Mathematics - III
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Lecture - 4
Linear Differential Equations of Second Order-Part-1

Welcome to lecture series and differential equations for under graduate students, today's topic is Linear Differential Equation of Second Order, till now we had learnt about the first order differential equations and their solutions. Now, we will go for higher order, so we will start with second order and we will start with linear differential Equation of second order.

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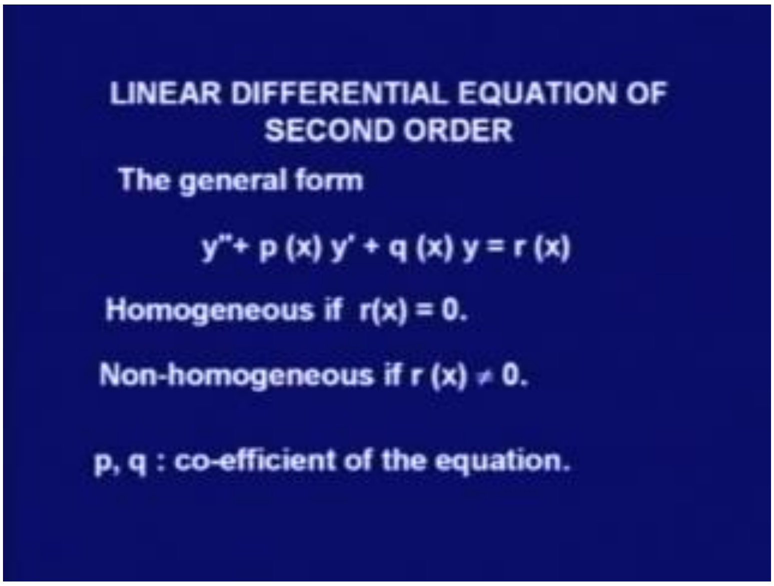

What are they? The general form is y double dash plus p x y dash plus q x times y is equal to r x, here y is the unknown function, y dash is first derivative of y and y double dash is second derivative of y. We are seeing here is that is the highest order occurring in this equation is this second order, so this is the second order equation. More over we are finding it out that y, y dash and y double dash, they are occurring in first degree only and separately, none of the terms contain the two things simultaneously, so this is a linear equation. Now, right hand side r x and the coefficients p x and q x they are function of x only, so we have this as a, so any equation effect can be written in this form this is called the linear differential equation of second order.

Here if this right hand side is 0, then we call this a homogeneous equation, if r x is any function of x we call it non homogenous; moreover this p x and q x they are called coefficients of the equation; let us see some example of the second order equation.

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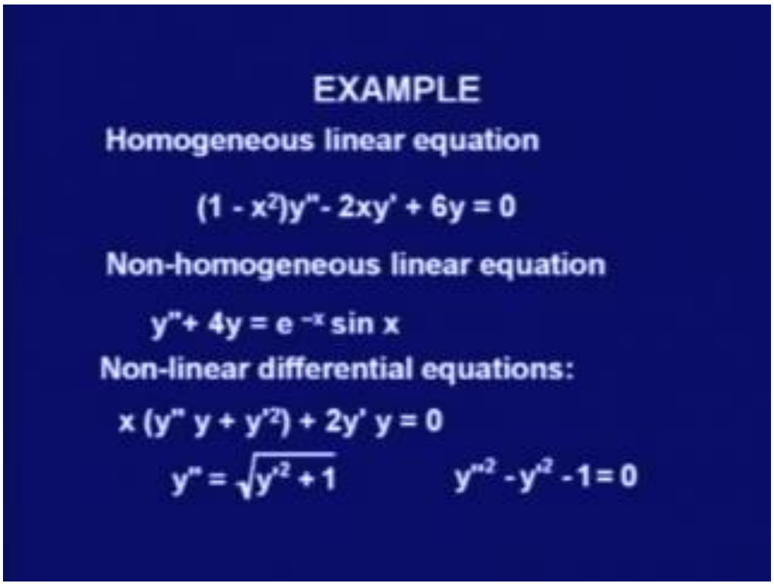

First see homogenous linear equation, 1 minus x square y double dash minus 2 x y dash plus 6 y is equal to 0, we see here is that is y, y dash and y double dash they are occurring in first degree and separately. The coefficients are 1 minus x is square 2 x and 6 they are none of them are containing y and the right hand side is 0, so this is a homogenous linear equation, of course this is not in a standard form.

Let us see another example non homogenous linear equation, y double dash plus 4 y is equal to e to the power minus x times sin x. Again we see here that y and y double dash they are occurring in first degree and separately and the right hand side is a function of x which is not 0, so this is non homogenous and linear. Let us see some example of non linear differential equations see here the first example, x times y double dash y plus y dash square plus 2 y dash y is equal to 0. We see here that y double dash and y they are occurring in the same term, again y dash is having the second degree, and here again we do have y dash and y that is occurring in the same terms, so this is non linear.

Now, let us see another example y double dash is equal to square root of y dash square plus 1, this is not in rational form, so see if I rationalize it I do get y double dash square minus y dash square minus 1 is equal to 0. We find it here out that of course, this equation is of second order, but both y dash and y double dash are occurring in the second degree, so this is not a linear equations. So, both of these are examples of non-linear equations. Now, we will try to learn about first about the homogenous linear equations, so let us first revisit some of the definitions which we already have done in the first order and in general. We will revisit in the terms of these homogenous linear differential equations of second order.

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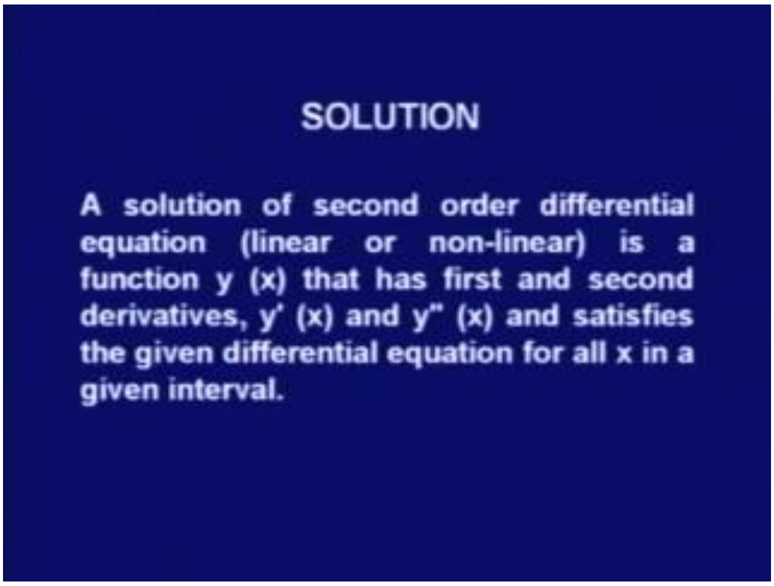

So, first definition we will see of solution, a solution of second order differential equation is a function y x that has first and second derivatives as y dash x and y double dash x. And satisfies the given differential equation for all x in a given interval that function we will call a solution, let us see one example.

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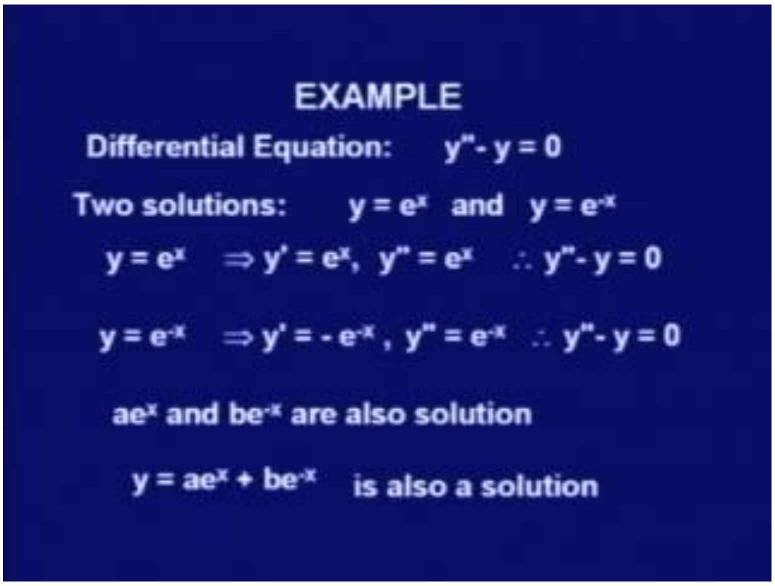

Let us take a differential equation y double dash minus y is equal to 0, this is a second order equation, we can see that e to the power x and e to the power minus x, these two are the solution of this equation. Why they would be solution, if I substitute y, y dash and y double dash in this equation they should satisfy the equation, so let us see it one by one. First take the y is equal to e to the power x, it is says y dash would be also e to the power x, as well y double dash will also be e to the power x.

So, we are getting y double dash and y they are same, so we will satisfy the equation y double dash minus y would be 0, so this is a solution. Similarly, we can check with e to the power minus x, what will be it is derivative minus e to the power minus x, second derivative e to the power minus x, again we are getting the second derivative and the function they are same. So, their difference would give me 0 that is again it is satisfying our equation, so both are solution of this differential equation.

Moreover we can see, if I take a to the times e to the power x and b to the times and e to the power minus x, they are also solution of the same equation, you can check it here. Moreover if I take a linear combination of these two solutions, that is a times e to the power x plus b times e to the power minus x, this will also be a solution of this equation, let us try to see this function.

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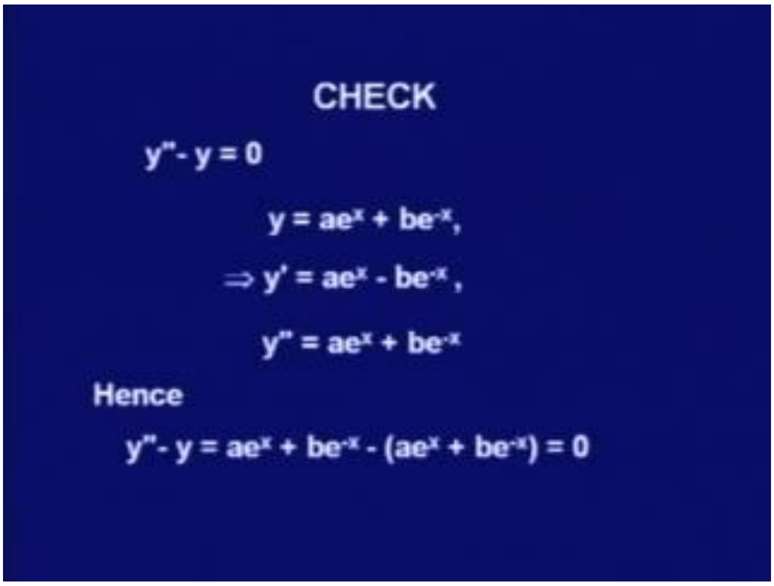

Our equation is y double dash minus y is equal to 0, what we want to check that a times e to the power x plus b times e to the power minus x is a solution, so we have to substitute the function and it is derivative. What is first derivative, a times e to the power x minus b times e to the power minus x, second derivative a times e to the power x plus b times e to the power minus x, again we find out that the function and it is second derivative they are same.

So, their difference would be 0 that is it is satisfying this given equation, so we have got actually what we have seen, if a function is a solution of a given differential equation it is constant, multiple, is also solution of it is this equation, moreover. If two functions are solution of a differential equation they are linear combination was also becoming a solution of differential equation, so this differential equation was homogenous. Now, from here we get our first result, that is the basic property of the fundamental theorem about the homogenous linear differential equation of second order.

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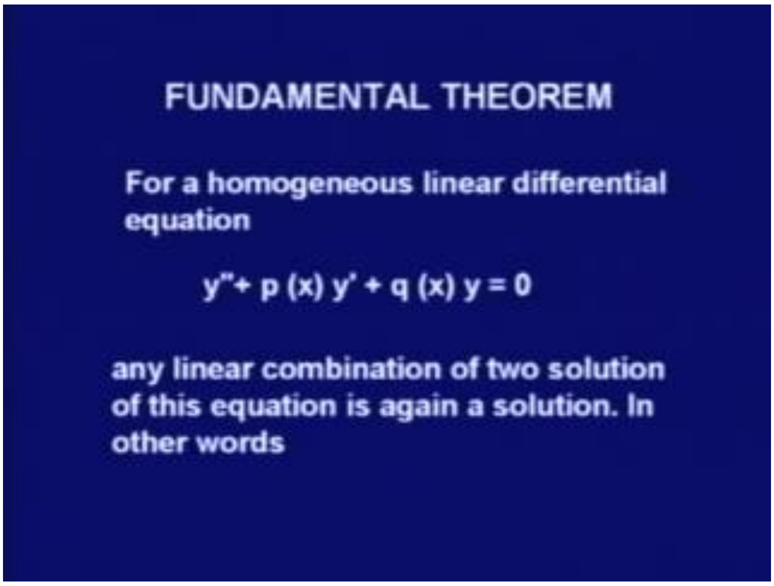

For a homogenous linear differential equation, y double dash plus p x y dash plus q x y is equal to 0, any linear combination of two solution of this equation is again a solution.

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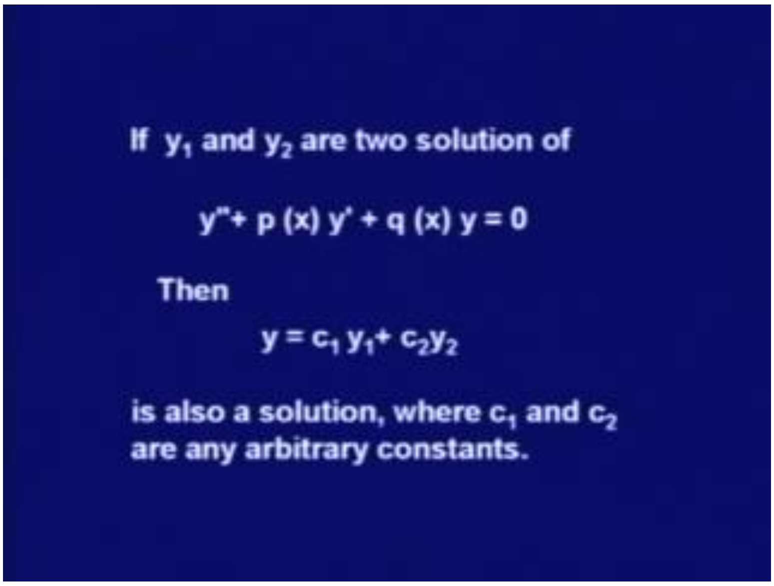

Or in other words, if y 1 and y 2 are two solution of the given equation that is y double dash plus p x y dash plus q x y is equal to 0. Then c 1 y 1 plus c 2 y 2 that is the linear combination of both solution will also be a solution, where this c 1 and c 2 are any arbitrary constants.

