Mathematics - III
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Lecture - 6
Euler-Cauchy Equation

Welcome to the Lecture series on differential equations for under graduate students. Today's lecture is on Euler-Cauchy equations. Till now we have learnt how to solve second order linear differential equations with constant coefficients, that is p x and q x were constant not the function of x. Of course we had learnt both the methods, that is for right hand side r x to be 0 and not to be 0, that is both the cases homogeneous equations and non homogeneous equations. Now, we will learn the method to solve linear differential equations, where the coefficients p x and q x are not constants. So, first we will discuss one special kind of equation, which has many engineering applications that is called Euler-Cauchy equation, what is an Euler-Cauchy equation?

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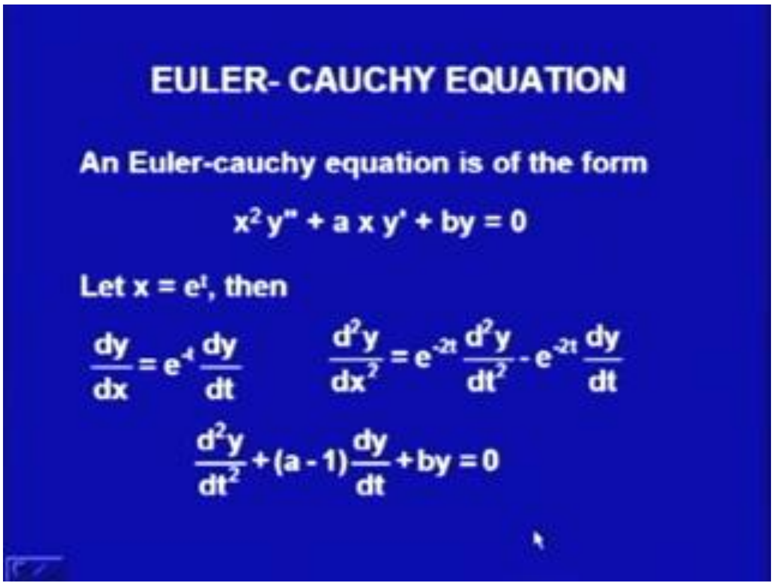

An Euler-Cauchy equation is of the form x square y double dash plus a x y dash plus b y is equal to 0, where the we see that, this is a second order linear differential equation, where the coefficient of y double dash is x square coefficient of y dash is a x and coefficient of y is b; here this a and b they are constants, but the coefficient of y dash that is a x and coefficient of y double dash that is x square, they are not constant but the function of x.

We can change this equation to a linear differential equation with constant coefficient with certain substitutions and that substitution is in the variable x. So, let us see that substitution say x is equal to e to the power t then d y over d x that is y dash would be nothing but, e to the power minus t d y over d t. And the second derivative d 2 y over d x 2 will be e to the power minus 2 t d 2 y over d t 2 minus e to the power minus 2 t d y over d t.

Now if I substitute this d y over d x and d 2 y over d x in the d y over d t and x as e to the power t over here in this equation, we will get this equation as d 2 y over d t 2 plus a minus 1, d y over d t plus by is equal to 0. Now, we see this is a second order linear differential equation, where the coefficients are constants a minus 1 and b both are constants, so now we do know that how to solve this equation and we can get the solution; however, the solution of this equation will be a function of y as a function of t.

This will be, that is y is a function of t while is we want to a solution y as the function of x. So, what we get solution as y as the function of t, we will substitute there t as log x and we will get the solution in the terms of x, there is one alternative method to get.

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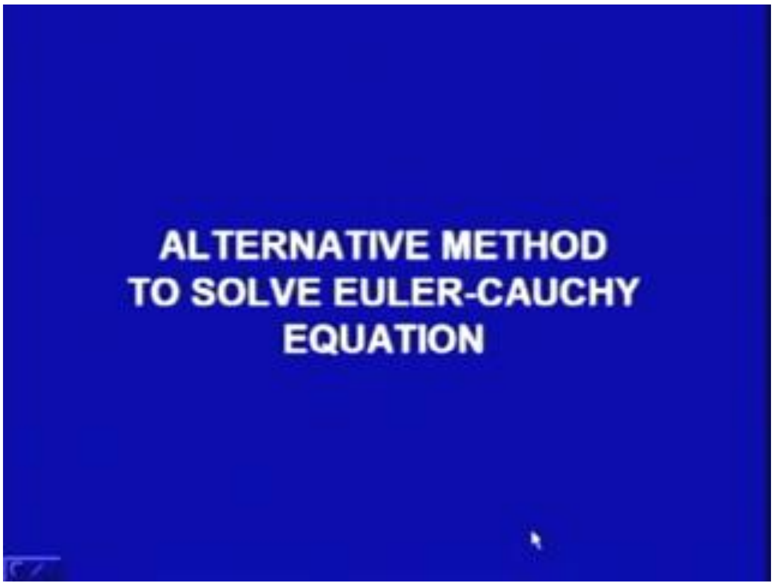

That alternative method to solve Euler-Cauchy equation.

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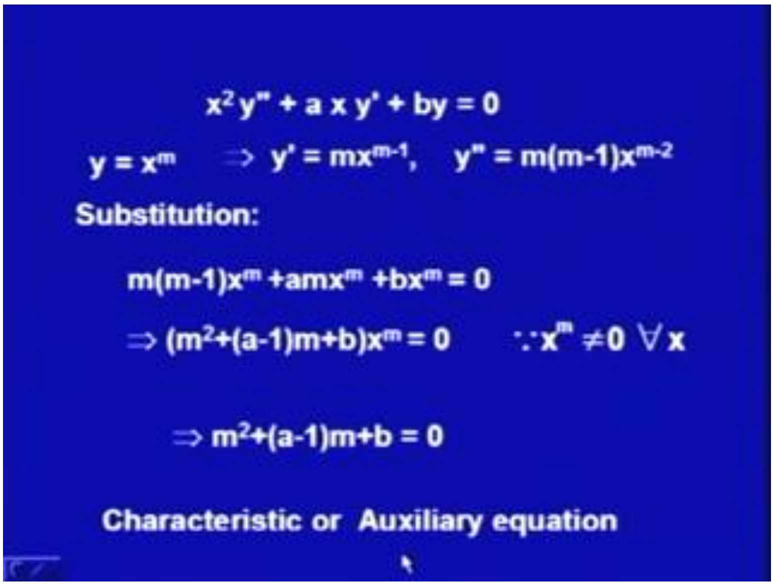

See, my equation is x square y double dash, plus a x y dash, plus b y is equal to 0, you see here that coefficient of y dash is x and coefficient of y double dash is x square, so if I put y as x to the power m, we would be able to we think is that it could satisfy this equation y, we see that y dash would be m times x to the power m minus 1, y double dash would be m into m minus 1 times, x to the power m minus 2.

So, if y is equal to x to the power m this is the solution of this equation, these things must satisfy this equation so, let us substitute it, if we are substituting we would get m into m minus 1, x to the power m, plus a times m x to the power m, plus b times x to the power m is equal to 0. Let us rewrite this equation again as m square plus a minus 1 m plus b times x to the power m is equal to 0, now if this x to the power m is a solution of this equation we want, that is this should not be trivial solution, that is this should not be 0 for all x.

And if this is a solution, it must satisfy this equation that is this must be 0, this will be 0 if either x to the power m is 0 or this m square plus a minus 1 m plus b is equal to 0, this cannot be 0, hence this would be solution only if m square plus a minus 1 m plus b is equal to 0. So we would get that is this would be a solution if it is satisfied, what it says this equation which we have got in m, this is called characteristic or auxiliary equation. Let us compare this method with the previous method there we have change it to the linear differential equation with constant coefficient, in that one what we have got that equation.

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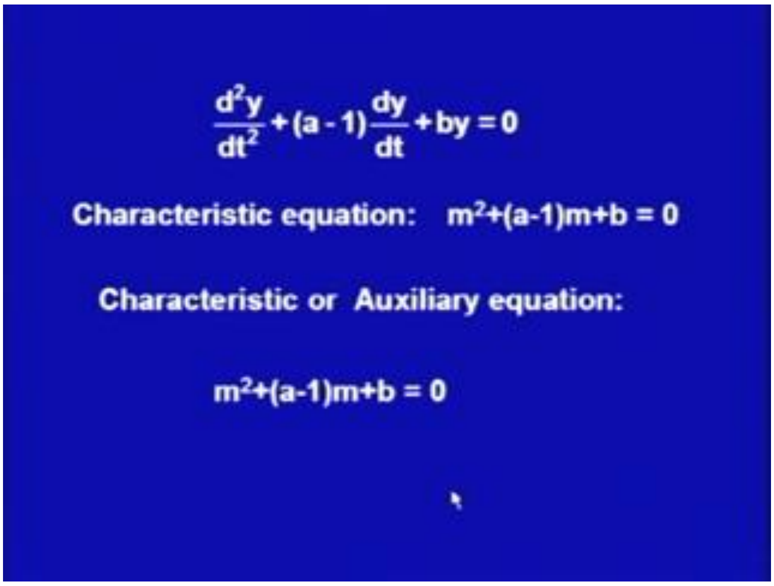

We have got d 2 y over d t 2 plus a minus 1, d y over d t plus b y is equal to 0, to solve this we require the characteristic equation, what is the characteristic equation? Characteristic equation would be m square plus a minus 1 m plus b is equal to 0. Now, let us compare what just now we have got the auxiliary equation, that was also m square plus a minus 1 m plus b. So, we have got that characteristic equation for both these methods, we are getting the same and after that what we will get, we will get the solution as the roots of this one.

So, here if we are getting, we would get the solution as e to the power m t kind of things and here we would get x to the power m kind of thing and here we have to get the solution in the terms of x that is e to the power t, I have to substitute as x. So, again we will get x to the power m and here also as x to the power m, so that says is both methods will give me the same kind of solutions, so see how we get the solution.

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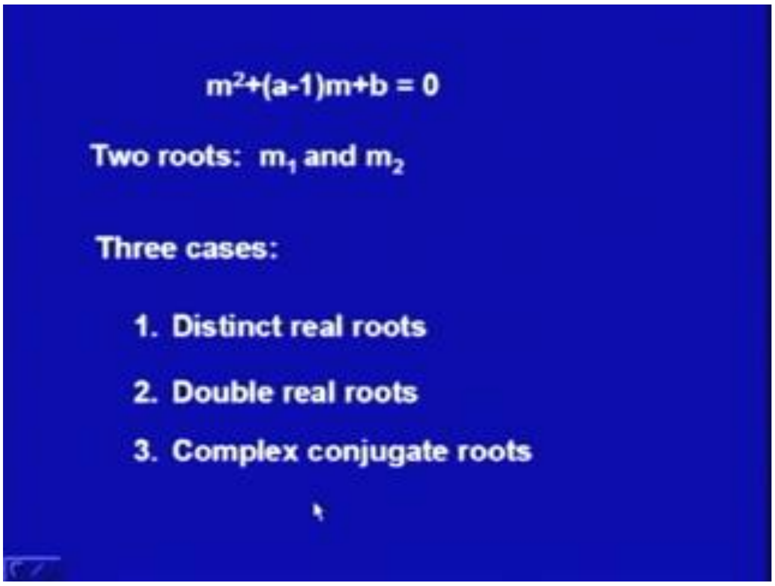

The equation is m square plus a minus 1 m plus b is equal to 0, this is a quadratic equation in m. So, of course it will have two roots of this equation and the solution would be depending on those two roots, and those two roots are depending upon what we are getting the two roots, so they would be 3 cases, the 2roots m 1 and m 2 and we will have 3 cases.

