Mathematics III
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Lecture - 13
One Dimensional Heat Equation

Dear viewers the topic of my lecture today is One Dimensional Heat Equation. In my last lecture, we had discuss the one dimensional wave equation and it is solution by the method of separation of variables. The method of separation of variables is also called as the product method. In our lecture today, we shall be discussing the one dimensional heat equation and it is solution by the same method that is the method of separation of variables. We shall consider a uniform bar and the flow of heat in the uniform bar under the assumptions.

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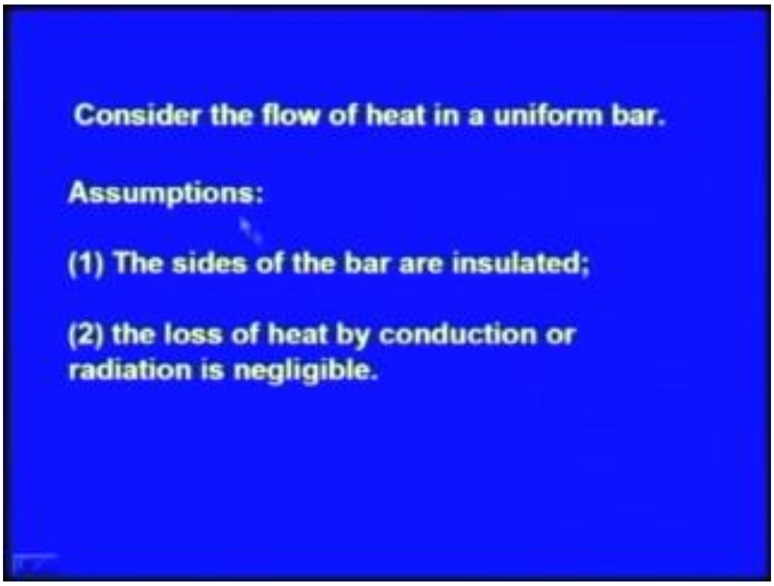

That the sides of the bar are insulated and the sides the loss of heat by conduction or radiation in the bar is negligible.

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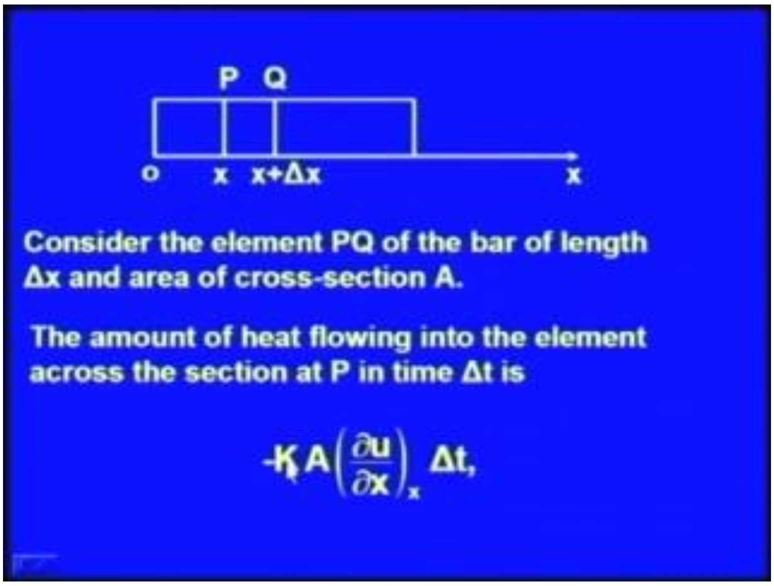

So, let us consider a small element PQ of the bar of length delta x and area of cross section A. Then the amount of heat that flows into the element across the section at the point P in time delta t is given by minus K into A into gradient at x delta u by delta x at x into delta t, where A is the area of cross section at P and K is the thermal conductivity of the material.

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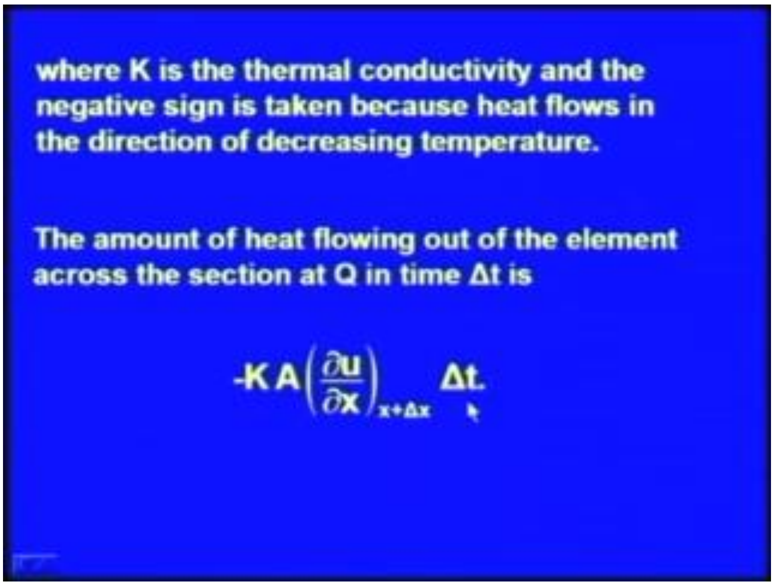

The negative sign is taken because the heat flows in the direction of decreasing temperature. The amount of heat that flows out of the other Side of the element PQ that is across the section at Q in time delta t is then minus K into the gradient at x plus delta x into delta t.

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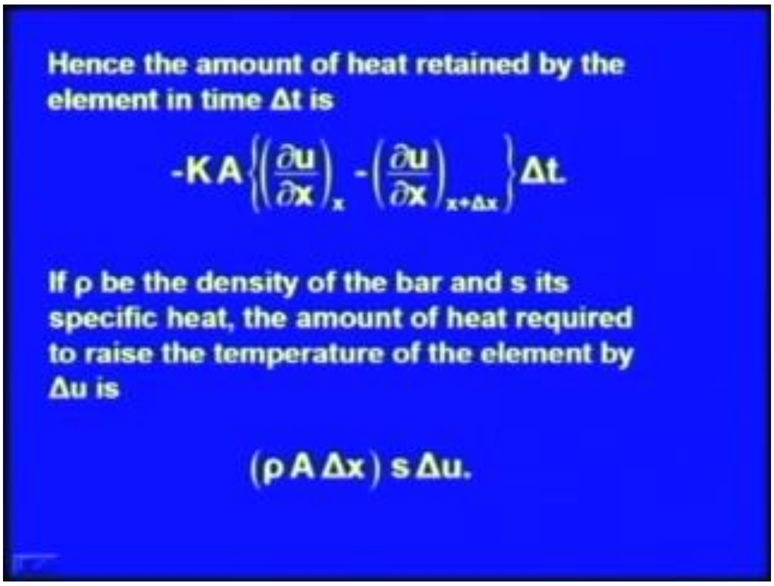

Hence the amount of heat retained by the element in time delta t is the heat that enters the element and minus the heat that leaves the element PQ at Q. So, we have minus K times delta u over delta x at x minus delta u over delta x at x plus delta x multiplied by delta t. Now, the amount of heat returned by the element, I mean raises the temperature of the element PQ.

So, let us say in time delta t the temperature of the element a PQ is raised by say an amount delta u. Then, if rho is the density of the bar and s its specific heat the amount of heat required to raise the temperature of the element PQ by delta u is rho into A into delta x into s delta u.

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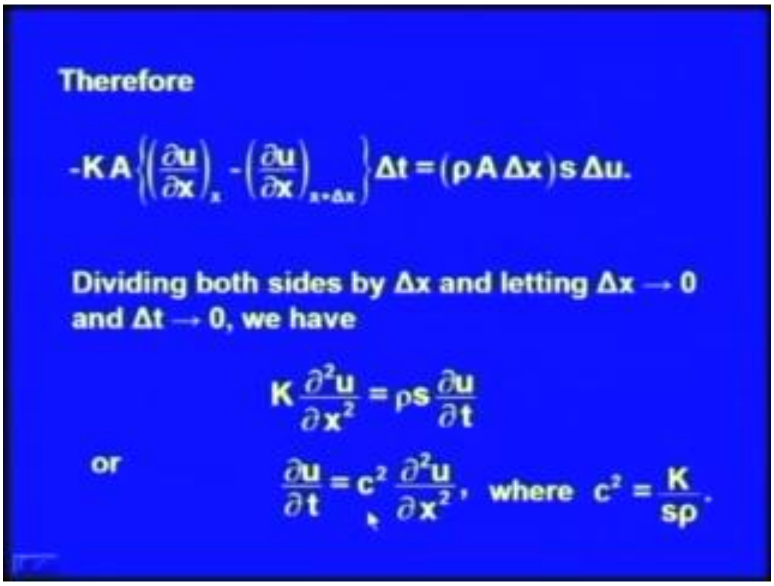

And therefore, we have minus KA into delta u over delta x at x minus delta u over delta x at x plus delta x into delta t equal to rho into A into delta x into s delta u. Now, let us divide both Sides of this equation by delta x and then let delta x and delta t go to 0, we then have; now when you divide this expression by delta x, you have taking minus Sign in Side. We have delta u over delta x at x plus delta x minus delta u over delta x at x divided by delta x gives you delta over delta x of delta u by delta x, that is delta square u by delta x square.

So, the left hand Side becomes K into A into delta square u by delta x square after dividing by delta x and the right hand Side becomes rho into A into s delta u and when you divide by delta t and let delta t go to 0. You have the differential equation delta u by delta t will gain go to delta u over delta t as delta t 10ds to 0 and we shall have rho into s.

