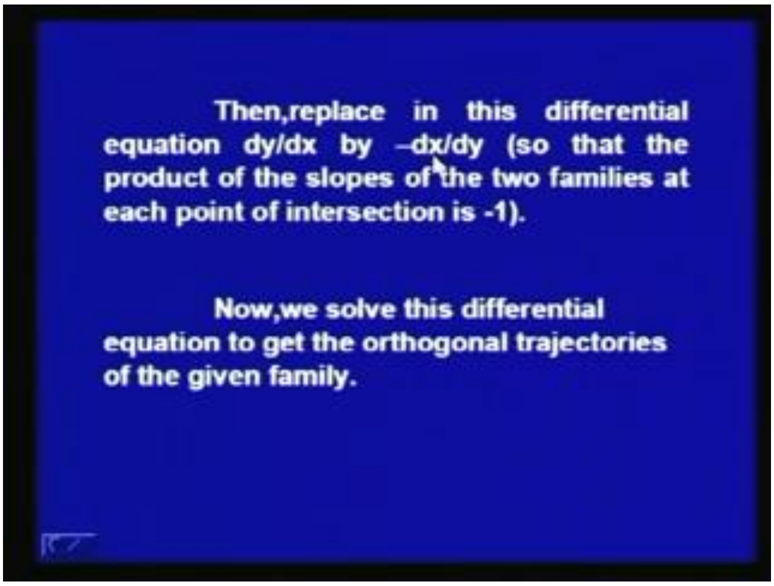

And then replace in the differential equation d y by d x equal to g (x, y), d y by d x by minus 1 upon d y by d x which can also be written as minus d x over d y. Because, at each point of intersection the product of slopes of the two families of curves will be equal to minus 1 and so we replace d y by d x by minus d x over d y and arrive at the differential equation for the family of orthogonal trajectories. So, let us consider the family of parabolas given by y equal to c x square and find it is orthogonal trajectories.

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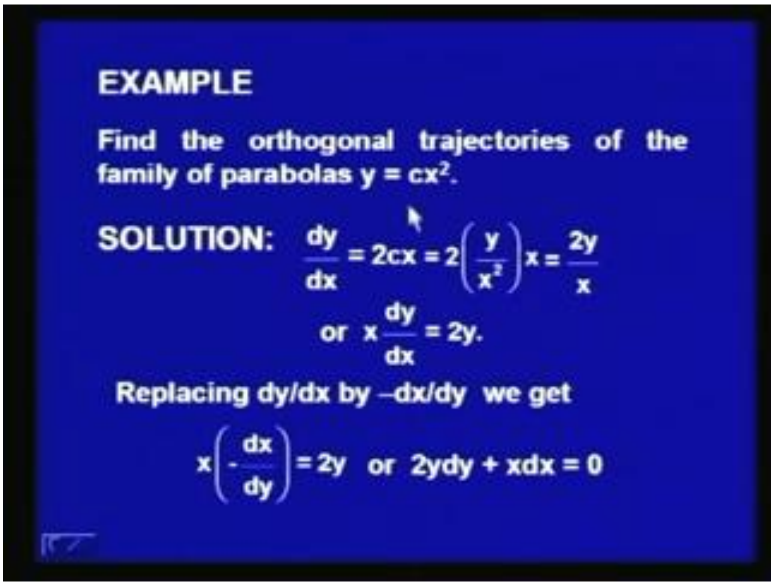

Now, y equal to c x square gives us d y by d x equal to 2 c x, so we can eliminate c from the equation d y by d x equal to 2 c x using the equation y equal to c x square, it will become d y by d x equal to 2 into y over x square into x or 2 y over x or we shall have x into d y by d x equal to 2 y. So, this is the differential equation corresponding to the given family of parabolas, it is of the form f (x, y, y dash) equal to 0. Now, in this differential equation we shall replace d y by d x by minus d x over d y to arrive at the differential equation corresponding to the orthogonal trajectories of y equal to c x square, we shall have x into minus d x over d y equal to 2 y or 2 y d y plus x d x equal to 0.

