Mathematics - III
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Lecture - 5
Linear Differential Equation of Second Order – Part - 2

Welcome to lecture series on differential equations for under graduate students. Today's lecture is in continuation of Linear Differential Equations of Second Order. In last lecture, we had learnt about these equations, moreover we have learnt how to solve linear differential equation with constant coefficient of second order. We will learn something more about these equations, as well we will also learn in today's lecture about non-homogeneous equations. So, first we will learn about the existence and uniqueness of the solution of initial value problem.

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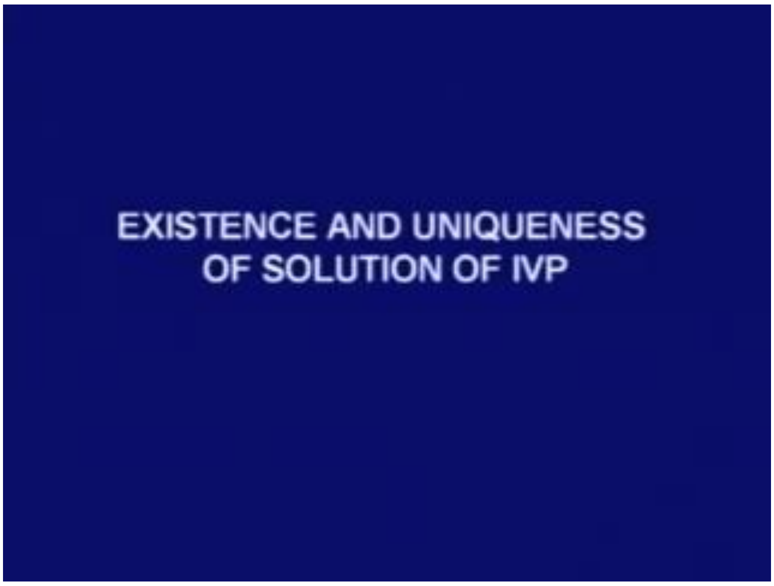

Here we are not going to find out, that is what these conditions are rather, we will state a result about under which conditions the solution of initial value problem, does exist, and under what circumstances that solution would be unique.

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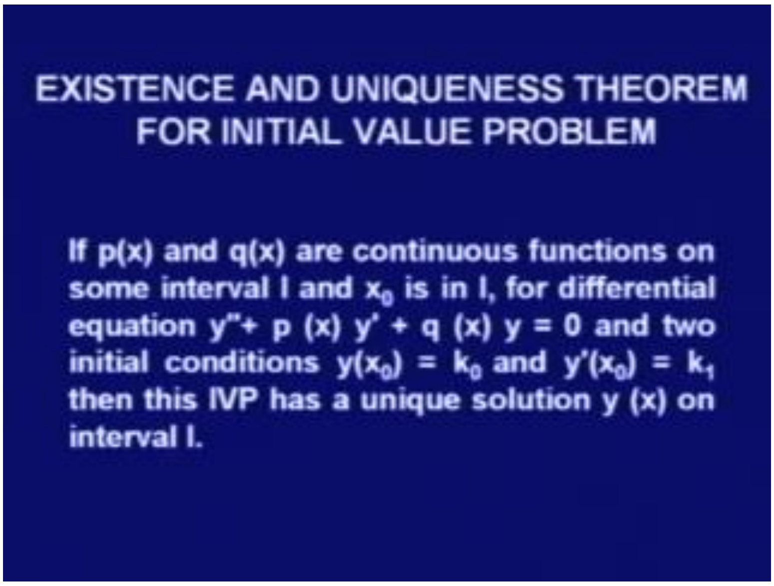

So, the result here is Existence and uniqueness theorem for initial value problem. If p x and q x are continuous functions on some interval I, and x naught is in I. For differential equation y double dash plus p x y dash plus q x y is equal to 0, and two initial conditions, that is function y at x naught is k naught. And the derivative of y at x naught is k 1, then this initial value problem has a unique solution y x on interval I.

We have learnt it that this solution unique solution of this initial value problem, is called the particular solution, and this particular solution, we can obtain, by giving specific value, to the general constants, in the general solution. How we are obtaining those constant values or this, particular solution, what we are doing is, we are putting this initial condition, to the general solution and then find out the values for those constants.

So, now we will learn that one, we had learnt that general solution, for second order equations, contains two linearly independent solutions, those two linearly independent solutions, we are obtaining by some methods. We are checking that those solutions, which we have obtained for the second order equations, whether they are, independent or not, for that, either we had implied the basic definition of linear independents of function or we had learn one more thing, that is those two solutions are not proportional. Today we will learn something more about this, linear independence of solution, which is known as Wronskian or Wronski determinant.

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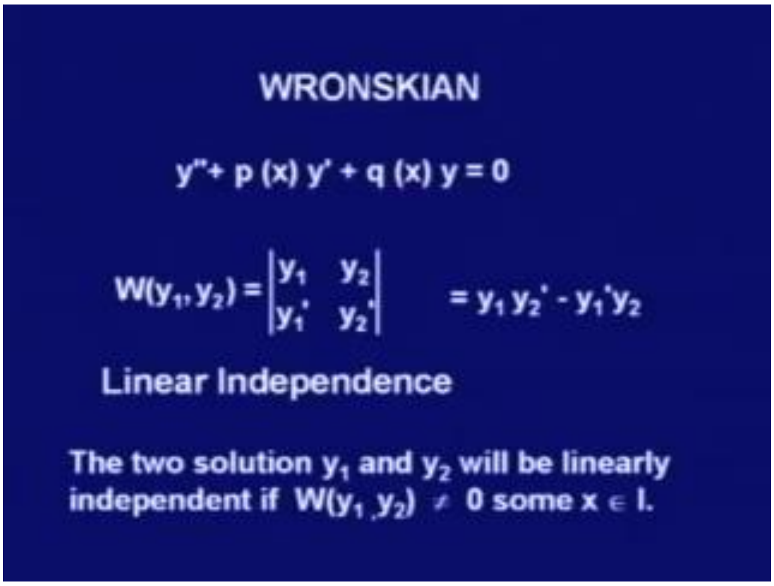

What we do call the Wronskian, Wronskian for the, second order equation y double dash p x y dash plus q x y is equal to 0. If it has two solutions y 1 and y 2, then the Wronskian of y 1 and y 2 is being given, as a determinant whose first row contains, y 1 and y 2 the two solutions. And the second row contains the derivative of those two solutions, of course, we do know the is equal to y 1 y 2 dash minus y 1 dash y 2, this is called the Wronskian, and how you are checking the linear independence. If this Wronskian is, not 0 for any value of, x naught in the interval I, then we say that, on that interval I, the two solutions y 1 and y 2 are linearly independent.

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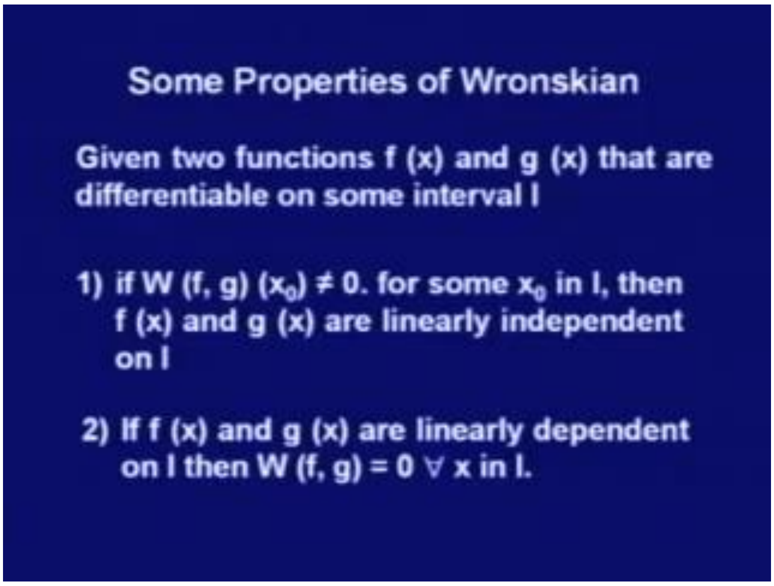

Some more properties of this Wronskian, given two functions f x and g x, that are differentiable, on some interval I. The Wronskian of f and g at some point x naught, is not 0 for some x naught in I, then we call that f x and g x are linearly independent, on I. Moreover, if f x and g x are linearly dependent on I, then this Wronskian of, f and g would be 0 for our x in I, so we see is, that is, if it is linearly dependent we will get, Wronskian of f and g to be, 0 for all x, and if it is linearly independent I could find out, at x naught some x naught, where it is naught 0. Now, one special property, which we are using for the differential equation that is called Abel's theorem.

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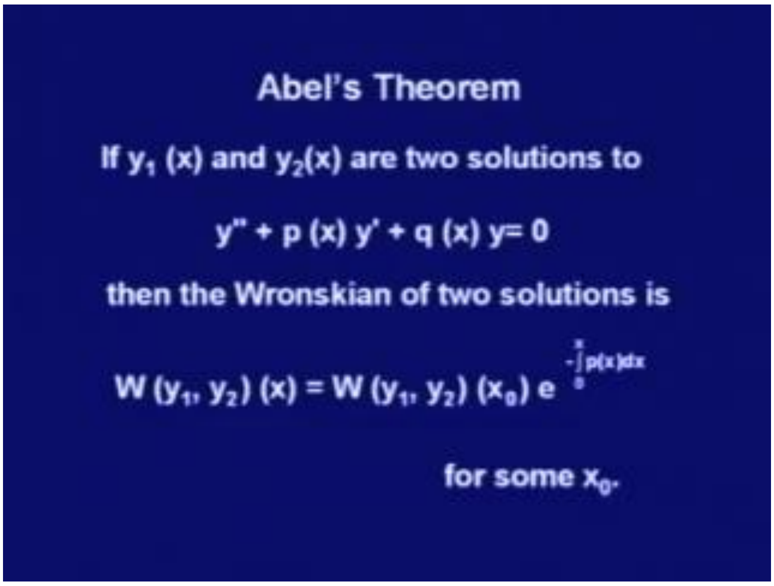

What is that, if y 1 and y 2 are two solutions to the, differential equation y double dash plus p x y dash plus q x y is equal to 0. Then, the Wronskian of the two solution is, at some point x that is, W y 1 y 2 at x is equal to W y 1 y 2 at x naught into e to the power integral minus 0 to x p x d x. We can show this result also, so we have going to, proof this.

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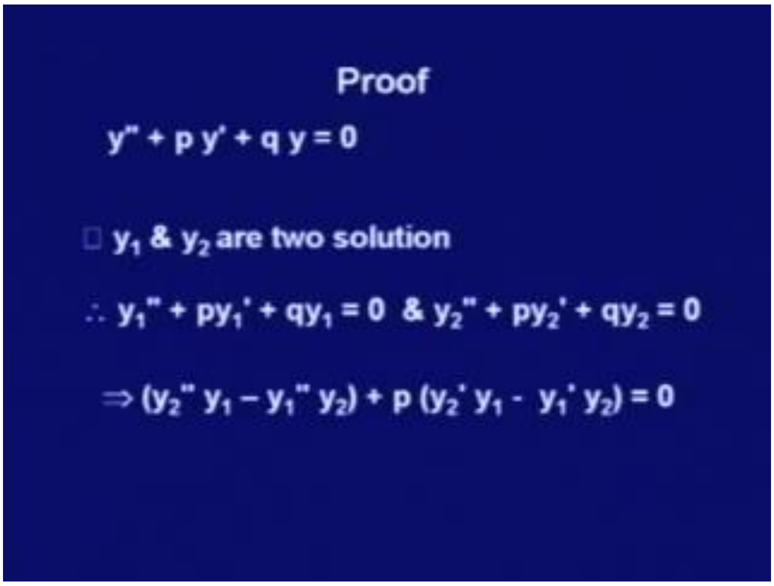

The differential equation y double dash plus p y dash plus q y is equal to 0, we have been given, that it has two solutions y 1 and y 2. Since, they are two solutions, so they would satisfy, our differential equation, what it says is that, y double dash plus p y 1 dash plus q y 1 is equal to 0 and y double dash plus p y 2 dash plus q y 2 would be 0. Now, from these two equations, if I eliminate, the term containing q, that is I would multiply the first equation by y 2, and the second equation by Y 1 and subtract. Then what we would get, we would get y 2 double dash y 1 minus y 1 double dash y 2 plus p y 2 dash y 1 minus y 1 dash y 2 is equal to 0.