So, K into delta square u by delta x square equal to rho s into delta u over delta t or we may say that delta u over delta t is equal to c square times delta square u over delta x square. Because K over s rho is a positive quantity, we can write it as the square of A constant c.

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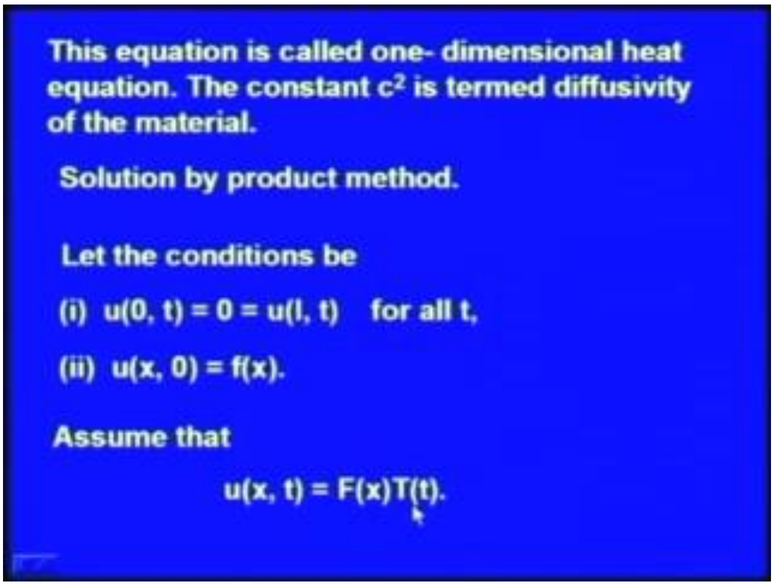

Now, this equation is called the one dimensional heat equation, the constant c square is termed the diffusivity of the material. We are now going to find the solution of this one dimensional heat equation by product method or the method of separation of variables. So, let us assume the conditions as follows, let us assume that at the end x equal to 0 of the uniform bar, the temperature is 0 for all the time t and same is to at the other end, that is at x equal to l, the temperature is again 0 at for all the time t.

So, the ends of the bar are kept at 0 temperature throughout and the initial condition is assumed as u x 0 equal to f x, that is the bar is kept at initially is at temperature given by the function f x. Initial temperature distribution function for the uniform bar is assume to be f x here, so as we have done in the case of the vibrating string, here also we assume that u x t the temperature at a distance x from the end x equal to 0. And at end t is assume to be product of two functions, one is a function of x, another one is a function of t.

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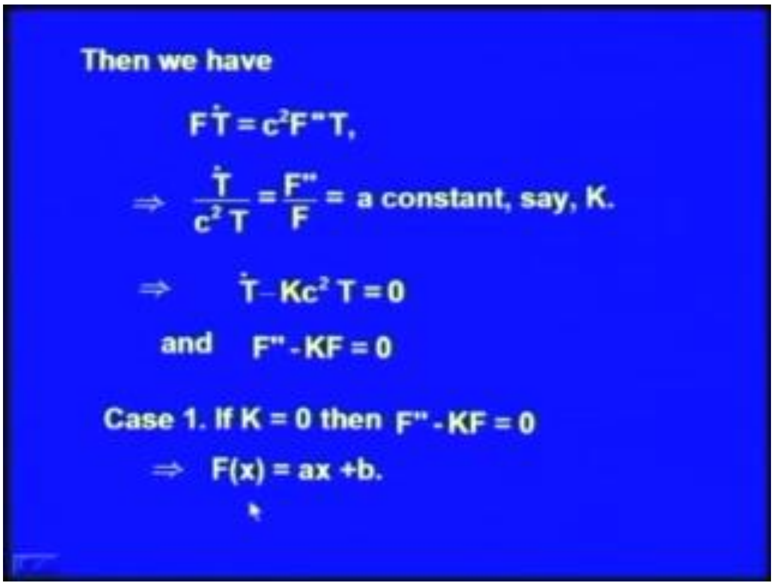

Let us substitute u x t equal to f x into T t in the one dimensional wave equation, we then have F into T dot equal to c square into F double dash into T, where T dot denotes the partial derivative with respect to T. In fact, here it is the T is the function of just one variable T, so it is ordinary derivative. So, T dot denotes the derivative of T with respect to T and dash denotes the derivative of F with respect to x, F double dash means d square F over d x square.

Now, let us separate the variables here we can write this equation also as T dot over c square into T equal to F double dash over F. Now, let us again the as just as in the case of the vibrating string, here also we can see that T dot over c square T is a function of T alone, while F double dash over F is a function of x alone and 2 will be equal if and only if each one is equal to a constant. So, T dot over c square T is a constant, F double dash over F is a constant and they are equal, so there equal to a constant say K.

Now, this leads us to two differential equations, ordinary differential equations T dot minus K into C Square into T equal to 0. It is differential equation ordinary differential equation of order one and another one F double dash minus K F equal to 0 which is an ordinary differential equation of order 2. We follow the same procedure to arrive at which value of K will give us the solution of the problem; we start with K equal to 1 with K equal to 0.

So, K is 1, if K is equal to 0, then F double dash minus K F equal to 0 reduces to F double dash equal to 0; that means, d square F over d x square is equal to 0. When, we integrate it twice with respect to x, we arrive at the general solution of F double dash equal to 0 as F x equal to ax plus b.

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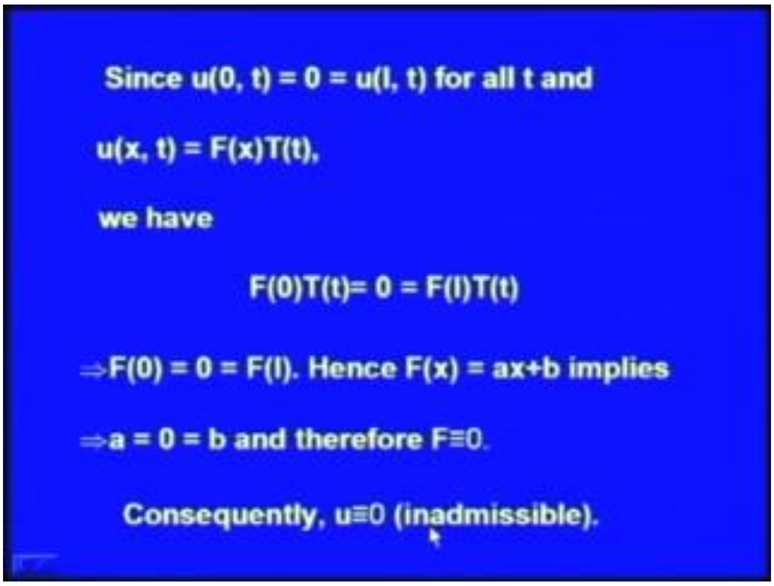

Now, let us use the given boundary conditions, that is u 0 t equal to 0 and u l t equal to 0 for all t, then and also that u x t is equal to F x into T t. So, then we shall have F 0 into T t equal to 0 and F l into T t equal to 0 for all the time T and this will imply that F 0 has to be 0 and F l has to be 0. Because, otherwise if you assume that T t is equal to 0 then u x t will be 0 for all values of x and t and which is an uninteresting case.

So, F 0 has to be 0 and F l also has to be 0 and if you make use of these two values, that is F 0 equal to 0 and F l equal to 0 in F x equal to a x plus b, what you get is F 0 equal to 0 gives you b equal to 0 and F l equal to 0, then gives you a l equal to 0. So, a is equal to 0 and therefore, a and b both are zeros and hence we get F as identically 0, when F is identically 0, clearly u is also identically 0, which is inadmissible. So, u i consequently u is identically 0, which is inadmissible and therefore, the case K is equal to 0 is discarded.

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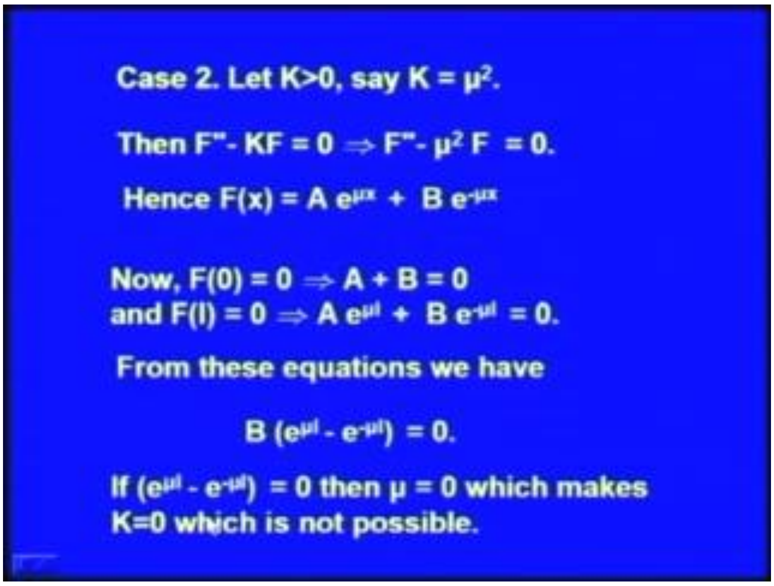

Let us, as now consider the case, when k is a positive constant, we may write it then K equal to mu square, with equal to mu square the second order differential equation F double dash minus K F equal to 0, implies F double dash minus mu square F equal to 0 and this is homogeneous differential ordinary differential equation of order 2. So, complementary function for this will be m square minus mu square equal to 0 and therefore, the values of m will be plus minus mu, which are both real and distinct.