The case 1, when I do have both these m 1 and m 2 they are distinct and real, the 2nd case would be that is when this m 1 and m 2 are real, but they are equal that is double real root and the 3rd situation will have when this both m 1 and m 2 are complex numbers. So of course, they would be in pair that is the complex conjugate roots, let us discuss the general solution in each case one by one.

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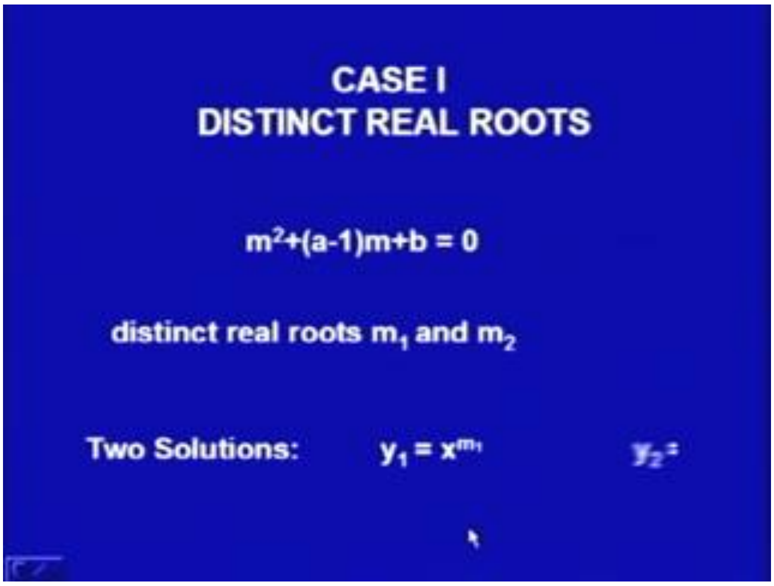

So, the case 1 distinct real roots, when this equation m square plus a minus 1 m plus b is equal to 0, has two distinct real roots m 1 and m 2, then the two solutions we would have as x to the power m 1 and x to the power m 2, these two solutions would be linearly independent.

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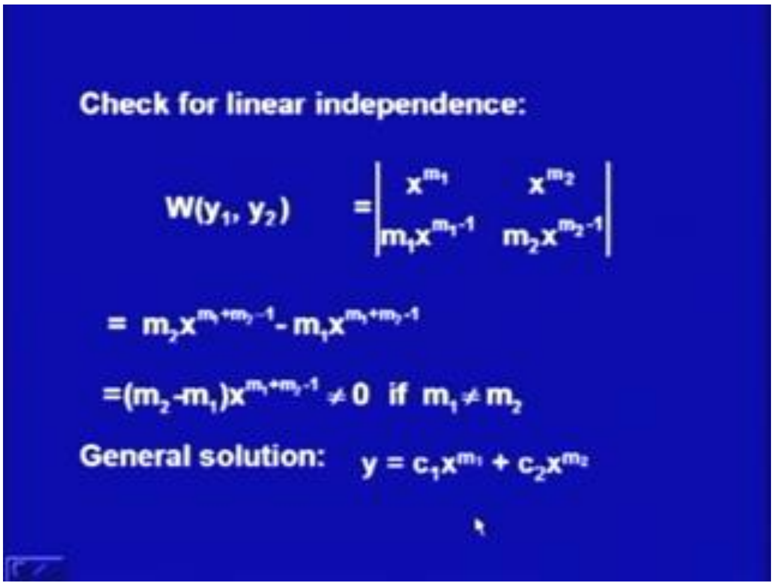

We can check this linear independence using the Wronskian, do you remember the Wronskian, Wronskian of the two solutions y 1 y 2 was given as determinant, containing the solutions y 1 and y 2 and their derivatives that is the determinant x to the power m 1 x to the power m 2 and the second row m 1 x to the power m 1 minus 1 and m 2 times x to the power m 2 minus 1.

What will be the value of this determinant, this would be simply m 2 times x to the power m 1 plus m 2 minus 1, minus m1 times x to the power m 1 plus m 2 minus 1, that is m 2 minus m 1 into x to the power m1 plus m 2 minus 1. Now, this x is not 0 and since m 1 and m 2 are distinct, they are not equal this will not be 0 for m 1 not equal to m 2, that is when they are distinct these two solution x to the power m 1 and x to the power m 2 they would be linearly independent. Hence the general solution would be c 1 x to the power m 1 plus c 2 x to the power m 2 this we have discussed the case 1.

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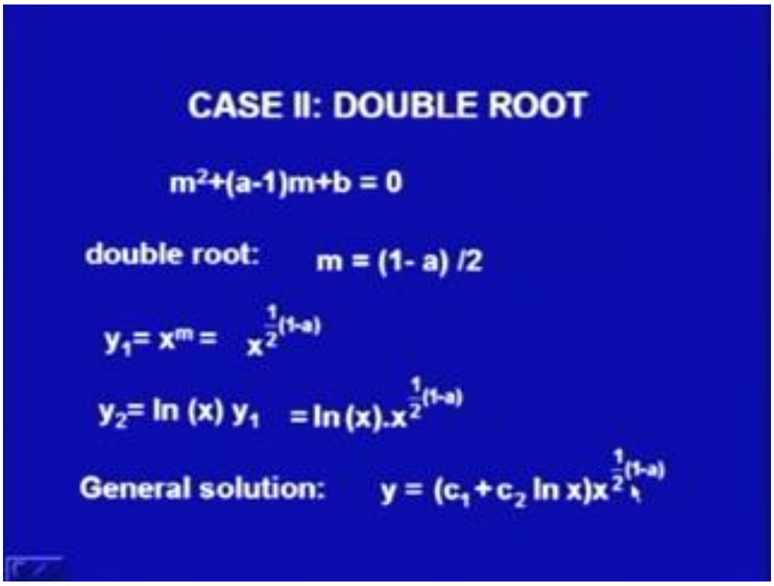

Now, go to the case 2, Double Root that says is that when we do have this equation m square plus a minus 1 m plus b, this has same real root that what will be that root, that root would be 1 minus a by 2, so the one solution we would get as x to the power m as x to the power 1 by 2, 1 minus a. Now to get the second linearly independent solution from here, we will use the technique of variation of parameter that is we will take the second solution y 2 as u times y 1 and then we will find out what is my u. So, by doing that technique we would get actually y 2 as log x times y 1 that is u we have got log x, so the second solution would be log x into x to the power 1 by 2, 1 minus a.

And the general solution would be then c 1 plus c 2 log x into x to the power 1 by 2, 1 minus a. So, when this characteristic equation has double root we do get the general solution is of the form c 1 plus c 2 log x, x into to the power half 1 minus a.

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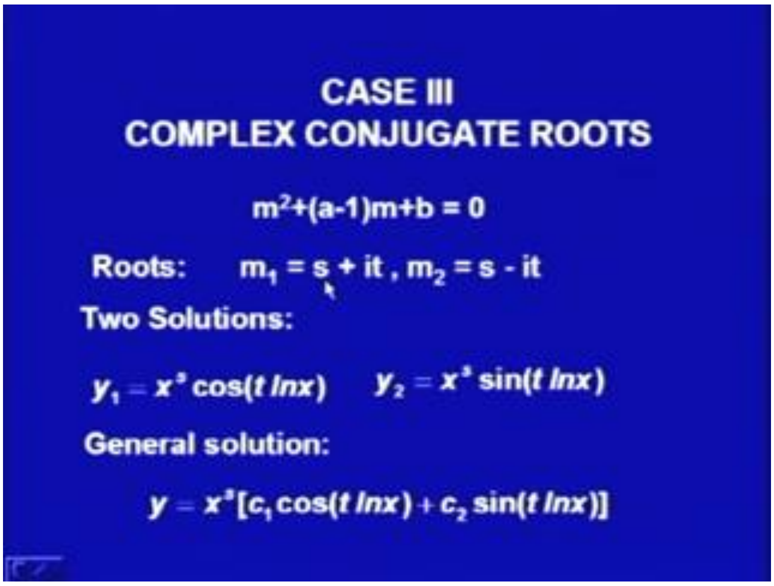

Now, the 3rd case complex conjugate roots, when this equation m square plus a minus 1 m plus b is equal to 0, has complex roots they would appear in the pairs, that is let us say once first root is s plus i t, then the second root would be of the form s minus i t. If these are the roots then the general solution are the two solutions we would get as first as x to the power s cos t log x and the second solution as x to the power s sin t log x.

Again, we can find it out this is the similar kind of thing as we have done with the differential equation with constant coefficients, these will be linearly independent again you can check it using the Wronskian. And the general solution will be of the form x to the power s c 1 cos t log x plus c 2 sin t log x where, this your s is the real part in the complex root and t is the imaginary part in the constant in the conjugate this complex roots. So, we have discussed these 3 cases, let us do some examples to understand this method and how to solve this Euler-Cauchy equations.

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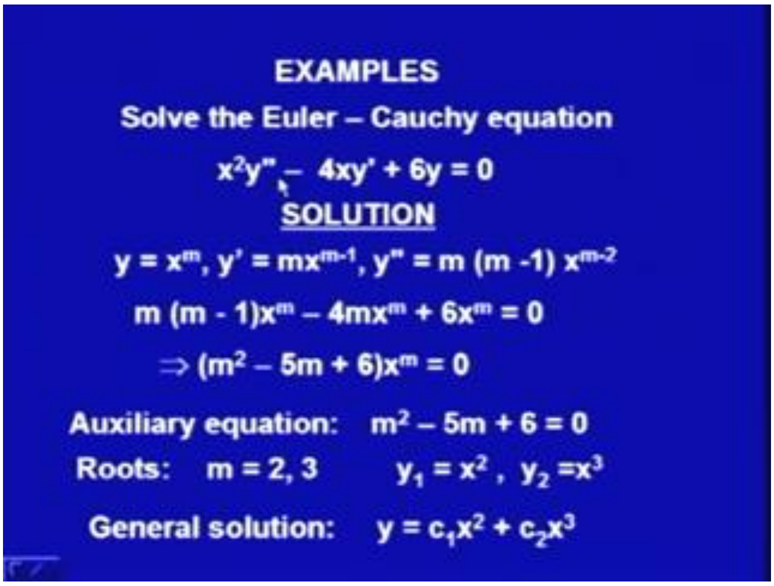

Let us do the 1st example, solve the Euler-Cauchy equation x square y double dash minus 4 x y dash plus 6 y is equal to 0, we will substitute y as function of x to the power m. So, we will substitute y as x to the power m thus y dash would be m times x to the power m minus 1 and y double dash would be m into m minus 1 times x to the power m minus 2.