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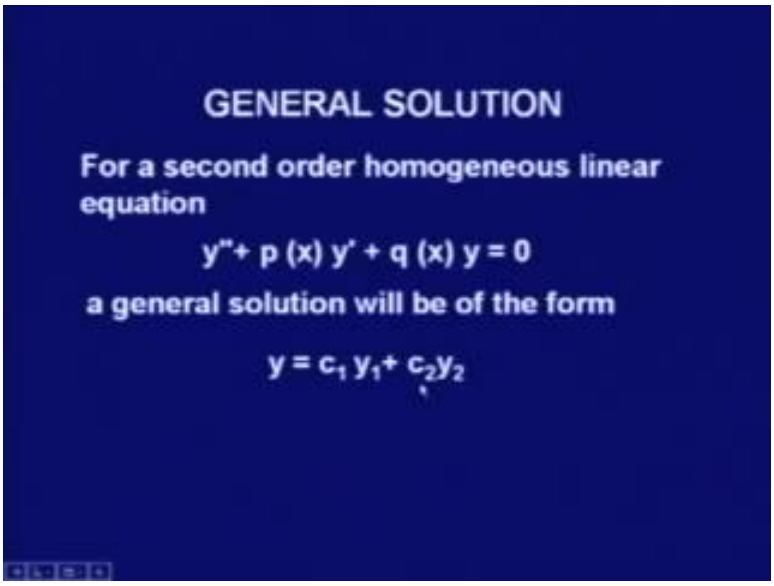

From here we just come to the general solution, for a second order homogenous linear equation, the equation as again same in the standard form y double dash plus p x y dash plus q x y is equal to 0. A general solution will be of the form c 1 y 1 plus c 2 y 2, where y 1 and y 2 are two solutions of this given homogenous equation, and c 1 c 2 are any constants and moreover particular initial value problem.

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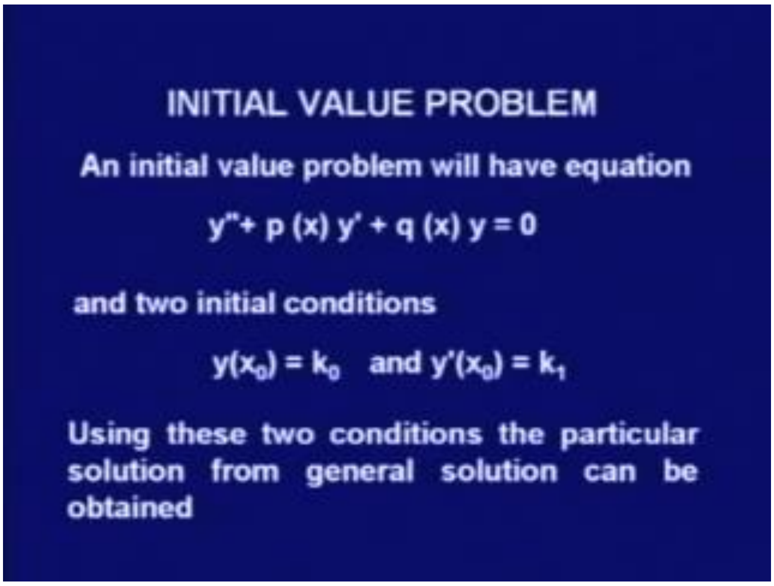

A initial value problem will have the equation that y double dash plus p x y dash plus q x y is equal to 0 and two initial conditions y x naught is equal to k naught and y dash x naught is equal to k 1. If you do remember in first order equation, we had only one initial condition, now it is a second order equation and we will have two initial condition that is the value of the function at a particular point and the derivative of the function at that particular point. The solution of these two conditions if I use in the general solution, we can obtain the solution of this particular problem or this particular solution of initial value problem.

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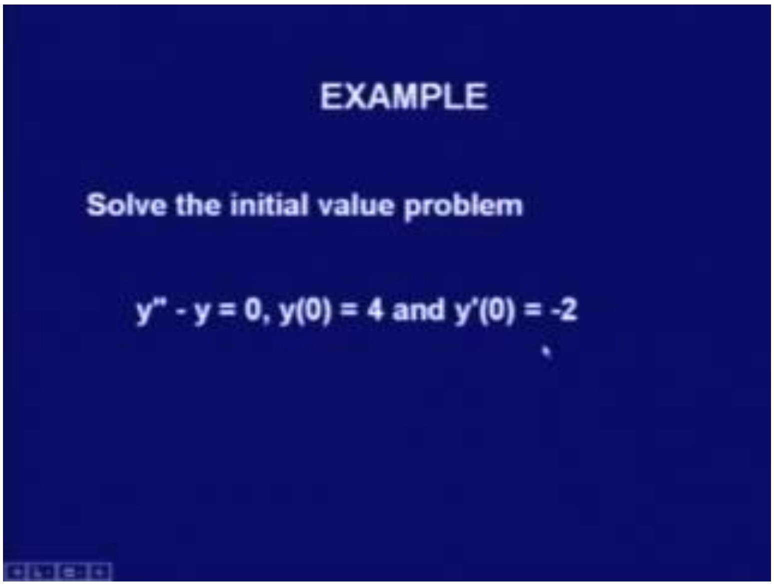

Again let us see this by example, again I have use the same equation y double dash minus y is equal to 0 and two initial conditions are y at 0 is 4 and the derivative at 0 is minus 2.

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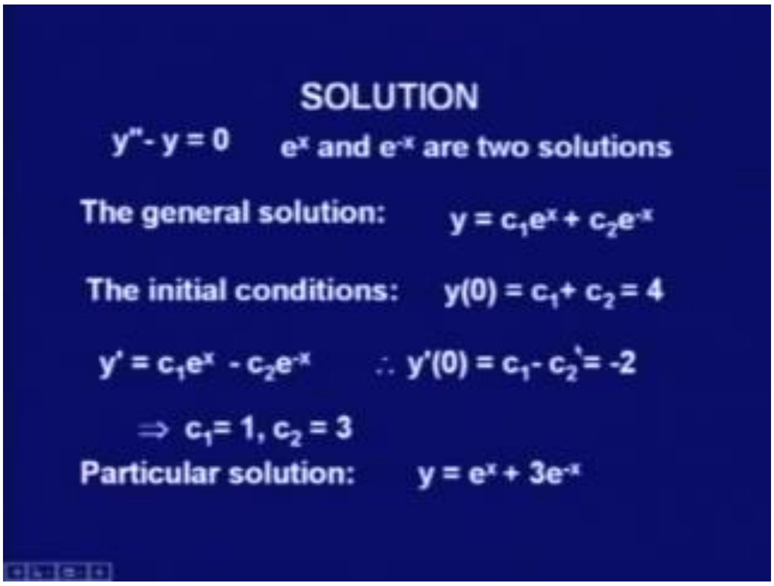

See we had already known that this equation we had seen that e to the power x and e to the power minus x was two solutions, so the general solution will be of the form c 1 times e to the power x plus c 2 times e to the power minus x. Now, we will use the initial conditions that at 0, what will be at 0, 0 it would be c 1 1 plus c 2 that is c 1 plus c 2 and the condition given is y 0 is 4, so we have got the first equation that c 1 plus c 2 is equal to 4.

Second was that derivative at 0, what is the derivative of this, c 1 e to the power x minus c 2 e to the power minus x, now if I put at x is equal to 0 I would get it again as c 1 minus c 2 which is given as minus 2. So, what we have got, we have got these two equations into one is these are linear equations we can solve it, and what will be solution that c 1 is 1 and c 2 is 3. So, what is the particular solution we have got that y is equal to e to the power x plus 3 times e to the power minus x, that is I have put the values of c 1 and c 2 in the general solution, we can check see the graph of this function also.

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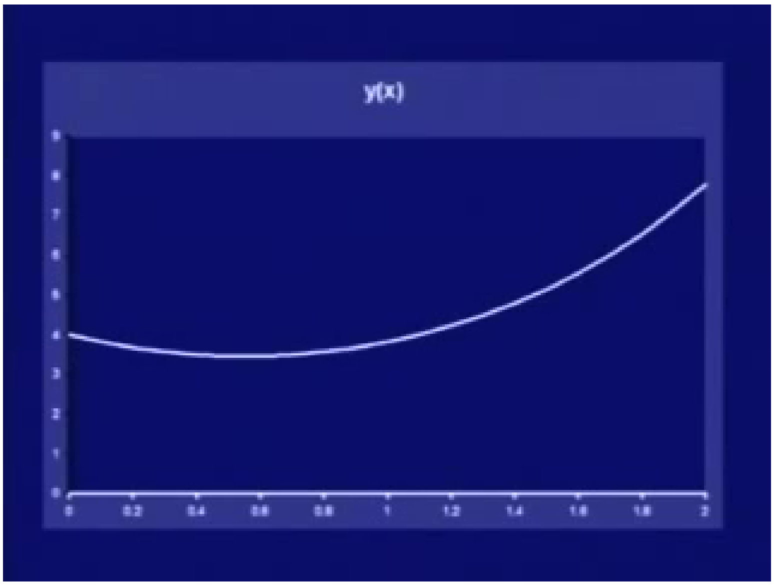

You see this is the graph of the function e to the power x plus 3 times e to the power minus x, we see that here at 0 the value of this function is 4 and the slope of this at this point if you see the slope would be this way, so that is the negative minus 2 slope. Now, let us do some more experimentation in this example.

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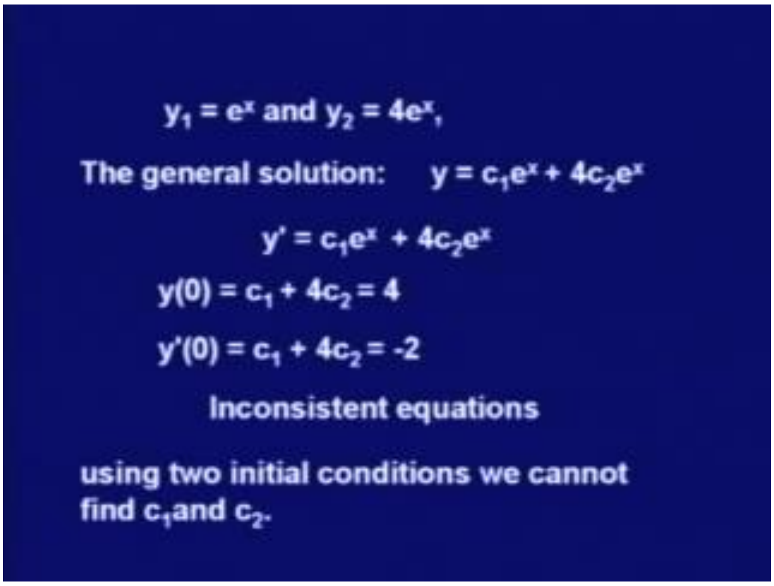

Let us say I choose these two as a solution that is one solution e to the power x, we do know that a times e to the power x is also a solution, so I have chosen here 4 times e to the power x this will also be a solution. So, now I have chosen the two solutions y 1 and y 2 as e to the power x and 4 e times e to the power x, if I am choosing these two as a solution what will be the general solution, that would be c 1 times e to the power x plus 4 c 2 times e to the power x.

Now, I would like to find out the particular solution using those initial conditions, so I would find out what is y dash that would be c 1 e to the power x plus 4 times c 2 e to the power x. Now, put initial condition at 0 it gives c 1 plus 4 c 2 and y dash at 0 will give me again c 1 plus 4 c 2, conditions given a y 0 is 4, so I have got the first equation c 1 plus 4 c 2 is equal to 4 and second equation c 1 plus 4 c 2 is equal to minus 2.

Now, you see I have got two equations, which are not consistent that means, I cannot find out the solution of these, I cannot find out c 1 and c 2 which satisfy these two equations. What it says given initial conditions I cannot find out the values of c 1 and c 2 what is the problem, see here these choice of y 1 and y 2.

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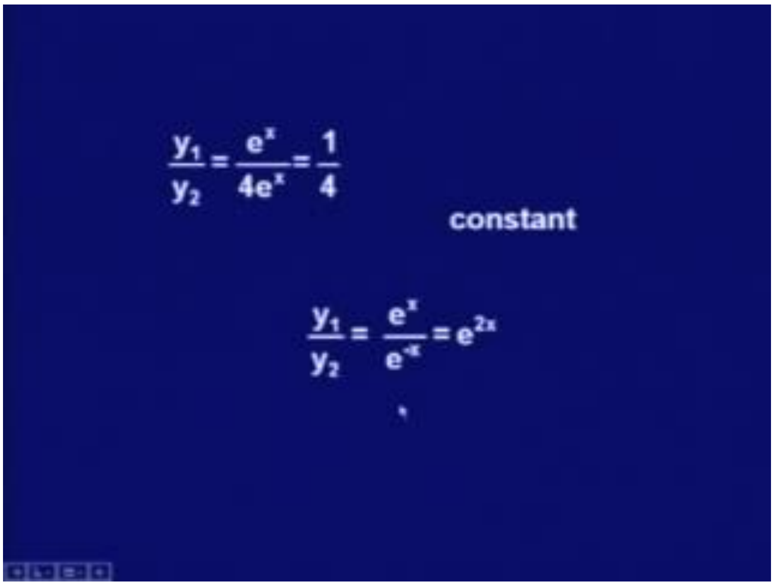

What is this ratio, if I take this ratio this ratio is coming out to be 1 by 4 a constant for all x, while as earlier what was our choice, was e to the power x and e to the power minus x which was e to the power 2 x, that is it is a function for all x. Now, here is the point, when I am getting this as a constant I am not able to get the particular solution. But, when it was not a constant I was getting it as a the particular solution I was able to find it out, that is says is now we require to modify our definition of general solution.