And therefore, the complementary function will be A e to the power mu x plus B e to the power minus mu x particular integral is 0 here, because the right hand Side of the differential equation is 0. So, general solution of the differential equation is then given by F x equal to A e to the power mu x plus B e to the power minus mu x. Now, making use of the conditions on F that is F 0 equal to 0 and F l equal to 0, we arrive at two conditions involving A and B, one is A plus B equal to 0, another one is A e to the power mu l plus B e to the power minus mu l equal to 0.

Now, when you solve these two equations, what you do is put A equal to minus B in this equation, second equation reduces to B times e to the power mu l minus e to the power minus mu l equal to 0. Now, either B is equal to 0 or e to the power mu minus e to the power minus mu l is equal to 0, in case we take e to the power mu l minus e to the power minus mu l equal to 0, we will have e to the power 2 mu l equal to 1, which implies that mu is equal to 0.

And, when mu is equal to 0, K being equal to mu square gives us K equal to 0, the case K equal to 0, we have already resolved, we have seen that it is inadmissible. So, mu equal to 0 is not possible and therefore, A B has to be equal to 0. So, when B is equal to 0 A plus B equal to 0 gives us A equal to 0 and hence again F is identically 0.

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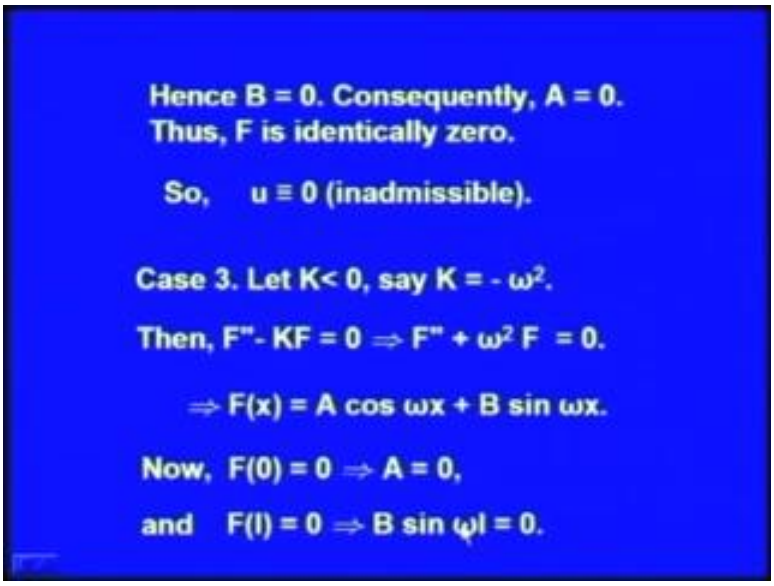

And, when F is identically 0, naturally u is also identically 0, which is inadmissible, so this case is also discarded. Let us, consider the third case, where K is a negative real constant, so let k is less than 0, then we may write it as K equal to minus omega square. Now, when F double dash minus K F equal to 0 with K equal to minus omega square becomes F double dash plus omega square into F equal to 0.

And, we know that this differential equation will give us the auxiliary equation as m square plus omega square equal to 0 and the roots of the auxiliary equation will then we plus minus i omega. So, the complementary function would be A cos omega x plus B sin omega x, particular integral is 0, because the right hand Side of these differential equation is 0, it is homogeneous. And so F x being the sum of complementary function in particular integral will be F x equal to the complementary function, that is A cos omega x plus B sin omega x.

Now, let us use the conditions on F, that is F 0 is equal to 0, F 0 is equal to 0 gives you when you put x is equal to 0 here, sin 0 zero. So, this term vanishes and we have the right hand Side s A only. So, F 0 equal to 0 gives A equal to 0 and when you put A equal to 0 here and x equal to l, then F l equal to 0 gives you B sin omega l equal to 0.

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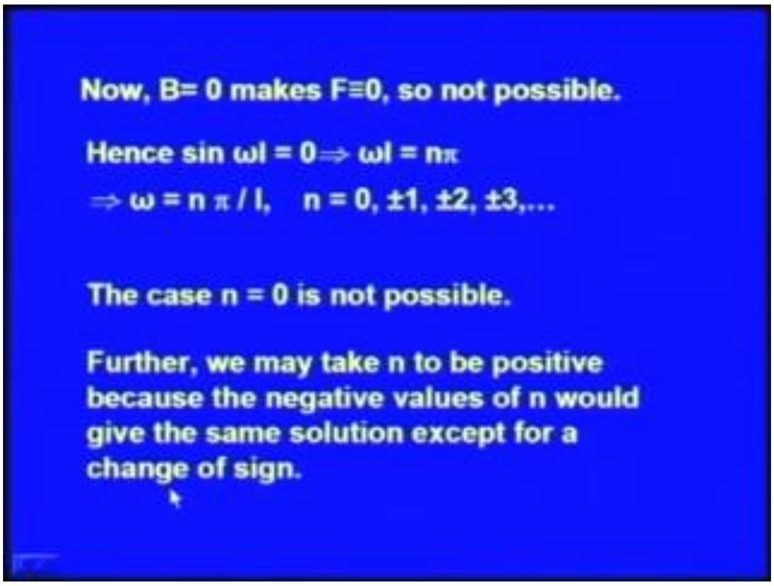

Now, B sin omega l equal to 0 in that if we take B as 0 then A and B both being 0 makes F identically 0, which is not possible as we have seen earlier also. So, we have to take sin omega l as 0, when sin omega l as 0, clearly omega l is equal to n pi and so omega is equal to n pi over l, where n takes all integral values, that is n is equal to 0 plus minus 1, plus minus 2, plus minus 3 and so on.

Now, when you take n as 0 here, means we are taking omega as 0 and when omega is 0, K being minus omega square, next K equal to 0 and K equal to 0, we have already resolved, we found that it is inadmissible, so it was discarded. So, the case n equal to 0 is not possible, now when you take the negative integral values of n, because F x becomes here B sin omega x.

So, if you replace omega there, if you take n to be negative, because of sin minus theta equal to minus sin theta, the solutions will be just negative of the solution, that we get for positive integral values of n and so we may just consider a sufficient to the positive integral values of n. So, we may take n to be positive, because the negative values of n would give us the same solution except for a change of sign.

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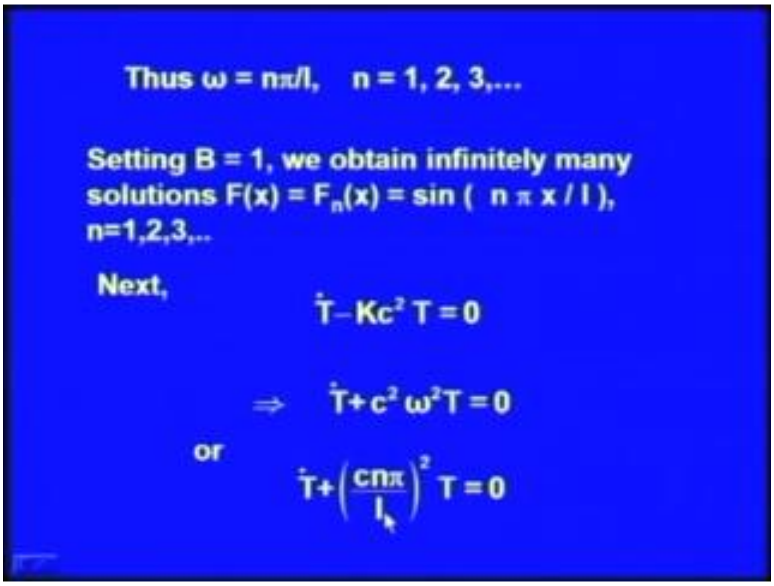

Thus, we may take omega as n pi over l, where n takes only positive integral values n equal to 1, 2, 3 and so on. we may set B equal to 1 for convenience, then we shall have infinitely many solutions of the differential equations of order 2, when F the solutions are F x equal to F n x equal to sin n pi x over l. For each value of n, we will get a solution of that differential equation, so we will have a sequence of solutions, we have denoted that by F n x, F n x is then sin n pi x over l, where n is taking values 1, 2, 3 and so on.

Now, having dissolved the second differential equation for F, now let us and found, we will now find the solution of the other differential equation, that is T dot minus K c square T equal to 0. We have seen, while finding the solution of the differential equation for F, that K has to be taken minus omega square.

So, making use of K equal to minus omega square, this differential equation transforms into T dot plus c square omega square plus T equal to 0. Now, we have also noted that omega has to be taken equal to n pi over l. So, let us put the value of omega here, this differential equation, then a transforms into T dot plus c n pi over l whole square into T equal to 0, now for convenience c n pi over l, we are denoting by lambda n.

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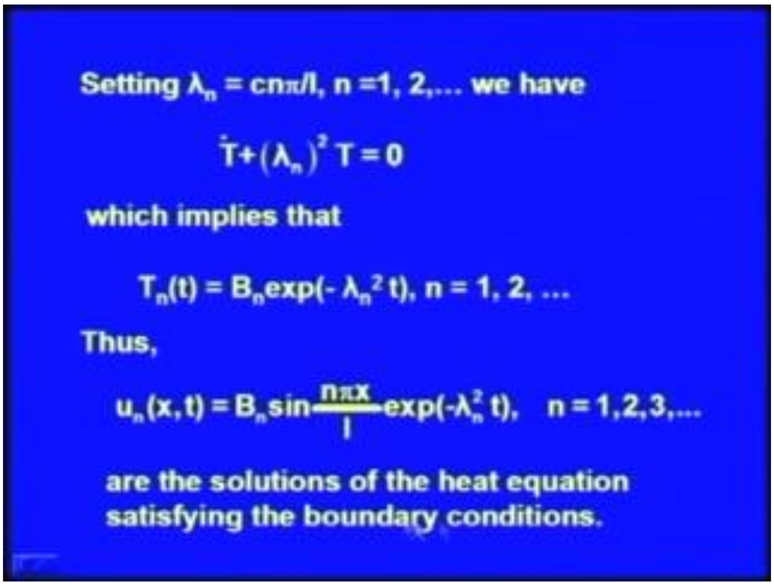

So, let us say lambda n denotes c n pi over l, where n equal to 1, 2, 3 and so on. We may write the differential equation resulting differential equation as T dot plus lambda n square into T equal to 0, which implies that, we can now separate the variables here. This d T equal to minus lambda n square into T equal implies that d T over T equal to minus lambda n square into d T.