If we substitute this in this given equation, we get m into m minus 1 times x to the power m, minus 4 m times x to the power m plus 6 times x to the power m is equal to 0. Now, rewriting these terms we get m square minus 5 m plus 6 times x to the power m is equal to 0, so we will get the auxiliary equation or the characteristic equation as m square minus 5 m plus 6 is equal to 0, we say the factors would be m minus 2 into m minus 3, so the roots would be m is equal to 2 and 3 we do have two real roots which are not equal that is real distinct roots.

So, the two solutions would be y 1 as x square, y 2 as x cube, so the general solution would be of the form c 1 x square plus c 2 x cube, this is a solution of this differential equation, x square y double dash minus 4 x y dash plus 6 y is equal to 0, we can check that this would be the solution. So, if let us say the checking this for x square, if I take x square as the solution, what will y double dash that will be 2.

So, I will get here 2 x square and y dash would be 2 x y, I would be getting minus 8 x square plus 6 x square, so that is satisfying this equation, similarly we can check with x cube or we can check with this general solution, let us do one more example.

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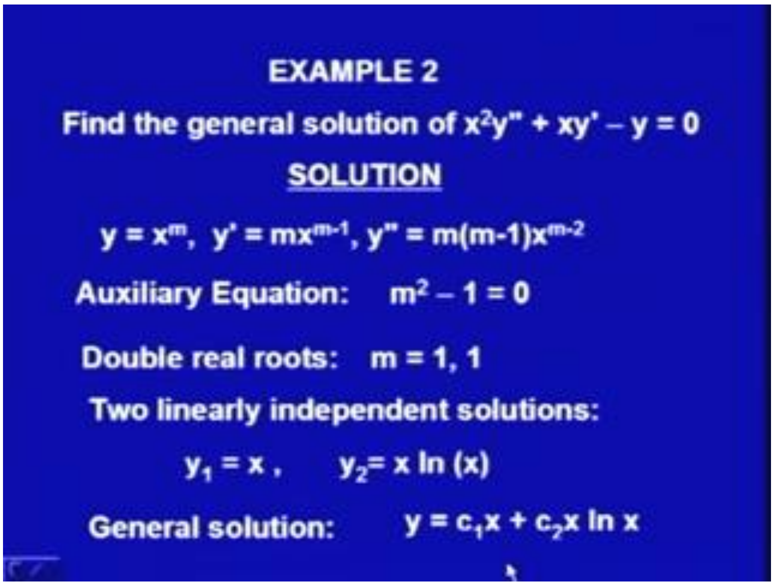

Example 2, find the general solution of x square y double dash plus x y dash minus y is equal to 0, again we will go with the same kind of function that is y is equal to x to the power m y dash as m times x to the m minus 1 y double dash as m into m minus 1 x to the power m minus 2. Substituting this in this equation and finding out the auxiliary equation, we would get from here we could see that is auxiliary equation should be m square into a minus 1 so here a is 1 so a minus 1 would be 0.

So, I would be getting the auxiliary equation as m square minus 1 is equal to 0, of course, it has two roots m as 1 and 1 that is double root m as 1 and 1. Now, this is the 2nd case so, what will be my two solutions the one solution y would be x and another solution would be x times log x, we can again check by putting this solutions that they are solutions and so, the general solution would be c 1 x plus c 2 x log x. Now, let us discuss one more example.

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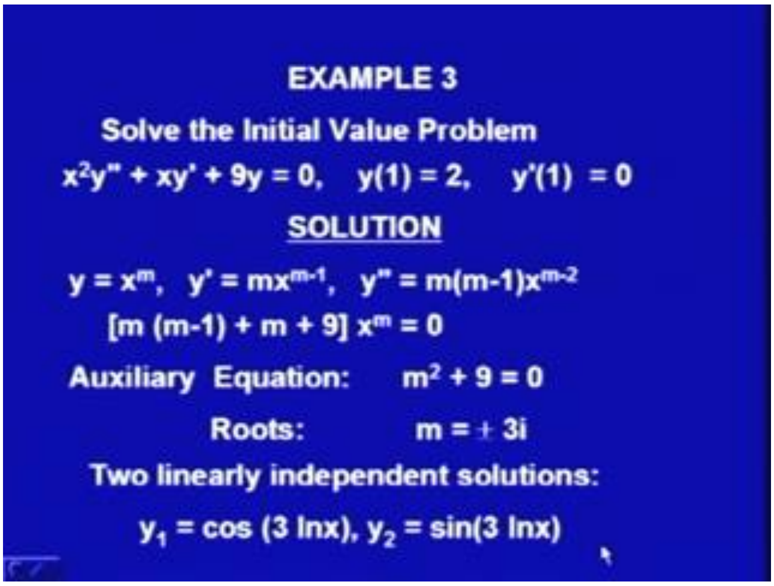

Example 3, solve the initial value problem x square y double dash plus x times y dash plus 9 y is equal to 0, with initial conditions y at 1 is equal to 2 and y dash at 1 is equal to 0. So, first we will solve this equation again by the same method that is y is equal to x to the power m, y dash as m times x to the power m minus 1, y double dash as m times m into minus 1 into x to the power m minus 2.

Substituting this we would get, m into m minus 1 plus m plus 9 is into x to the power m is equal to 0, so we will get the auxiliary equation as m square plus 9 is equal to 0. Now, we see that this will have complex, so that is m square is equal to minus 9, so I would have the roots as plus minus 3 i. Now, if I compare it with our original one that is s plus i t, so here the real part is 0 and the t is 3 and we are having the two roots plus 3 i and minus 3 i, so what will be out two solutions, our two linearly independent solutions would be cos 3 log x and sin 3 log x.

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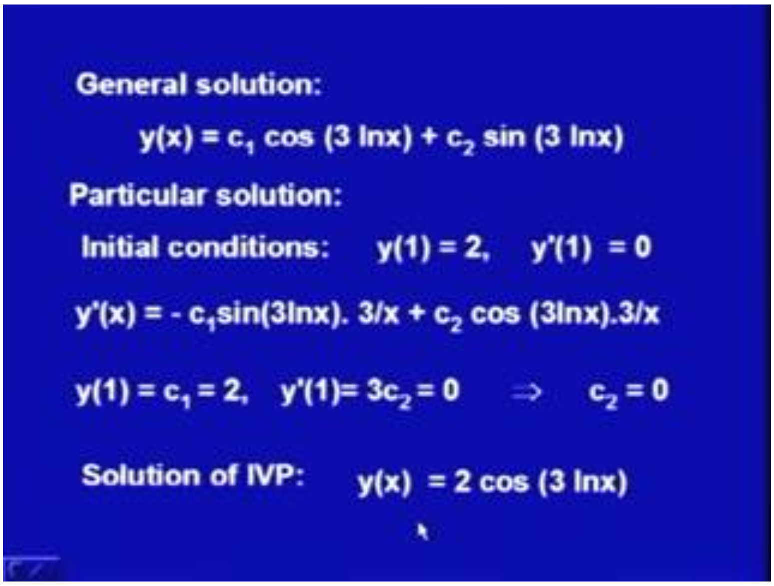

So, the general solution, we would get as y x plus c 1 cos 3 log x plus c 2 times sin 3 log x, to get the particular solution for the initial value problem, what are the initial values? Initial values are given that the function at value one is 2 on the derivative at 1 is 0, so we get what is y dash x from here that would be minus c 1 sin 3 log x into 3 by x plus c 2 cos 3 log x into 3 by x. Now, if the first initial condition is that x is equal to 1, so here if I put x is equal to 1, what I will get log 1 would be 0, so I would be getting here cos 0 and sine 0, cos 0 would be 1 and sine 0 would be 0, so what we would be getting y at 1 we would be getting is c 1 and that is given as equal to 2, so this is what we have got from the first initial condition.

In the second initial condition we do have y dash at 1, y dash at 1 that is here if I put x is equal to 1, we will get this term to be 0, and this term we would be getting as one and this we would be getting as 3, so what we would be getting y dash at 1 as 3 c 2 which is given as to be 0, which says is that c 2 would be 0. So, we have got the two constant values that is c 1 as 2 and c 2 as 0, so what will be my particular solution here we will substitute the values of c 1 and c 2, so the solution of initial value problem would be 2 times cos 3 log x. Now, you can check that you can put this solution in the given equation this will satisfy the equation, moreover this solution will also satisfy these two initial conditions.

So, thus we have discussed the Euler-Cauchy equation of second order, we have learnt that its solution is depending on the roots of its auxiliary or the equation and with they would be arising 3 cases. Now let us do the practical life application of these equations.

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So, we now discuss some application. Linear differential equation with constant coefficients have many engineering applications, here we will discuss one such an application which is...

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The second order linear differential equation in mechanical vibration, here what we are discussing very basic mechanical system, that is called a mass on an elastic spring. This is motion of basic mechanical system, this is mass on an elastic spring and this is called Mass Spring System, what is this system, let us first understand this system.

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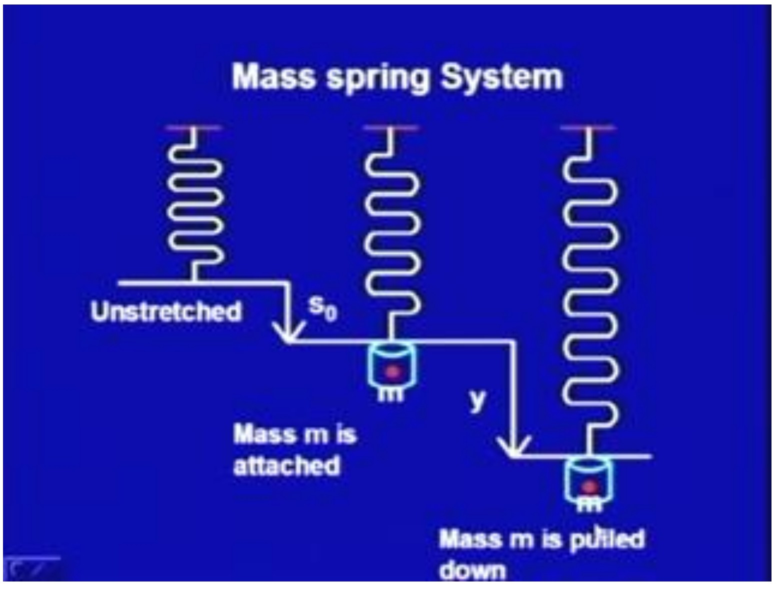

We do have spring this is an Unstretched spring, if I hang a mass of size m, on this one you see this is Unstretched spring, I had this is here on the one side and on this side I have attached a mass of size m, this will stretch this string. So, let us say this stretching is s naught, now what I will do is that is after this is stretching this is at the static one, so if I pull this mass little bit downwards. And then release it, that is I pull this mass downwards if I am pulling it again this string will get more stretched one and then if I release it, what will happen?