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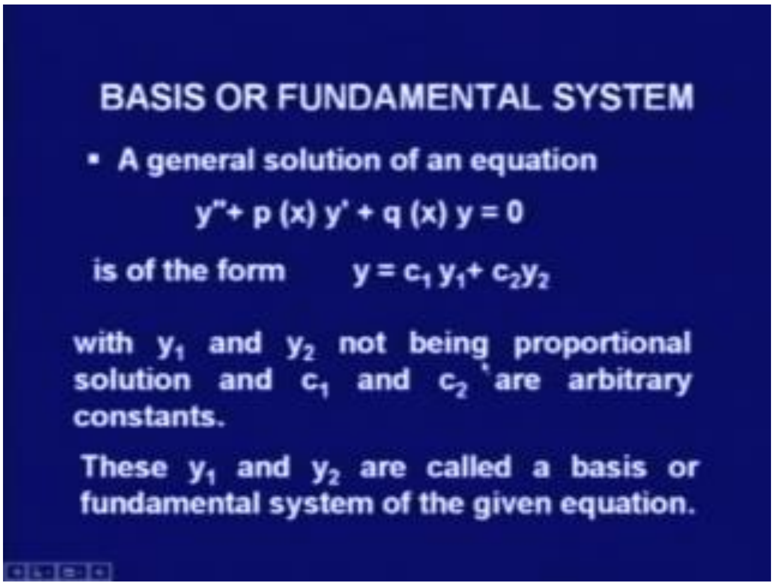

A general solution of an equation y double dash plus p x y dash plus q x y is equal to 0 is of the form c 1 y 1 plus c 2 y 2 with y 1 and y 2 not being proportional solution and c 1 and c 2 are arbitrary constants. Because, we have got problem when the y 1 and y 2 are having ratio as constant that we are calling a proportional, so let us just define this proportional solution.

So, these y 1 and y 2 which we are obtaining us not being proportional they are called the basis of the fundamental system of given equation. So, now what we have got the general solution consist of linear combination of two solutions, and those two solutions has to form the basis or they must be the fundamental system of the given equation, now let us see more definitions about this.

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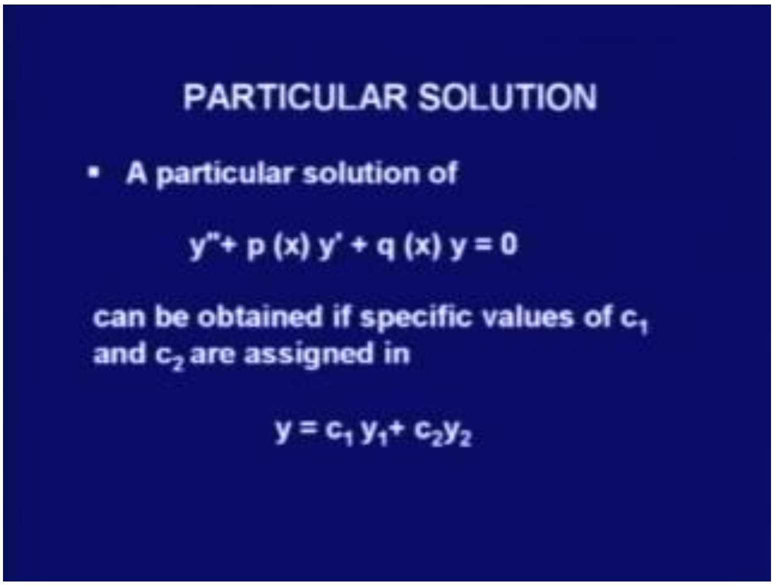

So, particular solution, particular solution of differential equation y double dash plus p x y dash plus q x y is equal to 0 can be obtained, if it is specific values of c 1 and c 2 are assigned in the general solution c 1 y 1 plus c 2 y 2.

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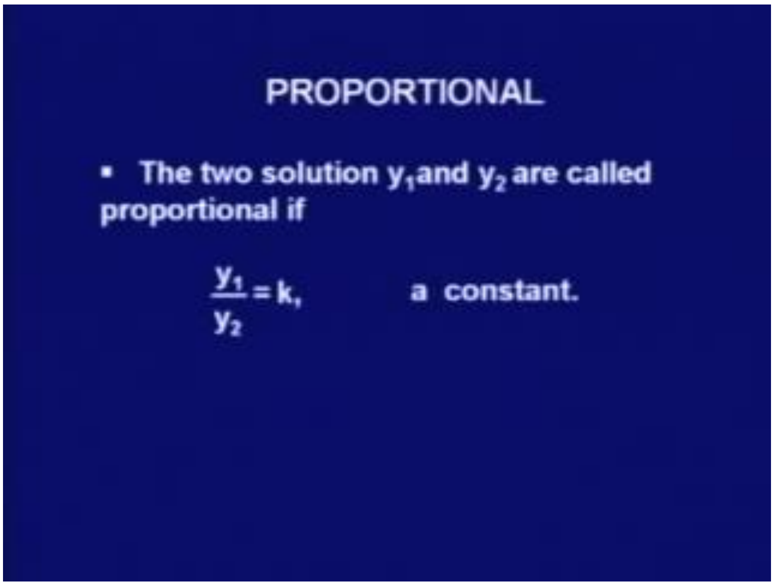

Now, let us define proportional, we call the two solutions y 1 and y 2 as a proportional, if the ratio y 1 by y 2 is a constant. The other term which we are using for being proportional and not being proportional that are more standard in mathematics, they are linear independence and linear dependence.

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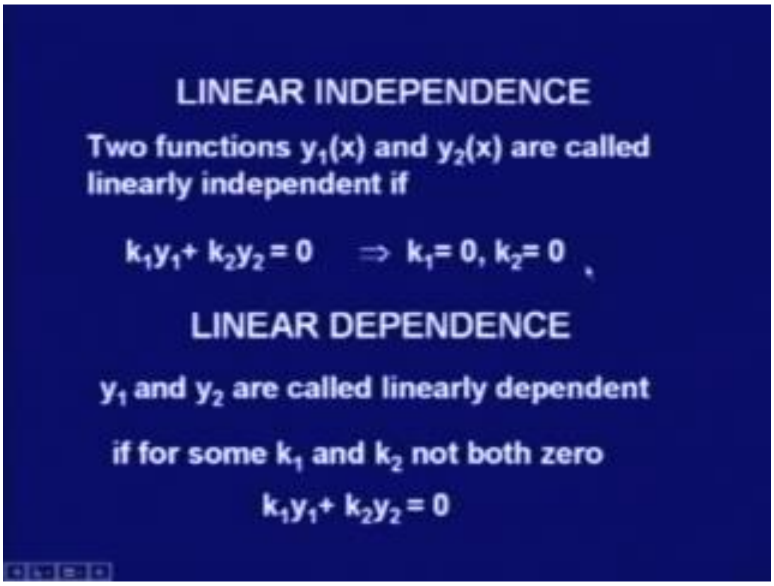

So, let us first see linear independence, two functions y 1 x and y 2 x are called linearly independent, if the linear combination k 1 y 1 plus k 2 y 2 is 0, only if k 1 is 0 1 and k 2 is equal to 0. That is if the two functions are such that, I cannot obtain this linear combination to be equal to 0 unless I do take both the constants to be 0, they are called linearly independent.

The other term is linear dependence, the two functions y 1 and y 2 are called linearly dependent, if for some k 1 and k 2 not both zero this liner combination turns out to be 0. So, in our example if I just see, e to the power x and 4 time e to the power x when I have taken y 1 and y 2 as that, then if I choose k 1 as minus 4 and k 2 as 1, I will get that where k 1 and k 2 both are not 0.

But if I take e to the power x and e to the power minus x, I will not be able to find out any constant other than 0, so that I can make this as a 0. So, we have got this the definition of linear independence and linear dependence, so let us reformulate our definition of basis or the fundamental system.

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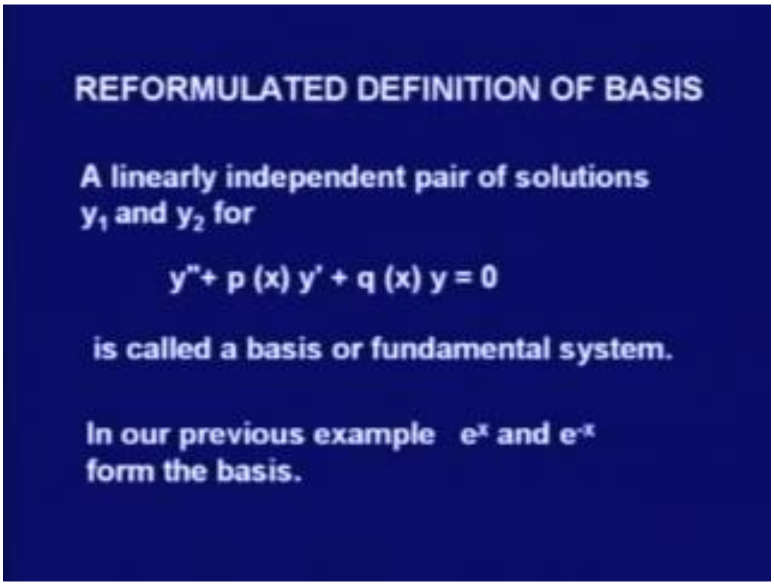

A linearly independent pair of solutions y 1 and y 2 for differential equation y double dash plus p x y dash plus q x y is equal to 0 is called a basis or fundamental system. So, the basis or fundamental system is linearly independent solutions of the second order differential equation. So, in our previous example e to the power x and e to the power minus x, they form the basis or they are the fundamental system.

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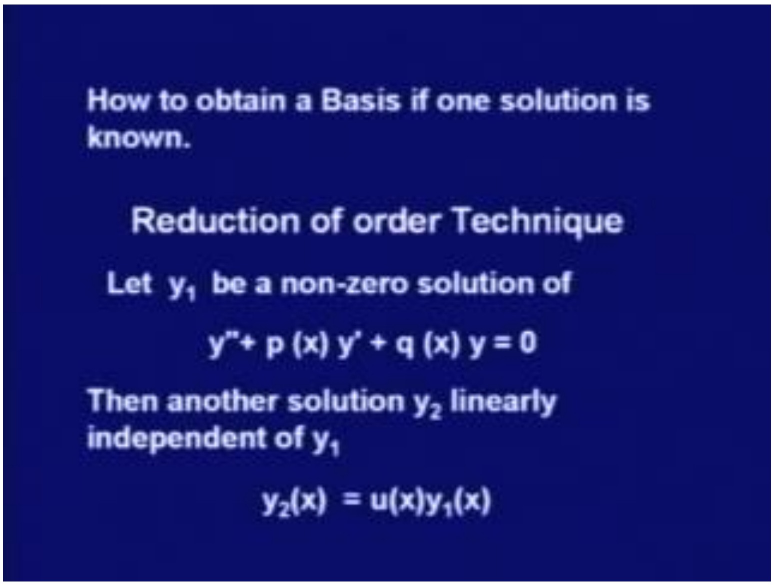

Now, first we will discuss how to obtain a basis if one solution is known, this is a very important one, because the one solution we may be able to guess or we are knowing from some other method. So, of course we would like to know what is the fundamental system, that means we would be interested in finding out the basis.

So, the technique which we are going to learn here is that is the reduction of order technique, see what is this technique. Let y 1 be a non 0 solution of given equation y double dash plus p x y dash plus q x y is equal to 0, then the second solution y 2 which is linearly independent of y 1 can be obtained as u x times y 1 x, now question is how to get it.

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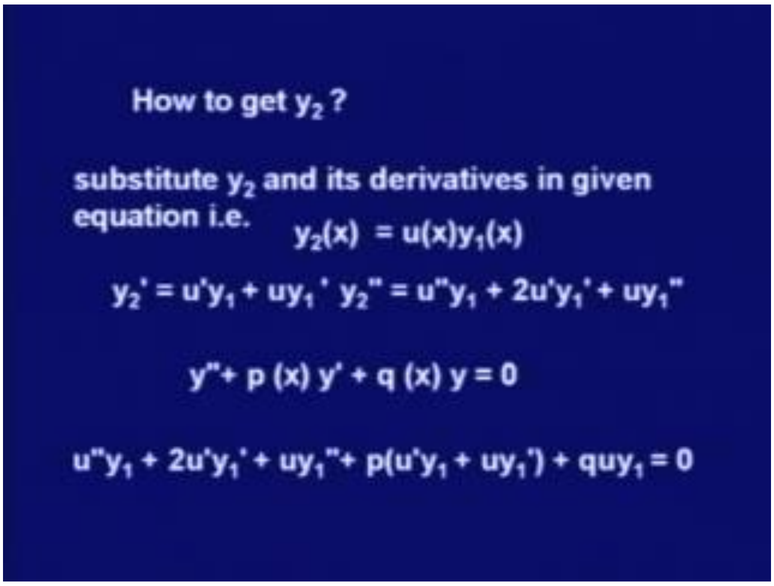

What we do is we simply put y 2 and it is derivative in the given equation, that is y 2 which we are having is u x times y 1 x and y 2 dash u dash y 1 plus u y 1 dash y double dash of course, q double dash y 1 plus 2 u dash y 1 dash plus u y 1 double dash will put all these in the given equation, y double dash plus p x y dash plus q x y is equal to 0. So, we put one by one, we get u double dash y 1 plus 2 u dash y 1 dash plus u times y 1 double dash plus p times u dash y 1 plus u y 1 dash plus q u y 1 is equal to 0, now let us call it the terms in u double dash, u dash and u.

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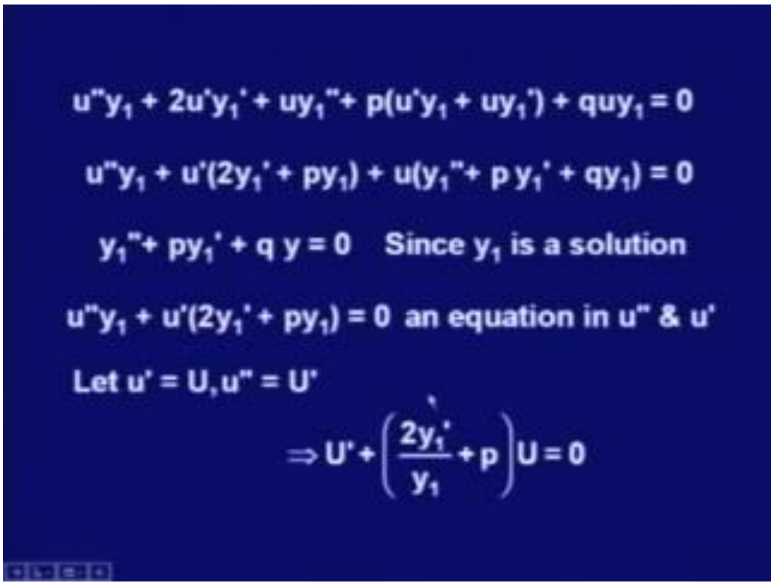

So, the equation is this now we are collecting the term, so the coefficient of u double dash is only y 1 that is the first term, co-efficient of u dash 2 y 1 dash plus p y 1 coefficient of u y 1 double dash plus p y 1 dash plus q y 1, now we are equating it to 0. Now, we see this coefficient of u, this is what is our given equation y double dash plus p y dash plus q y and y 1 is the solution, so it is satisfies, it should satisfy the given equation that means, this equation should be 0.