We can then integrate on both Sides and we shall find that the solution general solution of this differential equation may written as T n t equal to B n into e to the power minus lambda n square into t, n taking values 1, 2, 3 and so on, B ns are arbitrary constants here, for each value of n. Now, let us substitute the value the solutions the values of F x and T t in u x t, so each value of n gives us one set of solutions for the two differential equations and therefore, it gives us a sequence of solutions u n x t of the heat equation.

So, here we will have u n x t equal to B n sin n pi x over l into exponential of minus lambda n square into t n taking values 1 2 3 and so on. So, these are the solutions of the heat equation, which satisfy the given boundary conditions, that is the ends of the bar are kept at temperature 0 throughout.

Now, since the heat equation delta u over delta t equal to c square delta square u over delta x square is linear and homogeneous, just as in the case of the wave equation.

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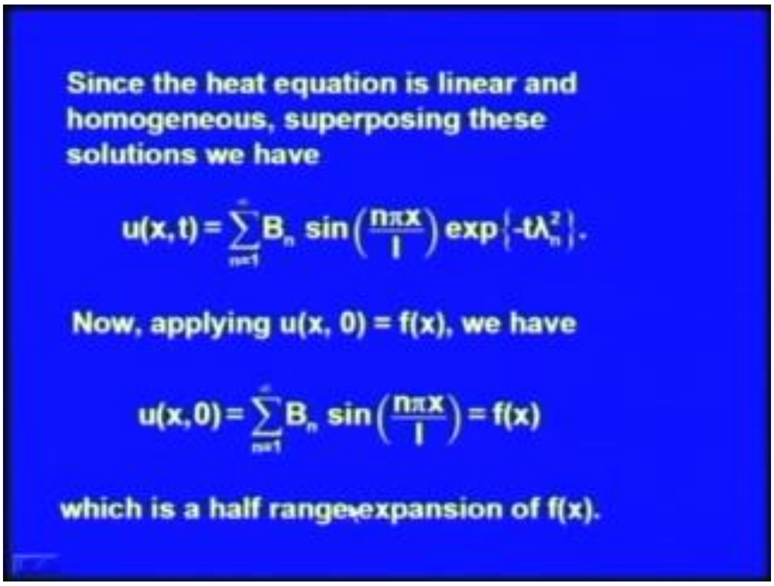

Here, also we can superpose the solutions u n x ts and have the solution and have u x t as sigma n equal to 1 to infinity, B n sin n pi x over l into exponential of minus t lambda n square. Now, let us apply the given initial condition, that is the initial temperature distribution in the bar is assume to be given by the function F x.

So, when you put t equal to 0 here, e to the power 0 is 1, so we shall have u x 0 equal to sigma n equal to 1 to infinity, B n sin n pi x over l, which is equal to the given fun then. Now, if you look at this equation F x equal to sigma n equal to 1 to infinity B n sin n pi x, you can see that it represents the half range expansion of the function F x.

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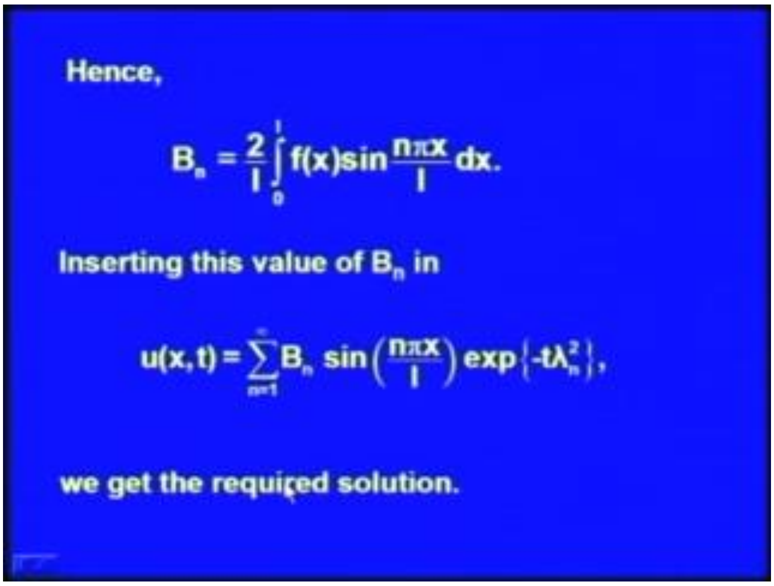

And therefore, B n will be given by 2 over l, 0 to l, f x sin n pi x over l d x, now let us insert this value of B n in the series for u x t, u x t is equal to sigma n equal to 1 to infinity, B n sin n pi x over l e to the power minus t lambda n square. So, when you insert the value this value of B n in this equation, you get the required solution.

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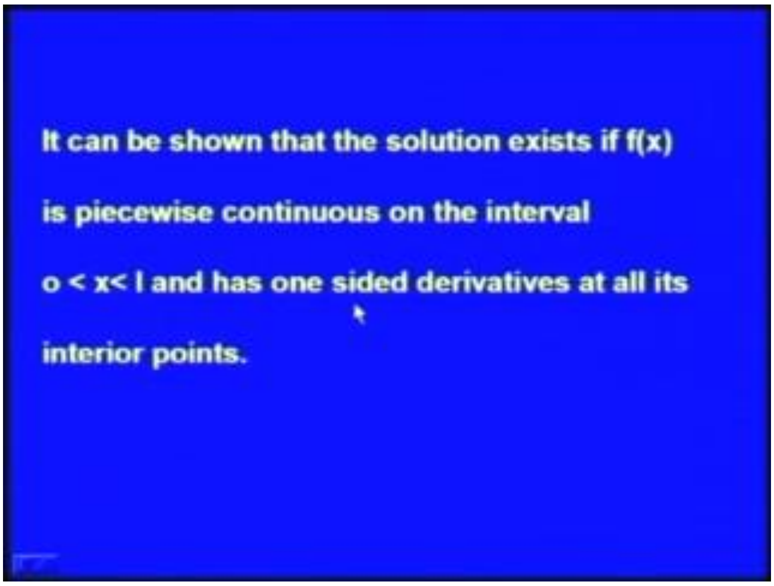

Now, it can be shown that the solution u x t exist, if f x is piecewise continuous on the interval 0 less than x less than l and f x has one sided derivatives at all its interior points.

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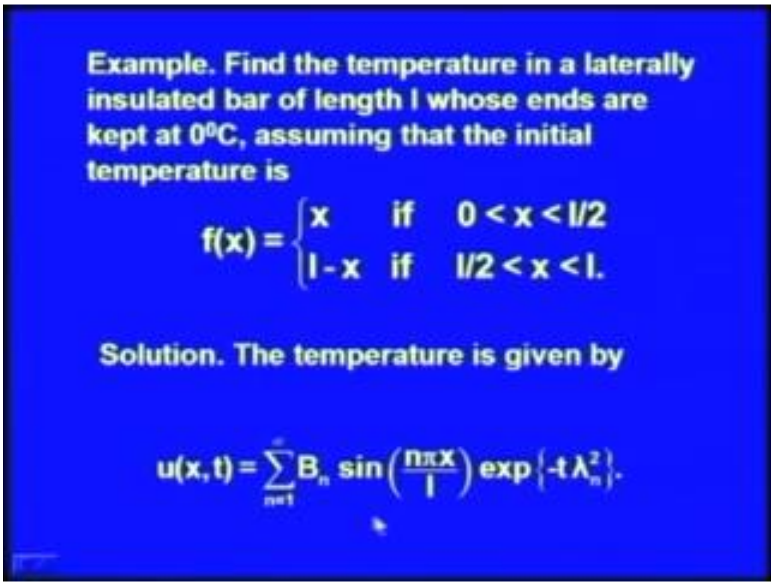

Now, let us take an example on this problem, so let us find the temperature in a laterally insulated bar of length l, whose ends are kept at 0 degree centigrade, assuming that the initial temperature of the bar is given by the function f x equal to x. If 0 is less than x less than l by 2 and l minus x, if l by 2 is less than x less than l. We know that the temperature in the laterally insulated bar, uniform bar of length l is given by u x t equal to sigma n equal to 1 to infinity, B n sin n pi x over l into exponential minus t lambda n square.

If we assume that, the ends of the bar are kept at 0 temperature throughout and the initial temperature of the bar is assume to be f x. So, with that here f x is given as x F 0 is less than x less than or by 2 and by l minus x, if l by 2 is less than x less than l, we shall be finding the value of B n.