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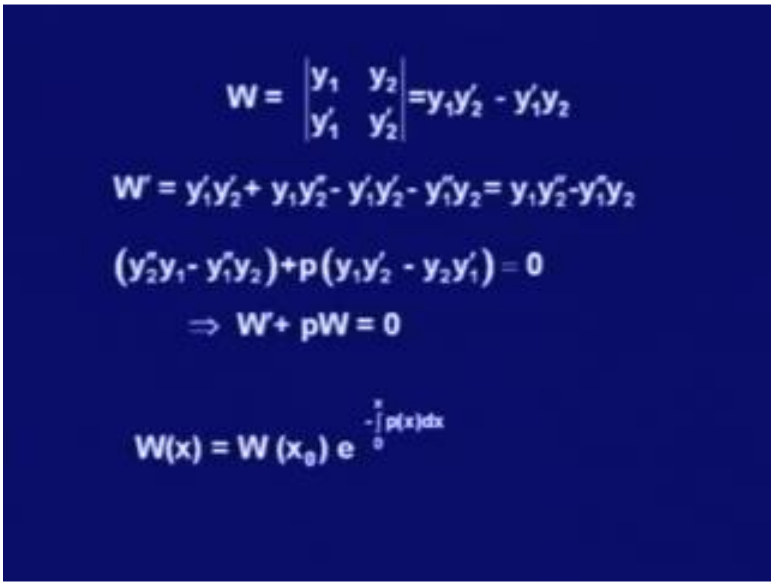

Now, let us see what we have obtain, again come back to our Wronskian, it is y 1 y 2, that is y 1 y 2 dash minus y 1 dash y 2. Now, if I differentiate this function, because this is a function of x, if I differentiate with respective x, what I would get, y 1 dash y 2 dash plus y 1 y 2 double dash minus y 1 dash y 2 dash minus y 1 double dash y 2, that is y 1 Y 2 double dash minus y 1 double dash y 2. Since, y 1 dash y 2 dash is, getting cancel it out.

Now we see, what the equation, we had obtained by eliminating q, we had obtained, y 2 double dash y 1 minus y 1 double dash y 2 plus p times y 1 y 2 dash minus y 2 y 1 dash is equal to 0. Now, we see that, the coefficient of p is nothing but, the Wronskian, and what we are getting, here the constant term that is nothing but, the derivative of the Wronskian, now, these are function of x.

So, we can write this equation as, W dash plus p w is equal to 0, where W dash is the derivative of W with respective of x, that says is that, we are getting, a differential equation with respective w. Now, what we are having is, now this is first order equation, we do know how to solve it, the solution of this, is W x naught e to the power minus 0 to x integral p x d x. We do know that, the solution we do know, and we do know this is, what we have replaced W x naught, as the constant, which we are having, that we had replaced with the particular value, x naught.

How we could this particular value, if I been given the value of, W at a particular value x naught, so this is what, we are getting as a solution this we do know, that is what that Abel's theorem says. Now, from here, what we are getting the result that is, if w at some x naught is not 0, I would get that, W x will never be 0, since this whatever we this part, this will never be 0 for any value of x, that says is, for linearly independent I require, w x to be not 0 for, only a single point, while when and they are linearly dependent, they would be 0 for all x. Now, let us seem one example how we are using this, Wronskian to find out, the linear independence.

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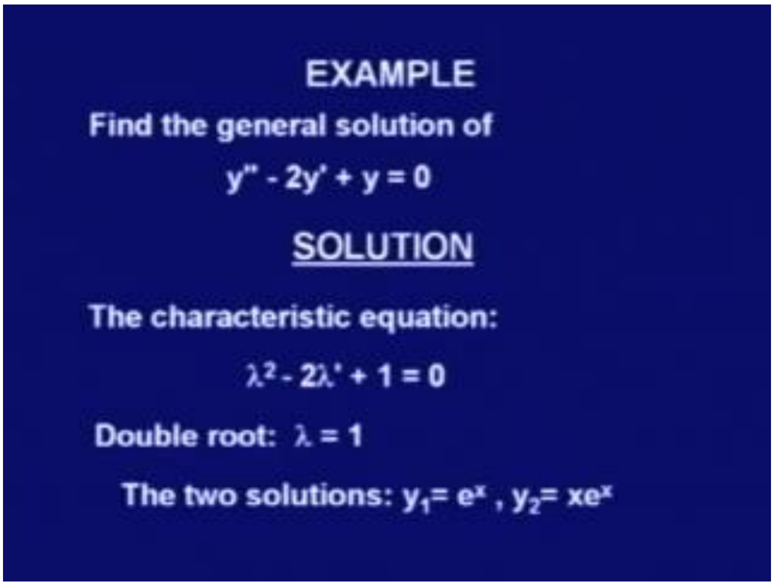

Find the general solution of, y double dash minus 2 y dash plus y is equal to 0, this is, a linear differential equation with constant coefficients, we do know how to solve it. So, solution let us see, what is the characteristic equation, that would be lambda square minus 2 lambda plus 1 is equal to 0. Now, we do know, this is nothing but, the lambda minus 1 whole square, so I would the double root lambda is equal to 1. We already know that, the two solutions for this, we are taking as, e to the power x and x times e to the power x. Now, we have wanted to check, that these two solutions are linearly independent and we would imply the method of Wronskian.

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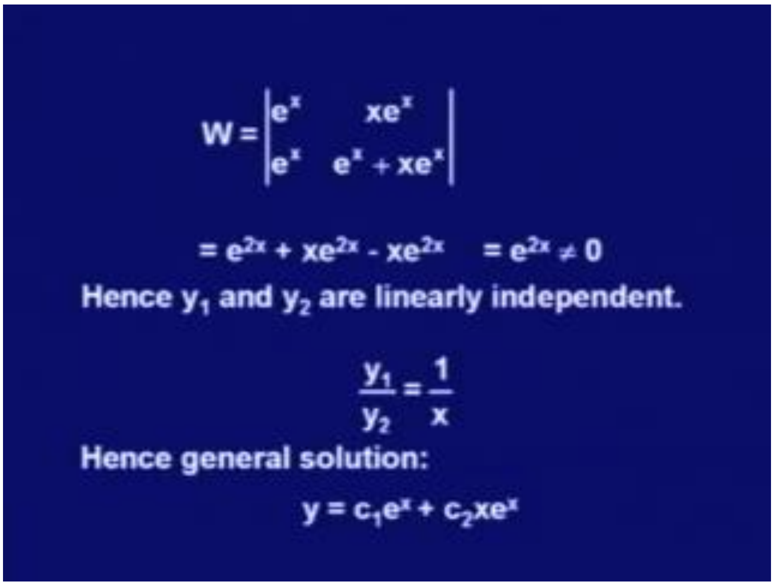

So we will find out the, Wronskian determinant for these two solutions, so what it would be, e to the power x that is the first solution, then the second solution x times e to the power x. And second row, the derivative of e to the power x, that is e to the power x, derivative of x times e to the power x, that is e to the power plus x times e to the power x, if I calculate, what is the value of this determinant, that would be e to the power 2 x plus x times e to the power 2 x minus x times e to the power 2 x, that is equal to e to the power 2 x.

That is, not 0 for any value of x, this says is that the two solutions e to the power x and x times e to the power x they are linearly independent. We know that, we had already checked it, using the method of constant proportion, so if I take, the ratio of e to the power x and x times e to the power x, that is coming out to be, 1 by x, which is not a constant. So, this also we do know, from the first method, of not proportion that, they are linearly independent.

So, what will be the general solution of this given equation, that would be c 1 e to the power x plus c 2 x times e to the power x. Now, we had learned about the, linear independence by one more method, that is called Wronski determinant or Wronskian, now we will move to the, Non Homogeneous equations.

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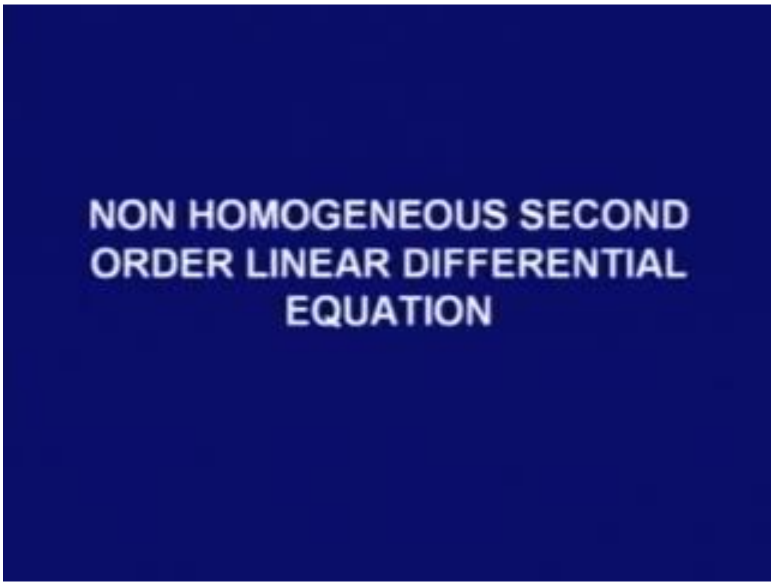

So, first we will do, non homogenous second order linear differential equations, what they are, we do know that they are, second order linear differential equations, with right hand side not to be 0.

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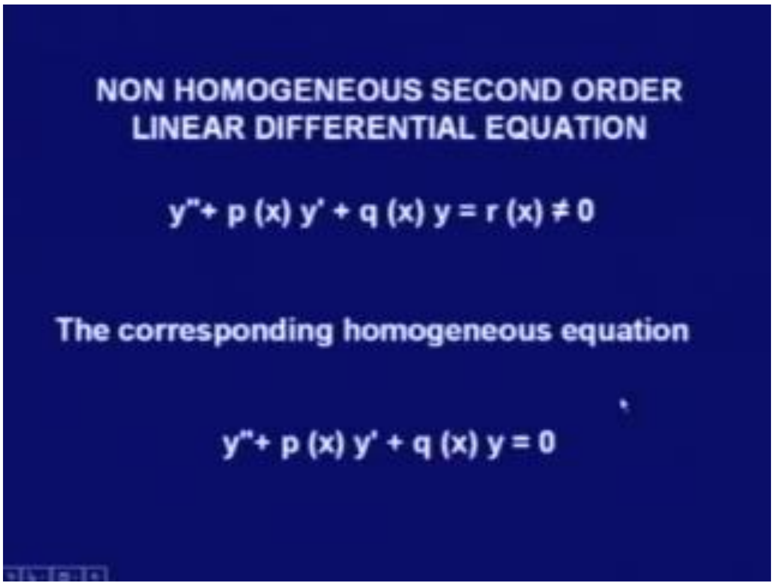

That is, y double dash plus p x y dash plus q x y is equal to r x and I should have, that r x should not be 0, this we do, now we learn about that is, what we are going to solve this. So, what we say is, one more term we are associating with this, that is corresponding homogenous equation, where I will take the equation as such, but the right hand side as, 0. This is called the corresponding homogenous equation, for this non homogeneous equation, or sometimes it also referred as, associated homogenous equation. So, this equation, we are obtaining as the same equation, but the right hand side r x, we have made it here 0.

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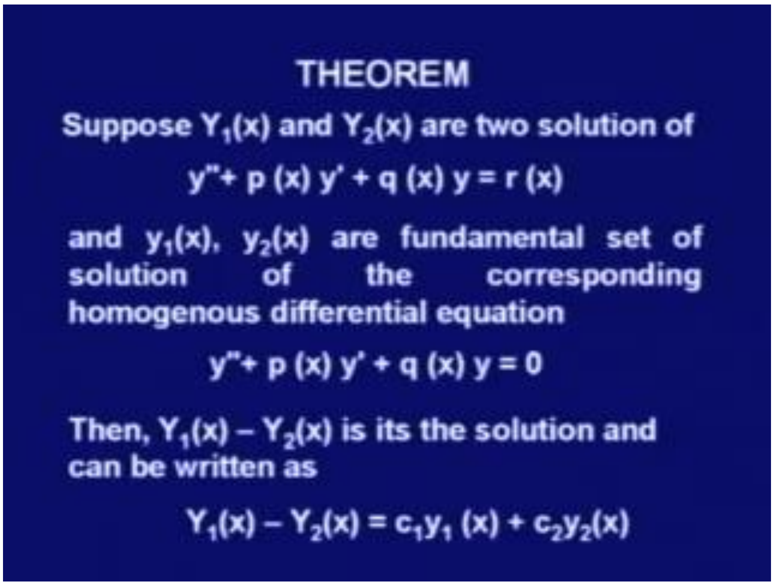

What we, get one result, that suppose, capital Y 1 and capital Y 2 are two solutions, of non homogeneous equation y double dash plus p x y dash plus q x y is equal to r x. And, suppose is small y 1 and y 2 are fundamental set of solution of, corresponding homogeneous differential equation, that is y double dash plus p x y dash plus q x y is equal to 0. Here, what we have taken, that is y 1 and y 2 are the fundamental set that means, they are linearly independent solution of, this homogenous equation.

Then capital Y 1 minus Y 2, is also a solution of, this homogenous equation, and since it is a solution of this homogenous equation. It can be written as a linear combination of the fundamental set of solution, that is y 1 x and y 2 x, that is I could write, capital Y 1 minus capital Y 2, as c 1 times y 1 plus c 2 times y 2. Again, this is again a simple result, which we can show here, by the knowledge, whatever we are knowing, so let us proof it.