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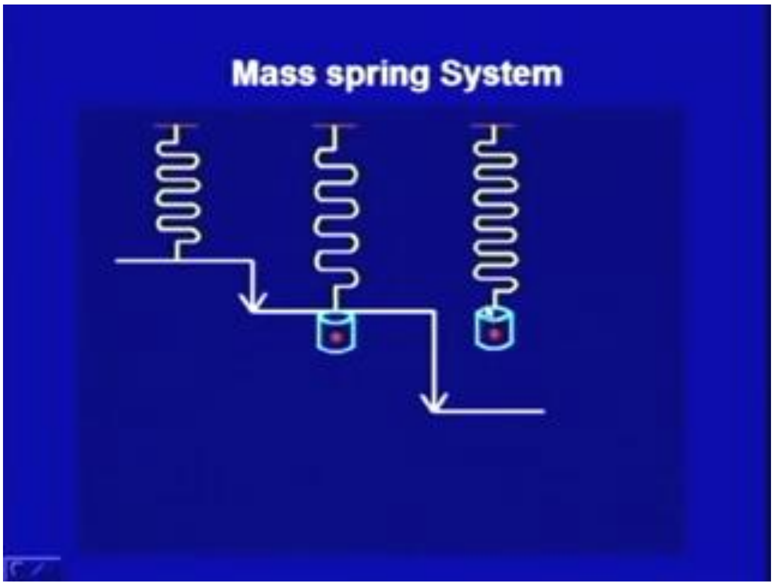

This will go up and then it will go under the motion like this one, that is where we are having is that once I had pulled it down and then left it, It will go in a motion we want to discuss or we want to find out this basic motion of this spring. So, we would see, that is how we are going to model this system, model this system means we will find out what are the differential equations which are governing this motion. Let us see how we are going to do.

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So, we are going to setting up this model for this Mass Spring System.

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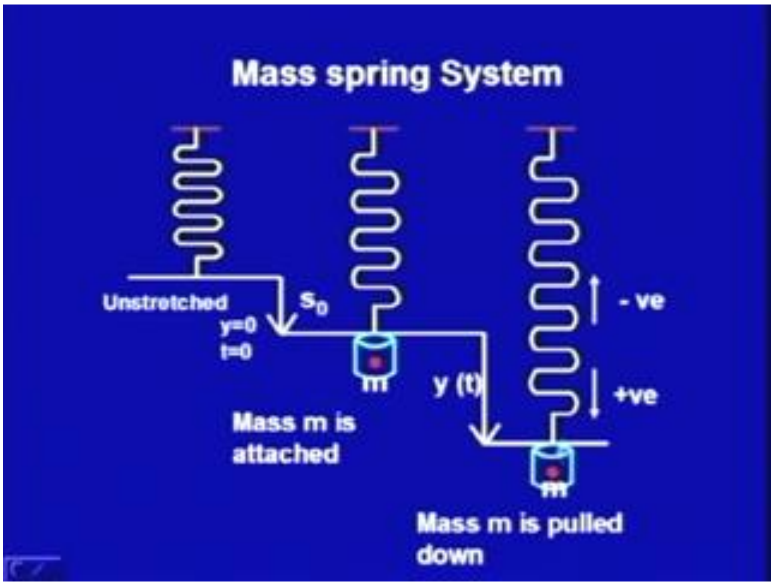

Here we have taken a spring, which would be going to be the whether it will be compressed or it would be stretched it will resist it, that is this is simple spring. It is attached with one hook on the one side and the second side we have put a mass of size m. Now, that will give it is the stretching and this stretching, let say is that is this stretch is s naught, here this is in the this is not in the motion, once I had put this mass over there this will just hang on. This position I am taking as the initial position, this initial position and then this mass been pulled down if it been pulled down, it will again stretch it and then when we are releasing it will go to the motion, so now we will make certain basic assumptions for setting up this motion.

The first setting up is that we take that motion should be strictly vertical, that is we are not talking about this kind of motion, we will talk about only a strictly vertical motion. We are taking that direction towards the down side as positive and direction towards the upside as the negative. This is the static condition that is from here, initially we are starting that is when the mass has been hanged that is this initial stretch of s naught we have taken and then we are pulling it down and then releasing it. So, after time t wherever this thing is, that is suppose it is here, then this is the displacement from the initial condition y is equal to 0 to at y t at time t. Now how this system is being governed let us see this system would be governed Newton's second law.

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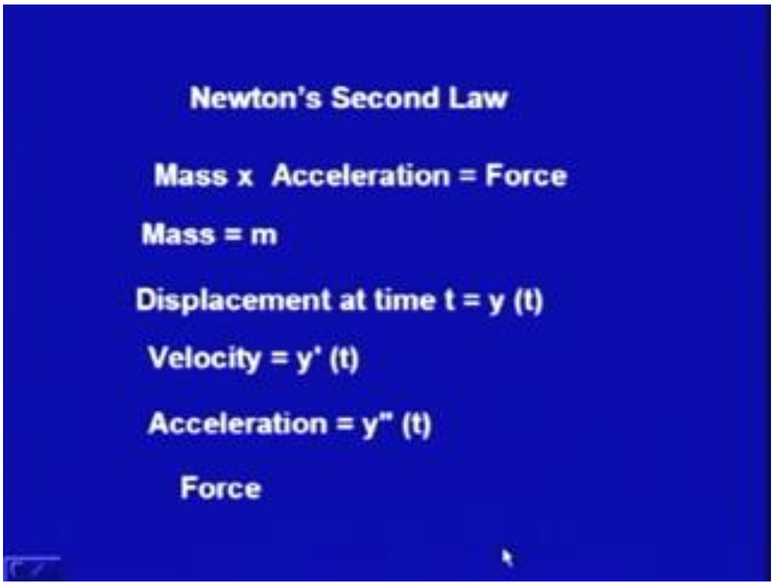

What is the Newton's second law that says is mass into acceleration is equal to force. What is the mass, mass is being given as m. Acceleration since we have seen in the last one that is y is the displacement we do know if y is the displacement at time t is y t, then velocity would be y dash t that is the derivative of y with respect to t at time t that would be the velocity and acceleration would be the second derivative y double dash t. So, we have got the mass we have got the acceleration, now what is the force? Let us see the force, the force which is governing this complete system.

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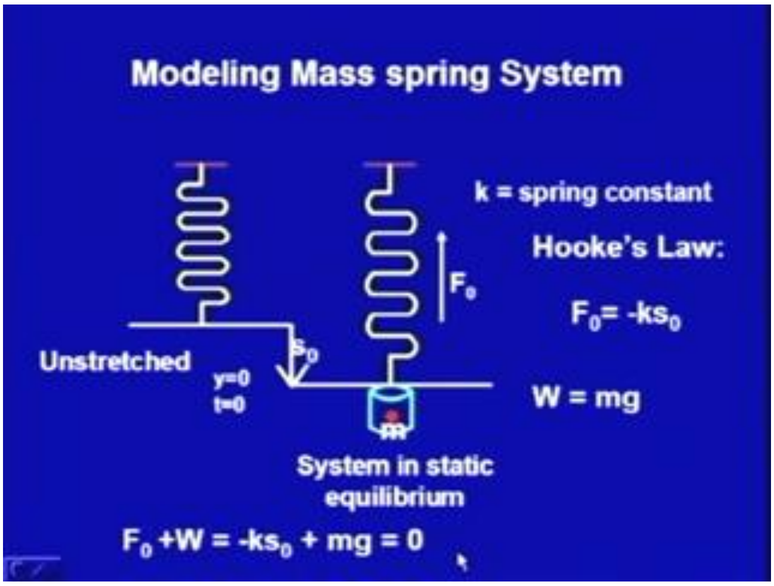

Let us see, when this is Unstretched spring, when a mass m is being attached it has been stretched, when it has been stretched it says is there is a force acting upwards say F naught which is balancing it. So, this force F naught would be under upward side and this is not moving, that is it is balancing the weight of this mass, what will be the weight of this mass.

By the Hooke's law this force F naught would be actually minus k s naught that is this is against that is whatever the stretching this k here is called the spring constant. And we are taking this because this is on the upward side this is balancing the weight, weight is m into g now this system is in the static equilibrium, that it is not moving, that says is we are having is F naught plus w that is minus k s naught plus m g this should be 0, this is not giving any motion, this force is 0.

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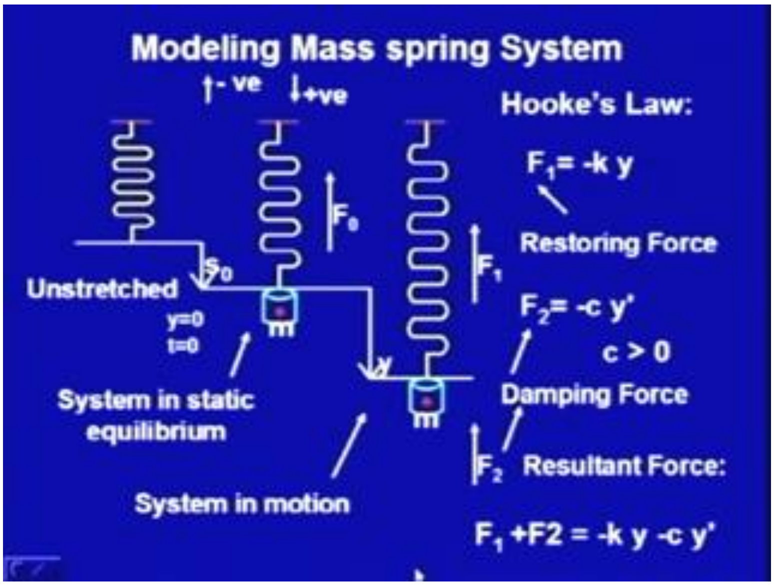

Now, when it goes under the motion that says is when we are pulling it and then releasing it, when we are pulling it one more force that is it is being stretched again and what is this stretching one, this is stretching from this initial condition y we have taken 0, now this stretching extra stretching is of y that will generate another force F 1 and what this would be again by Hooke's law this would be minus k y. Again the sign would be minus, since this force is upwards and the k is the spring constant and y is this stretching at the time t, this is called the Restoring force.

Moreover every system does has a damper, if the damping is not there, that is there should be some force which is governing this motion then this motion will go on forever we will learn this little bit after. Let us say that damping force is F 2 this will again be in the upward direction, this damping force is going to govern the velocity, so that it would reduce it. So, a very good approximation of this F 2 is minus c y dash that is y dash that is the velocity and minus c, this c this is called the damping constant and again the sign is minus since this would force will also act upon on the upward side.

So, now the resultant force what would be this c we are also taking as positive this k is also positive, the resultant force would be F 1 plus F 2, so the total force would be minus k y minus c y dash, now we are ready to get the model of this system.