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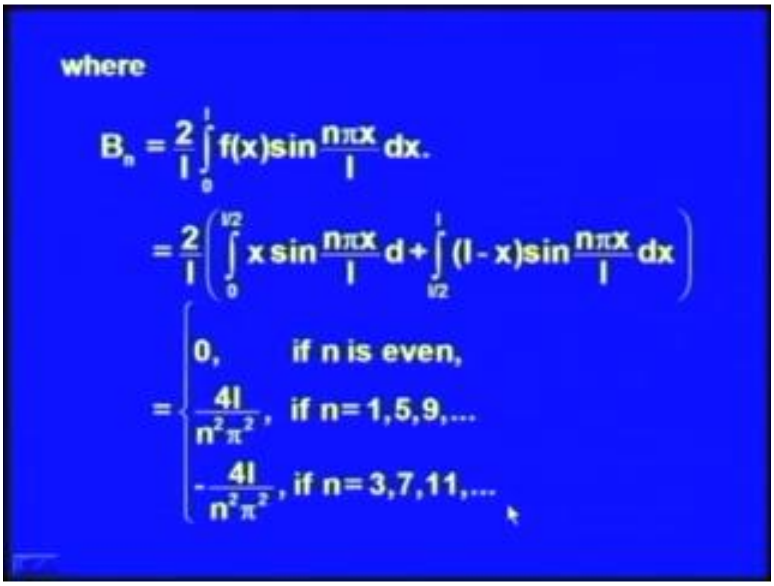

So, B n we know is given by 2 over l, integral 0 to l, f x sin n pi x over l d x. Let us, put the values of f x as x and l minus x over the interval 0 to l by 2 and l by 2 l and then when you integrate by parts and simplify you can verify that B n is equal to 0. If n is an even integer and it is 4 l over n square pi square, if n takes value 1, 5, 9 and so on. It is minus 4 l over n square into pi square, if n takes values 3, 7, 11 and so on.

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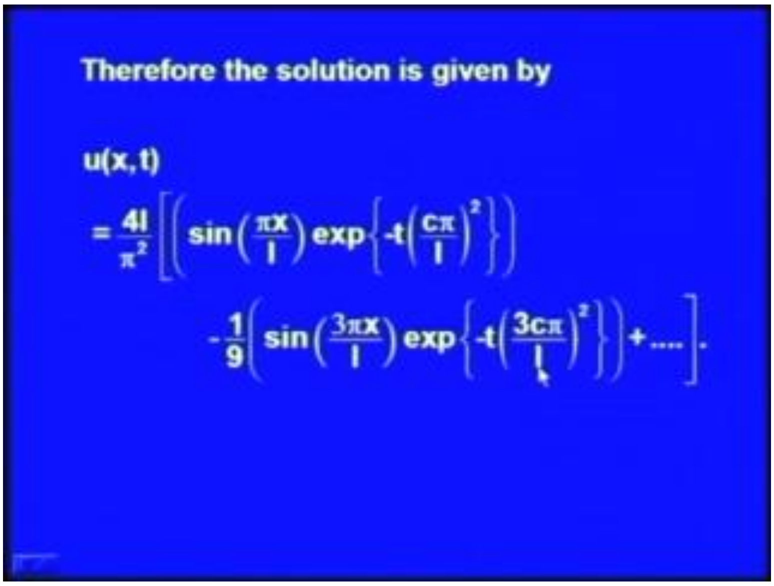

And, so let us put these values of B n in the series for u x t, we can see that u x t is equal to 4 l over pi square, which is common to all the terms multiplied by sin pi x over l. So, into exponential of minus t c pi over l whole square this the term corresponding to n equal to 1. Then, the term corresponding to n equal to 3 is this minus 1 over 9 sin 3 pi x over l exponential minus t 3 c pi over l whole square, this lambda ns which is c n pi over l, so here n is 1, here n is 3.

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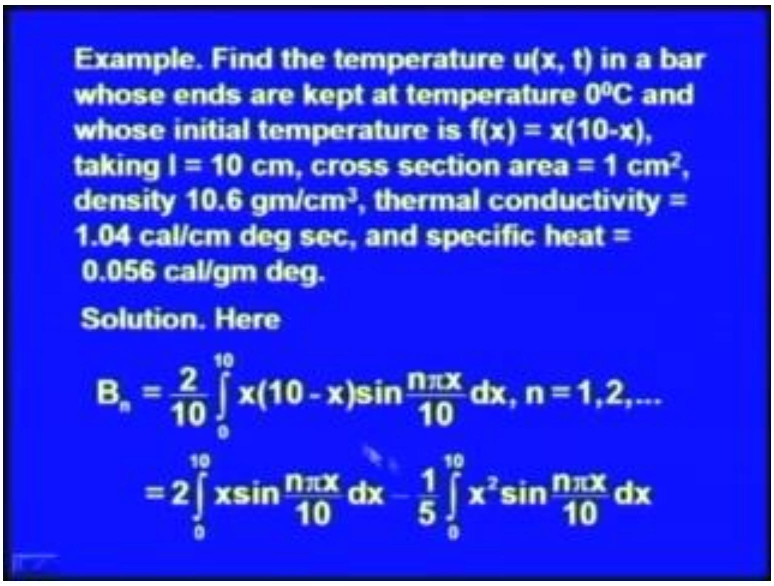

So, this was the problem where we had taken the two ends of the bar at 0 temperature, in this problem again, we take the ends at 0 temperature and but with a different function. Here, we have function f x given by x into 10 minus x, we are given the specific values of l, the cross section area in form, l is assume to be 10 centimeter cross section area is assume to be 1 centimeter square density is given as 10.6 gram per centimeter cube thermal conductivity is this given a specific heat is given.

So, here straight away we shall find the value of the constants B n, that occur in the infinite series for u x t. So, we have B n equal to 2 over 10 integral 0 to 10 x into 10 minus x sin n pi x over 10 d x, where n takes value 1, 2, 3 and so on.

Now, let us evaluate this integral, so we bracket into two parts and we then have B n equal to 2 over 2 times integral 0 to 10 x sin n pi x over 10 d x minus 1 over 5 integral 0 to 10 x square sin n pi x over 10 d x. For convenience, let us call this first term on the right side as I 1 and the second term on the right side this one as I 2.

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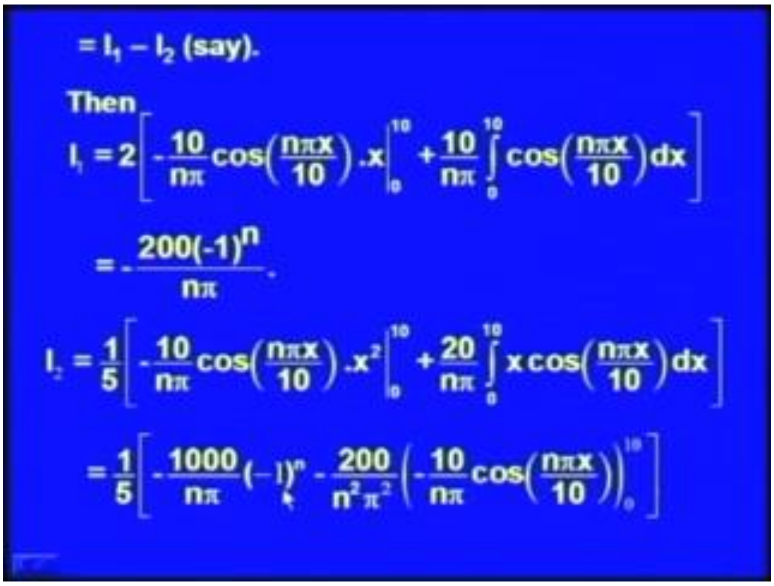

So, we write B n as I 1 minus I 2 and first we start the value at the integral I 1 using integration by parts, it comes out to be 2 times minus 10 over n pi cos n pi x over 10 into x, evaluated at 0 and 10 plus 10 over n pi integral 0 to 10 cos n pi x over 10 d x and when you substitute the lower and upper limits, you and simplify, you find that I 1 is equal to minus 200 into minus 1 to the power n over n pi.

Again, we using integration by parts I 2 comes out to be mi 1 by 5 into minus 10 over n pi cos n pi x over 10 into x square evaluated at 0 and 10 plus 10, 20 over n pi integral 0 to 10 x cos n pi x over 10 d x. And when you put the lower and upper limits here and carry out the integral here and substitute the limits we have this.

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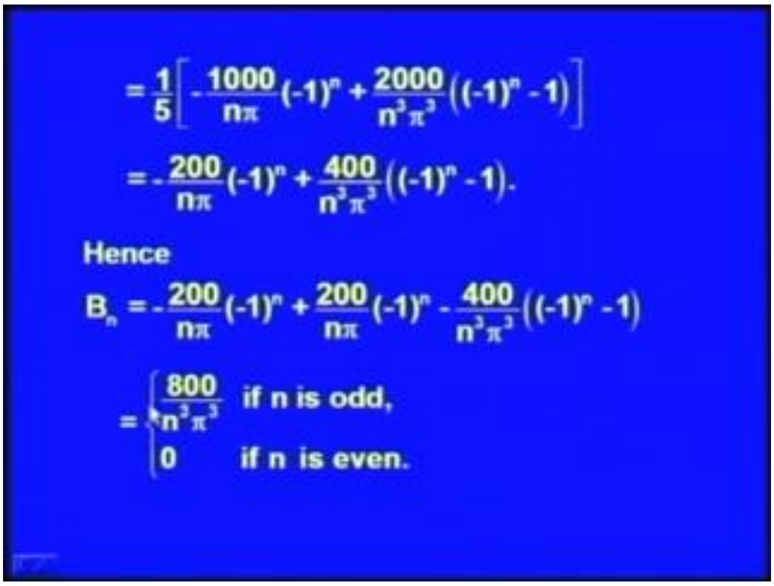

Further, after simplification gives us 1 over 5 minus 1000 over n pi into minus 1 to the power n plus 2000 over n cube pi cube into minus 1 to the power n minus 1. And when we multiply by 1 over 5, it is n simplify further, I can we can write it as minus 200 over n pi minus 1 to the power n plus 400 over n cube pi cube into minus 1 to the power n minus 1. And hence B n is equal to I 1 minus I 2 gives us minus 200 over n pi minus 1 to the power n plus 200 over n pi minus 1 to the power n minus 400 over n cube, pi cube into minus 1 to the power n minus 1.