Because, y 1 is the solution of the given equation, that is says is what is remaining to us u double dash y 1 plus u dash 2 y 1 dash plus p y 1 is equal to 0. Now, this is an equation in u dash and u double dash, let us take u dash as capital U, then u double dash would be capital U dash. Now, do you write this given equation in the terms of this capital U and capital U dash, writing in a standard form we will get U double dash plus 2 y 1 dash upon y 1 plus p times U is equal to 0.

Now, you see this is the first order equation in capital U, that is why we are calling this reduction of order technique. Now, we have got a differential equation of first order, for which we do know the methods to solve it, so we will solve it by using the variable separable method.

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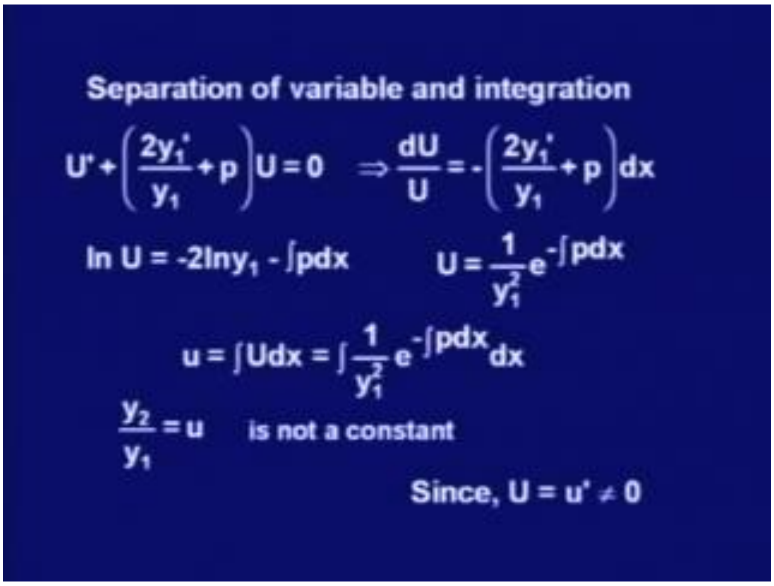

That is first separating the variables and then, integrating it, so separating the variables we get d U upon U is equal to minus 2 y 1 dash upon y 1 plus p times d x. Now, integrating on both the sides we get log U is equal to minus 2 log y 1 minus integral p d x, taking antilog U is 1 upon y 1 square into e to the power minus integral p d x.

Now, U is nothing but, U dash, so what will be this small u integral of capital U with respect to x that is integral of 1 upon y 1 square e to the power minus p d x with respect to x. Now, we see that this function u, this is the ratio of y 2 and y 1 we have got the second solution as y 2, the first solution was y 1, this ratio is u this is not a constant why, we see that u dash that is this capital U, this is 1 upon y 1 square e to the power minus p d x.

Whatever be this p it may be the 0 even then, this exponential function is never 0, that is says is I will never get this capital U or that is U dash as 0, since U dash is not 0 u will never be a constant, thus we are getting that y 2 and y 1 would be linearly independent.

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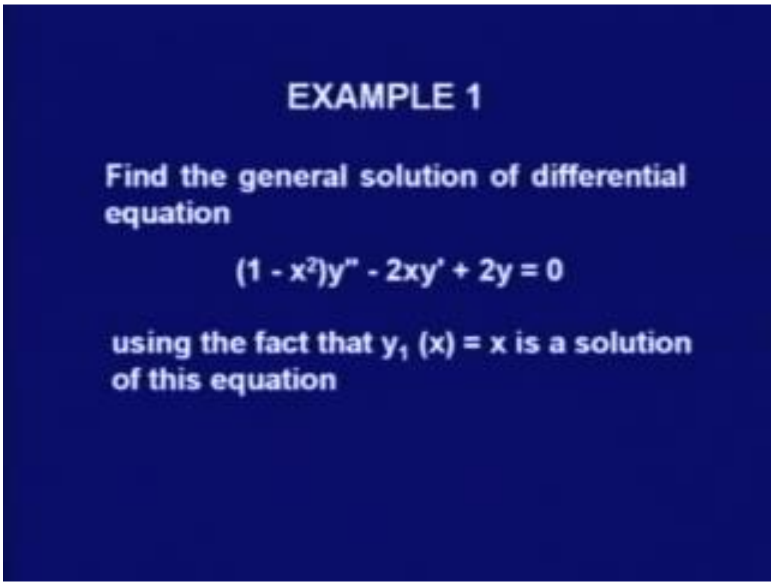

Let us see one example, find the general solution of differential equation 1 minus x square y double dash minus 2 x y dash plus 2 y is equal to 0, using the fact that y is equal to x is a solution of this equation. I am going to use this just now which we have learnt that technique of reduction of order we have given that x is a solution.

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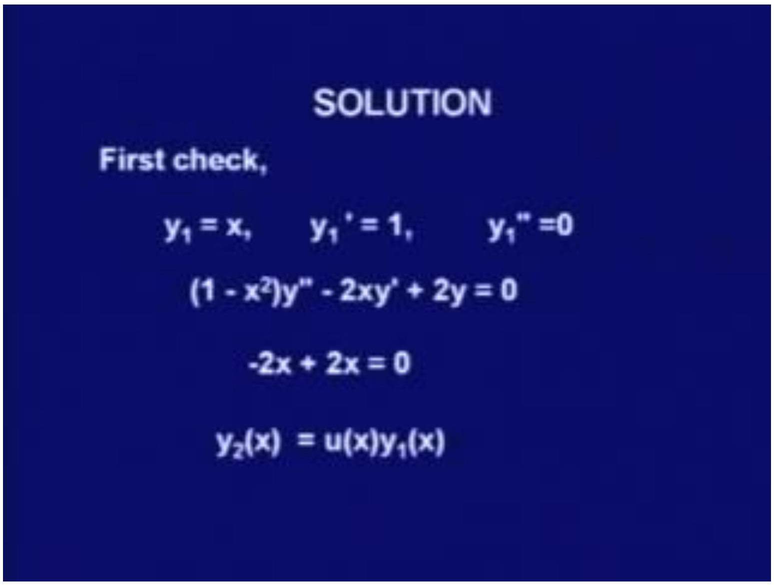

So, first we will check whether it is a solution or not y 1 is x, so y 1 dash would be 1 and y 1 double dash would be 0. If I substitute this in the given equation I would get that the first term is 0, second term will give me minus 2 x, third term also give me plus 2 x that is it is 0 it is satisfying, so this is the solution. What will be the way to find out the second solution will use the technique u x times y 1 x.

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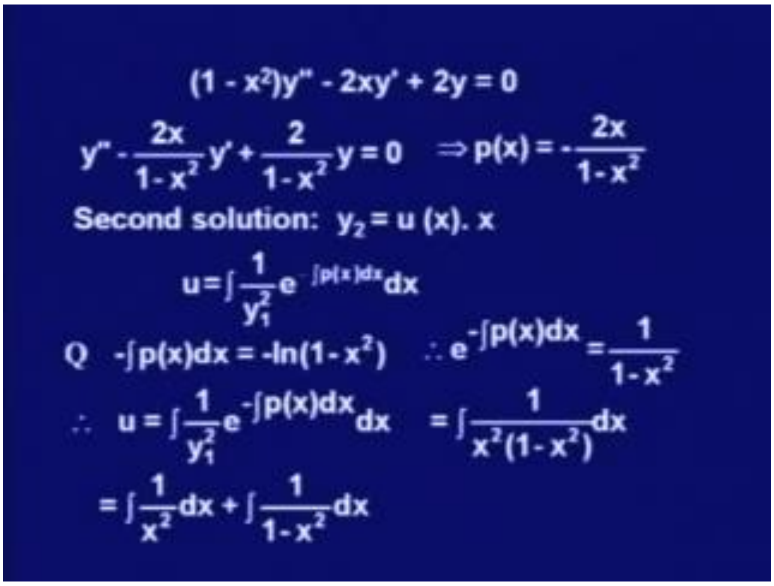

Now, my given equation is 1 minus x square y double dash minus 2 x y dash plus 2 y is equal to 0 let us rewrite it in standard form, y double dash minus 2 x upon 1 minus x square y dash plus 2 upon 1 minus x square y is equal to 0. What I have done, I have just divided this equation by 1 minus x square to make it in the standard form, in a standard form the coefficient of y double dash has to be 1.

This stats that in standard form if I write, what will be the p here is minus 2 x upon 1 minus x square, now the second solution would be u x times x, that is the x was the first solution. Now, we can obtain this u by substituting this y 2 y 2 dash y 2 double dash in this given equation, but I am just leaving for you; I am just going to do the formulation just now we had obtained the formula for finding you, that is integral of 1 upon y 1 square e to the power minus p x d x.

So, what is integral minus p x d x this is minus log 1 minus x square, hence e to the power minus p x d x would be 1 upon 1 minus x square, so what will be u I am substituting all these in the formula y 1 is x square, so it should be 1 upon. X square 1 minus x square it is integral with respect to x, integrating of course we have to do first the partial fractions 1 upon x square d x plus 1 upon 1 minus x square d x.

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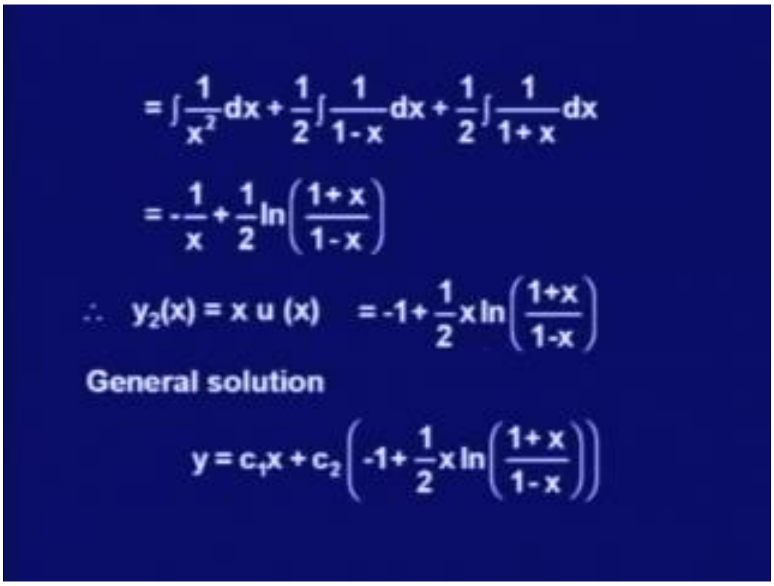

Again one more partial fraction, 1 upon x square d x 1 integral second integral half 1 upon 1 minus x d x, third integral half integral 1 upon 1 plus x d x. Now, integrating these things we get minus 1 by x plus half log 1 plus x upon 1 minus x, so what will be the second solution x times u x, that would be minus 1 plus half x times log 1 plus x upon 1 minus x.

Thus we have got the general solution as c 1 times x plus c 2 times minus 1 plus half times x into log 1 plus x upon 1 minus x. So, given one solution x we had obtained the second solution which is linearly independent of this solution, and we had obtained the general solution, let us try one more example here.

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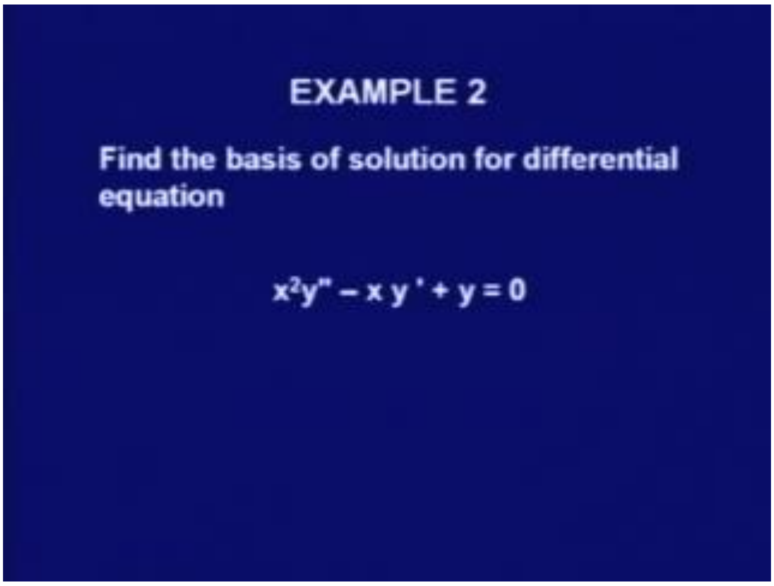

Find the basis of solution for differential equation x square y double dash minus x y dash plus y is equal to 0, now here we have not been given any solution. So, what we will do I am just going to do is that is guessing one solution, and then obtaining the second solution as linearly independent of the one guessed solution.

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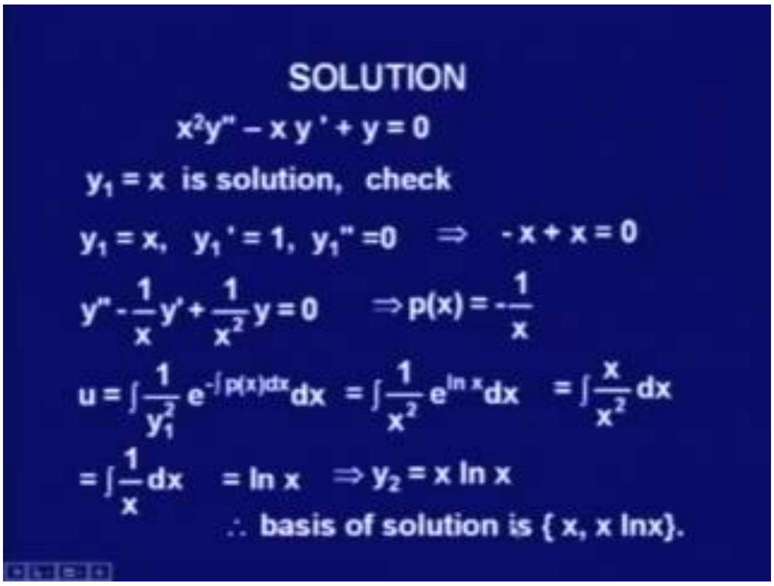

The given equation is this, for guessing one what we just think is that is here, if I take y as x, then I would get here x y dash would be 1, so this would be minus x and y double dash would be 0, so this term would be ruled out and I will get that this should satisfy the solution. So, let us check y 1 is equal to x and checking as a solution, y 1 is x, y 1 dash would be 1 and y 1 double dash will be 0, substituting I would get minus x plus x that is satisfying the given equation, so it is the solution.