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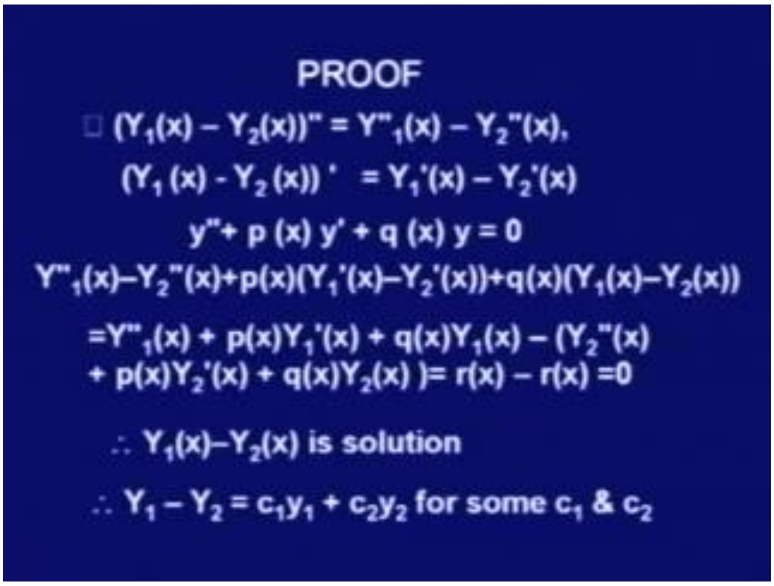

Since Y 1 x and Y 2 x, second derivative would be nothing but, the second derivative of Y 1 minus second derivative of Y 2, and the derivative of Y 1 minus Y 2 is nothing but, the derivative of Y 1 minus derivative one of Y 2. we will substitute this, in our homogeneous equation, that is y double dash plus p x y dash plus q x y is equal to 0, since we want to show that, Y 1 minus Y 2 is a solution of the homogenous equation.

Let us substitute it, what do we get, Y 1 double dash minus Y 2 double dash plus p times Y 1 dash minus Y 2 dash plus q times Y 1 minus Y 2. Now, rearrange the terms, first I write the terms, corresponding to Y 1, that is Y 1 double dash plus p x times Y 1 dash plus q x Y 1, then minus Y 2 dash Y 2 double dash plus p x times Y 2 dash plus q x Y 2. Now, the first term, which we are having this is, since capital Y 1 is the solution of non homogenous equation that means, this first, terms what we are having, that would be, that would be, equal to the right hand side of non homogenous equation, that is r x.

Similarly, the second term will also be the, non homogenous, solution of non homogenous equation, that is it should also be equal to r x, that says is r x minus r x will get 0. What it says, if I am substituting in this equation, Y 1 minus Y 2, I am getting right hand side equal to 0, that says, Y 1 minus Y 2 is satisfying this equation, which says that, it is a solution. Now, since it is a solution, we can write it is as a, linear combination of, the fundamental set of solutions, that is i could write it as, c 1 y 1 plus c 2 y 2 for some c

1 and c 2. Now, from here, what we could get, we could get one more result, see what we could do, let us say.

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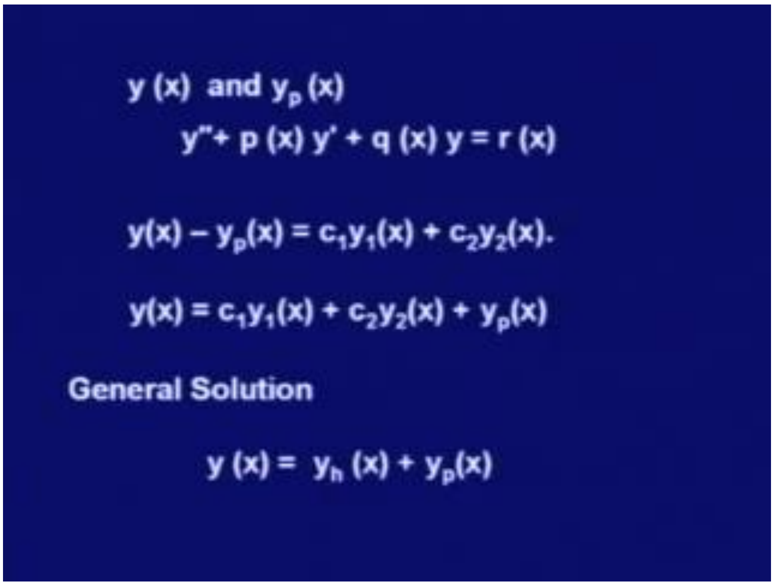

Let us consider, the two solution y and y p x, what are they, they are actually, we are taking, this non homogeneous equation y double dash plus p x y dash plus q x y is equal to r x. And let us say, y is a general solution of this non homogeneous equation, and y p x is a particular solution of this non homogeneous equation, so now what we are getting; we are getting, this as capital Y 1, this as capital Y 2. The two solutions of non homogeneous equation, then by the previous result, we do know, that y minus y p, would be solution of, corresponding homogeneous equation.

Let us say that is, the fundamental set of homogeneous equation, solution of homogeneous equation, would small y 1 and y 2 that means this would be equal to c 1 y1 plus c 2 y 2. Now, from here, I could write, y as c 1 y 1 plus c 2 y 2 plus y p x, so the first term c 1 y 1 x plus c 2 y 2 x is nothing but, the general solution of the corresponding homogeneous equation. And, y p x is the particular solution of the non homogeneous equation.

That says is, the general solution of non homogeneous equation y x, I can write as y h x plus y p x, where y h x is the, solution of homogeneous equation, that is c 1 y 1 x plus c 2 y 2 x, this is the general solution of associated homogeneous equation. And y p x is, a particular solution of, the non homogeneous equation, thus we are getting, now you see, we had already learnt, how to find out, or that is these solutions y 1 and y 2 they are linearly independent. Now, this y p, which we are saying, if I am keeping this in the general solution, this y p should also be linearly independent of y 1 and y 2. So, this is important thing over here, now we come to our, main theorem of non homogeneous equation.

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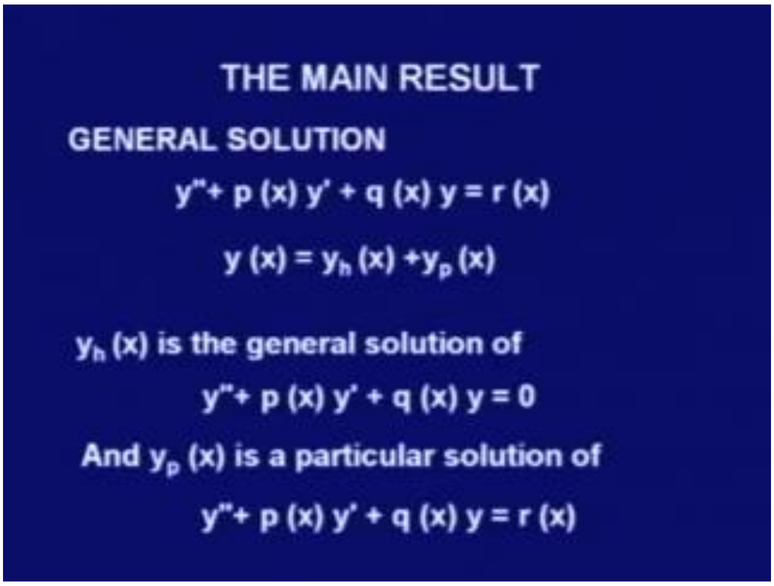

This is about general solution, for a second order linear differential equation, which is non homogeneous, that is y double dash plus p x y dash plus q x y is equal to r x. It is a solution, general solution can be given as, general solution of associated homogeneous equation plus particular solution of the, non homogeneous equation, that is y x can be written as, y h x plus y p x, where y h x is the general solution of, corresponding homogeneous equation, y double dash plus p x y dash plus q x y is equal to 0 and y p x is a particular solution of, our non homogeneous equation. Now, what will be, the result about the particular solution.

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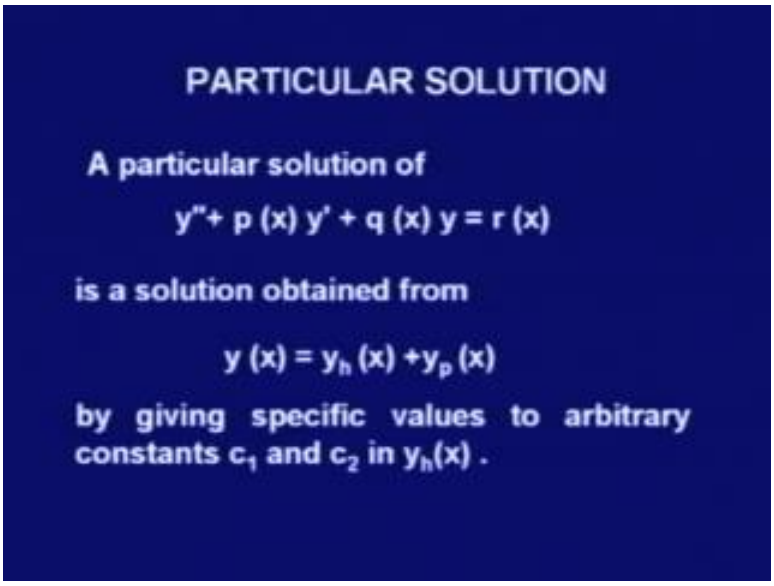

The particular solution we do know, can be obtained from the, general solution y h x plus y p x, by giving this specific values c 1 c 2 in the y h x. Since, we do know that, y h x is the, general solution of associated homogeneous equation, y p x is a particular solution, so it is non containing any arbitrary constant. The arbitrary constants are over here, now if I keep this, c 1 and c 2 0, I will get only y p x, that is the particular solution, which we are keeping over here, but that is not the only particular solution. We can obtain some other particular solutions as well, by giving some values to the c 1 and c 2. So, what will happen to the initial value problem that is from there we will obtain this c 1 and c 2, by putting the initial values in the general solution.

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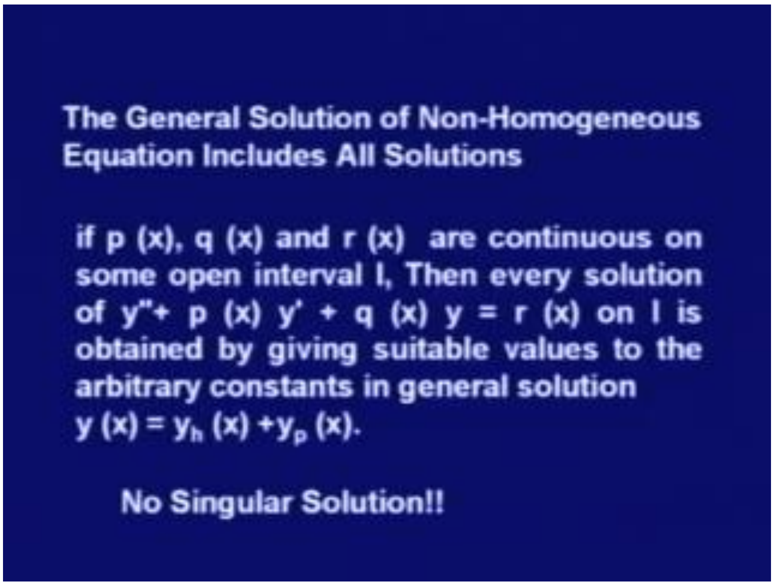

So, the general solution of, now one more result over here, the general solution of, non homogeneous equations, include all solutions, what this is what says, this says, if p x q x and r x are continuous, on some open interval I. Then every solution of y double dash plus p x y dash plus q x y is equal to r x on I, is obtained by giving, suitable values to the arbitrary constants, in general solution y h x plus y p x. That is, every solution of this non homogeneous equation, can be obtained through this general solution, it may imply, there is no singular solution.

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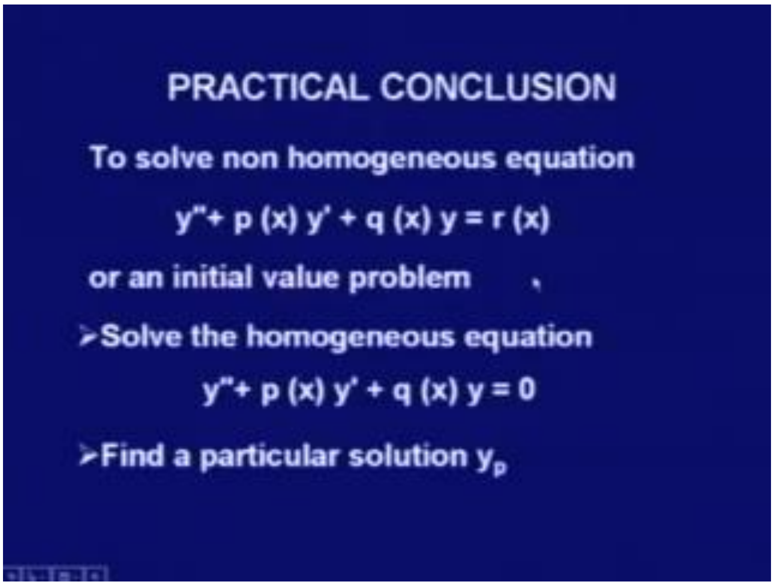

Now, what we have drawn the practical conclusion out of these results, we have obtained that, to solve a non homogeneous equation, y double dash plus p x y dash plus q x y is equal to r x. First, or an initial value problem what we do, first we do, solve the homogeneous equation, y double dash plus p x y dash plus q x y is equal to 0, and find a particular solution y p of non homogeneous equation, and then we do, keep it over there. We had, learnt the techniques to solve, homogeneous linear equations of course, there we have not done it, for general p x and q x, we have done only for the, constants p x and q x.

That's says is now, so let us, do first the examples we will do, about this constant, then we will learn about, functions as well. So, we do know this constants, we do know how to solve this homogeneous equations, but the things remain is, how to find out the particular solution for this non homogeneous equation, we will start with the basics.