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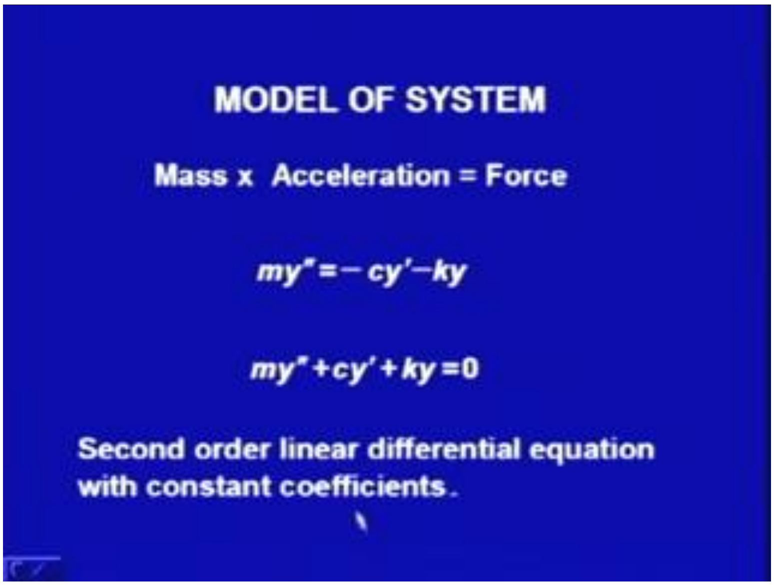

That is by second law of Newton's second law, Mass into Acceleration is equal to Force mass is m, acceleration is y double dash, the force we have find it out is minus cy dash minus k y, that is we have got a differential equation m y double dash plus c y dash plus k y is equal to 0. This is a second order linear differential equations with constant coefficients since all these are constants.

So, this is governing this mass spring system this equation and this is a homogeneous one. So, we just say is that is what we do get how to get the solution of this and how this motion we are trying to understand. So, first we will discuss the case of Undamped system.

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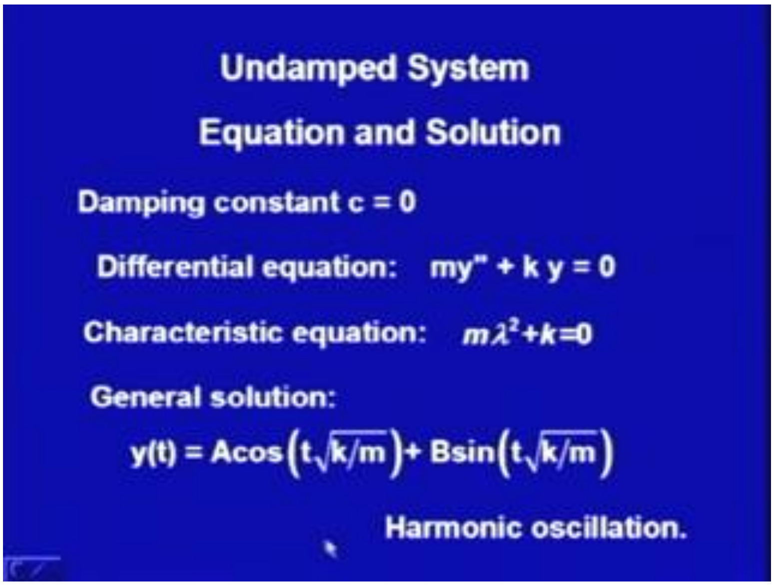

Undamped system that is there is no damping is present, that is the damping constant c would be 0, that is what we would get our equation, our equation would be ((Refer Time: 27:54)) m y double dash plus k y is equal to 0, that is the term involving c y dash since c is 0, that is not present there ,now here m and k both are positive. So, we would get what will be our characteristic equation for this linear differential equation, that is m lambda square plus k is equal to 0.

Its root would be of course, since both k and m are positive its root would be complex one and those roots would be minus square root of k by m i plus and minus, so the general solution would be of the form, A cos t times square root k by m plus b times sin t times square root of k by m, t here I am taking is the time t that is I am having the equation in y and t. This is the general solution. Now, let us see that is what this general solution is let us see in the graph, this is called Harmonic Oscillation why?

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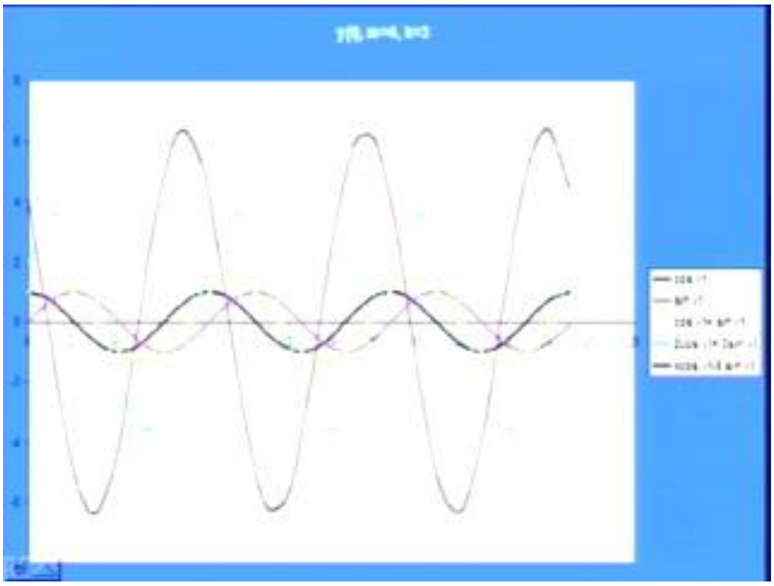

You see here, we are having is that is for different values of a and b, so let us say a is equal to 1 and b is equal to 0, for here I have taken m as 4 and k as 3 that is the spring constant as 3 and mass as 4. Then what we are having is that is this motion first this black line, you see this is just going like this one, now you see what we are having is its starting the motion is being started that is displacement is going on, it goes up at a distance then it goes down with the same distance, then it goes up again with the same distance.

And you see the time it is taking between the cycles that is from here to here, here to here, here to here the cycle is also same. And it is also taking that the displacement that is the distance it is covering, the displacement it is making that is also equal, that is what I said is when we are not having a damper, the motion will go on forever, so this will go on forever with equal magnitude.

Similarly, if I take my a to be 0 and b is to be 1, then you see is that again I am getting this harmonic motion this pink kind of line, again you are having is that the cycle is same the magnitude is also same whether the upwards or the downwards and this will go on. Similarly, if at the different values if a and b, I have taken as 1 we are getting is that is the magnitude is differing here and but cycle is different cycle different magnitude, but it is going on.

Similarly, with other values as a as 2 and b as 3 we are getting it again, that is the magnitude is getting hard one and the cycle is again in the similar manner, smaller one. And then if I had a as 4 and b as minus 5, we have got that the magnitude is again little bit higher and the cycle is something little bit smaller, and we are getting, that this will go on this is never going to down that is the motion will go on forever. So, let us see another example over here

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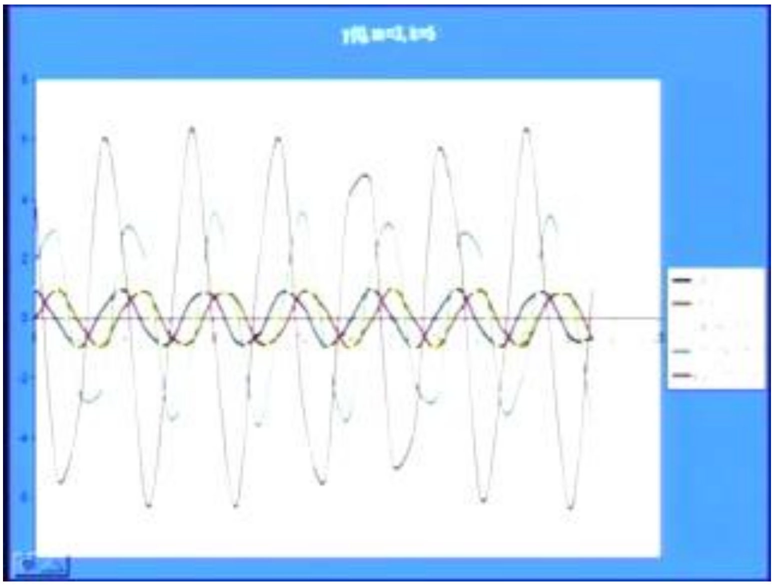

Here I have changed that mass to be 3 and spring constant k is 5, that is its more resistant string, again you see the similar kind of pattern we are having, again I have taken this a b as the similar kind of thing this is a is equal to 1 b is equal to 0; that is this first one. We are getting is this, this motion then if a is equal to 0 and b is equal to 1 we are getting this motion and so on we are having again we are having is this motion is going on with the equal cycle in one motion and with the equal magnitude whether upwards or downwards. That is what we are calling it as harmonic oscillation and without damping this motion will go on forever, forever time.

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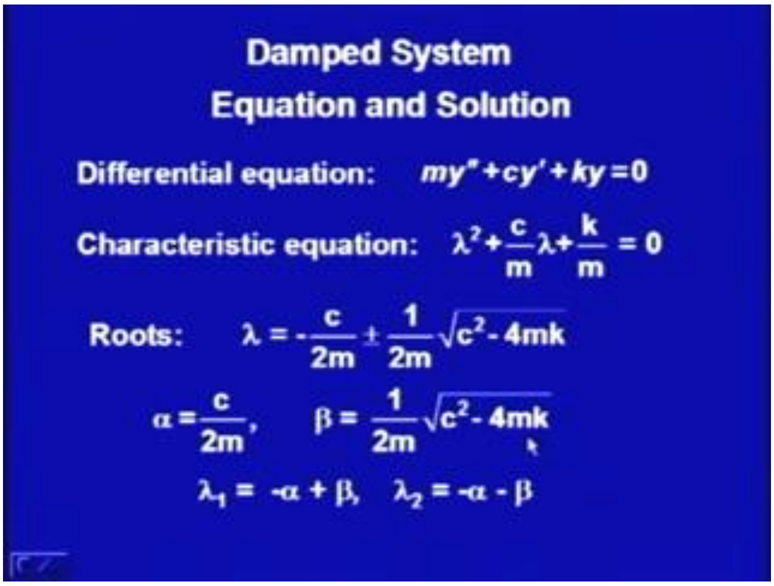

Let us see the damped system, that is when the damping is present, so first we see what is the equation and the solution and then will discuss this motion. So, the differential equation would be m y double dash plus cy dash plus k y is equal to 0, here this m is the mass, c is the damping constant and k is the spring constant. This equation will have the characteristic equation as lambda square plus c by m lambda plus k by m is equal to 0, since this is a quadratic equation it will have two roots lambda 1 and lambda 2.

And those roots would be of the form minus c by 2 m plus minus 1 by 2 m square root c square minus 4 m k, that is we will have one root as minus c by 2 m plus 1 upon 2 m square root of c square minus 4 m k, another root we would have minus c by 2 m minus 1 by 2 m square root of c square minus 4 mk. Let us rewrite it as in the simplified manner, let us say this first thing as the alpha c by 2 m and the second part that is beta as 1 upon 2 m square root of c square minus m k, then we would have the two roots, lambda 1 as minus alpha plus beta and lambda 2 as minus alpha minus beta.