Now, for second solution again I am going to use the formulation, so I have to write this equation in the standard form, that is I have to get the coefficient of y double dash as constant 1, so I have divide it by x square. Now, this we could do if x is not 0, so that means, this solution which we are going to obtain that would be only true when x is not 0, so we are making it in a standard form, it gives me p x minus 1 by x.

Now, for you I am just using the formula, so integral p d x is minus log x, so I would get it first solution 1 upon x square e to the power minus half minus p x d x that is log x, this if I rewrite again, I would get x upon x square d x that is integral of 1 upon x d x which is log x. So, I have got this second solution as x times log x, so what we have got the basis of solution x and x log x.

So, of course, what I have done, I have guessed here one solution and obtain second linearly independent solution as x log x that is giving me the basis. Now, we will learn how to obtain the general solution of a second order differential equation, that is we are not going to do the guessing job or anything and then, finding out the second solution by the technique which we have learnt. Now, we will do in general how to solve this one, so for that first we will do the second order homogenous linear equations with constant coefficients.

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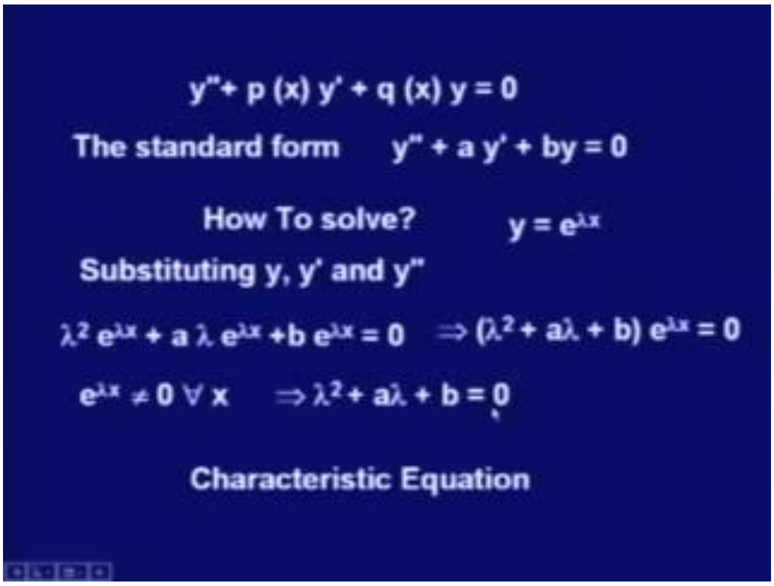

What are they a second order equation is, y double dash plus p x y dash plus q x y is equal to 0, here this coefficients p x and q x they are function of x. Now, I want this coefficients to be constant not a function of x, that is if I change this p x and q x to the constants a and b, we would get the second order differential equation with constant coefficient.

So, the standard form for this would be y double dash plus a y dash b y is equal to 0, we have a and b are some constants, how to solve this equation, we will again go with the things which we already know. We have done the first order equations, there we know that linear equations have the solution of the form e to the power lambda x, so we will try with this kind of function.

So, what we will try, we will try that let us see or let us assume that the solution will be of the form e to the power lambda x, we will substitute y, y dash and y double dash in this given equation. What will be y dash that would be lambda times e to the power lambda x and y double dash as lambda square times e to the power lambda x, so we are going to substitute this y, y dash and y double dash.

This equations, so what we are getting is y double dash lambda square times e to the power lambda x plus a times lambda times e to the power lambda x plus b times e to the power lambda x. Now, take this e to the power lambda x as common, what we get lambda square plus a lambda plus b times e to the power lambda x is equal to 0. Now, see what we have started we have started that e to the power lambda x is a solution, if it is the solution this cannot be a 0 function.

And if it is this solution this must satisfy the equation that says is I must get this coefficient lambda square plus a lambda plus b is equal to 0, so that e to the power lambda x should be a solution of this given equation. This is called the characteristic equation, that is this equation, which we have worked in this is called the characteristic equation, so let us have a formal definition of this term also.

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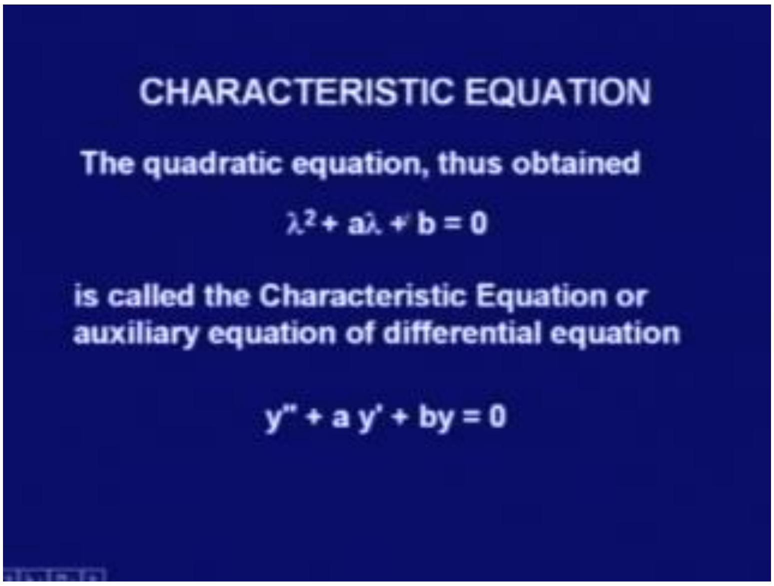

The quadratic equation thus obtained lambda square plus a lambda plus b is equal to 0 is called the characteristic equation or auxiliary equation of the differential equation y double dash plus a y dash plus b y is equal to 0. What we have seen here, that if we are having a second order differential equation with constant coefficients, then a corresponding characteristic equation, we are getting as lambda square plus a lambda plus b is equal to 0. That is we are having it is a quadratic equation, it is first coefficient as same as the constant one or the coefficients of this equation and this equation are same, this is called the characteristic equation.

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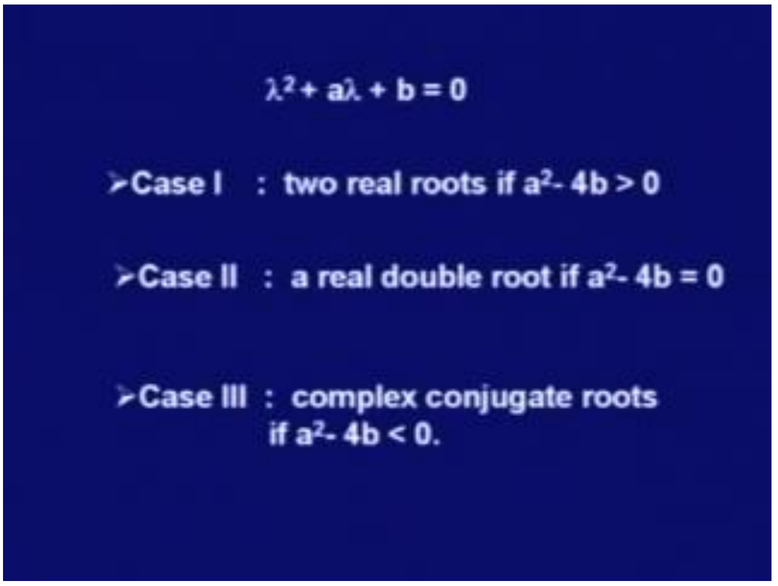

Now, we see that this equation is a quadratic equation, so what will be the solution of our differential equation, they will depend upon what the solution of this quadratic equation we are getting. Solution of quadratic equation that we do not know we are also calling them the roots of this quadratic equation, we do know the roots of this quadratic equations depend upon the discriminant function.

A square minus 4 b, accordingly we get three cases and accordingly we will get the solutions on the different cases, so let us make those three cases. The case one, if my discriminant function a square minus 4 b is positive we will get two real roots, and the second case is if this discriminant function a square minus 4 b is 0 we will get real root double one that is I will get the same roots. And the case three if my a square minus 4 b is less than 0, we will get the complex conjugate roots; we will discuss these cases one by one and the solution therefore, for the given differential equation.

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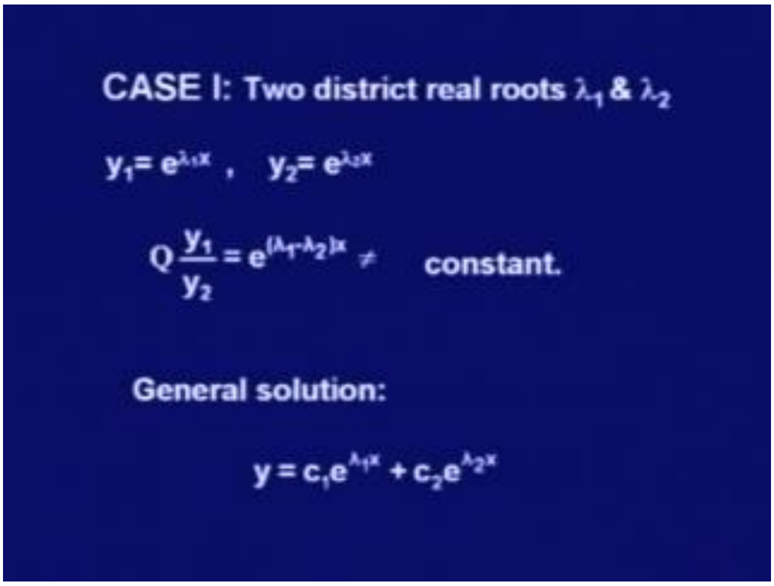

Case 1, there is a two distinct real roots lambda 1 and lambda 2, we will say that the two solutions we can make, because we have got the two roots lambda 1and lambda 2, so we can make the two solutions. One e to the power lambda 1 x and second y 2 as e to the power lambda 2 x, we can see that these two solutions are linearly independent, they form the basis we can check it by the proportionality.

Y 1 upon y 2 is equal to e to the power lambda 1 minus lambda 2 x, now this we are getting is lambda 1 minus lambda 2, since they are distinct real roots, so lambda 1 minus lambda 2 this is not 0, that is says it will never be a constant. Thus they are linearly independent solution, so the general solution will be of the form c 1 e to the power lambda 1 x plus c 2 e to the power lambda 2 x.

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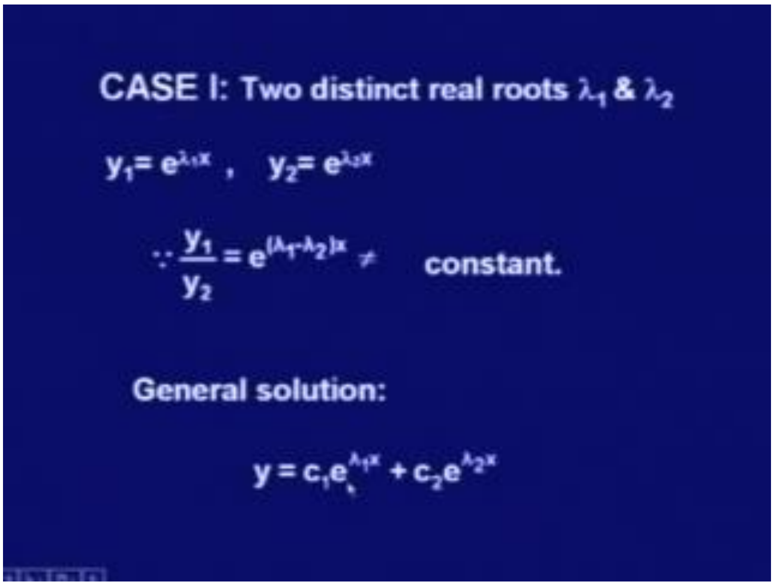

Case 1 two distinct real roots lambda 1 and lambda 2, so we do have the two solutions y 1 e to the power lambda 1 x and y 2 x e to the power lambda 2 x, we see that these two solutions are linearly independent. We can check by the, taking the ratio of these two solutions since y 1 upon y 2 is e to the power lambda 1 minus lambda 2 x, now we do know that lambda 1 and lambda 2 are distinct they are not equal.

So, lambda 1 minus lambda two will not be 0 that is this will not be a constant, so they are linearly independent, that is these two solutions form the basis are the fundamental system. Hence in this case the general solution of the given differential equation will be of the form c 1 e to the power lambda 1 x plus c 2 e to the power lambda 2 x.

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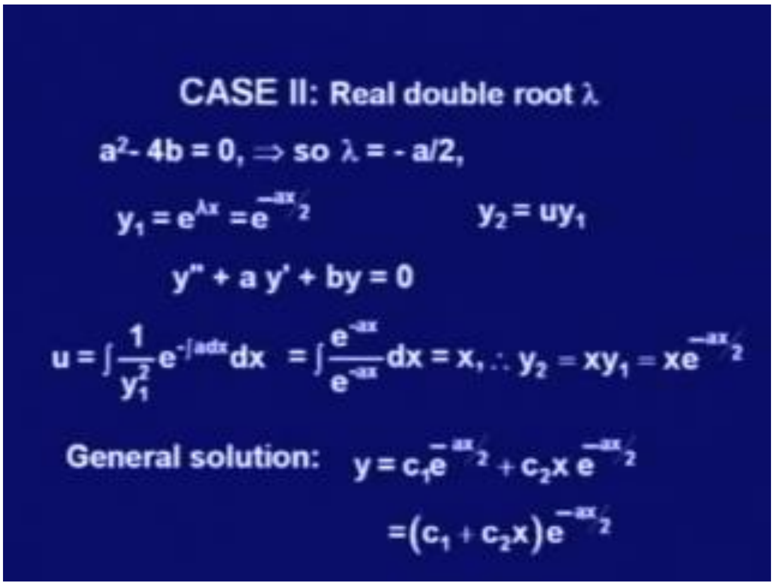

Let us discuss the case 2, that real double root lambda, when a square minus 4 b is 0, then the single root we do get lambda as minus a by 2, that means we get only one solution e to the power minus a x by 2. How to get other solution which is linearly independent of this, we will use again the technique which we do know that reduction of order technique.

So, we will use y 2 as u times y 1, now our equation is y double dash plus a y dash plus b y is equal to 0, this is in the standard form p is the constant a, so we will get u by the formula 1 upon y 1 square e to the power minus p x is a, so a times d x. Now, what is the integral that y 1 is we are getting is e to the power minus a x by 2, so when I substitute this y 1 and this a.