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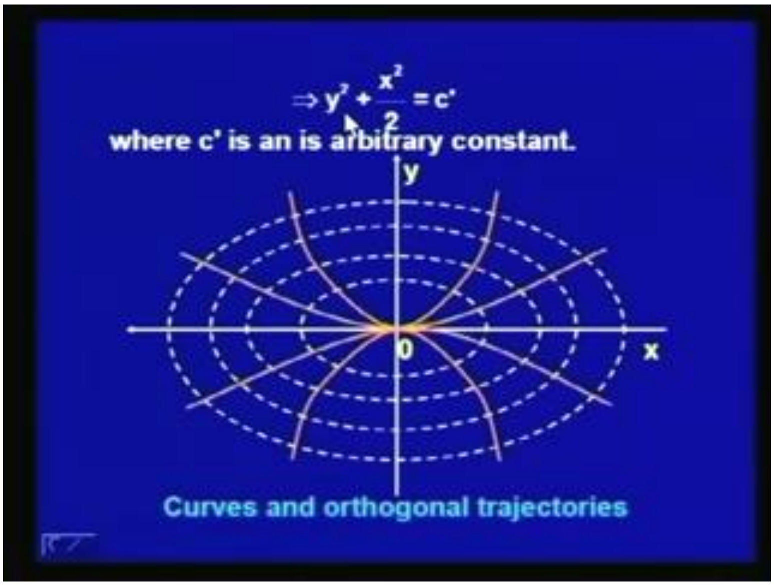

When we integrate on both sides of this differential equation, we can see that we will have y square plus x square over 2 equal to c dash, where c dash is an arbitrary constant. Now, you can see that this is a family of ellipses where the semi major axis has length square root 2 c dash and the semi minor axis as length square root c dash. We have shown this, this is the family of parabolas and these are ellipses and you can see that at each point of intersection every member of the family of parabolas intersect, every member of the family of ellipses at right angles, so they are orthogonal trajectories of each other.

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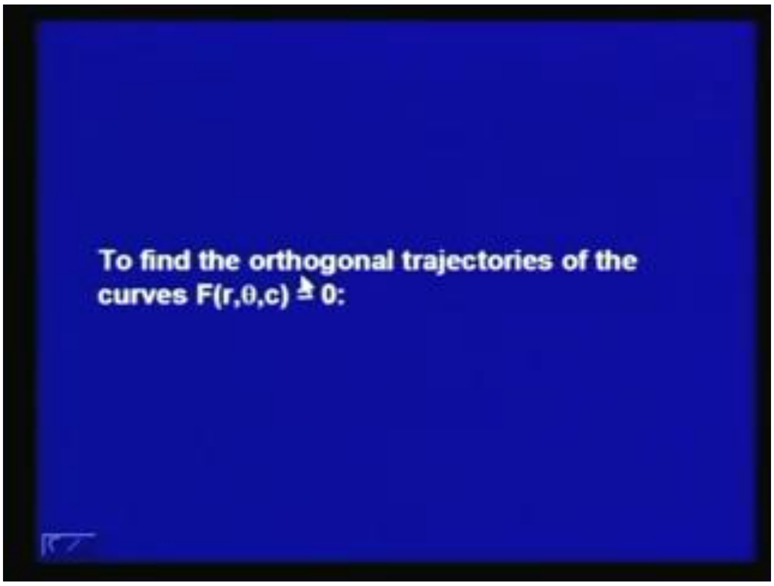

Now, let us take the case of a polarate equation of F (r, theta, c) equal to 0 and let us find the orthogonal trajectories of a polar family of curves F (r, theta, c) equal to 0.