Now, you see in the beta we are having this term square root of c square minus 4 m k, that is we are finding out square root of some quantity where the c m and k all they are constants. So, of course we will have our roots will be depending upon what is this inside this square root that is c square minus 4 m k, whether it is positive quantity, its 0 quantity or its negative quantity, accordingly we would have different roots. So, we will have actually with 3 cases.

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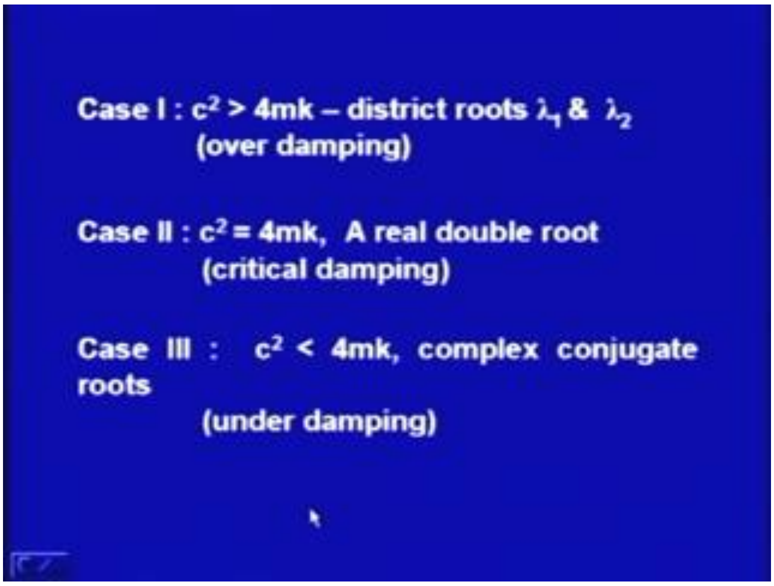

The case 1 is, when c square is greater than 4 m k, in that case we will have two different root, distinct real roots lambda 1 and lambda 2 this case is called over damping. Then 2nd case when c square is equal to 4 m k will have double real roots, this is called critical damping. We will learn all these terms why we are calling them and the 3rd case when c square is less than 4 m k, that is its imaginary part beta would be imaginary and that is complex conjugate roots this we would call under damping. Let us discuss the solution in each case one by one.

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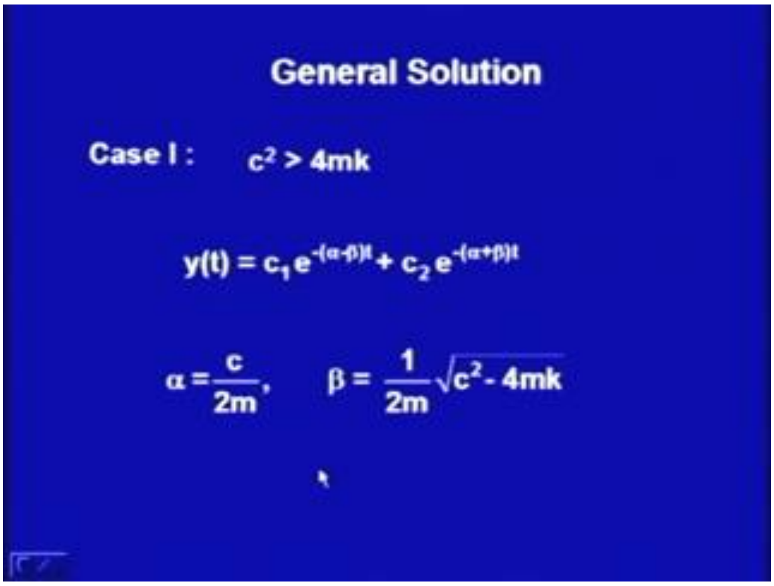

So, general solution in case 1, case 1 is when c square is greater than 4 m k that is we would have 2 roots as minus alpha minus beta and minus alpha plus beta, so the general solution we would have y t as c 1 e to the power minus alpha minus beta t plus c 2 e to the power minus alpha plus beta t. Here if I, this alpha is we do know is that c by 2 m and beta is 1 by 2 m c square minus 4 m k. Where m is the mass, c is the damping coefficient and k is the spring constant. Now, for different values of c, m and k we will get, so that the c square minus 4 m k is positive we will get the different roots and if I do give some initial conditions, so that we get some particular solution, so will discuss this case one by one.

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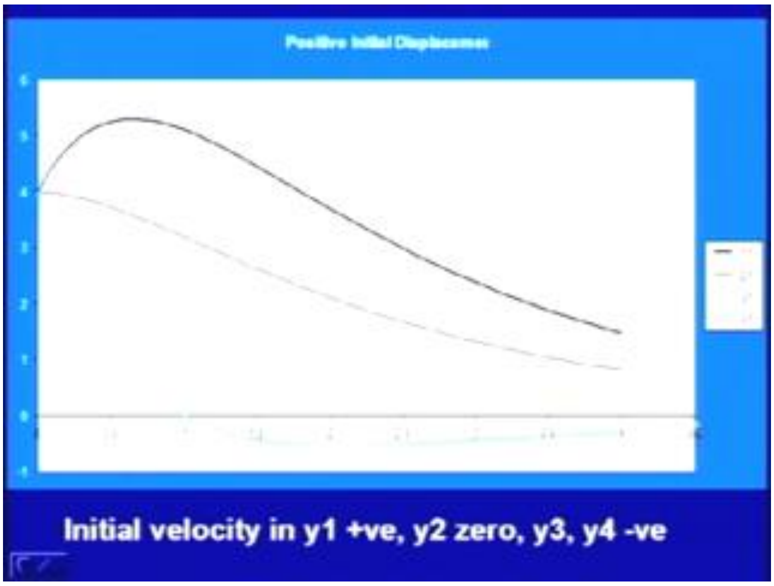

Here what I have taken, I have taken some values of c m and k such that c square minus 4 m k is positive and initial condition what I have taken is that is y at 0 is 4, that is it is positive initial displacement. Then we do have the another initial condition that would be governing y dash, that is y dash whether it is I have defined some values and I am classifying them into 3 parts that is my velocity, initial velocity y dash is positive, negative or zero.

Let us see first case y 1, this y 1 we are having is here as positive velocity, initial that is y dash one positive, I have taken some value so in that case how this motion will go on. Initially it has been displaced that is pulled downwards 4 at 4, then it has been released, it has generate it and with the damping, it was generated a velocity positive velocity, so we are getting it as that is its going little bit up and then it is dying down because of damping.

If initial velocity is 0, then it is started dying down from the initially it is not gone up. When the initial velocity is negative, that is initial velocity is upwards, then we are having we would have the two cases, one is that is it is going down like this one and then it is dying down to the displacement is coming to be 0 or it may happen that it comes to 0 and then it goes to the negative one, that is we are having is this motion upwards, but again it will come up and it will go to the 0. So, if the damping is present ultimately we will have that our motion would stop and after sometime we would go to the 0 displacement.

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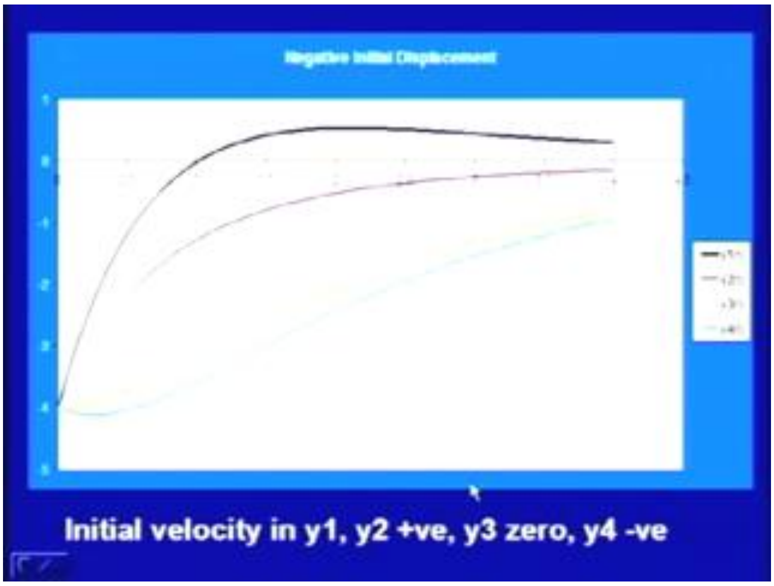

Now, if I take initial displacement to be the negative, that is if 0 is negative, so again I have taken a case when y 0 is minus 4. Again with the second initial condition about the velocity again I have divided positive negative and 0. Now, if the displacement is negative and my velocity is also negative, that is the last case y 4 this case. So, you see is that is it is first going down little bit and then coming up and it will ultimately reach to the 0, that is the displacement would be 0. If the initial velocity is 0, it is starting with the negative displacement and the displacement is going up that is its reducing and it will certainly reach to somewhere 0.

If initial velocity is positive, we could have again, the two cases one case is that is it has started little bit, that is the displacement has gone very up like this one, that is reduced certainly and then the reduction power is little bit less and then we are moving over here. We may have this in this positive case, that is it may go to the 0 and then it cross this line then it goes the to the positive displacement and then it is dying down. So, again we have got that, when damping is present we will not have this harmonic kind of oscillation that the motion goes on forever, rather we would be having is that after sometime the motion will die down and there displacement could be 0.

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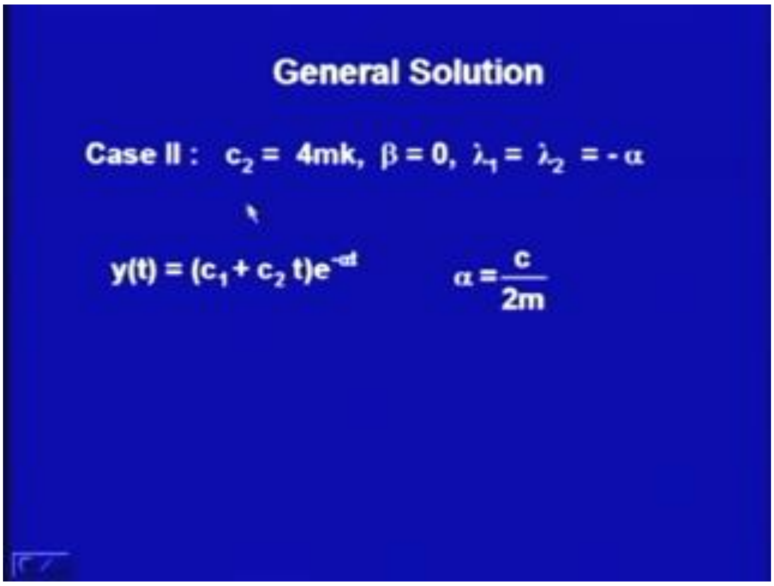

Now, let us discuss the 2nd case, that case 2 where c 2 is equal to 4 m k, that is beta would be 0 and we will have double root that is both lambda 1 and lambda 2, they are equal to minus alpha, in this case we do know that the solution would be of the form c 1 plus c 2 t times e to the power minus alpha t, alpha is of course our c by 2 m, in this case if I take some values of c and m such that c square is 4 m k.