So, what we get e to the power minus a x upon e to the power minus a x that is integral with respect to x, this is constant, so it is integral is just the function x, that is we have got the second solution as x y 1 is equal to x times e to the power minus a x by 2. So, we have got the two solution in this case, first solution is e to the power minus a x by 2 and second solution is x times e to the power minus a x by 2. So, you will get the general solution as y times y is equal to c 1 times e to the power minus a x by 2 plus c 2 times x times e to the power minus a x by 2 or we can write it as c 1 plus c 2 x times e to the power minus a x by 2.

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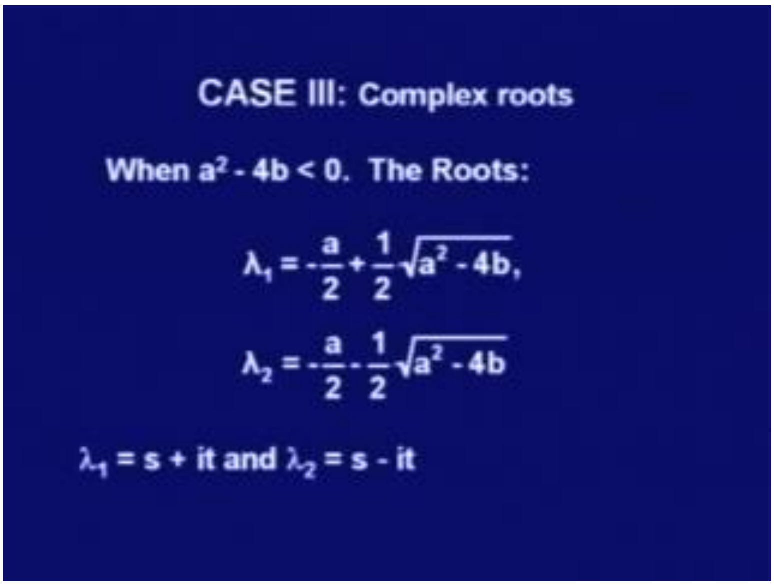

Now, let us see the 3rd case complex roots, when a square minus 4 b is less than 0, the roots would be complex numbers that is lambda 1 would be minus a by 2 plus half square root of a square minus 4 b. And the other root would be minus a by 2 minus half square root of a square minus 4 b, now since a square minus 4 b is negative, so this is square root is an imaginary number, let us say that imaginary number to be i t. So, let us say this lambda 1 and lambda 2 we are saying is s plus i t and s minus i t, where s is what is minus a i by 2 and t is square root of 4 b minus a square by 2, since I am using this imaginary minus 1 I am taking square root of minus 1, I am writing it as i.

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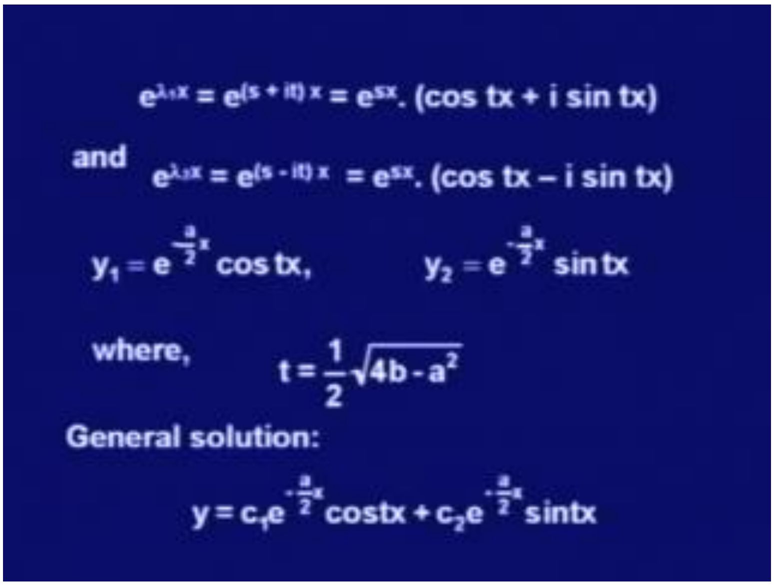

Then what will be with the two solutions, the two solutions would have one as e to the power lambda 1 x that is e to the power s plus i t x, which we could write as e to the power s x into e to the power i t x. This again we are using with De Moivre's theorem, we can write cos t x plus i sin t x, similarly the second solution e to the power lambda 2 x that is e to the power s minus i t x, we can get as e to the power s x into cos t x minus i sin t x, they would be linearly independent.

Actually we can get these are little bit learn the solutions, we can get a smaller one that is we can get a linear combination of these two solutions, which are again linearly independent. How to get that one, we do is that is we add these two solutions, if we add we do get is that 2 times e to the power s x cos t x and if I divide it by 2 i would get e to the power s x cost t x.

Similarly, if I subtract from second one to the first one I would get or from first one the second one I do get, 2 i times sin t x and of course, multiplied with e to the power s x 2 times, so if I divide it by 2 y I would get the two solutions as e to the power minus a by 2. Say this is minus a by 2 minus a by 2 x cos t x and another is e to the power minus a by 2 x sin t x, we can check again that they are linearly independent.

Because if I take the ratio what I will get y 1 upon y 2, I will get cos t x upon sin t x that is cot or cotangent t x which is a function of x not zero or not a constant. Thus we are getting that y 1 and y 2, they are linearly independent and what are these t is here is of course, half of a square root of 4 b minus a square. So, thus we will get general solution as c 1 times e to the power minus a by 2 x cos t x plus c 2 times e to the power minus a by 2 x sin t x.

So, we have seen that, we have got for a linear differential equation with constant coefficient, characteristic equation which has two roots. And depending upon the discriminant function we are getting three types of solutions, so let us summarize this method, how we are going to solve this equation.

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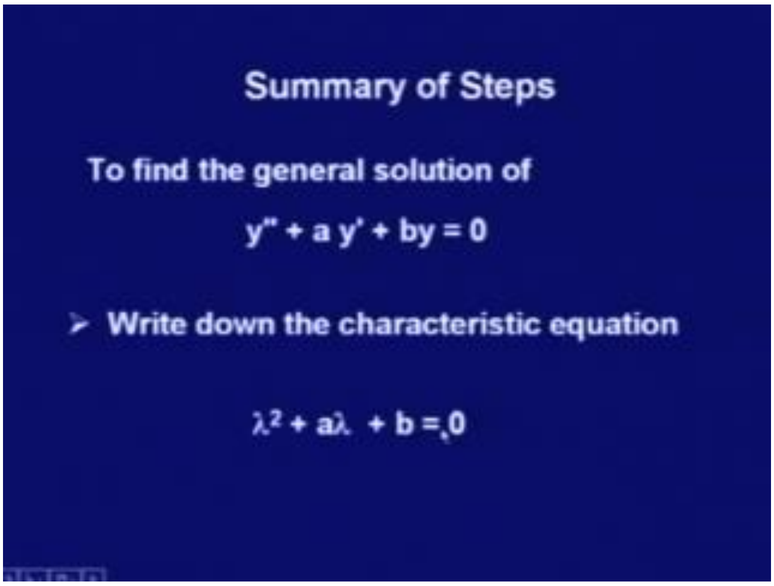

So, to find the general solution of y double dash plus a y dash plus b y is equal to 0, where a and b are constant, first we write the characteristic equation. The characteristic equation how we are writing, we are just writing is that is we write a second order quadratic equation in lambda, where the coefficients are same as the coefficients of given differential equation, that is lambda square plus a lambda plus b is equal to 0.

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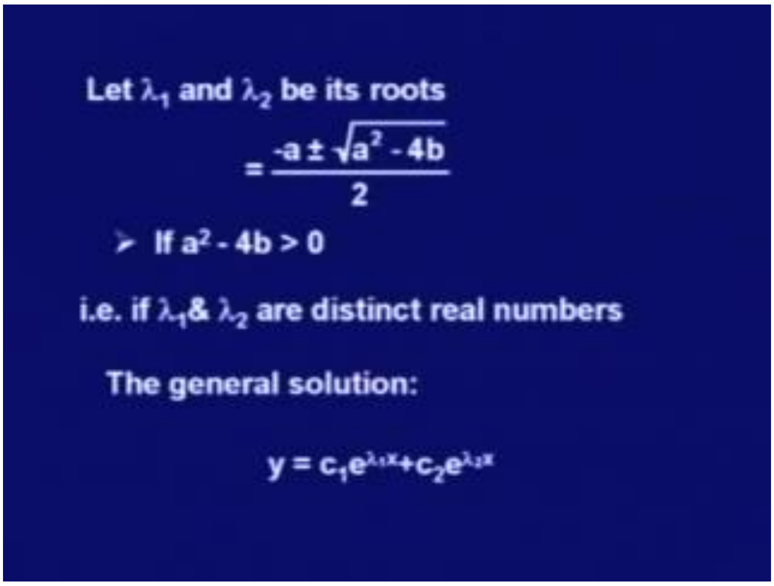

Then, we check it is roots lambda 1 and lambda 2, what they would be they are a minus a plus square root of a square minus 4 b by 2 and minus a minus square root of a square minus 4 b by 2. So, of course depending upon the value of a square minus 4 b we do have different cases, so in case a square minus 4 b is positive, we will get the two roots lambda 1 and lambda 2, they would be two distinct real numbers.

And the general solution would be c 1 e to the power lambda 1 x plus c 2 times e to the power lambda 2 x, where of course my lambda 1 would be minus a plus square root of a square minus 4 b by 2. And lambda 2 would be minus a minus square root of a square minus 4 b by 2, depending upon what is the value of this we would get is lambda 1 and lambda 2.

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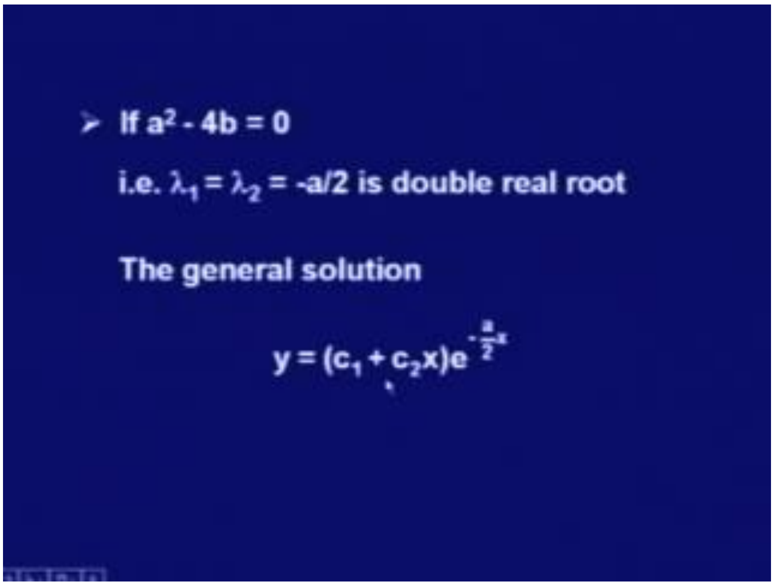

Then, if this a square minus 4 b is 0 that is the second part would be 0, I would get lambda 1 and lambda 2 both has minus a by 2, in this case the two solutions we get as e to the power minus a by 2 x and x times e to the power minus a by 2 x. Or in chart here we are just saying is that is, then the general solution will be of the form c 1 plus c 2 x times e to the power minus a by 2 x, where a is of course, the coefficient of y dash in the given differential equation.

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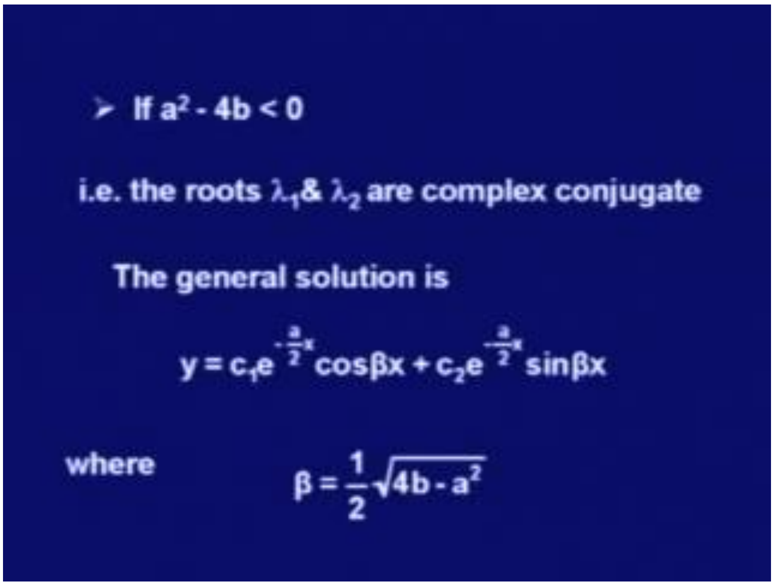

Third case was that if a square minus 4 b is less than 0, we get the roots lambda 1 and lambda 2 as the conjugate payer and the general solution is of the form c 1 e to the power minus a by 2 x cos beta x plus c 2 times e to the power minus a by 2 x sin beta x. Where a is the coefficient of y dash in the given equation and beta is nothing but, half times square root of 4 b minus a square. So, now let us to understand or to clear these steps, let us do some examples for solving second order linear differential equations with constant coefficients.

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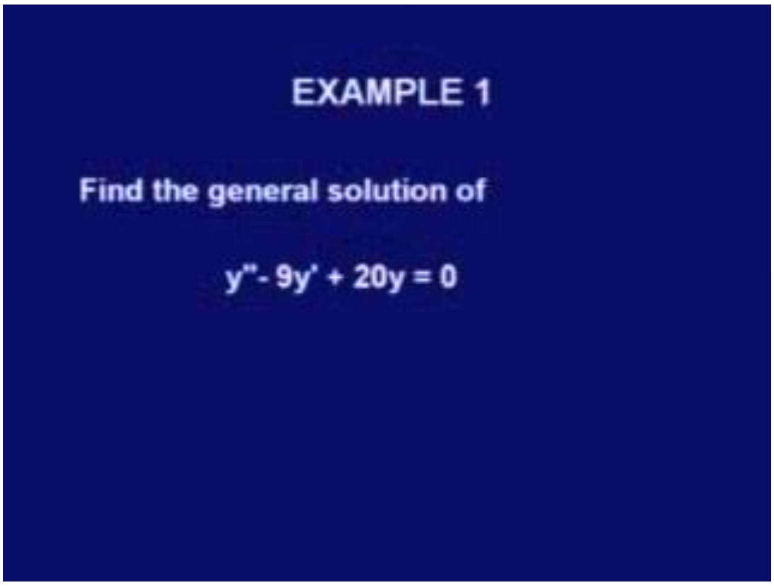

First example, find the general solution of y double dash minus 9 y dash plus 20 y is equal to 0.