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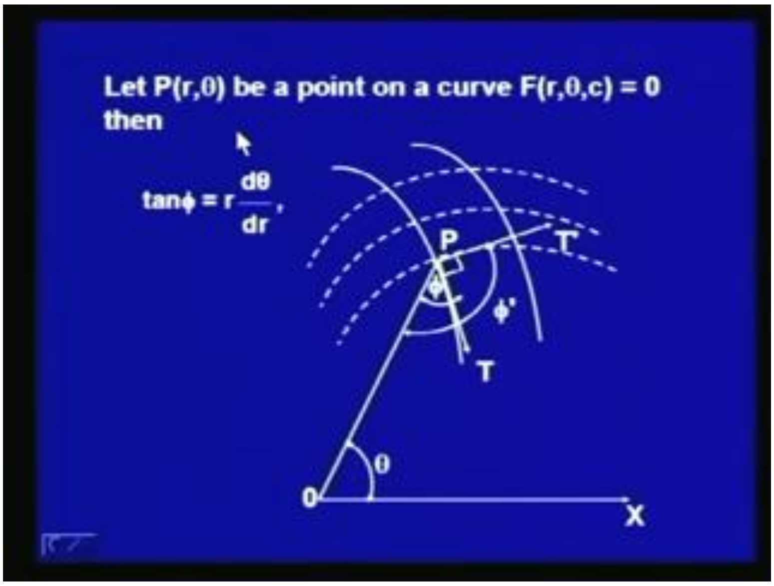

So, here let us see, let P (r, theta) be a point on the curve, F (r, theta, c) equal to c, this is the curve F (r, theta, c) equal to 0, P is a point here, we can see that the radius vector O P makes angle phi with the tangent P T to the curve F (r, theta, c) equal to 0. And these curves shown by dotted lines, this is their the family of orthogonal trajectories; so you can see that at the point of intersection P the angle between the tangents to the curve F (r, theta, c) equal to 0 and the orthogonal curve it is phi by phi by 2. P T dash is tangent to this curve while P T is tangent to the curve, F (r, theta, c) equal to 0, the angle between the 2 is phi by 2.

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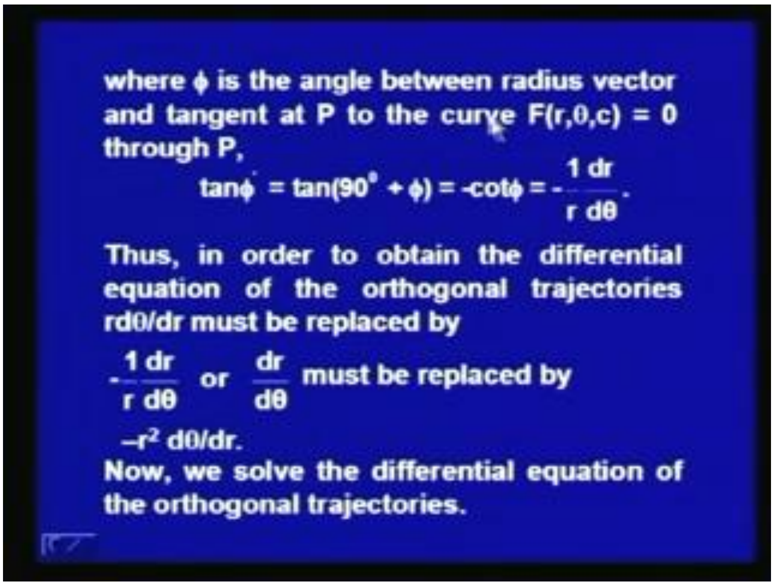

So, phi is the angle between the radius vector and tangent at P to the curve F (r, theta, c) equal to 0, now through the point P, now tan phi dash, since phi dash minus phi is equal to 90. So, tan phi dash will be equal to tan 90 plus 5 and tan 90 plus phi is minus cot phi, we know that tan phi is equal to r d theta by d r, so minus cot phi will be minus 1 over r d r by d theta.

And therefore, in order to obtain the differential equation of the orthogonal trajectories r d theta by d r in the given differential equation of the family of curves F (r, theta, c) equal to 0 must be replaced by minus 1 over d r by d theta. Or we can say that d theta by d r they are must be replaced by minus 1 over r square d r by d theta, or we can take the reciprocal and say that, d r by d theta in that differential equation of the family F (r, theta, c) equal to 0 must be replaced by minus r square d theta by d r. Now, we solve the differential equation of the orthogonal trajectories.