Now, if you take n to be a odd positive integer, then minus 1 to the power n minus 1 will become minus 2, so we shall have after simplification minus 1 to the power n be become minus 1. So, after simplification this will give us 800 over n cube pi cube because, when n is an odd integer this term and this term will cancel out. And this will give us minus 2s minus 2 into minus 1 mi plus 2, we have 800 over n cube pi cube and where n is even minus 1 to the power n becomes 1.

So, 1 minus 1 is 0, so this term becomes 0 and we get here 200 over n pi minus 200 over n pi which is again 0, so we get B n equal to 0, so when n takes values 2, 4, 6 and so on. We find that B n is 0 and when n takes values 1, 3, 5 and so on, we find that B n takes value 800 over n cube pi cube.

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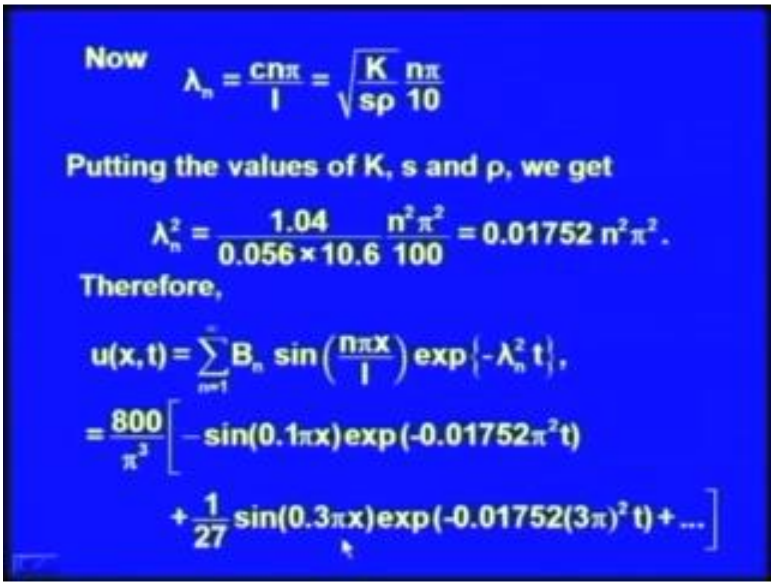

Now, we have c n equal to lambda n equal to c n pi over n and we know that c is equal to under root K over s into rho, because we had denoted k over s rho by c square. So, root K over s rho, we have into n pi by l, l is 10 here. Now, let us put the values given values of K as n rho here and determine lambda n. In fact, of we write lambda n square, lambda n square comes out to be K that is 1.04 as .056 rho1, 10.6 into n square pi square over 100.

When, we evaluate the value of this, it is 1.04 over .056 into 10.6 into 1 by 100, we com we get .01752 into and then n square pi square. So, lambda n square is gives as this value and therefore, u x t which is given by sigma n equal to 1 to infinity, B n sin n pi x over l exponential minus lambda n square t is given by this series, 800 over pi cube is common to all the terms we are taken it out.

And then, put the values of n, we have seen that, when n is n odd integer, it is having B n is having value at 800 over n cube pi cube. So, this is the term corresponding to n equal to 1, where we have made use of the value of lambda n square as this and this is the term corresponding to n equal to 3.

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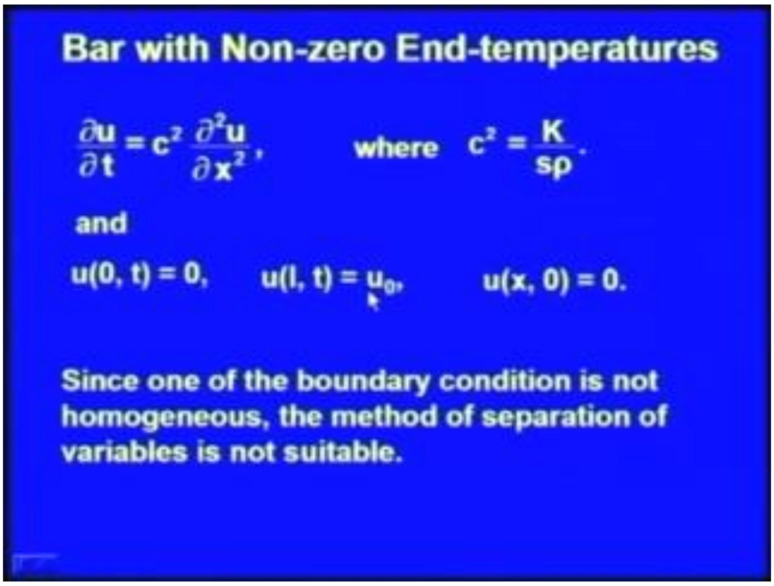

Now, we discuss the case of the uniform bar, where the both the ends are not at 0 temperature, so let us take up the case of a bar with non 0 end temperature. So, here we have again have the same heat equation, delta u over delta t equal to c square delta square u over delta x square, where we had assumed that c square the diffusivity of the material is given by k over s rho.

We here assume that the end x equal to 0 is kept at 0 temperature throughout, while the end at the temperature x equal to l is maintained at u naught u naught being a constant and initially the temperature of the bar is 0. So, 1 end of the bar is kept at 0 temperature, the end of the other bar is raise to the temperature u naught and is maintained their initially the bar was at 0 temperature.

Now, one of the boundary condition here that is the condition that at x equal to l, the end x equal to l is kept at 0 temp u naught temperature and maintained throughout. The boundary condition is not homogeneous here, so the method of separation of variables cannot be is not suitable here.

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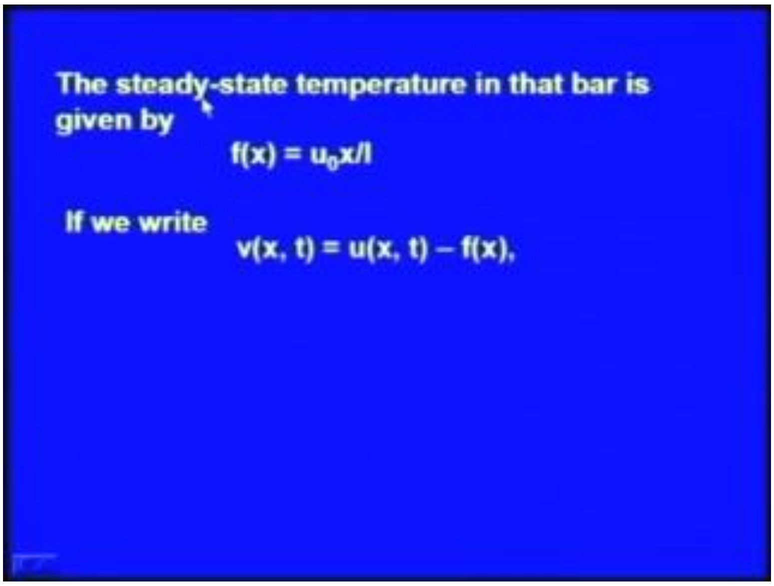

And therefore, what we will do is that, when the bar will come to the study state, what happens is that the rate of change of u with respect to time is 0, that is delta u over delta t is equal to 0. So, the heat equation becomes delta square u over delta x square equal to 0 and therefore, u x t is the linear function of x only, u x t becomes A x plus B. And if, you put in that the initial condition the boundary condition that at x equal to 0, the u is 0 and at x equal to l u is u naught, then what you get is u x t becomes u naught x over l.

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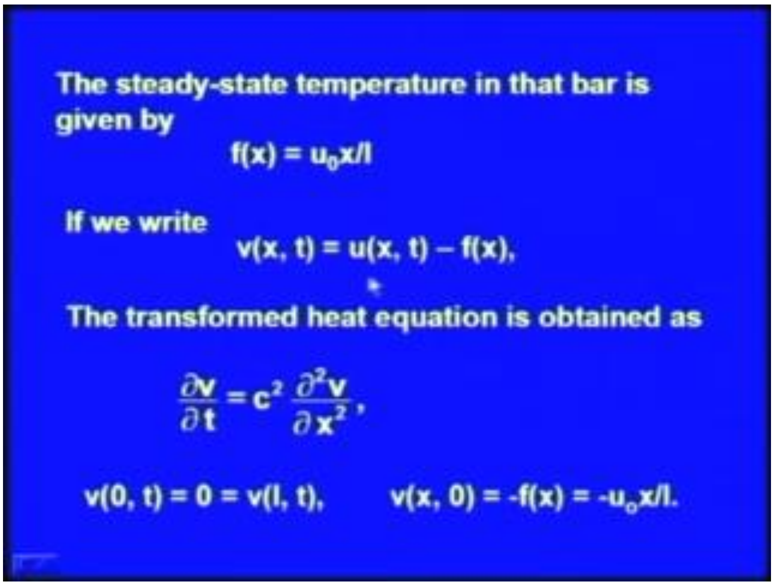

So, that u x t we have written as f x, we write v x t as u x t minus f x, then the transformed heat equation is obtained as delta v over delta t equal to c square delta square v over delta x square. Because, delta v over delta t gives you delta u over delta x, the partial derivative of f x with respect to t is 0.

So, delta v over delta t is equal to delta u over delta x and when you differentiate v with respect to x partially twice, that is u delta square v over delta x square, you find then you get delta square u over delta x square minus delta square over delta x square of f x. But, f x being a linear function of x, that is u naught x over l, when you differentiate f x twice with respect to x, it will give you 0.