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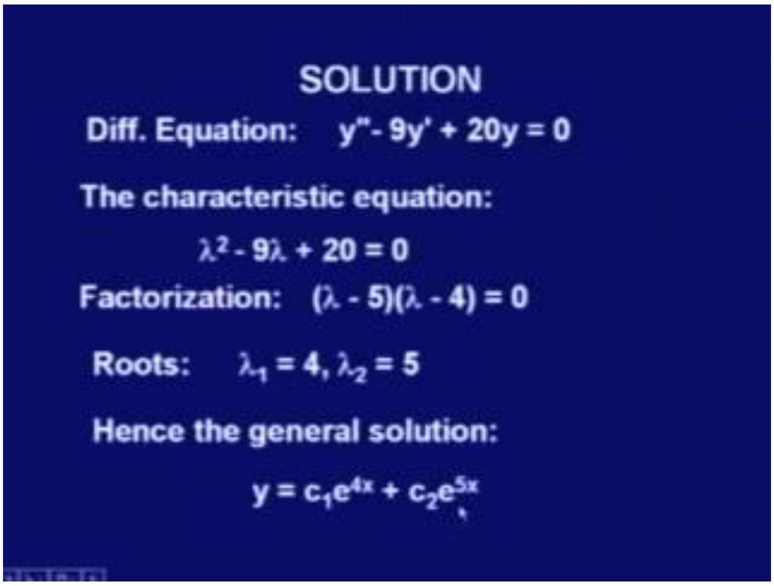

See given differential equation is y double dash minus 9 y dash plus 20 y is equal to 0, so corresponding characteristic equation would be lambda square minus 9 lambda plus 20 is equal to 0, to find it is root will just do the factorization here. So, factorization gives me lambda minus 5 into lambda minus 4 is equal to 0, thus roots are 4 and 5 they are real and distinct, so this is the first case.

What will be the general solution, general solution would be c 1 e to the power 4 x plus c 2 times e to the power 5 x, so what we have got, the general solution of the differential equation y double dash minus 9 y dash plus 20 y is equal to 0, the general solution is c 1 e to the power 4 x plus c 2 e to the power 5 x, let us do one more example.

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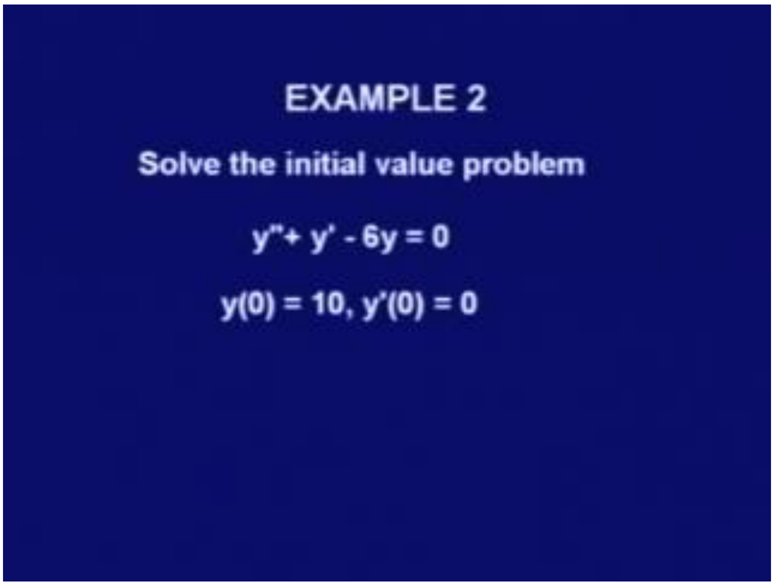

Solve the initial value problem y double dash plus y dash minus 6 y is equal to 0 with the initial condition y at 0 is 10 and y dash at 0 is 0, how you are going to do is.

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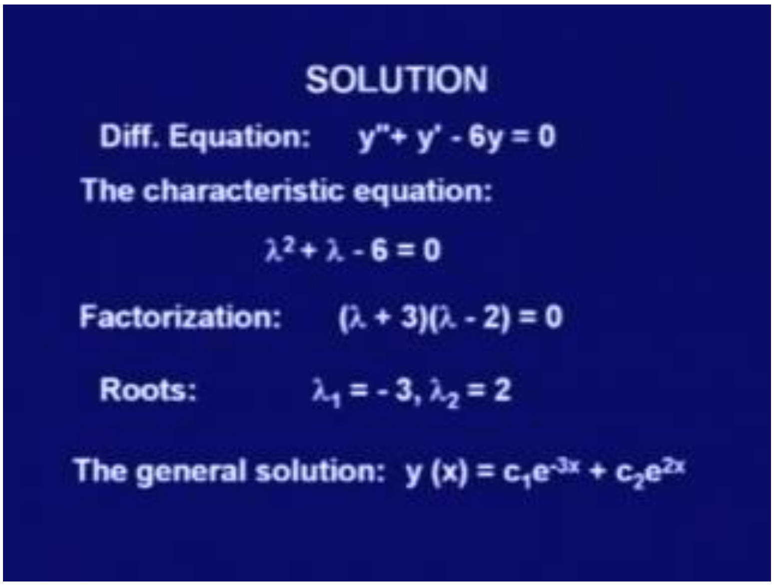

The given differential equation is y double dash plus y dash minus x y is equal to 0, so the characteristic equation lambda square plus lambda minus 6 is equal to 0 factorize it, we get lambda plus 3 into lambda minus 2 is equal to c that is says the roots are minus 3 and 2. Again what we are getting is real and distinct roots, so the general solution would be c 1 e to the power minus 3 x plus c 2 e to the power 2 x.

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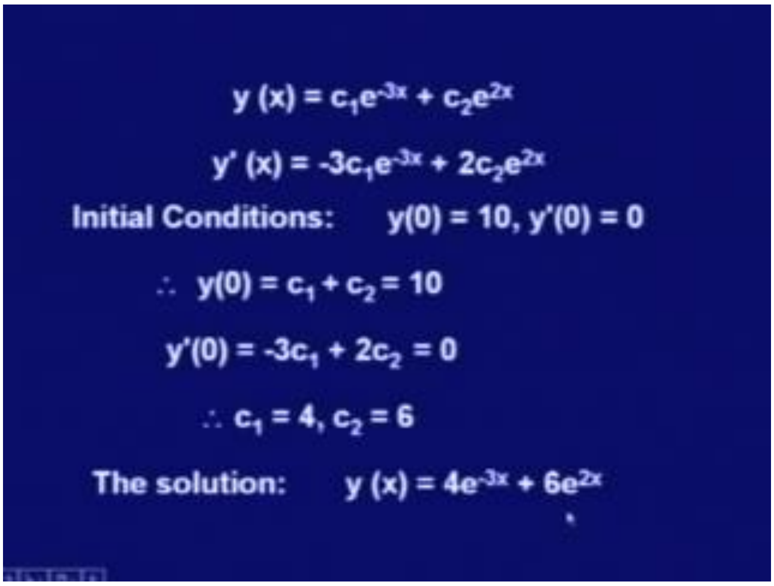

Now, for the initial problem initial value problem the particular solution, general solution is c 1 e to the power minus 3 x plus c 2 e to the power 2 x, this gives y dash as minus 3 c 1 e to the power minus 3 x plus 2 c 2 e to the power 2 x, the initial condition is y at 0 is10 and y dash at 0 is 0. So, we will put x is equal to 0 in the first one, we get is here c 1 plus c 2 this is given as 10, in the second y dash if I put x is equal to 0 what I get minus 3 c 1 plus 2 c 2 this is given as 0.

Now, this is linear equations into one knows we can solve it and what we will get, we will get the solution as c 1 is equal to 4 and c 2 is equal to 6, this is the unique solution. So, will get only single solution, particular solution for the initial value problem and that is c 1 we are putting as 4 c 2 we are putting as 6 we are getting is 4 times e to the power minus 3 x plus 6 times e to the power 2 x. So, we have got this is the particular solution of our initial value problem, see some more examples, so that we can practice all the way in which all those three cases we can complete.

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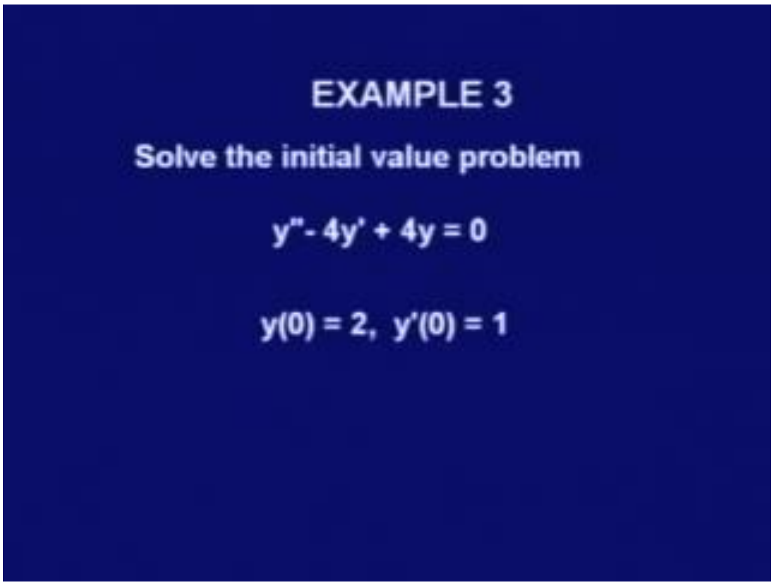

Solve the initial value problem y double dash minus 4 y dash plus 4 y is equal to 0 with the initial conditions y at 0 is 2 and y dash at 0 is 1.

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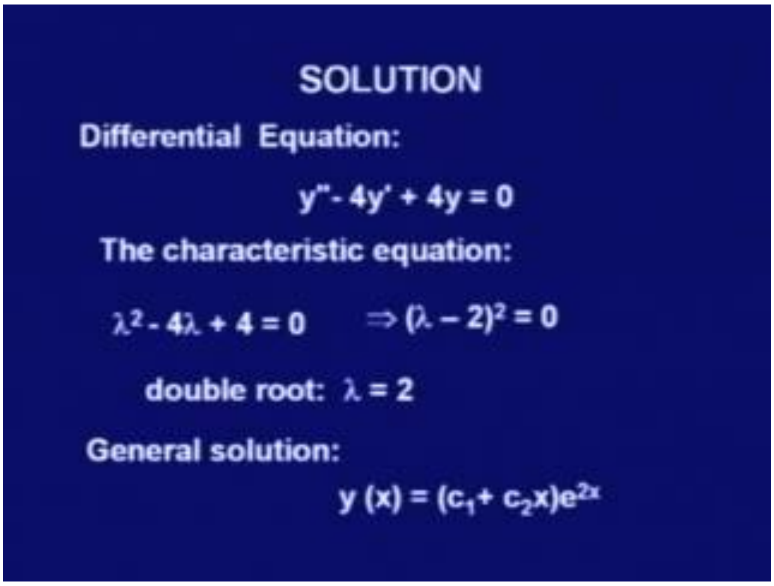

Let us solve it with differential equation is y double dash minus 4 y dash plus 4 y is equal to 0, so the characteristic equation that will be lambda square minus 4 lambda plus 4 is equal to 0. You see this is the whole square of lambda minus 2, that is says I do get the double root lambda is equal to 2, so the one solution I would get e to the power 2 x, what should be the other solution if you do remember, we have got x times e to the power 2 x.

So, the general solution we will get as c 1 plus c 2 x times e to the power 2 x, now we have to solve the initial value problem, so we have to use initial conditions for that.

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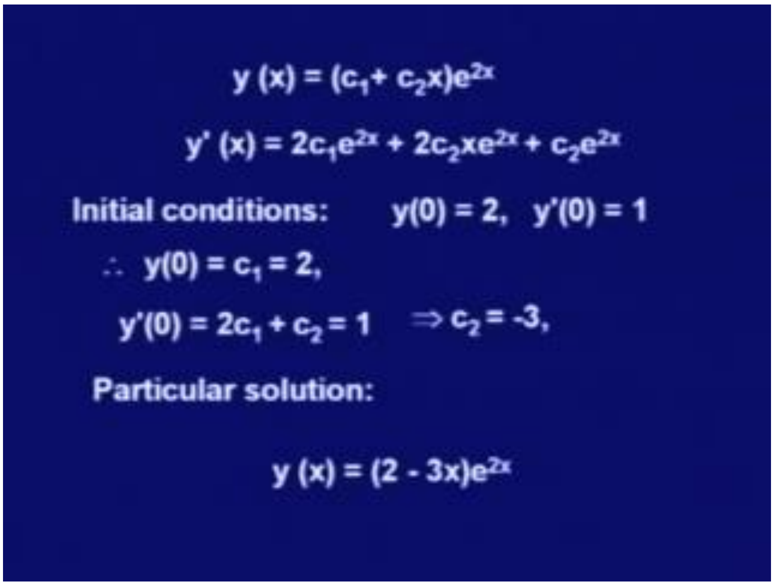

We find out the y x and it is derivative, what will be derivative of y x, y x is c 1 plus c 2 x e to the power 2 x, so it is derivative is 2 c 1 times e to the power 2 x plus 2 c 2 x times e to the power 2 x plus c 2 times e to the power 2 x the derivative of x is 1. Now, use at 0 the initial conditions are given at 0, that is y 0 is at 2 and y dash 0 is 1, when I put here in the first one x is equal to 0 I do get the only term c 1, so we get c 1 is equal to 2.

When I put in y dash x is equal to 0 I do get the two terms, 2 c 1 and c 2 this is given as 1 such 2 c 1 plus c 2 is 1, since c 1 is 2, so c 2 we would be getting as minus 3. So, what the particular solution we have got, now we will put this values c 1 and c 2 in this given in this general solution, particular solution would be 2 minus 3 x e to the power 2 x. So, we have got the particular solution of this initial value problem as 2 minus 3 x e to the power 2 x, let us see one more example.