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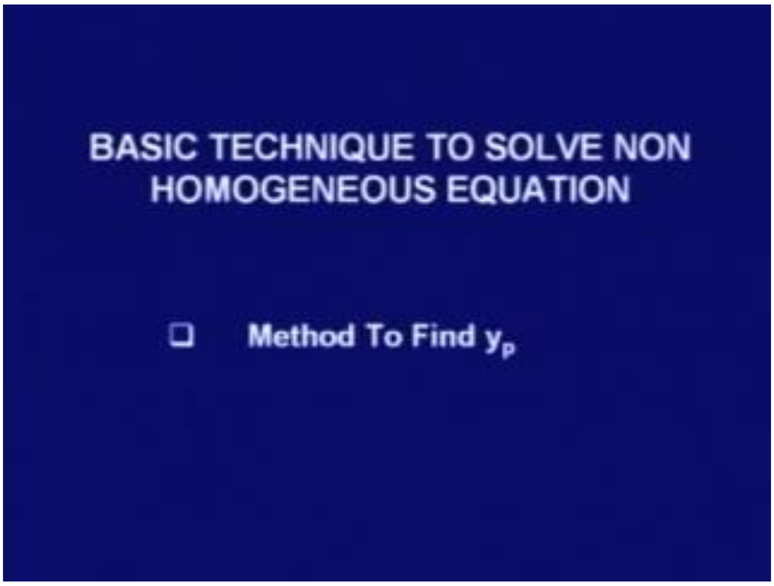

So, let us, first Find out that is, basic technique to solve the non homogenous equation, it is, method to find y p, let us start with the as I said, with basics, so let us, start with an example.

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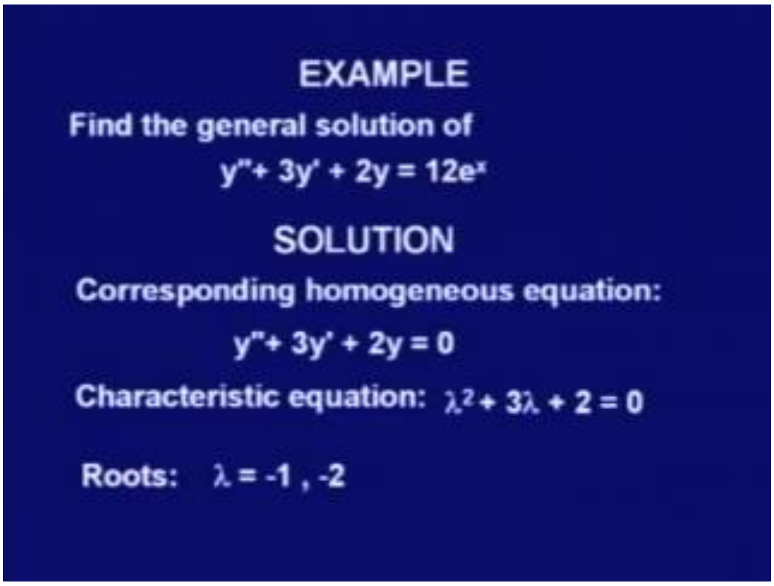

Find the general solution of y double dash plus 3 y dash plus 2 y is equal to 12 times e to the power x, we see, this is a linear equation, it is coefficients are constants, but the right hand side is, not 0 that is, it is non homogeneous. So, what we do is, first we will solve the, associated homogeneous equation, what is the corresponding homogeneous equation, y double dash plus 3 y dash plus 2 y is equal to 0. We do know, how to solve it, first we find out the characteristic equation, what is the characteristic equation, lambda square plus 3 lambda plus 2 is equal to 0. What are it is roots of course, it is lambda plus 2 into lambda plus 1 is equal to 0, so it has two roots, minus 1 and minus 2, they are distinct real numbers. So, what will be the, corresponding general solution of the homogeneous equation.

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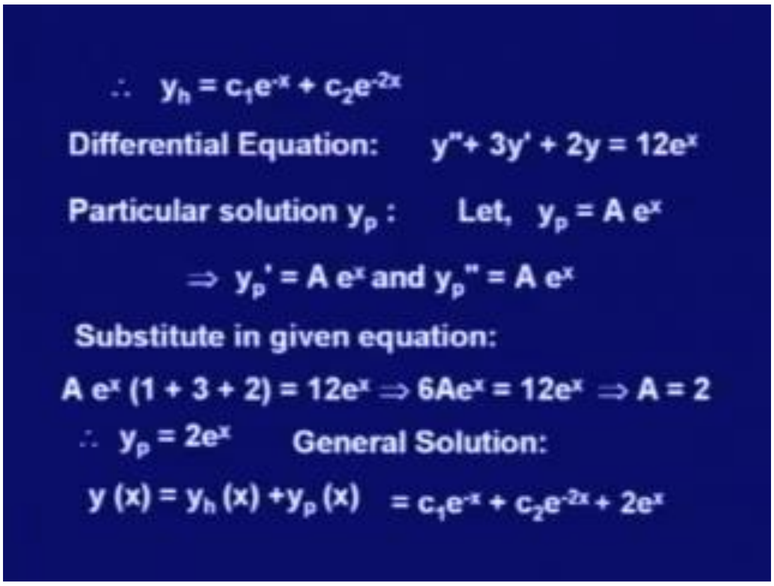

That will be c 1 e to the power minus x plus c 2 e to the power minus 2 x, now the differential equation was, y double dash plus 3 y dash plus 2 y is equal to 12 time e to the power x. Now, I have to find out the particular solution, particular solution means, is that is it should be a solution of this equation, now let us see right hand side we are having, the term e to the power x. So, now if I experiment with, e to the power x we do know, it is derivative, as well as double derivative, all are e to the power x.

So, if I am substituting, so we can try like this one, so what we are doing is, we are taking particular solution as, A times e to the power x. Because, I do not know, what will satisfy this equation that means, it should if I am putting e to the power x, I should get equal to 12 times e to the power x. So, I have put, I have taken that as a constant, because e to the power x, I can obtain by, putting e to the power x, but this constant we have to obtain, so I have put here, a how to obtain this A.

Now, if I am assuming that, this is a solution this must satisfy this equation, so now I will, substitute y p y p dash and y p double dash in this equation. So, what is y p dash A times e to the power x and y p double dash, that is again A e times e to the power x, let us substitute it. We would be getting is, A times e to the power x, into 1 plus 3 plus 2 is equal to 12 times e to the power x, what it says, that 6 A times e to the power x is equal to twelve 12 e to the power x, which implies that, my a must be 2.

That is the particular solution should be, 2 times e to the power x, we can check that is satisfying this equation, so this is a particular solution. So, now what will be the, general solution of our non homogeneous equation, that would be the general solution of, homogeneous plus the particular solution. So, I get c 1 e to the power minus x plus c 2 e to the power minus 2 x plus 2 times e to the power x, so this is the general solution of homogeneous equation.

We can check, that if I substitute this function and its derivatives in this equation, for any value of c 1 and c 2, that is take the general constant c 1 and c 2, I will obtain, that is this will satisfy this equation, so this is a general solution of the non homogeneous equation. So, this is here we have done with, an example and the basic distinct, we have got, that is right hand side we had seen, that it is containing the term e to the power x and we had already experimented from the beginning of this course, with the function e to the power x, so we could do it.

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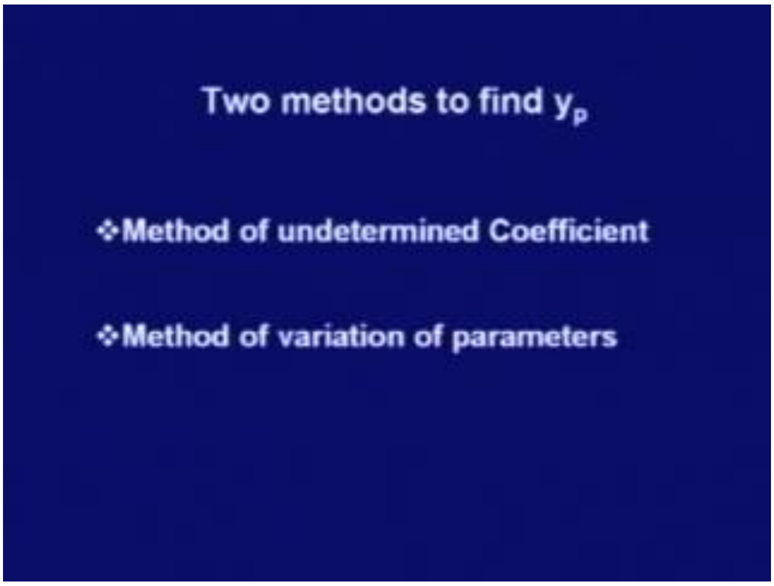

Now, let us learn, the methods to find out y p, we will learn here the two methods to find out, this particular solution. The one method, method of undetermined coefficient, that is something like that, what we have done, in this example and there is another method, called method of variation of parameter. So, first we will learn, the method of undetermined coefficients.

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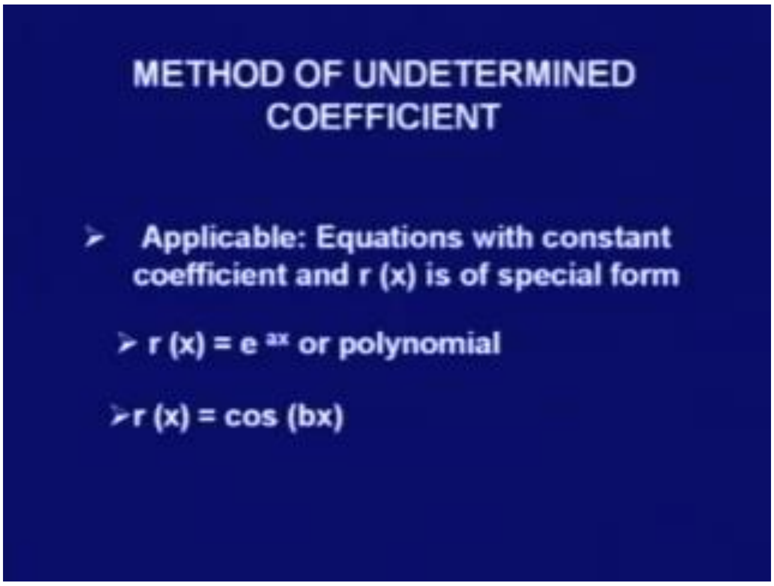

What this method says is, this method is first let, let us learn, that is where this method is applicable this method is applicable in the equations with constant coefficient c. We would be dealing with, only linear equations with constant coefficients, and right hand side r x that is also of a special form, that is special form, that is either my r x is of the form, e to the power a x or it is a polynomial, or it may be a function of cosine or sine something like that, then only this method is applicable. Now, let us learn, what this method is, this method says is that, if r x is of a special form, I would choose that, particular kind of y p, how we are doing it.

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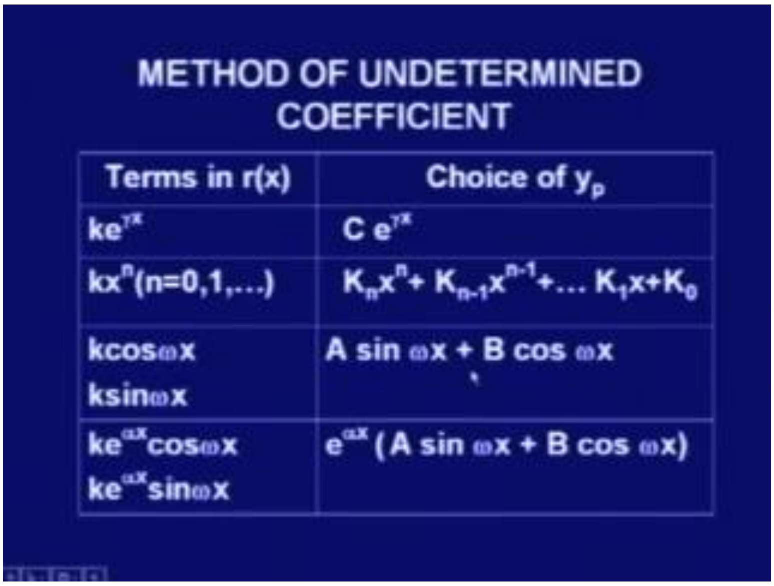

Let me summarise it, by a table, we say that is, if I do have the r x as, k times e to the power gamma x, that as in our example, I was having 12 times e to the power x, that is gamma was one. And, what I have choosed, I have choosed, the y p x e to the power x and a here, that says is that, if r x has of the form k times e to the power gamma x, choose y p as c times e to the power gamma x, if my r x is containing the polynomials, that is here I have given the example as, k times x to the power n, then choice of y p will be a polynomial of degree n.

See here, we are not taking, only k times x to the power n, we are taking the complete polynomial k naught k 1 x plus so on, k n x to the power n. So, if it is the polynomial, or if it is some terms of polynomial, in the choice of y p, we will always choose the complete polynomial of degree n. Then if, the terms in r x are cosine or sine, that is k times cosine omega x or k times cosine sine omega x, we would choose y p as, a sine omega x plus b cosine omega x.