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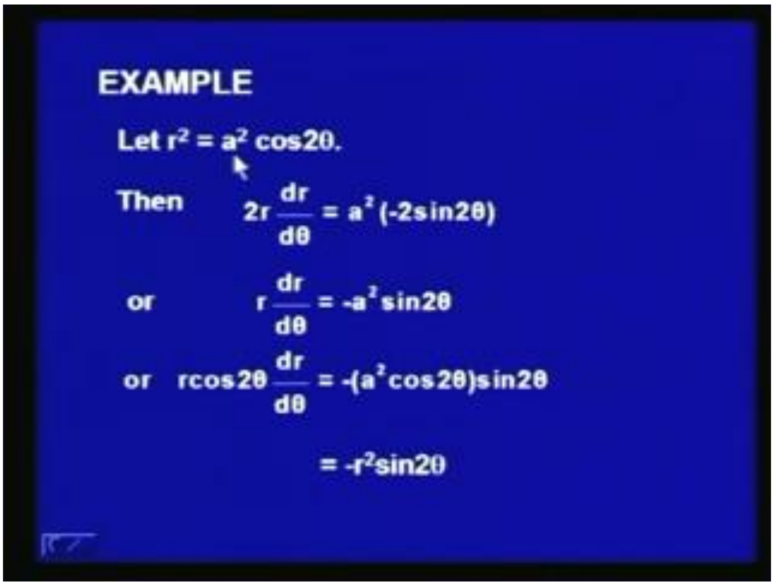

Let us consider the equation r square equal to a square cos 2 theta, so when you differentiate with respect to theta on both sides we get 2 r d r by d theta equal to a square into minus 2 sin 2 theta, or we can say r into d r by d theta equal to minus a square sin 2 theta. Let us multiply by cos 2 theta both sides, we want to eliminate a square from here to arrive at the differential equation of the family of curves r square equal to a square cos 2 theta; it is known that this r square equal to a square cos 2 theta gives us a lemniscates, so this is a family of lemniscates.

So, we multiply both sides of r d r theta equal to minus a square sin 2 theta by cos 2 theta and get r cos 2 theta d r by d theta equal to minus a square cos 2 theta into sin 2 theta. We replace a square cos 2 theta by r square and so we will get the right hand side as minus r square sin 2 theta.

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Or we can say cos 2 theta d r by d theta is equal to minus r sin 2 theta, now let us replace r d r by d theta by minus r square d theta by d r to get the differential equation of the orthogonal trajectories of the family of lemniscates. We will have minus r square cos 2 theta into d theta by d r equal to minus r sin theta or we will have d r by r equal to cot 2 theta d theta.

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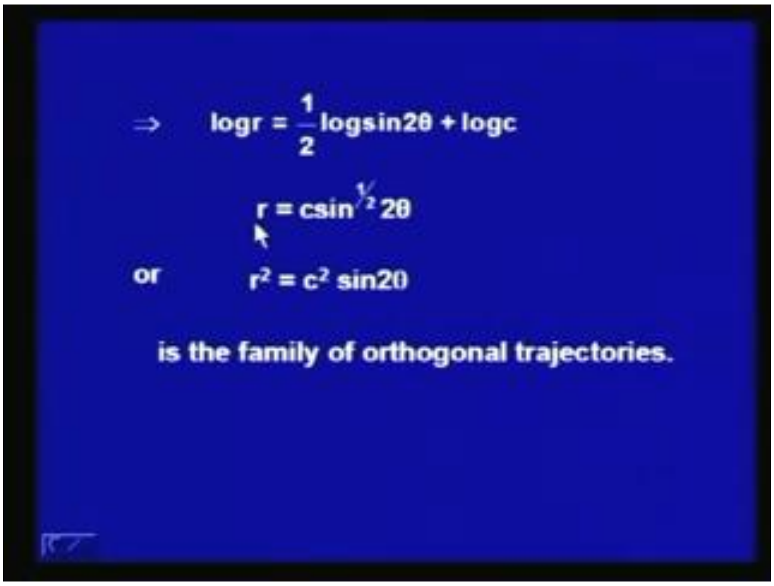

We can integrate with both sides, we will have log r equal to half log sin 2 theta plus log c or we can write r equal to c times sin 2 theta raised to the power half, squaring both sides we have r square equal to c square sin 2 theta, which is the family of the orthogonal trajectories of the family of lemniscates.