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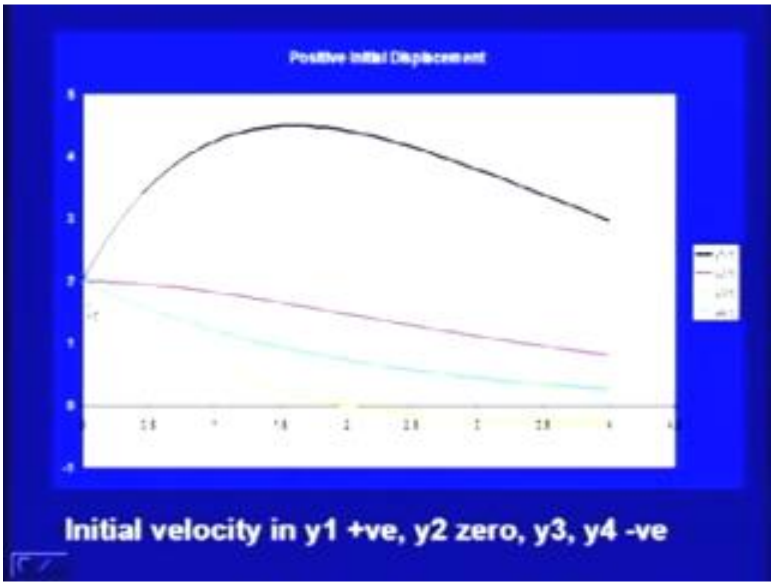

Again I have taken initial condition that my initial displacement y is positive and that is 2, again with the second initial condition about the velocity we have divided it into positive, negative and zero. So, when the displacement is positive and velocity is also positive, this is the second case where I am having c square is equal to 4 m k, that is we are having double real root. Again it is going little bit up and then it is coming down, so of course, after some time it will come down, when initial velocity is 0, it is starting from here it is going down very slowly, but it is going down, then when the initial velocity is negative it is going down, first it goes down very fast, then it is going down slowly and it will go to the 0.

It may have another case where it is going down to the 0 and then it is going down to the negative displacement, then it will come up and it will go to the 0, that is again in this case also the motion will die down, that is the displacement would be 0, after sometimes the motion will stop.

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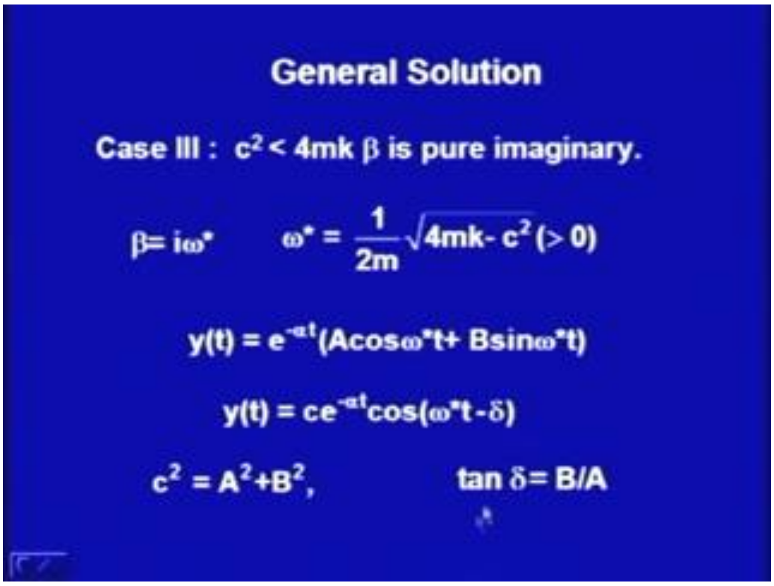

Let us discuss the third case, case 3 is when c square is less than 4 m k, that is the beta is pure imaginary. Let us say beta is i omega star, where this omega star is one upon 2 m square root of 4 m k minus c square that is this thing is positive, so omega star would be positive. In this case, the general solution will be of the form e to the power minus alpha t A cos omega star t plus B sin omega star t. By the trigonometry, we do know that this thing we can rewrite as c times e to the power minus alpha t cos of omega star t minus delta. Where this c is such that c square is a square plus A square and this delta is such that tan of delta is B upon A.

Now, we see this is the general solution, this is telling us the displacement at time t, this is having e to the power minus alpha t and cos of something, now the cos of any this cos function we do know this lies between minus 1 and plus 1. So, what will have this motion will lie between minus c e to the power minus alpha t and plus c e to the power minus alpha t. Let us see what kind this motion would be look like.

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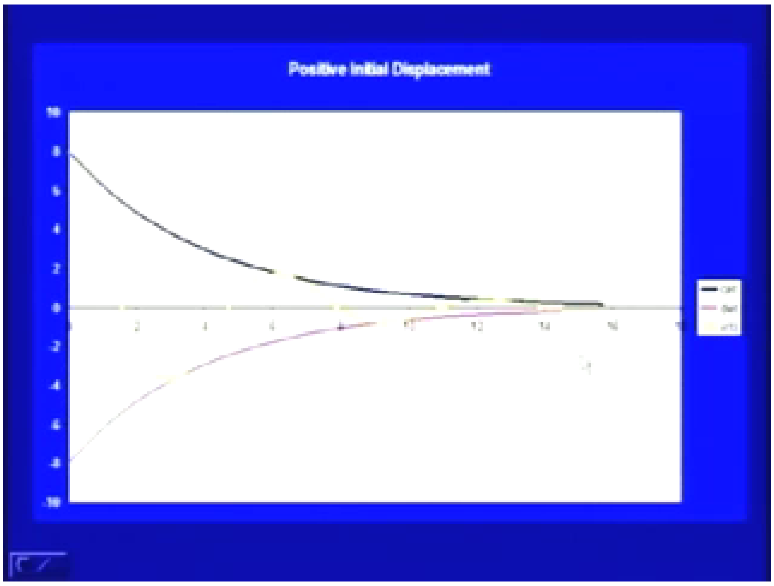

You see here, this is the line c e to the power minus alpha t and this is the line minus c e to the power minus alpha t, again I have taken this some c and m such that c square is less than 4 m k, you see this motion is actually this one, this is going first it is starting with here initially and then it is going down. Then it is coming up and like this you see each time the magnitude of the motion, you are getting is getting lesser and lesser and afterwards it is going to be die down its coming to 0.

So, this is what is the motion, when we do have the under damping, you see is that is when we had over damping we had got the motions which are just going down, when we had critical damping again the motions were just down. And here what we are having is that motion is going down and up and down and up and then they are dying down, that is they are stopping. So this is what is called the under damping.

So, we have discussed the 3 cases of this differential equation, which is governing this motion of basic mechanical system, that is called the mass spring system. Now, it may happen that is this motion is being governed by some force, that is we are calling Forced Oscillation or Resonance.

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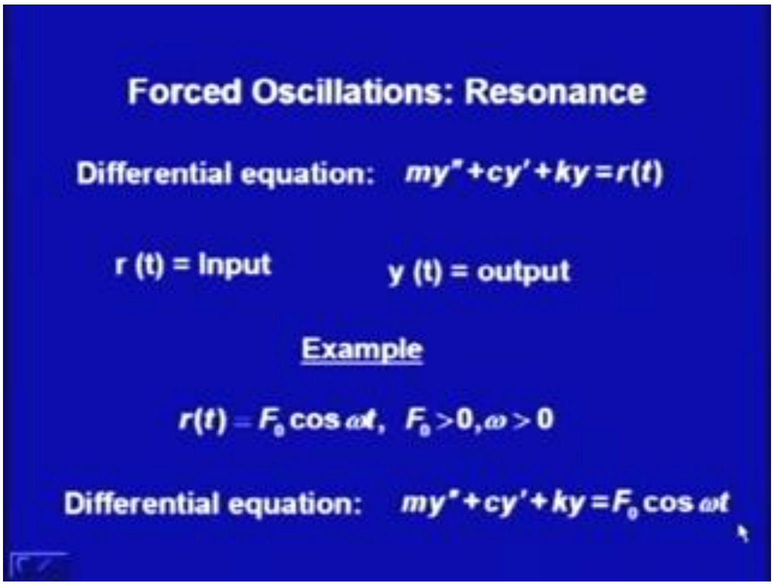

What is this, this says is that this motion is been governed by a force or force is affecting the motion that means, till now what I was having that is this was the equation of the motion which was equal to 0, that is this was balanced by balanced equation. Now, what we are having is that this motion is governed by one force called r t, that is now we are having non homogeneous equation. m y double dash plus cy dash plus k y is equal to r t.

Now, what is r t which is governing this motion, this r t is called input and what will be the solution of this equation y, that would be called output. This is we are giving, that is sometimes we says is that, this is what is the input and or this what the force is being applied, and this output we are calling the response to this force which is governing this motion, so this is also called the response.

For example, let us take this my r t, that is the force, which is governing the motion is of the form F naught cos omega t, this omega is something different. So, F naught is a constant and the force is of this form, you see I have taken this special form that is cosine function, if you do remember we have to solve this non homogeneous equation, where this both F naught and omega are positive.

Then what will have the equation our differential equation would be m y double dash plus cy dash plus k y is equal to F naught cos omega t. Now, this is a non homogeneous equation, it has solution, which is having the solution of homogeneous equation and at particular solution, which is governed with this right hand side, that is called the particular solution and then the general solution we do have as the homogeneous solution plus the particular solution.

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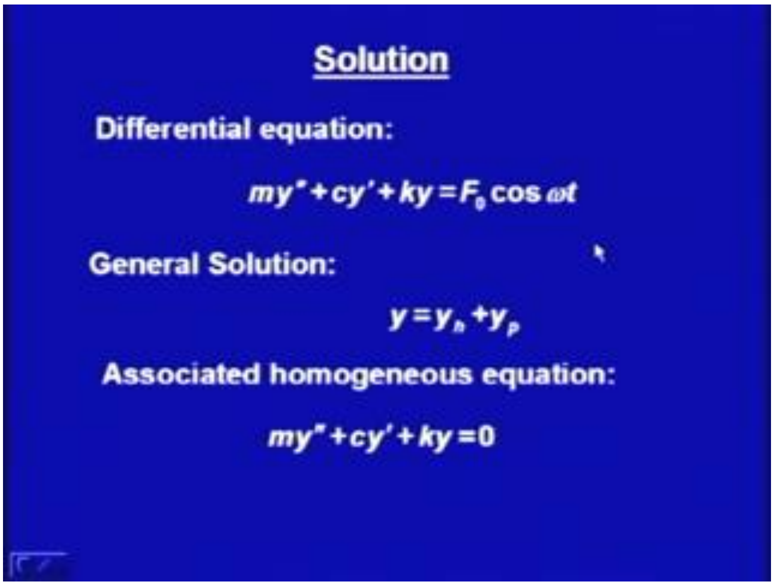

So, we do have this differential equation, its general solution is of the form y h plus y p, where y h is the general solution of associated homogeneous equation m y double dash plus cy dash plus k y is equal to 0 and y p is the particular solution, now we will concentrate on this y p. If you do see, that is what we are having here is that our r t is of the form F naught cos omega t, that is we are having a special form. So, we will use this method of undetermined coefficient, do you remember that method we are having one table, which is telling us what is the choice of the particular solution y p, that table is this one.