Again you see, my r x may contain only cosine or only sine, but in the y p, we would choose, both sine and cosine function. Now, if the term in the r x is having, a multiple of e to the power alpha x and cosine or sine terms, we would be choosing y p as e to the power alpha x a sine omega x plus b cosine omega x. That is now, what are the rules to apply this method, so we will learn some basic rules.

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Rules of this method are, first is called the basic rule.

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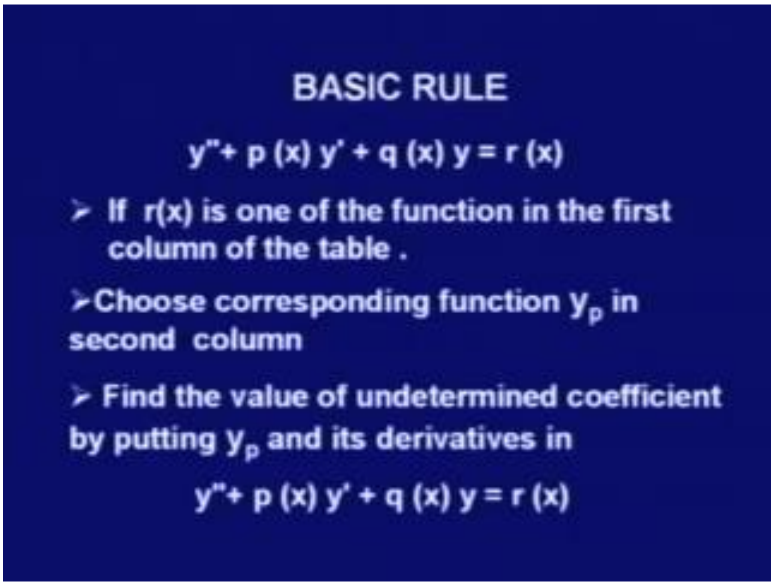

Basic rule, just now as I had explained you, that is for second order linear differential equation, which is non homogeneous y double dash plus p x y dash plus q x y is equal to r x. If r x is one of the function in the first column of the table, just now as I had explained, I would choose, corresponding function y p in the second column. And, find the value of, undetermined coefficient by putting y p and it is derivative in the given equation.

So, we had got, that is by using that table, we can choose y p, and as in our example what we have done, we had find out, the value of A, by putting y p y p dash and y p double dash in the equation and then we have determined what is A. So, here, whatever the undetermined coefficients k c k 1 k naught and so on, we could find out, all of them by putting y p and it is derivatives, in the non homogeneous given differential equation, and solve for the, unknown coefficients that is why, it is called the method of, undermined coefficients.

Now, you do remember, I said that, the general solution contains two things, one is general solution of associated homogeneous equation y h x, and another is y p x. Now, it would be the general solution that is says is, I should have y p also linearly independent of the solutions, which we are having as Y 1 and Y 2 of the homogeneous equation. That says, sometimes it may happen, that is whatever I am getting, as the choice of y p, that may turn out to be, one of the solution of the homogeneous equation.

Then of course, because here I am choosing it a constant time that function, that will not be, linearly independent of, the solutions of homogeneous one. So, how to solve this kind of problem, that says is the second rule, as modification rule.

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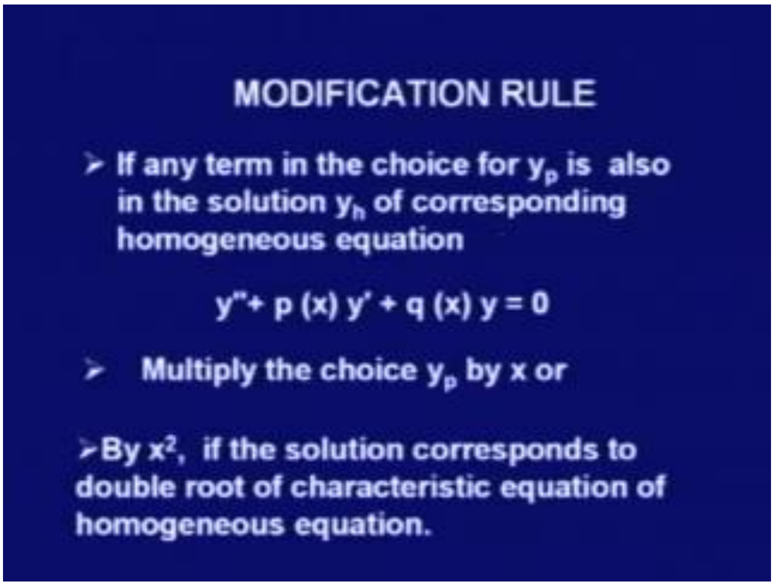

It is says, if any term, in the choice for y p is also in the solution, y h of corresponding homogeneous equation, y double dash plus p x y dash plus q x y is equal to 0. Then, multiply the choice y p by x or by x square, if the solution corresponds to the, double root of the, characteristic equation of the homogeneous equation, why we are doing it, say we can understand this very well, since, that particular solution is, also coming as, in the solution of homogeneous equation.

So, with a constant coefficient of course, we cannot get it as, linearly independent, so we have to multiply with x. Now, if it is solution corresponding to the double root then we do know that, x times that solution is also, coming as the fundamental solution. So, I have to multiply by x square, then there is one more rule, that is called Sum rule.

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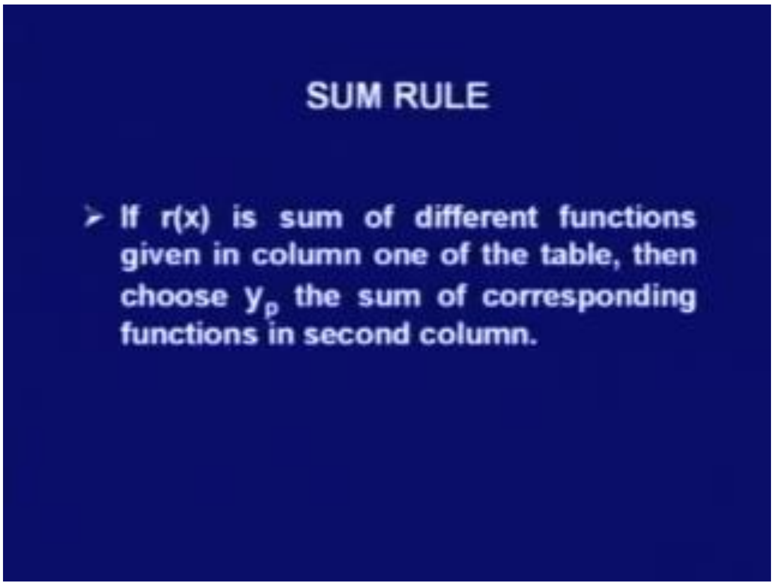

What it says is, I may have r x, that is r x is sum of different functions given in the column one of the table, then choose y p as the, sum of corresponding functions in the second column. Now, let us see, that is, how to use these rules and how to use this particular method, to find out the general solution of, non homogeneous equation, that we will see through some examples, so let us go with the examples.

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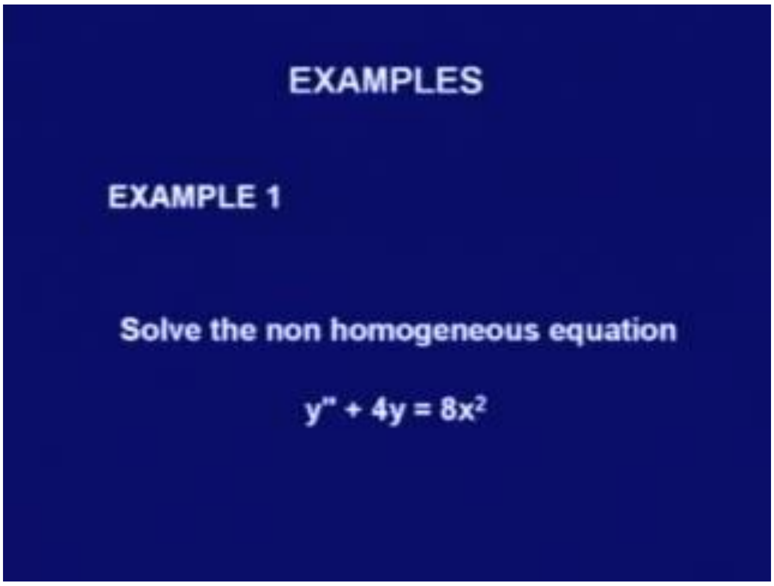

First example let us say, solve the non homogeneous equation y double dash plus 4 y is equal to 8 x square, we see this is the second order equation, constant coefficients are constants, and right hand side is of the special form, that is it is a polynomial, that is 8 times x to the power 2.

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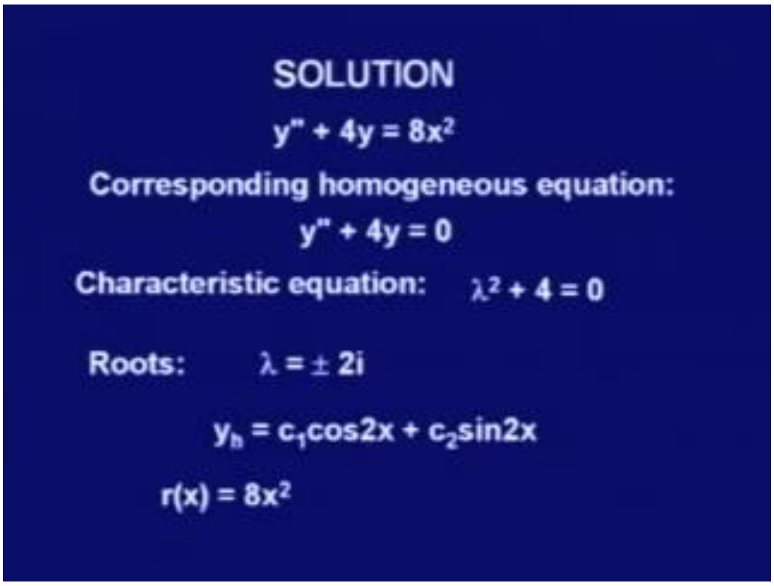

So, go for the solution, given equation is y double dash plus 4 y is equal to 8 x square, the corresponding homogeneous equation will be, y double dash plus 4 y is equal to 0, to solve this we require the characteristic equation, what is the characteristic equation, that is lambda square plus 4 is equal to 0. What will be it is root, it is lambda square is equal to minus 4 that means, it will have, complex conjugate roots, the roots would be plus minus 2 i, we do know in this case, what is the general solution of homogeneous equation, that is c 1 cosine 2 x plus c 2 sine 2 x.

Now, what is my right hand side, that is 8 x square, now if you are not remembering, we of course, go back to our table ((Refer Time: 35:13)). We are having 8 x square, that is here n is equal to 2, so what we would choose, we would choose k naught plus k 1 x plus k 2 x square.

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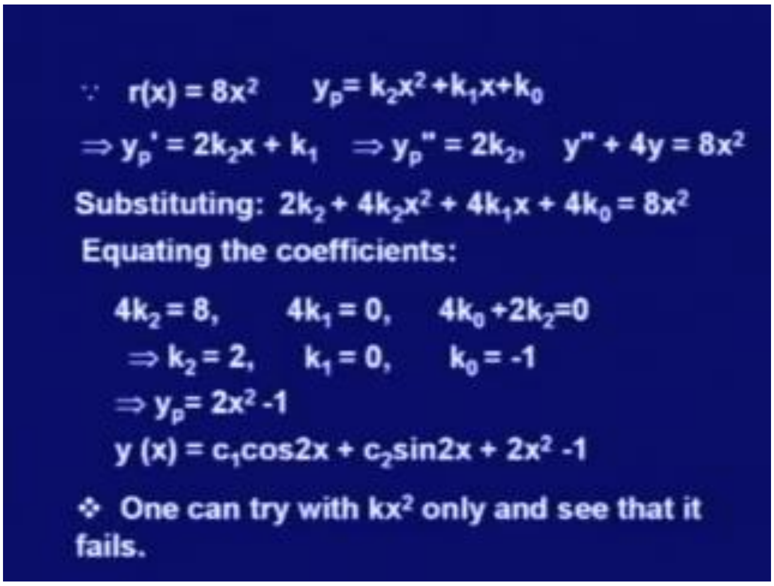

So, let us see, since r x is 8 x square, y p we would choose k 2 x square plus k 1 x plus k naught, now substitute its derivatives in the equation, that is first find out what is y p dash, that is 2 k 2 x plus k 1 y p double dash would be 2 k 2. Now, substitute this in the given equation, y double dash plus 4 y is equal to 8 x square, what we get, we get 2 k 2 plus k 2 square 4 k 2 x, x square plus 4 k 1 x plus 4 k naught is equal to 8 x square. Now, to find out these constants, we will compare the, coefficients of different powers.

So, equating the coefficients, we get, the coefficient of x square on the right hand side is 8, while the coefficient of x square on the left hand side is 4 k 2. The first equation, we are getting is 4 k 2 is equal to 8, then we are getting is, the right hand side the coefficient of x is 0, while on the left hand side the coefficient of x is 4 k 1. So, I get, 4 k 1 is equal to 0, now right hand side, I do not have any constant term, while in the left hand side, I do have, the constant term 2 k 2 plus 4 k naught, that says is 2 k 2 plus 4 k naught, is also 0.