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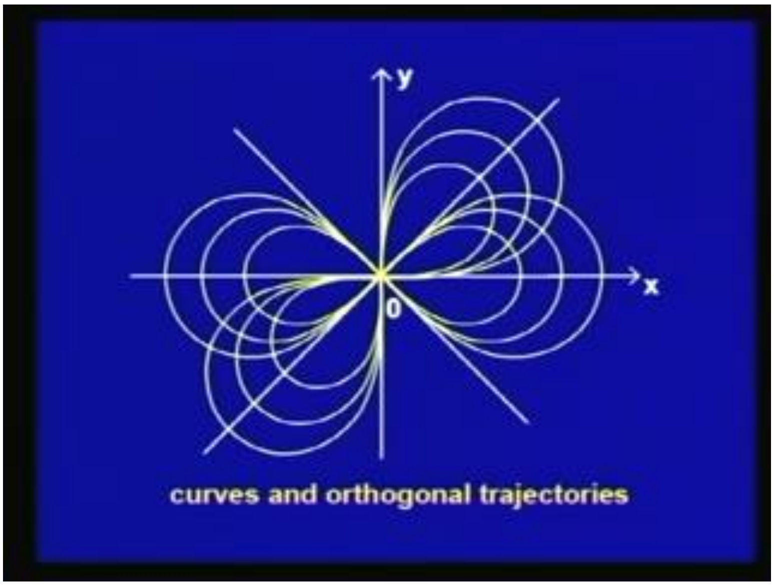

In this picture, these curves they are the curves corresponding to r square equal to a square cos 2 theta, they are family of lemniscates and they are orthogonal trajectories are these curves given by r square equal to c square sin 2 theta. You can see that at each point of intersection every member of one family cuts every member of the other family at right angles, it is their orthogonal trajectories of each other.

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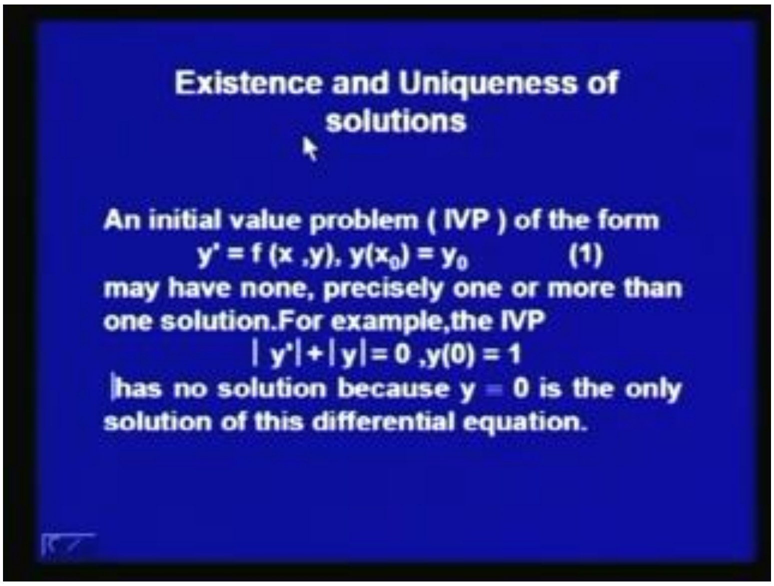

Next we discuss existence and uniqueness of solutions, so for the differential equations that we have considered each one had a general solution. In the case of an initial value problem, an initial value problem consist of a differential equation d y by d x equal to f (x, y) and a condition y x naught equal to y naught which the solution has to satisfy. We could obtain a particular solution of the differential equation by finding the value of the constant c, which occurs in the general solution using the initial condition y x naught equal to y naught.

But, in the general case an initial value problem y dash equal to f (x, y), where y x naught is equal to y naught may not have any solution or it may have an unique solution or it may have more than one solution. Let us discuss an example of each one of those cases, first we consider the initial value problem mod of y dash plus mod of y equal to 0 where we are given that y at 0 is equal to 1.

Now, the differential equation mod of y dash plus mod of y equal to 0 has an unique solution y equal to 0. Because, the left hand side of the differential equation mod of y dash plus mod of y is the sum of two non negative real valued functions, so their sum is 0 if and only if y is identically 0. Since we are given that y at 0 has to be equal to 1 y equal to 0 cannot be the solution of the given initial value problem, because y equal to 0 at x equal to 0 gives us value 0 and not 1. So, the given initial value problem mod of y dash plus mod of y equal to 0 where y 0 is equal to 1 has no solution.