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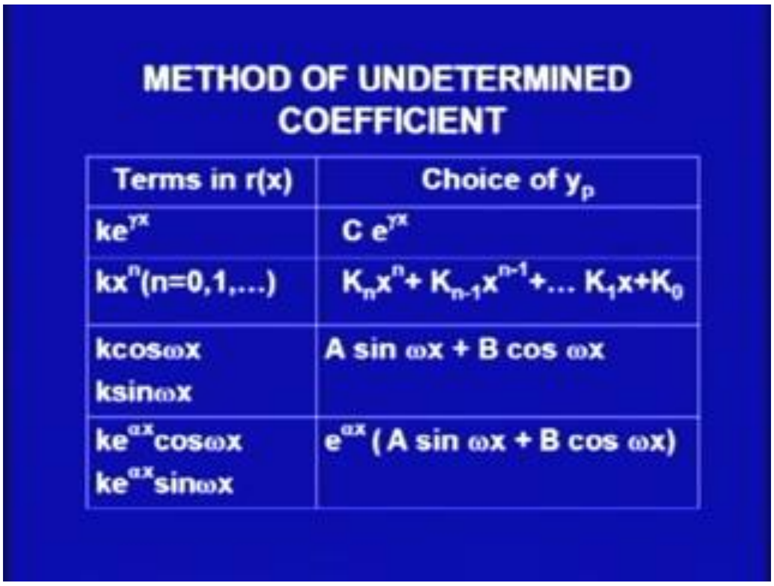

So here what we are having is, we are having a constant times cos omega x. So, my choice of particular solution should be of A times sin omega x plus B times cos omega x.

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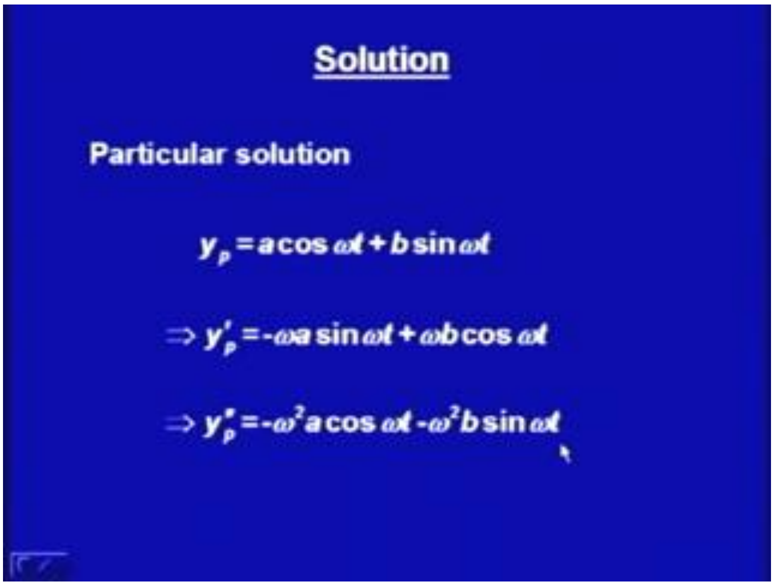

Let us so, we will have our particular solution as of the form a cos omega t plus sin b sin omega t, if this satisfies the equation then it will be a solution. So, we would have to find out y p dash as minus omega a sin omega t plus omega b cos omega t and y p double dash would be minus omega square a cos omega t minus omega square b sin omega t. We will substitute this in the given equation.

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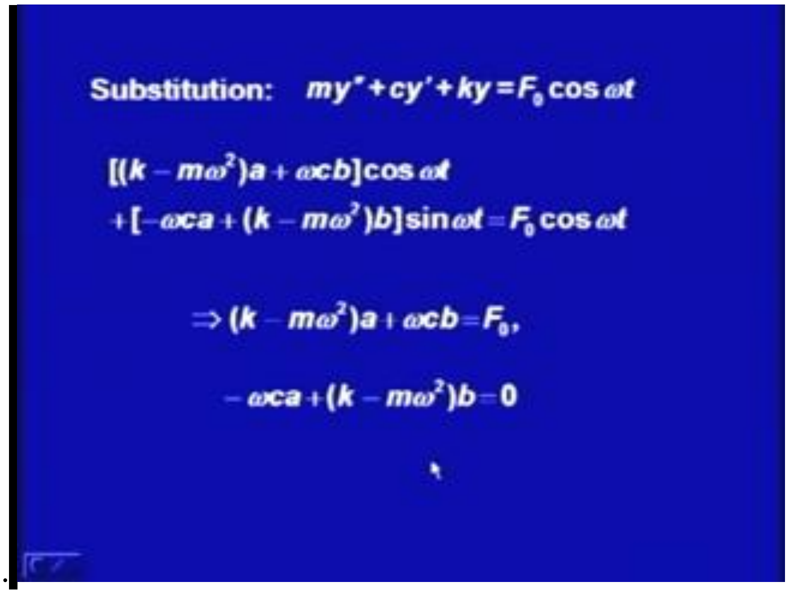

The equation is m y double dash plus cy dash plus k y is equal to F naught cos omega t, when I am substituting those y p, y p dash and y p double dash, we would get the equation of the form, k minus m omega square a plus omega c b and this whole is multiple of cos omega t. And then we do have a sin omega t and its coefficient as minus omega ca plus k times, plus k minus m omega square times b, which is equal to F naught cos omega t. To determine this a and b we would equate this coefficients, so what it gives is that is the coefficient of cos omega t on the right side is F naught and on the left side is this one.

So, we get the first equation as k minus m omega square a plus omega c b is equal to F naught. And we do not have any term of sin omega t on the right side, so we will get this second equation as minus omega ca plus k minus m omega square b is equal to 0. These are the two algebraic equations in two unknowns a and b, so we can solve it and we get the solutions, so we will get what is the particular solution, homogeneous equation will give me the solution of y h and we will get the general solution.

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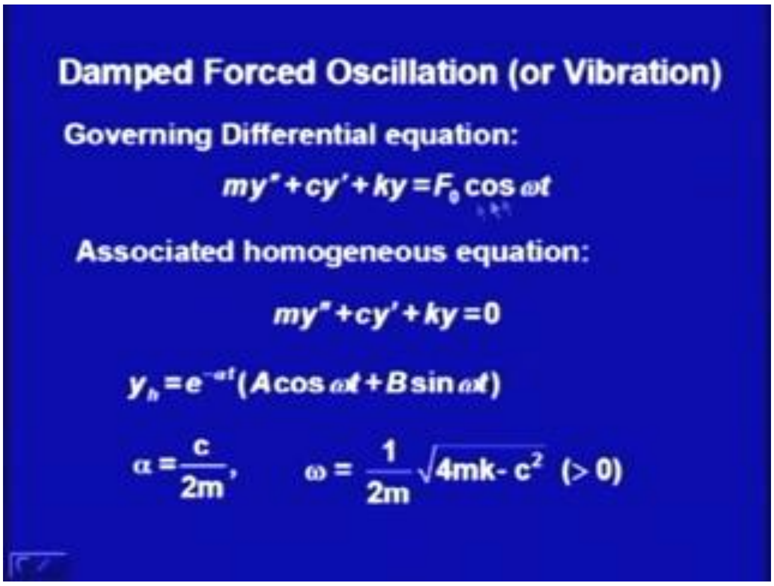

Now, we will discuss one special case, rather which is called the Damped Forced Oscillation or which is more generally known in the mechanical engineering as vibration. What is this one, this governing differential equation would be my double dash plus cy dash plus k y is equal to F naught cos omega t, I am just discussing that this is special one that is F naught cos omega t. Associated homogeneous equation would be my double dash plus cy dash plus k y is equal to 0. Its solution let us say, suppose that this equation has the complex roots, that is its characteristic equation has complex roots, then the solution would be e to the power minus alpha t a cos omega t plus b sin omega t, where this alpha would be c by 2 m and omega would be 1 upon 2 m square root of 4 m k minus c square. Now, this is what we are writing.

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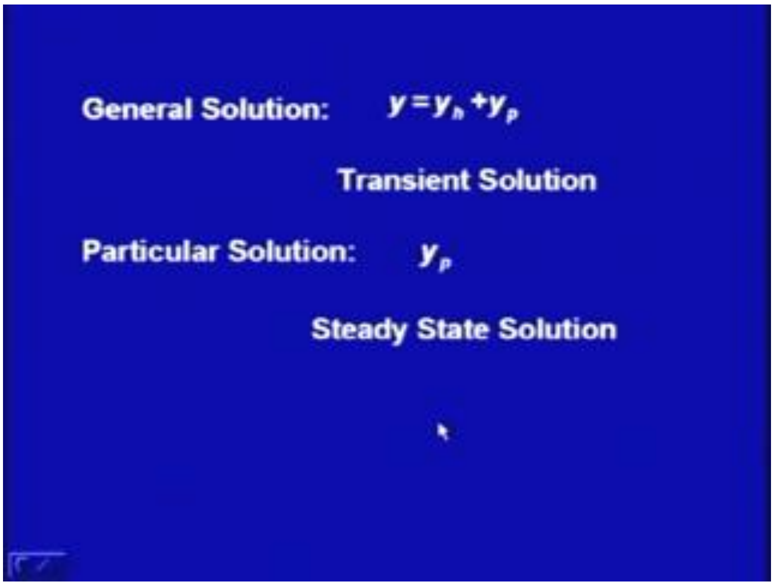

And then the Particular solution, we would be general solution would be of the form y h plus y p, this general solution is also known in this vibration case as the transient solution, and this particular solution y p this is known as steady state solution. Why this is called transient solution, and this is a steady state solution, this we will learn a little later on and we would actually learn these things with the help of an example.

This example we will do in the next lecture, where we would really model a vibration model and we will see that is how the solution is coming at, how this vibration is taking place and why we call one thing as transient solution, this general solution and this particular solution as a steady state solution. How this is the response to the input function, all these things we will see in the help of example and that example we will go with the next lecture.

Today, we have learnt a special kind of equation called Euler-Cauchy equations, this is second order linear differential equations where the coefficients were not constant, but the function of x. We had learnt homogeneous Cauchy-Euler equations, how to solve them. We had learnt some application of linear differential equation with constant coefficient. We had learnt one special application in the simple mechanical system called the mass spring system.

And we had learnt, when thus there is no force is governing the motion, both the cases damp is that when there is no damping and when there is damping is present. So, we have seen the damped oscillations and we have seen the un damped motion that is called harmonic oscillation. Now, next lecture we will learn one more system mechanical vibration.

Thank you.