Now, from the first equation I am getting, k 2 is equal to 2, second equation gives k 1 as 0, and the first equation, and the third equation gives me, that k naught is equal to minus 1. Now, substitute this in our choice of y p, what I would get, I would get y p as 2 x square minus 1, so what will be the general solution of our equation, y double dash plus 4 y is equal to 8 x square, that is c 1 cosine 2 x plus c 2 sine 2 x, that is the general solution of, homogeneous equation plus this particular solution, 2 x square minus 1.

Now, you can again, check with this solution, find out its derivative and double second derivative, substitute in this equation, and check that is, in general if I am taking, only c 1 and c 2, this will satisfy for, all values of c 1 and c 2. So, it there is a, this would be the interesting, exercises for you to do, you can check it that, this is a solution of this, non homogeneous equation.

Now, let us do one more example, one more thing here, you would be worrying that is, why I have chosen this as, i said is we will always choose polynomial. You can try, one more interesting exercise, you can try only with k x square, do not choose the complete polynomial. And, see that, what the solution, you are getting, that will not be actual solution of this, non homogeneous equation, this is another interesting exercise, which you can try.

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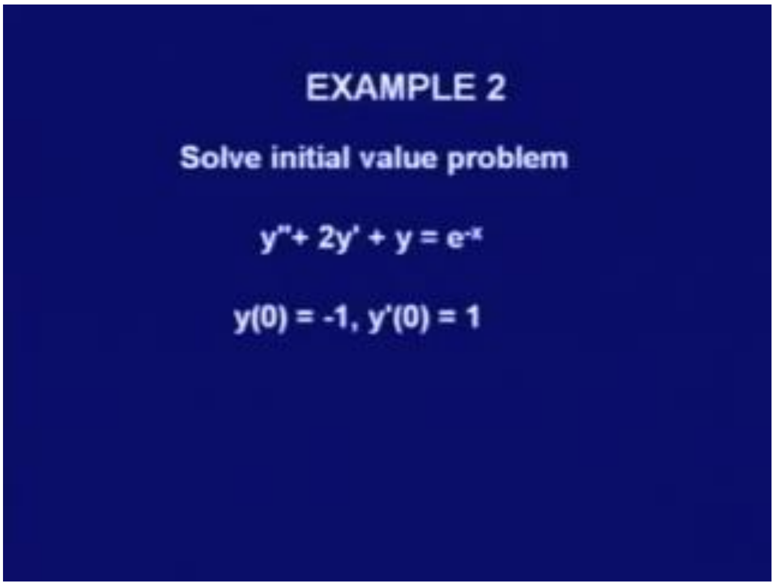

So, let us move to the second example, solve the initial value problem, y double dash plus 2 y dash plus y is equal to e to the power minus x, where the initial conditions are, that the function at 0 is minus 1, and the derivative at 0 is 1, move to the solution.

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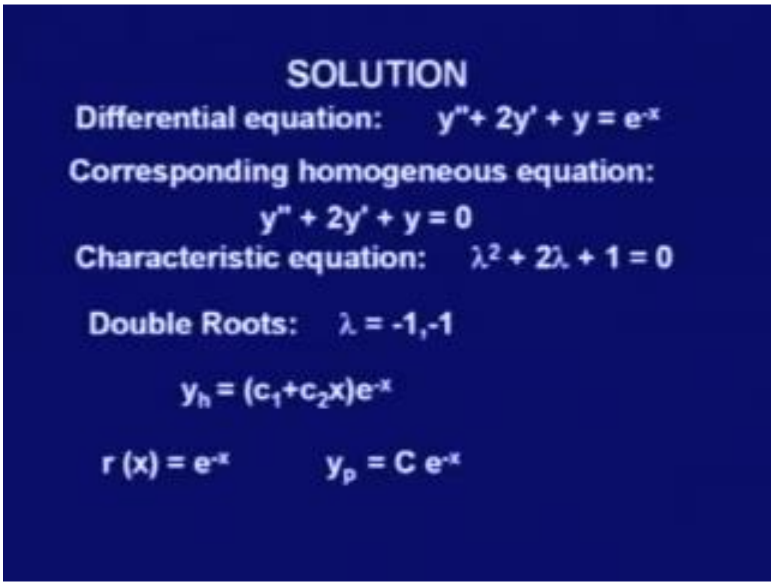

The given differential equation is y double dash plus 2 y dash plus y is equal to e to the power minus x. See again, we are having, the equation has constant coefficients, and right hand side is of a special form, e to the power minus x, first find out, the corresponding homogeneous equation, that is y double dash plus 2 y dash plus y is equal to 0. For solving this we require, it is associated characteristic equation, that is lambda square plus 2 lambda plus 1 is equal to 0.

We see, this is the whole square of lambda plus 1 that is we are going to get, the double root lambda as minus 1 and minus 1. We do know, in homogeneous equations, when we are having the double root, the solution is e to the power minus x, and the other solution we choose x times e to the power minus x. So, the general solution would be, c 1 plus c 2 x times e to the power minus x, now right hand side is e to the power minus x.

Now, you see, if I am using our table, I would get, y p as, a times e to the power minus x or c times e to the power minus x, but this e to the power minus x is also coming as, here in the e to the power minus x is also a solution of my homogeneous equation. Moreover, this is a solution corresponding to the double root, that is x times e to the power minus x is also, a solution so we have to apply the modification rule.

So, again I am repeating this, these steps, first we see, from our table, since I do have, e to the power minus x, I would choose c times e to the power minus x, that is gamma is minus 1. Since, again this e to the power minus x is coming as, a solution of the homogeneous linear equation.

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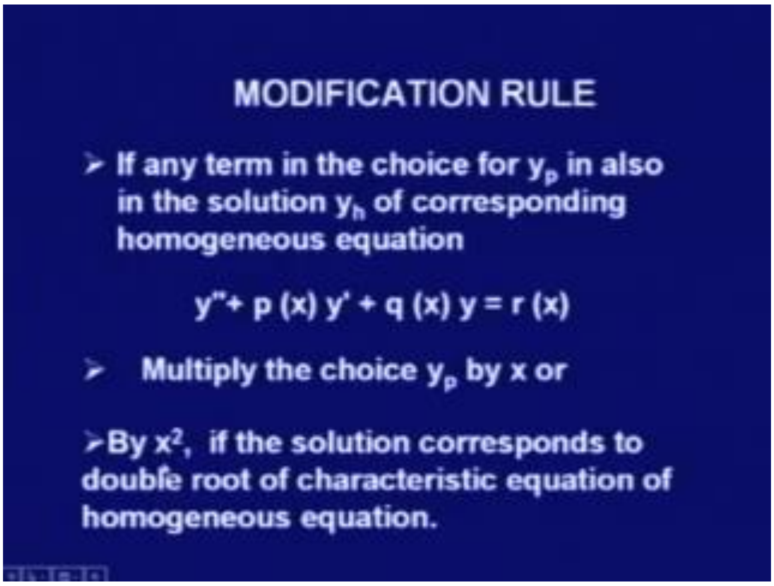

We do apply the modification rule, the modification rule says is, multiply the choice of y p by x or by x square. So, here my solution, e to the power minus x is, corresponding to the double root, so I have to multiply by x square.

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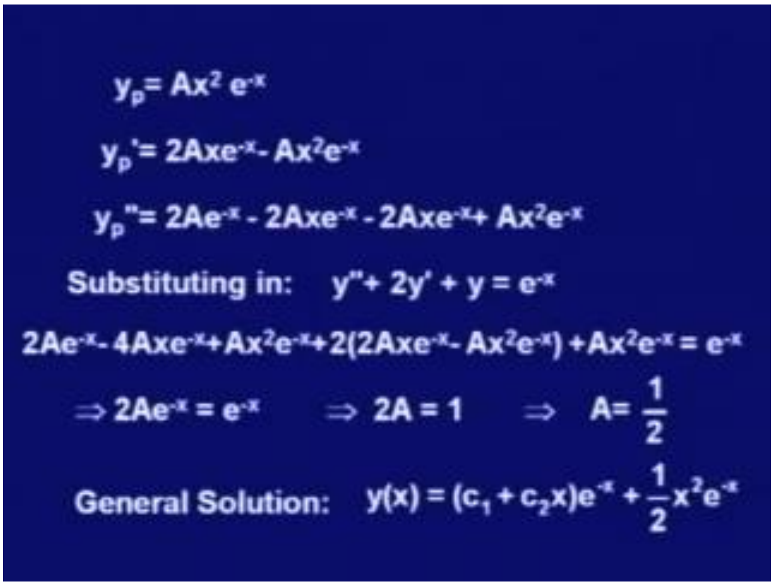

That says is, I should choose my y p as, a times e to the, a times x square times e to the power minus x find out, what is y p dash and y p double dash. So, y p dash it is derivative, 2 A x e to the power minus x, minus A x square e to the power minus x, Second derivative, 2 A e to the power minus x, minus 2 A x e to the power minus x, minus 2 A x e to the power minus x, plus A x square times e to the power minus x, that is it will be, 2 A e to the power minus x minus 4 A x times e to the power minus x, plus A x square e to the power minus x.

Substitute this in, given equation, that is y double dash plus 2 y dash plus y is equal to e to the power minus x. Left hand side what we would be getting, 2 a e to the power minus x minus 4 A x times e to the power minus x plus A x square e to the power minus x plus 2 times it is derivative, that is we are getting it as, 2 A x e to the power minus x minus A x square e to the power minus x plus y is A x square e to the power minus x, this must be equal to e to the power minus x.

Let us see, in the left hand side, here I am having is minus 4 A x e to the power minus x, this would be plus 4 A x e to the power minus x, that is this term is get cancel it out. Here, I am getting is a times x square e to the power minus x, that is x coefficient of x square e to the power minus x, here is a, here is minus 2 x this 2, and here is one more A plus A, again it is cancelling out. So, what is being left, I have been left with, 2 A e to the power minus x is equal to e to the power minus x, so comparing the coefficients we get, 2 A is equal to 1 or A is equal to half.

So, what will be my y p, y p would be half x square e to the power minus x, so the general solution would be, c 1 plus c 2 x e to the power minus x, that is the general solution of homogeneous equation, plus this particular solution half x square e to the power minus x. Now, you can actually again check with this, solution that you could find it out, that substitute these, in this equation and find out this is satisfying this equation, and moreover, here, we are getting is e to the power minus x, x times e to the power minus x, and x square e to the power minus x, all of them are linearly independent. Now, to find out the particular solution, because this is an initial value problem, which we had started.

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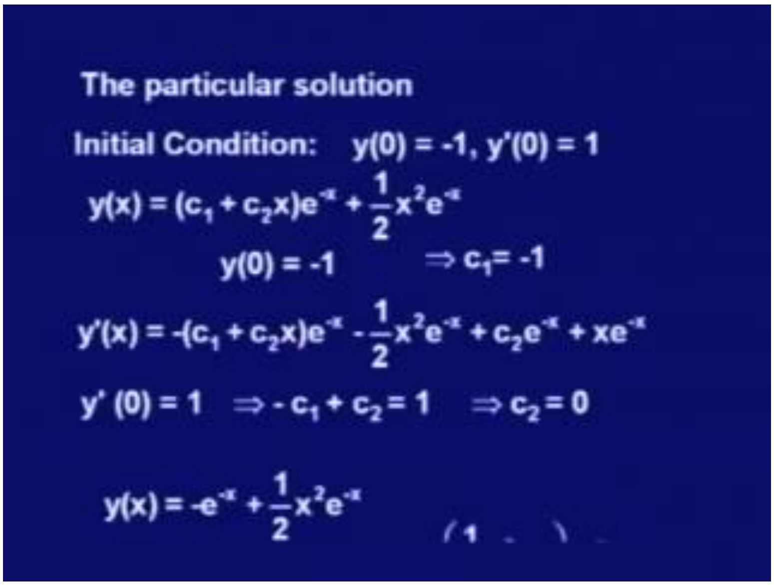

So, for finding out the Particular solution, what are the initial conditions given, we have been given that the function at 0 is minus 1, and it is derivative at 0 is 1. What the general solution we had got, c 1 plus c 2 x e to the power minus x, plus half x square e to the power minus x. Now, if I substitute this function, at x as 0, that is I find out the value of the function at 0, that is given as minus 1, from here in this solution, if I am substituting x as 0, I would get only c 1, so it gives c 1 is equal to minus 1.

Then find out it is derivative, derivative is minus c 1 plus c 2 x e to the power minus x minus half x square e to the power minus x plus c 2 times e to the power minus x plus x times e to the power minus x. Now, again the second condition is, y dash at 0 is 1, so again we are putting x is equal to 0, what I would be left, I would be left, minus c 1 plus c 2, which is equal to 1, from the first, result we have got c 1 as minus 1 thus gives, c 2 as 0.

So, we have got the particular solution that is I will put, c 1 is equal to minus 1 and c 2 is equal to 0, in our general solution of homogeneous equation. So, I would get the solution of, initial value problem as, minus e to the power minus x, plus half x square e to the power minus x. We can check that, this is satisfying both initial conditions, that is the function value at 0 is minus 1, and the function derivative of the function at 0 is plus 1.