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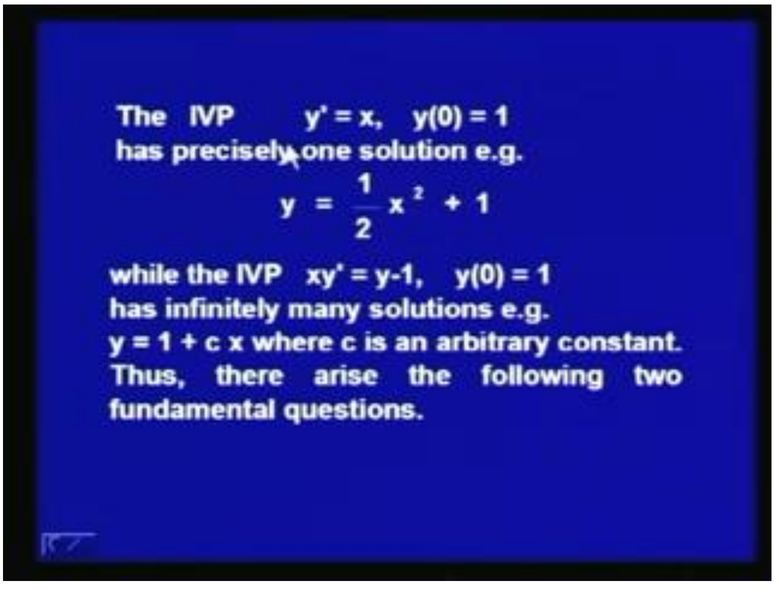

Next we consider the initial value problem y dash equal to x where we are given that y 0 is equal to 1, we shall show that it has precisely one solution, d y by d x equal to x we can write as d y equal to x d x, that is we are able to separate the variables x and y. After integrating both sides we shall have y equal to half of x square plus c, the value of c we can find using the initial condition y at 0 equal to 1, it will turn out that the value of c is 1.

And therefore, y equal to half x square plus 1 is the solution of the given initial value problem y dash equal to x, where y 0 equal to 1, so the given initial value problem has precisely one solution. Next let us consider the case of an initial value problem x y dash equal to y minus 1, where y 0 equal to 1 we can see that here in the differential equation x y dash equal to y minus 1, if you put x equal to 0 we find y is equal to 1.

So, y at 0 equal to 1 is automatically satisfied, so let us assume that x is not equal to 0, we can divide this equation x y dash equal to y minus 1 by x and then we shall have y dash equal to y over x minus 1 over x. We can bring it to the linear form y dash minus 1 upon x into y equal to minus 1 upon x, it will be a linear differential equation of first order in y where the integrating factor will be e to the power minus integral of 1 over d x, 1 over x into d x.

So, we will have the integrating factor as e to the power minus log x that is 1 over x and so we will be able to find the general solution as the dependant variable by into 1 over x equal to integral of minus 1 over x square plus c, which will give us y equal to 1 plus c x. Now, y equal to 1 plus c x is the solution of x y dash equal to y minus 1 for all x not equal to 0, but when you put x equal to 0 in this we see that y equal to 1.

So, y 0 equal to 1 that case can also be included in the solution y equal to 1 plus c x and therefore, we may say that y equal to 1 plus c x is the general solution of the initial value problem x y dash equal to y minus 1. Since c is an arbitrary constant, so y equal to 1 plus c x gives us infinitely many solutions of the given initial value problem. Thus an initial value problem in the general case may have none and unique or infinitely many solutions and so there arise the following two fundamental questions.