So, delta square v over delta x square is same as delta square u over delta x square and therefore, the heat equation which we had assumed as delta u over delta t equal to c square delta square u over delta x square is transformed into delta v over delta t equal to c square, delta square v over delta x square. The bound the boundary conditions which we had assumed as u 0 t equal to 0 and u l t equal to u naught odd transformed into null v 0 t equal to 0, because when you put x equal to 0 here v 0 t becomes u 0 t minus f x.

So, and v l t, v l t is equal to when you put x equal to 0 here, we get v 0 t equal to u 0 t minus f 0, f 0 is 0 when you put x equal to 0 here and u 0 t is 0. So, v 0 t is 0 and when you put x equal to l here, we get v l t equal to u l t, u l t is assumed to be u naught and then, here when you put x equal to l, f x also is equal to u naught. So, u naught minus u naught is 0, so we get v l t equal to 0.

So, the two boundary conditions for u are transformed into v 0 t equal to 0 and v l t equal to 0, the initial condition for u, we had assumed as u x 0 equal to f x. So, the initial condition, you had assumed as u x 0 equal to 0, we had started with initial temperature being 0. So, here v x t, when you put t equal to 0 we get v x 0 equal to u x 0 minus f x. So, that is we get v x 0 equal to minus f x, but f x is u naught x over l, so we get v x 0 equal to minus u naught x over l. So, now we transformed equation, which is again the equation of heat conduction type, we have the two end temperatures as zeros and the initial temperature distribution is given by a function of x minus u naught x over l.

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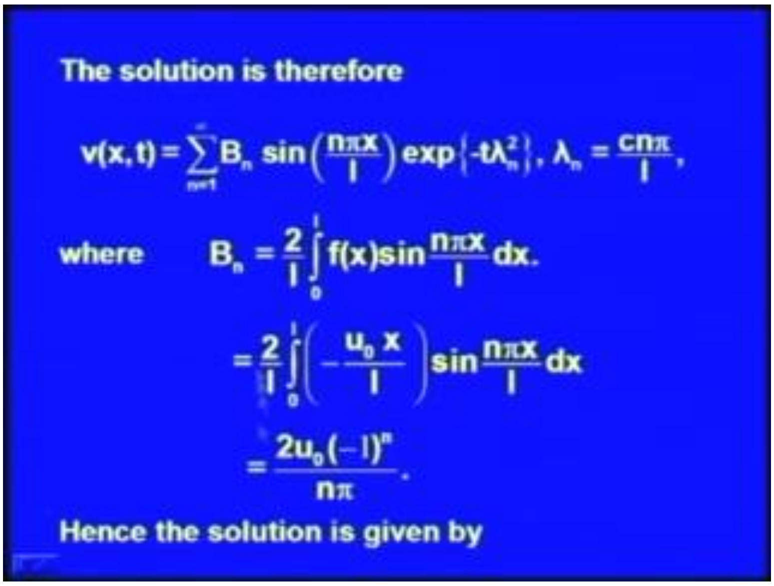

Such a case, we have already solved and we have seen that the solution is comes out to be v x t equal to sigma n equal to 1 to infinity, B n sin n pi x over l into exponential of minus t lambda n square, where lambda n is c n pi over l. And we know that the values of B n's are given by 2 over l, integral 0 to l f x sin n pi x over l. So, here f x is minus u naught x over l and therefore, B n is 2 over l, 0 to l minus u naught x over l into sin n pi x over l d x. When, you integrate this function and put the limits, what you get is B n equal to 2 into u naught minus 1 to the power n over n pi.

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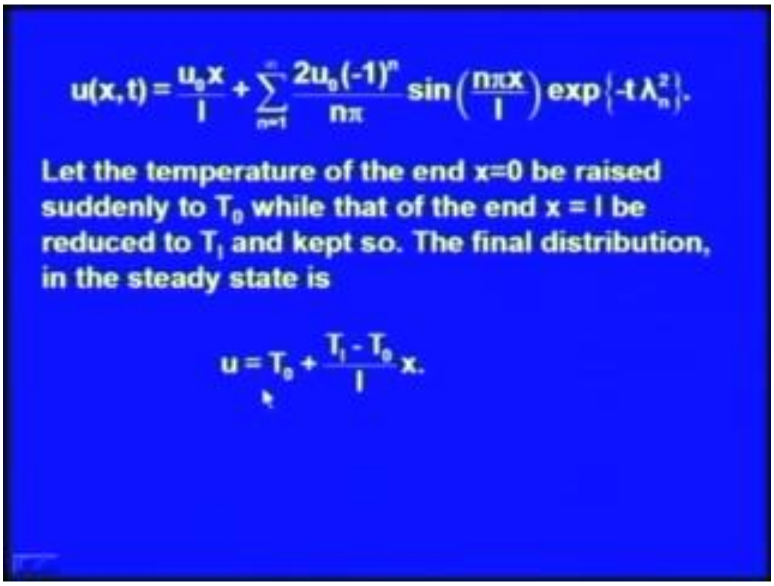

Hence, the solution is given by now v x t is equal to u x t minus u naught x over l, because we had defined v x t as u x t minus f x and f x had come out to be u naught x over l. So, v x t, we found as being equal to u x t minus u naught x over l, so let us put the value of v x t here and then take the term u naught x over l to the other side. We have u x t equal to u naught x over l plus sigma n equal to 1 to infinity, they may there is the value of B n 2 into u naught minus 1 to the power n over n pi into sin n pi x over l into exponential of minus t lambda n square.

Now, let us take the range the end, temperature of the end x equal to 0 suddenly to T naught and while the temperature of the end x equal to l be reduced to T 1 and kept so throughout. Then, when the bar comes back to be steady state, the final distribution in distribution of temperature will be given by u equal to T naught plus T 1 minus T naught over l into x, because again when the bar comes back to the steady state, you will be equal to a linear function of x.

So, you we you will be equal to A x plus B and when you will find the values of the constants A and B making use of these conditions that at x equal to 0, the temperature u is T naught and x equal to l, the temperature u is T l. Then, we will get the values of A and B, A will be equal to T naught, A plus B x, if you write u equal to A plus B x, then A will be equal to T naught B will come out to be T l minus T naught over l. So, u is equal to T naught plus T l minus T naught over l into x.

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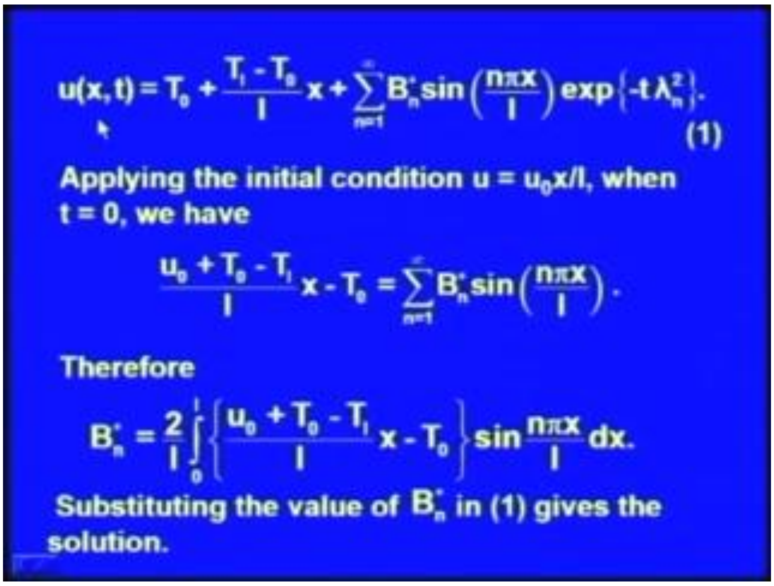

And then, the solution will take the form u x t equal to T naught plus T l minus T naught over l into x plus sigma n equal to 1 to infinity, B n star sin n pi x over l, exponential of minus t lambda n square, where a we have applied the initial condition, that u is equal to u naught x over l. Initially, we have started here, counting the time, we are from the stage, when the initial temperature distribution in the bar was given by u equal to u naught x over l.

Now, let us put t equal to 0 here, when you put t equal to 0 here u x 0 is u naught x over l and this u to the power e to the power 0 becomes 1 here. So, let us evaluate the values of the constants B n star here, the this equation trans implies the gives us u naught plus T naught minus T l over l into x minus T naught equal to sigma n equal to 1 to infinity, B n star sin n pi x over l. When, you make use of this initial condition, that is at t equal to 0, u x t is u naught x over l.

So, then again half range expansion for this function, so the value of B n star, we can write the B n star will be 2 over l integral 0 to l, then this function of x. So, u naught plus T naught minus T l over l into x minus T naught multiplied by sin pi n pi x over l, now when you evaluate the value of this B n star and substitute, it in this equation 1, you get the temperature distribution in the bar u x t.

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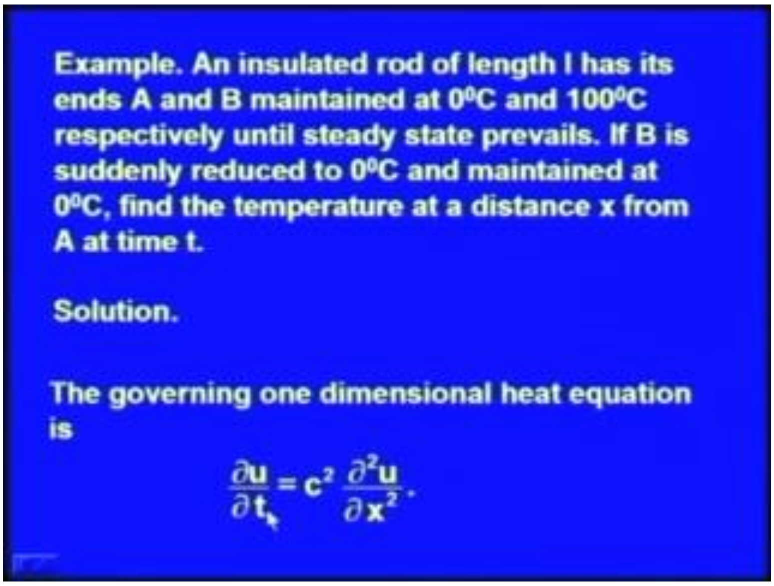

Let us take an example on this problem on this article an insulated rod of length l has its ends A and B maintained at 0 degree centigrade and 10 degree centigrade respectively.