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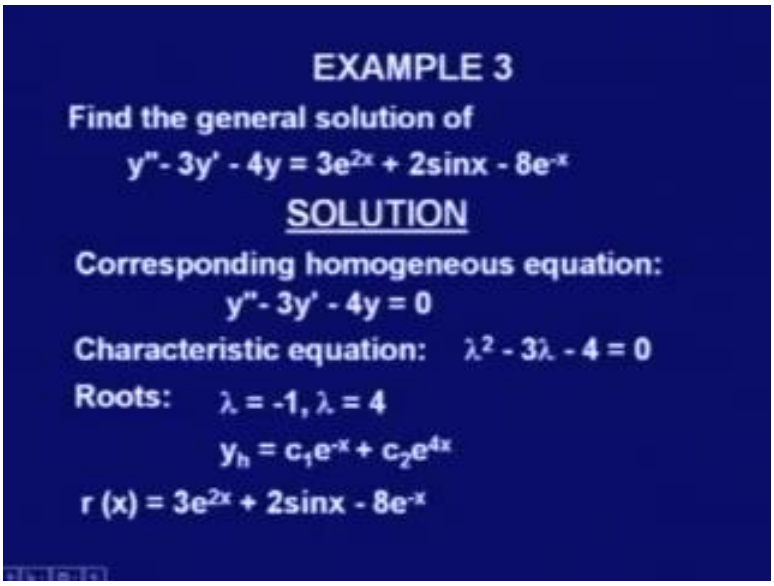

Let us do, one more example over here, find the general solution of y double dash minus 3 y dash plus sorry, y double dash minus 3 y dash minus 4 y is equal to, 3 times e to the power 2 x plus 2 times sin x minus 8 times e to the power minus x. Now, we see, our right hand side is actually, sum of many functions, which are in our table, e to the power 2 x e to the power minus x sine x and so on. So, it says is that, we have to use the, sum rule.

Let us see, if we have to use some other rules also, so first we go for the solution, what is the corresponding homogeneous equation, y double dash minus 3 y dash minus 4 y is equal to 0. To solve this, we require the characteristic equation, what is the characteristic equation lambda square minus 3 lambda minus 4 it is equal to 0. We do find it out, that it is nothing but, lambda minus 4 into lambda plus 1 the factors, so it will have two roots, minus 1 and 4, they are real and extinct.

So, the general solution of homogeneous equation will be, c 1 e to the power minus x plus c 2 e to the power 4 x. Now, we see right hand side, right hand side r x we have 3 times e to the power 2 x plus 2 sine x minus 8 times e to the power minus x, now so if I just go with the our table, if you do remember something that is, e to the power gamma x we do take, c times e to the power gamma x. So, here if I do take I have to choose, one y p as e to the power minus x.

And, that is here again, as a solution corresponding to the 1 root, that means, we have to use the modification rule, that is, in this one we would be going to use, the sum rule as well as the modification rule. For the sum rule we require, which functions we have to choose, we have here function of, e to the power gamma x and sine x, so let us consent our table ((Refer Time: 47:54)), e to the power gamma x choice means, I have to use constant times, e to the power gamma x, and sine omega x, I have to use the function, which is having a constant times sine omega x, and a constant times cosine omega x, with plus.

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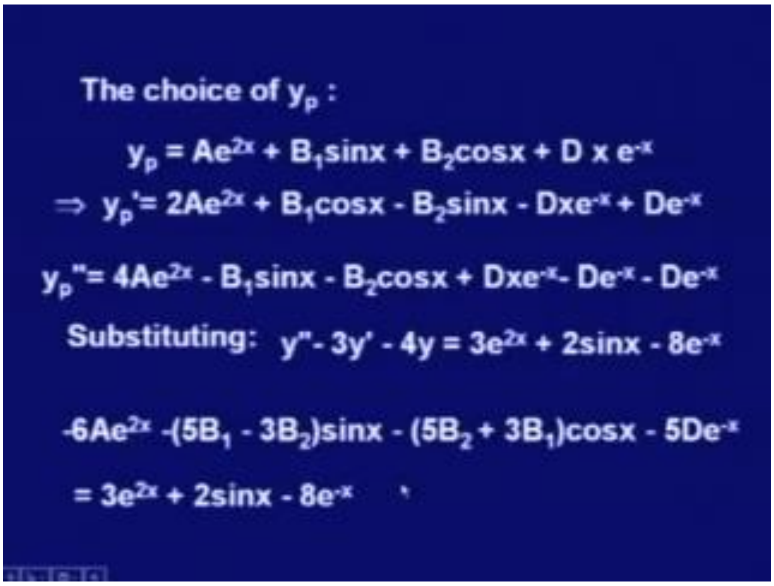

So, let us see, what will be our choice of y p, the choice of y p would be, A times e to the power 2 x plus B 1 sine x plus B 2 cos x plus D x times e to the power minus x, why I had chosen here since, e to the power minus x was already a solution. So, I have to modify it as, x times e to the power minus x, so here I had used the modification rule, and in this complete one, I had use the sum rule. So, we have got this as, the choice of y p.

Now, find out it is derivative, 2 A e to the power 2 x plus B 1 cosine x minus B 2 sine x minus D x e to the power minus x plus D times e to the power minus x. Second derivative, 4 A e to the power 2 x minus B 1 sine x derivative of sine x is cosine x, so B 2 cosine x, derivative of D x e to the power minus x is plus D x e to the power minus x minus D e to the power minus x and the derivative of this is minus D e to the power minus x.

Now, substitute this in the given equation, what was the equation, equation was y double dash minus 3 y dash minus 4 y is equal to 3 times e to the power 2 x plus 2 sine x minus 8 times e to the power minus x. So, substituting it, and of course, the calculations you can check it, we get minus 6 A times e to the power 2 x, minus 5 B 1 minus 3 B 2 sine x minus 5 B 2 plus 3 B 1 cosine x minus 5 D e to the power minus x, this should be equal to the right hand side, that is 3 times e to the power 2 x plus 2 times sin x minus 8 times e to the power minus x.

Now, equate the coefficients, we get the coefficient of here you see, e to the power 2 x, it is coefficient here is minus 6 A, here is 3. So, I would be equating this as this one, the coefficient of sine x here is, minus 5 B 1 minus 3 B 2, and here is plus 2, so I would get 5 B 1 minus 3 B 2 is equal to minus 2. The cosine x term, we do not have over here, so the this 5 B 2 plus 3 B 1 would be 0, the coefficient of e to the power minus x is minus 5 D e here, here it is minus 8, so I would get 5 D is equal to 8.

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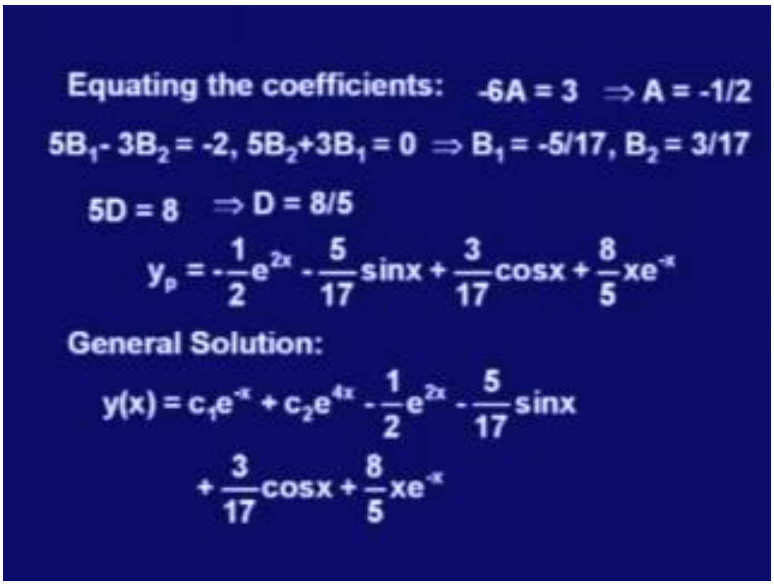

So, equating the coefficient, I am getting minus 6 A is equal 3, which gives me A is equal to minus half. Second, what we are getting for cosine and sine, 5 B 1 minus 3 B 2 is equal to minus 2 and 5 B 2 plus 3 B 1 is equal to 0, solving these equations we get, B 1 as minus d 5 by 17 and B 2 as 3 by 17 and last one, we are getting 5 D is equal to 8, gives me D is equal to 8 by 5. So, what will be our particular solution substitute this A B 1 B 2 and D, minus half e to the power 2 x minus 5 by 17 sine x plus 3 by 17 cosine x plus 8 by 5 times x into e to the power minus x.

So, the general solution would be, this is the general solution of, homogeneous equation c 1 e to the power minus x plus c 2 e to the power 4 x, this particular solution, that is same thing I have written here. And, of course, we get is that is, since here it is e to the power minus x here, I am getting is x times e to the power minus x, so now, this is what is the, general solution of this homogeneous equation, where now we had used, all the three rules, that is we have seen the basic rule, then we have seen the modification rule, we had used the sum rule also.

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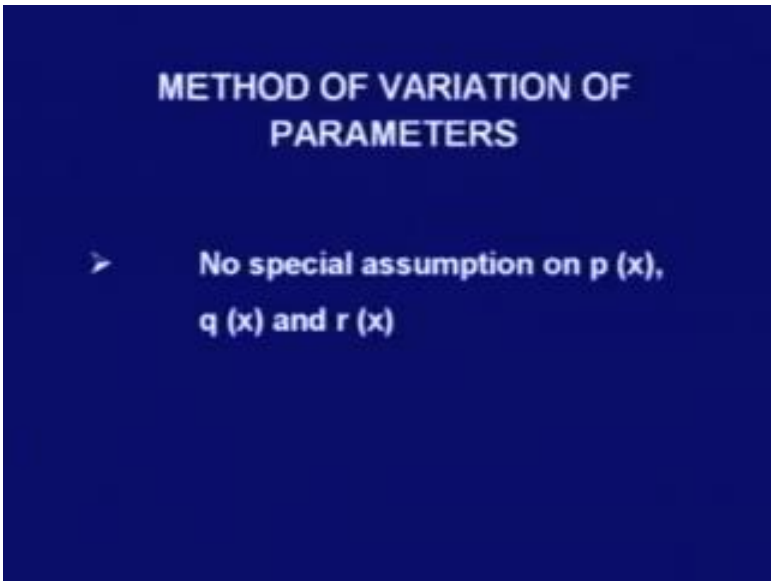

Now, we will learn the second method, of finding the particular solution that is, method of variation of parameters. This method is more general, it does not require in a special conditions, on your equation, neither it requires any condition, on your right hand side r x. It is more general, but little bit more complicated, let us see what this method says, is that is, may have p x q x as any, this no special assumptions on this, how this method is working, or what is this method.

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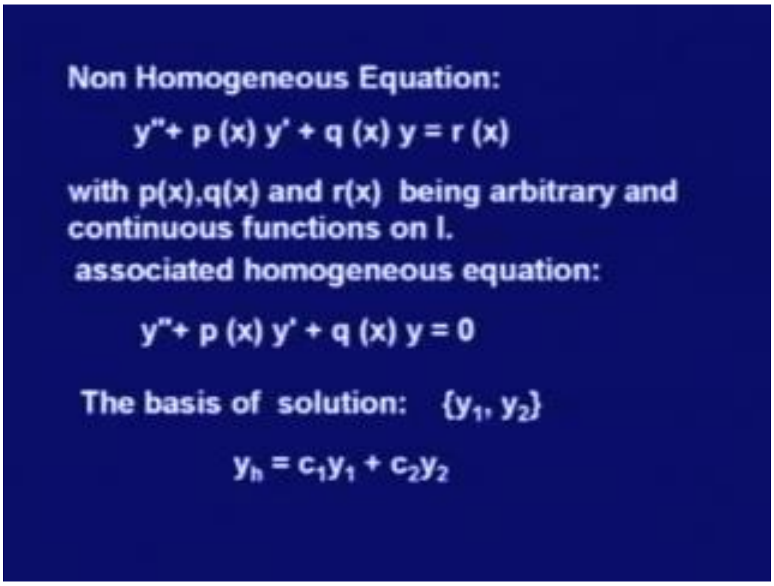

Let us say, a non homogeneous equation, y double dash plus p x y dash plus q x y is equal to r x, where I do have that, p x q x and r x are arbitrary, but continuous only, that is we require the condition only, that this coefficients on the right hand side has to be continuous and and some interval I. We also require, to know the, basis solutions of the corresponding homogeneous equation, so associated homogeneous equation y double dash plus p x y dash plus q x y is equal to 0. Let us say, that is the fundamental set of system or the basis of solution, if y 1 and y 2. Then how we are finding the particular solution, this would be the general solution of homogeneous equation.