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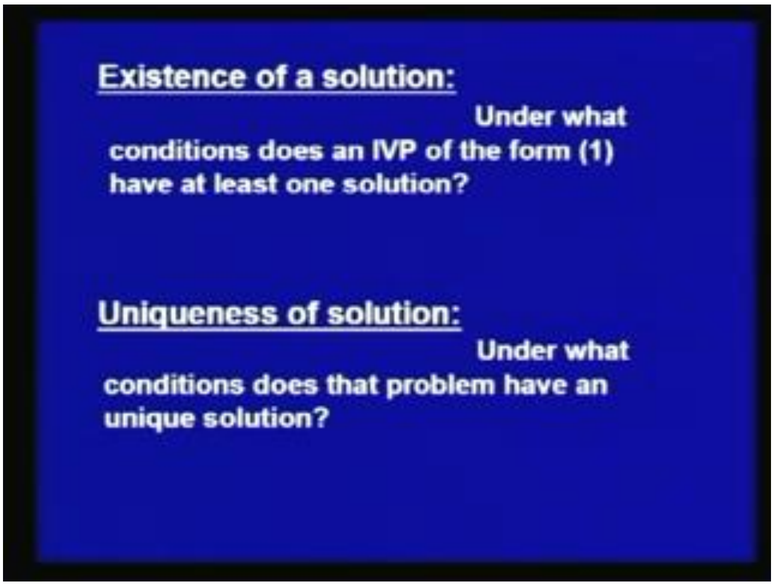

It is question is the existence of a solution, under what conditions does an initial value problem of the form one that is d y by d x equal to f (x, y), where y x naught is equal to y naught have at least one solution. The next question is uniqueness of solution, under what conditions does the initial value problem have an unique solution, the theorems that answers these two questions are known as the existence theorems and uniqueness theorems.

The examples considered above were so simple, we have considered three examples earlier, one example was the example of an usual value problem which had no solution. Then we considered an example of a initial value problem where we had precisely one solution. And then the third example was an initial value problem where we had infinitely many solutions, so those examples were so simple that we could find answers to the questions of existence of a solution, an uniqueness of the solution just by looking at the differential equation and making some simple calculations.

But, in the case of complicated differential equations that is the differential equations, which cannot be solved by elementary methods that is the techniques which we discussed so far; the existence an uniqueness theorems will be of greater importance.

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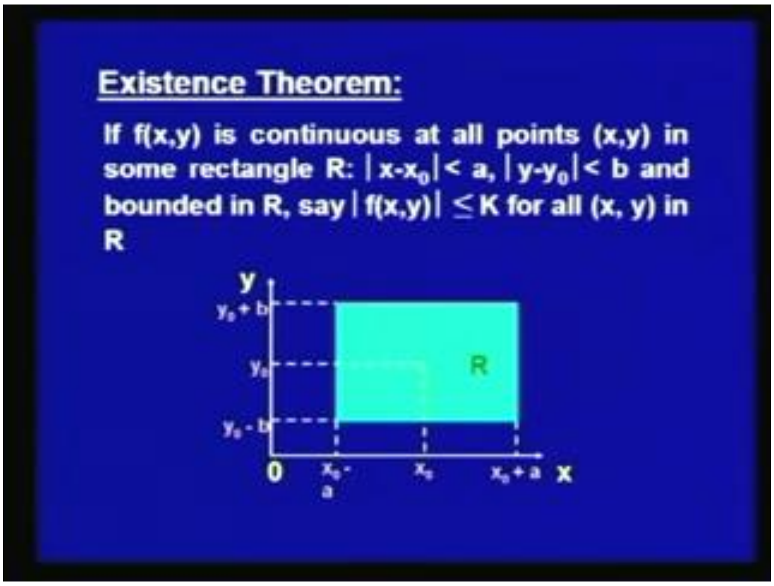

Let us look at the existence theorem, if f (x, y) is continuous at all points (x, y) in some rectangle R given by mod of x minus x naught less than a, mod of y minus y naught less than b and bounded in R say mod of f (x, y) less than or equal to k for all (x, y) in R.