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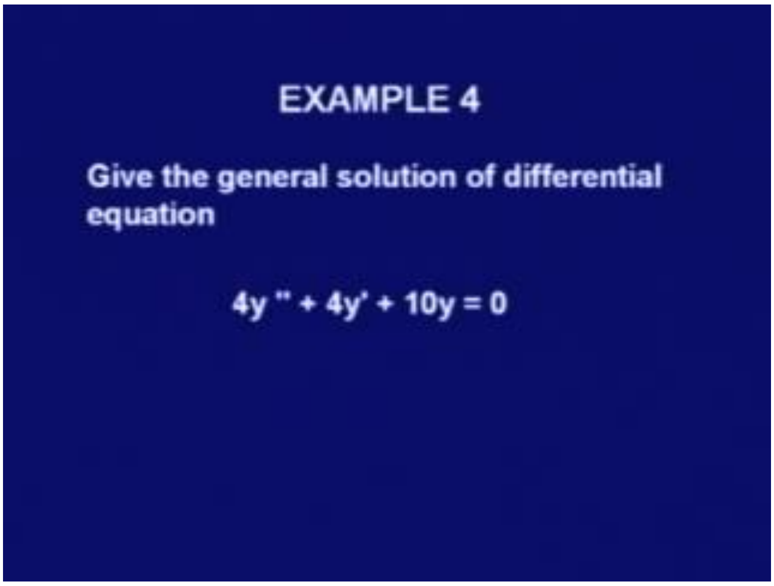

Give the general solution of differential equation 4 y double dash plus 4 y dash plus 10 y is equal to 0.

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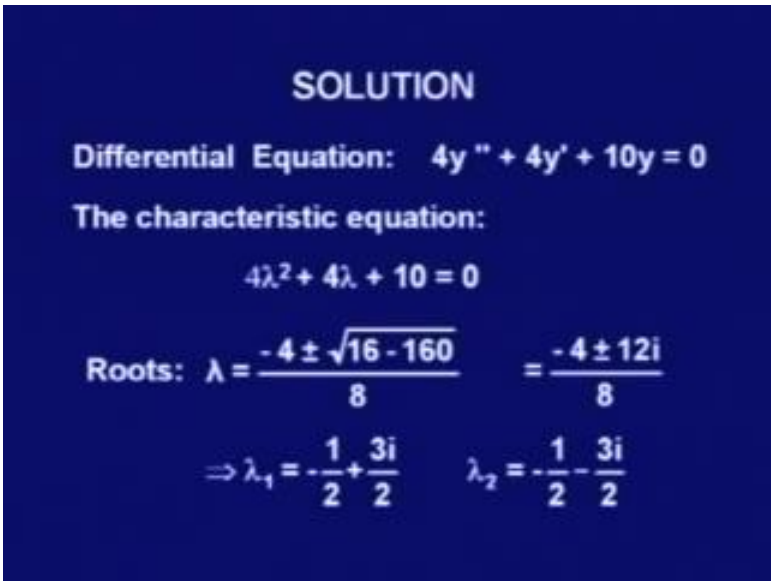

See how to solve it differential equation is 4 y double dash plus 4 y dash plus 10 y is equal to 0, the characteristic equation would be 4 lambda square plus 4 lambda plus 10 is equal to 0. You see here, we are not having this in a standard form that is the coefficient of y double dash is not 1, the characteristic equation we write with the same coefficients as they are in the given differential equation.

So, we get 4 lambda square plus 4 lambda plus 10 to find out it is roots, let us use the formula for finding out the roots that is minus b plus minus square root of b square minus 4 a c upon 2 a. So, this we are having is minus 4 plus minus b square is 16 minus 4 a c that is 160 upon 8, now you do get is that is this is negative minus 144, so we will get actually the complex roots minus 4 plus minus 12 i upon 8 are more simplified manner.

We are getting is the first root as minus half plus 3 by 2 i and the second root as minus half minus 3 by 2 i, so we are getting the complex roots, they are always coming as a conjugate payers. What will be our general solution, we see here s is my minus half and you see their s what we have got minus a by 2 as the coefficient of y dash, the coefficient of y dash when we were having the equation in the standard form.

Here the equation is not, if I write it in the standard form that is the coefficient of y double dash as 1, then I would get it as a as 1, because I will divide it by 4, so I will get coefficient of y dash as 1, so that is why it is minus 1 by 2 and so on, so we do get the...

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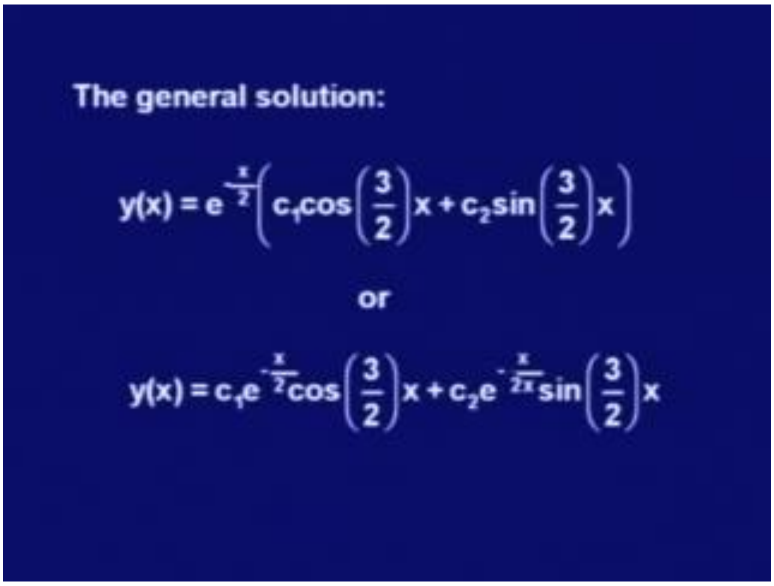

So, the general solution will be y x e to the power minus x by 2 c 1 times cos 3 by 2 x plus c 2 sin 3 by 2 x, or we can write it as c 1 e to the power minus x by 2 cos 3 by 2 x plus c 2 times e to the power minus x by 2 sin 3 by 2 x.

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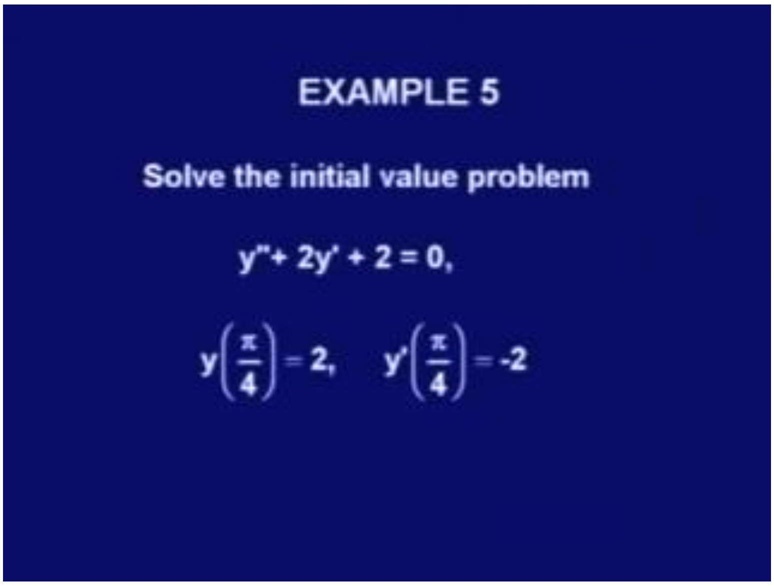

Now, next example solve the initial value problem y double dash plus 2 y dash plus 2 is equal to 0, the given initial conditions are y at pi by 4 is 2 and derivative of y that is y dash at pi by 4 is minus 2, see how to solve it.

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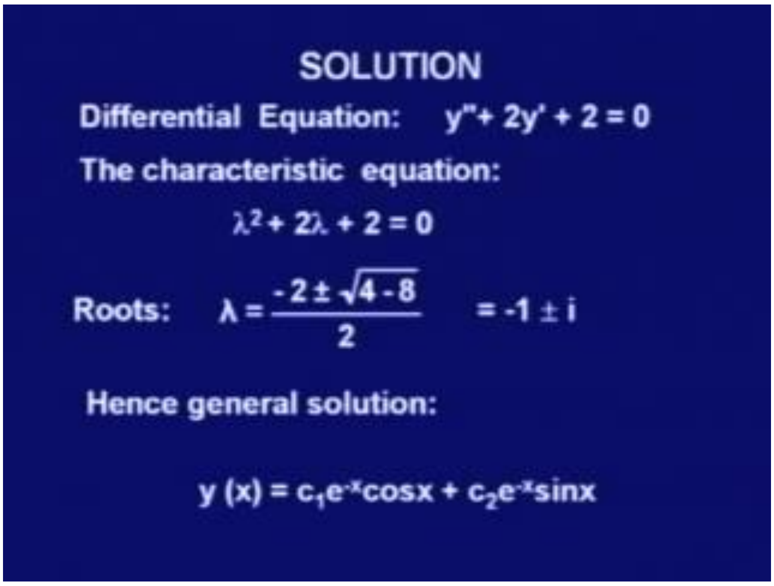

The given differential equation is y double dash plus 2 y dash plus 2 is equal to 0, the characteristic equation would be lambda square plus 2 lambda plus 2 is equal to 0, it is roots would be again will find out from the form directly minus 2 plus minus square root of 4 minus 8 by 2. Again we see this discriminant function is negative, so we will get the complex roots minus 1 plus minus i, that is the two roots we are getting as minus 1 plus i n minus 1 minus i. So, what will be the general solution, general solution will get c 1 e to the power minus x cos x plus c 2 e to the power minus x sin x. Now, for the particular solution, we have to find out the values of c 1 and c 2 for that we have to use the initial condition.

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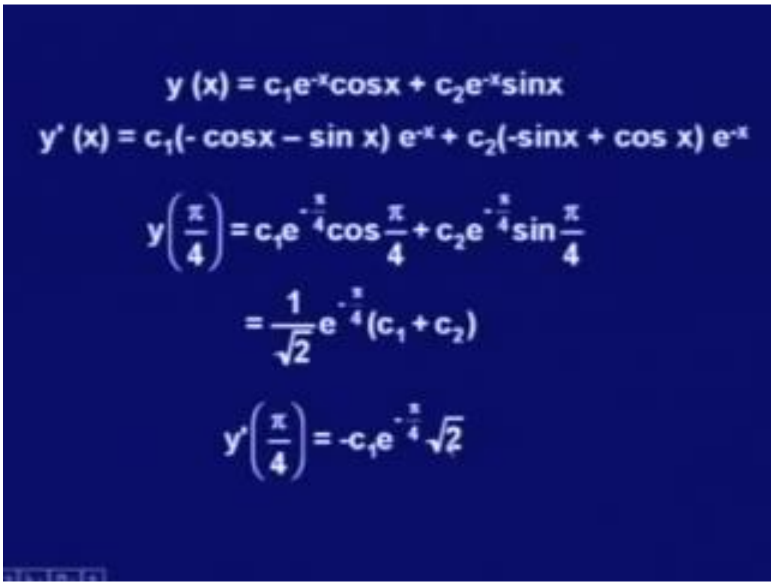

So, we will find out what is the y dash x, y dash x is c 1 e to the power minus x cos x is derivative would be minus cos x minus sin x e to the power minus x, similarly it would be c 2 minus sin x plus cos x e to the power minus x. Now, the initial conditions are being given at pi by 4, so what will be y at pi by 4, I will put the value of x as pi by 4, I will get e to the power minus pi by 4 and cos pi by 4 and here again sin pi by 4.

We do know the value of pi by 4 and sin pi by 4, they are 1 by square root 2, so we do get it 1 by square root 2 e to the power minus pi by 4 times c 1 plus c 2 this is given as, and similarly we can find out y dash at pi by 4, we do get here is that is because, we are getting is minus cos x plus cos x something and e to the power minus x is as common. So, we would be getting it actually and the coefficients, we would be getting it as actually c 1 e to the power minus pi by 4 times square root 2.

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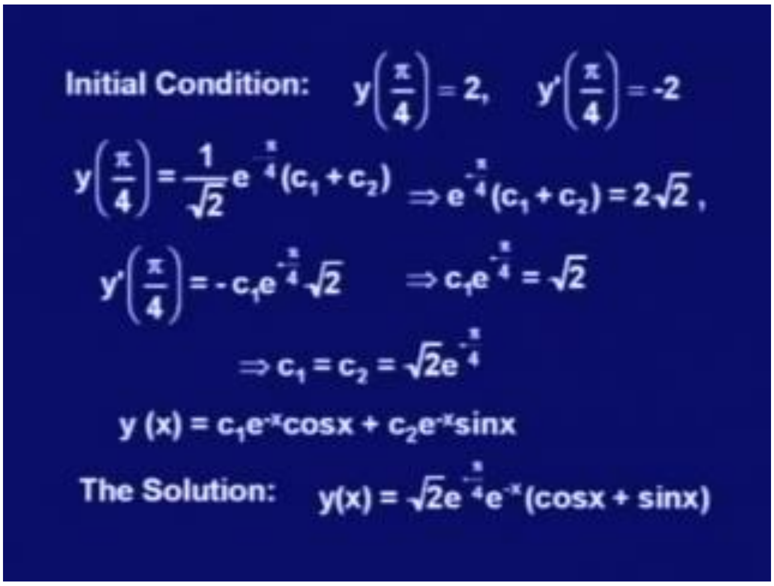

Now, put this in the given initial conditions, they have been given that y at pi by 4 is 2 and y dash at pi by 4 is minus 2, so substitute in those find out the values that is 1 by root 2 e to the power minus pi by 4 times c 1 plus c 2, this is equal to e to the power minus pi by 4 c 1 plus c 2 is 2 times square root 2. Similarly, the second one we would get that c 1 times e to the power minus pi by 4 is square root 2, so from here we are getting the value of c 1 and that will put over here and we will get the value of c 2,

And both we are getting as equal to square root 2 times e to the power minus pi by 4, so what will be our particular solution, we will put the values of c 1 and c 2 into the general solution. General solution was e to the power minus x cos x plus e to the power minus x sin x, so you will put c 1 and c 2 both as square root 2 e to the power minus pi by 4, we get the solution as square root 2 times e to the power minus pi by 4 e to the power minus x times cos x plus sin x.

So, we had learnt today about the second order linear differential equations, in that we had learnt that is what we mean by the solution, what we mean by the general solution, what we mean by the particular solution. And if one solution is known how to find out linearly independent and other solution, so that we can make the general solution and we had learnt that linear differential equations of second order with constant coefficients, how to find out the general solution. We will continue with this kind of equation in the next lectures also, today we have finishing up our lecture here.

Thank you.