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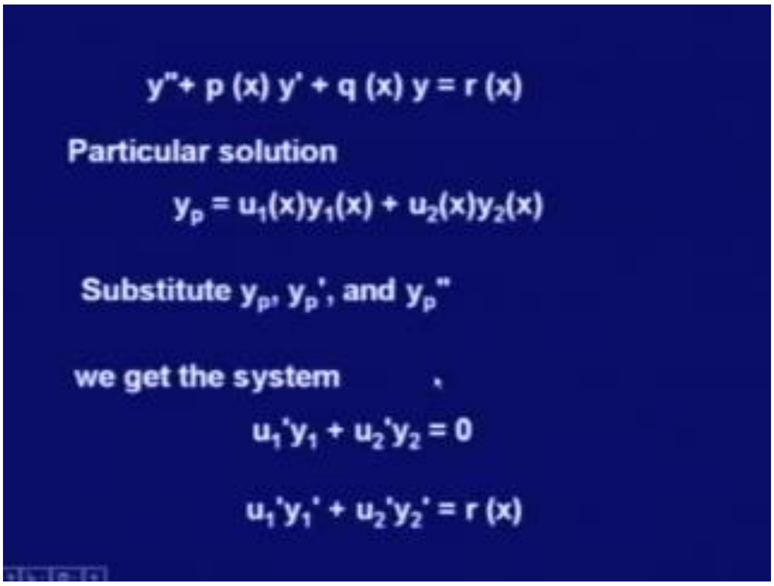

The particular solution we are saying is another function, as u 1 y 1 plus u 2 y 2, that is we had obtained, corresponding homogeneous equation, it is general solution. Then for particular solution, we had chosen u 1 y 1 plus u 2 y 2, where this u 1 x and u 2 x are, any arbitrary functions of x only continuous on differentiable. How to obtain this u 1 and u 2, for this what we are doing is, we are substituting this y p y p dash and y p double dash in this given equation.

So, when we substitute it, we do use this condition also, that is y 1 and y 2 is the solution of corresponding homogeneous equation, all those things when we do, we can find out this u 1 u 2, as the solution of the system of, this system. That is u 1 dash y 1 plus u 2 dash y 2 is equal to 0, u 1 dash y 1 dash plus u 2 dash y 2 dash is equal to r x, you see, here I am getting the two equations, which are having y 1 y 1 dash y 2 and y 2 dash.

But, they are both are having u 1 dash and u 2 dash, that is I would get, the solution of this, system of linear equation, as the u 1 dash and u 2 dash, and then u 1 and u 2, we can obtain as, integral of them, so what will be that solution.

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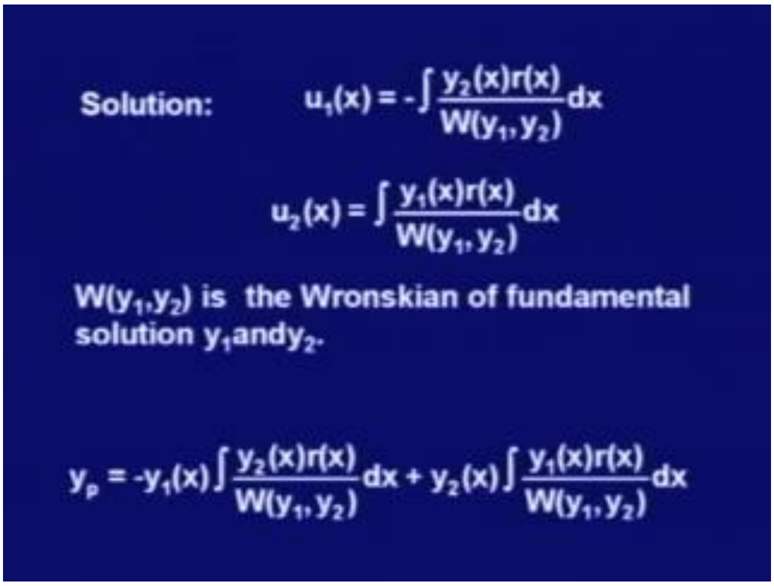

Let us see here, u 1 x is minus integral y 2 x r x upon, wronskian of y 1 y 2 W of y 1 y 2 is the wronskian of the solution y 1 and y 2, y 1 y 2 are the general solution, the fundamental set of solution of associated homogeneous equation, this integrated with respect to x. And, u 2 is y 1 x r x upon wronskian of y 1 y 2, integrated upon, with respective x, these are the two solutions, so what I will get that particular solution, we would get, so here, this is being written that, wronskian of W y 1 y 2 is the wronskian of the fundamental solutions y 1 and y 2, y p we would get minus, that is u 1 y 1.

So, minus y 1 integral y 2 r x upon, wronskian of y 1 and y 2, plus y 2 times integral y 1 r x upon wronskian of y 1 y 2, integrated with respective x. You see here, that we are now, there we had learnt the wronskian, and we said is we are using it for, showing the linear independence of the solution now you find it out, if they are linearly independent, we do know that this wronskian cannot be 0. Since, it cannot be 0, so in the division it would, create any problem, and we are seeing now here, that is how we are using wronskian, to find out the particular solution of non homogeneous equation.

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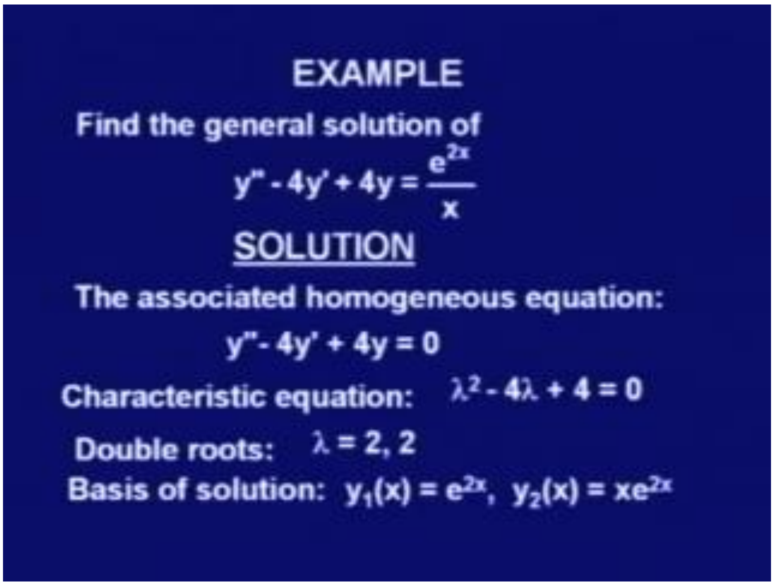

let us try to learn, that how to use this method, so let us learn one with one example, find the general solution of, y double dash minus 4 y dash plus 4 y is equal to, e to the power 2 x upon x. You see of course, we do not require any special conditions on the, coefficients of the differential equation, but since, till now we had learnt, only to find out the solution of linear differential equations, with constant coefficients.

So, in this example I have taken this, coefficients to be constant, but the right hand side is not of the, form in which I could use the, undetermined coefficient, that is method of undetermined coefficient. Since, what I am having in the right hand side, e to the power 2 x upon x, now it is not coming as the sum of any known functions, because it is the ratio of two functions, which are in that table. So, I cannot use that method, so I do not have any option, other than to use this method of variation of parameter.

So, let us try to solve this, first we will find out, associated homogeneous equation, that is y double dash minus 4 y dash plus 4 y is equal to 0, to solve it, we require, the characteristic equation, that will be lambda square minus 4 lambda plus 4 is equal to 0. Now, we see this is nothing but, lambda minus 2 whole squares that is again we are getting, the double root lambda is equal to 2 and 2. What will be the general solution of, this or what will be the fundamental set of solutions, the fundamental set of solutions would be e to the power 2 x and x times e to the power 2 x.

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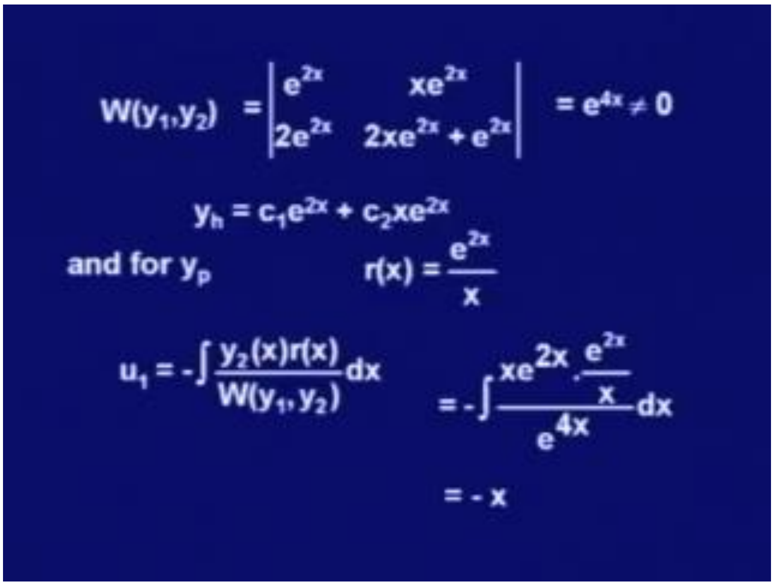

Now, so what should be the Wronskian of this fundamental solutions, that is I have got two solutions, e to the power 2 x and x times e to the power 2 x. So, wronskian, first row is as such, second row the derivative, that is two times e to the power 2 x, 2 x times e to the power 2 x plus e to the power 2 x, when we solve this determinant, we get since here we would be getting is 2 x e to the power 4 x plus e to the power 4 x minus 2 times x times e to the power x, I would get e to the power 4 x, which is not 0.

Since, these are linearly independent, I am getting this is not 0 for any value of x, so we have got e to the power 4 x, as the wronskian. So, what will be now, my u 1 and u 2 rather than, using that putting u 1 y 1 in those once and and finding out, what it is we would be using the formulae’s directly, y h is c1 e to the power 2 x plus c 2 x times e to the power 2 x, for y p r x is e to the power 2 x upon x. So, now what will be my u 1, the formula says, minus integral y 2 x r x upon, wronskian of y 1 y 2 integrated to respective x.

Substitute this values of y 2 r x and so on, y 2 is x e to the power 2 x, r x is e to the power 2 x upon x, wronskian W y 1 y 2 is e to the power 4 x. Let us substitute, x e to the power 2 x into e to the power 2 x upon x upon e to the power 4 x, we are getting is, that this x is being cancel out by this, e to the power 4 x is being cancelled out with this. Am getting is integral d x, that is the constant with respect to x I would get, minus x, so u 1 I have got minus x.

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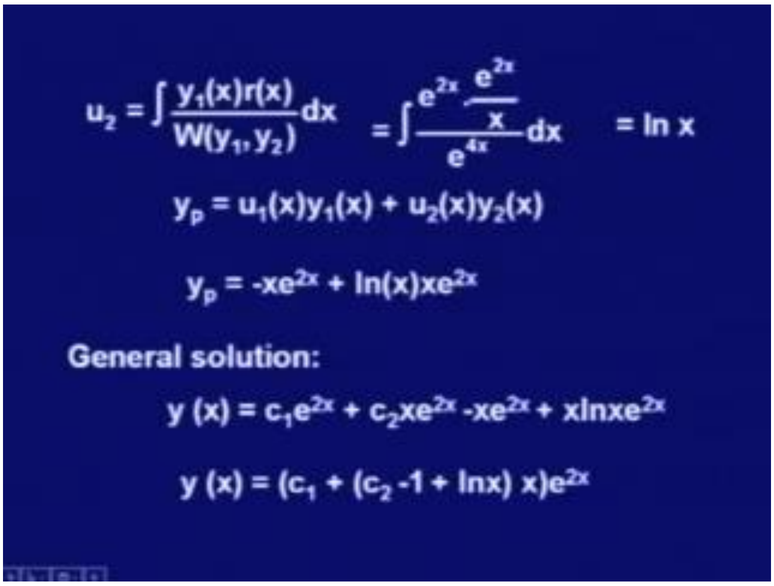

What will be u 2, u 2 is y 1 x r x, wronskian of y 1 y 2 integrated with respective x, y 1 is e to the power 2 x, r x is e to the power 2 x upon x; wronskian is e to the power 4 x. What I am getting is e to the power 4 x 4 x is cancelling out, I would be getting integral of 1 by x d x, which we do know is, log x. So, what will be my particular solution, u 1 y 1 plus u 2 y 2, u 1 we have got minus x, and what was y 1, e to the power 2 x, so minus x e to the power 2 x, u 2 is x times e to the power 2 x, Why we have got, log x.

So, we are getting log x times x e to the power 2 x, so what will be the general solution, general solution would be c 1 e to the power 2 x, plus c 2 x times e to the power 2 x, minus x times e to the power 2 x, plus x log x e to the power 2 x. Now, we can rewrite it, because we are having e to the power 2 x, all the places, so we can rewrite it as, c 1 plus c 2 minus 1 plus log x, whole multiplied with x, then whole multiplied with, e to the power 2 x.

So, this is what is, now here we have learnt, one method, which says is method of variation of parameter, how we are obtaining, the particular solution, when the right hand side is, not of the special form, and we are not getting, that as from the undetermined coefficients. Of course we have learnt, only about differential equations, which have constant coefficients, now this is what, all about we had learnt about, linear differential equations, with constant coefficients, homogeneous equations, we had learnt, how to solve them.

We had learnt about, non homogenous equations as well, where we had learnt, that is all the methods which we had learnt, in the non homogeneous equations, they were applicable, both with whether the coefficients are constant, or the coefficients are, not constant. They are general functions, they are not changing, but all the examples, which we had done, they were having all the coefficients as constants, why, since we had learnt in the, linear differential equations only to solve the constant coefficients.

That is why, in the non homogeneous, we could not do any example, where the coefficients are, not constants. So, next lecture, we would learn about, the linear differential equations, which have the coefficients as, non constant that is, my p x and q x were, are functions rather than the constants. So, that is all for the today's lecture.

Thank you for this.