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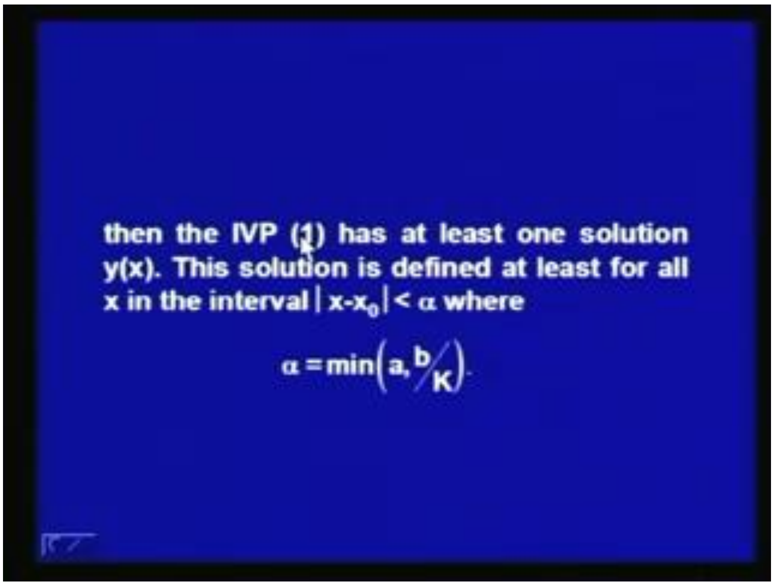

Then, the initial value problem 1 has at least one solution this solution is defined at least for all x in the interval mod of x minus x naught less than alpha, where alpha is equal to minimum of a and b by K. Now, ((Refer Time: 47:29)) this theorem tells us that if the function f (x, y) is continuous in some region of the (x, y) plane and it is a bounded function for all (x, y) in that region, then the initial value problem always has a solution, so this theorem is known as an existence theorem.

In this picture you can see that this is the rectangle R, which is given by the inequalities mod of x minus x naught less than a and mod of y minus y naught less than b and containing the point x naught y naught here. So, if the function f (x, y) is continuous in this region and bounded in R, then the initial value problem will all ways have a solution. And the solution will be defined at least for all values of x in the interval, mod of x minus x naught less than alpha, where alpha is the minimum of the 2 numbers a and b by K.

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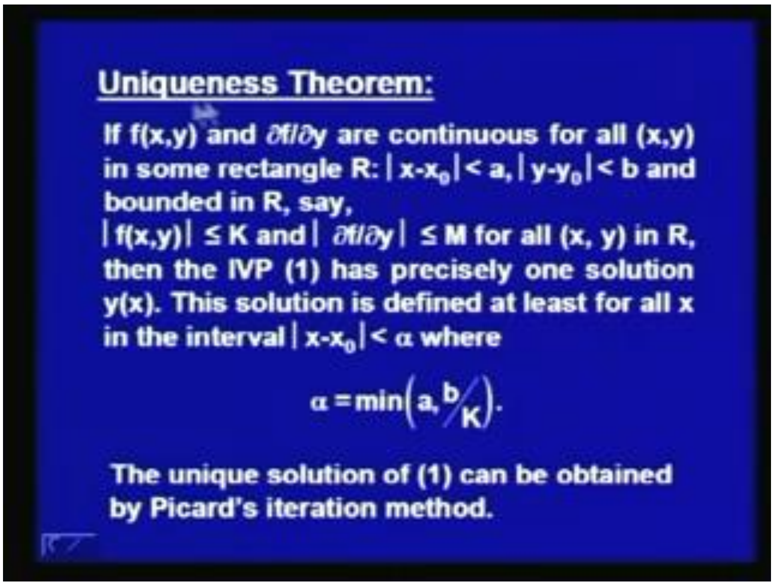

The next theorem which is the uniqueness theorem tells us that, if f (x, y) and delta f over delta y are continuous for all (x, y) in some rectangle R, given by mod of x minus x naught less than a mod of y minus y naught less than b and bounded in R. That is to say mod of f (x, y) less than or equal to k and mod of delta f over delta y less than or equal to M for all (x, y) in R. Then the initial value problem one has precisely one solution y x and this solution is defined at least for all x in the interval mod of x minus x naught less than alpha where alpha is the minimum of a and b by K.

The unique solution of the differential equation 1, that is the initial value problem d y by d x equal to f (x, y), where y x naught equal to y naught can be obtained by Picard's iteration method. So, in our next lecture we shall discuss the geometrical interpretation of the existence and uniqueness theorems and we shall discuss the Picard's iteration method. And then certain numerical methods, like Euler's method and modified Euler's method to arrive at the numerical solution of an initial value problem d y by d x equal to f (x, y), where y x naught is equal to y naught.

Thank you.