Until a steady state conditions prevail, if B suddenly reduce to 0 degree centigrade and maintained at 0 degree centigrade, find the temperature at a distance x from A at time t.

So, here we have a problem where the end A is kept at 0 degree centigrade, the end B is kept at 100 degree centigrade. So, here u naught is equal to 100 degree centigrade, until a steady state conditions prevail. So, when this bar comes back to the steady state suddenly what is done is the end B is cooled to 0 degree centigrade. When, the end B is cooled 0 degree centigrade a fresh flow of heat takes place in the bar and before it comes back to the a steady state. We have to find the temperature at a distance x and from the end A at time t in the intermediate stage, that is in the tangent stage. Now, we know that the governing one dimensional heat equation, we know it is delta u over delta t equal to c square.

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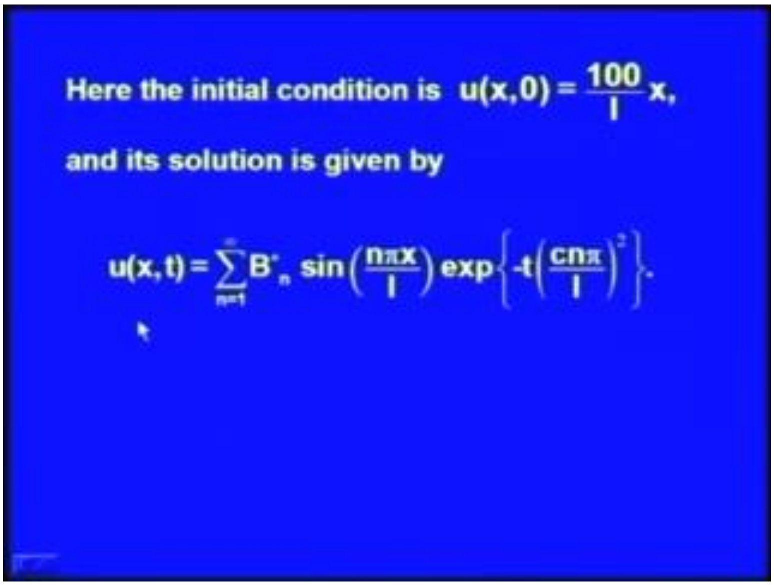

Delta square u over delta x square and initial condition here is that u x 0 is u naught x by l, we have seen that u naught is 100 here. So, u x 0 is equal to 100 x by l and its solution is given by u x t equal to sigma n equal to 1 to infinity, B n star sin n pi x over l exponential minus t c n pi over l whole square. Because, here we have the end x equal to a 0 is kept at temperature 0 and where the end B is cooled to temperature 0. So, here T 0 is 0 and T l is also 0, so if you put these values of T 0 and T l in the article and then, you can see that a u x t becomes this infinite series.

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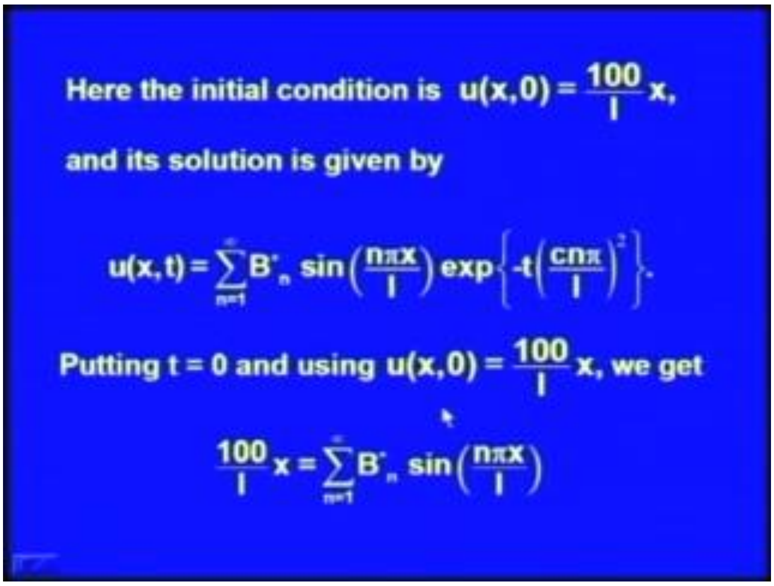

Where, when you put t equal to 0, you get u 100 x by l equal to sigma n equal to 1 to infinity, B n star sin n pi x over l.

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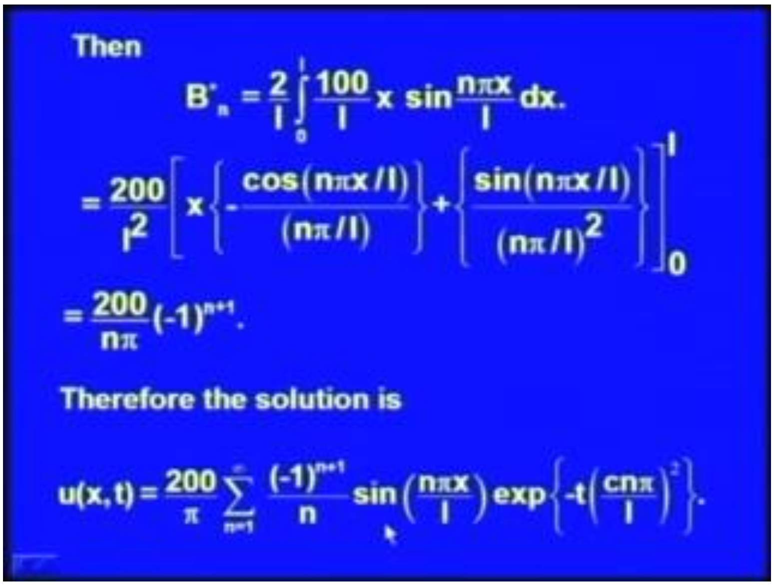

So, the values of B n star, we can find B n star will be 2 over l, 0 to l and u naught x over l, that is 100 x over l here into sin n pi x over l d x and when you integrate this by parts. And then, you get 200 over l square multiplied by x into minus cos n pi x by l over n pi by l plus sin n pi x by l over n pi by l whole square evaluated at 0 and l.

And, when you put these values that the lower and upper limits and simplify the values of B n star are given by 200 over n pi minus 1 to the power n plus 1, where n takes values 1, 2, 3 and so on. So, let us substitute those values of B n star in this infinite series for u we get u x t equal to 200 over pi sigma n equal to 1 to infinity minus 1 to the power n plus 1 over n into sin n pi x over l into exponential of minus t c n pi x over l whole square.

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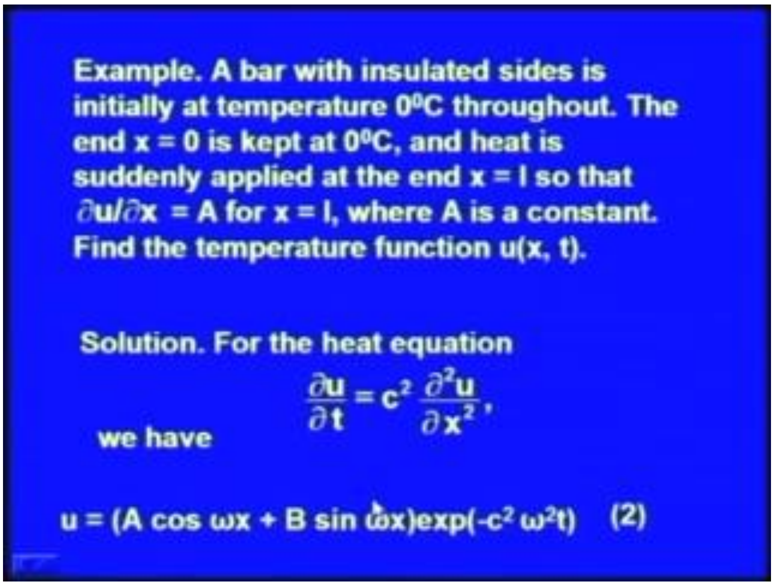

Now, let us take another problem, where we are given different boundary conditions, then the boundary conditions that we are discussed so far, a bar with insulated sides is initially at temperature 0 degree centigrade throughout. So, we are assuming here that u x 0 is equal to 0 for all x and the and x equal to 0 is kept at 0 degree centigrade and heat is suddenly applied at the x equal to l. So, that delta u over delta x is equal to A for all for x equal to l.

So, the gradient is kept at kept a constant A for at the end x equal to A, where A is a constant we have to determine the temperature function u x t here. Now, so let we see we should let us see, how do we solve such a problem, where the boundary conditions are not the as we have taken in the case of the article here. We are taking, we given the gradient instead of the value of u at x equal to l, so how will do such a problem.

Now, we have seen that when you solve this per differential equation delta u over delta t equal to c square delta square u over delta x square, what you do is you write u as f x into

T t. And then, you discuss the cases K equal to 0, K greater than 0, K less than 0, what you do note is that K less than, K equal to 0 and K greater than 0 are to be discarded.

We have to take K negative and when you take k equal to minus 1 omega square, you get u x t equal to A cos omega x plus B sin omega x into exponential minus c square omega square into t. We have to find the values of A and B making use of the given boundary conditions here.

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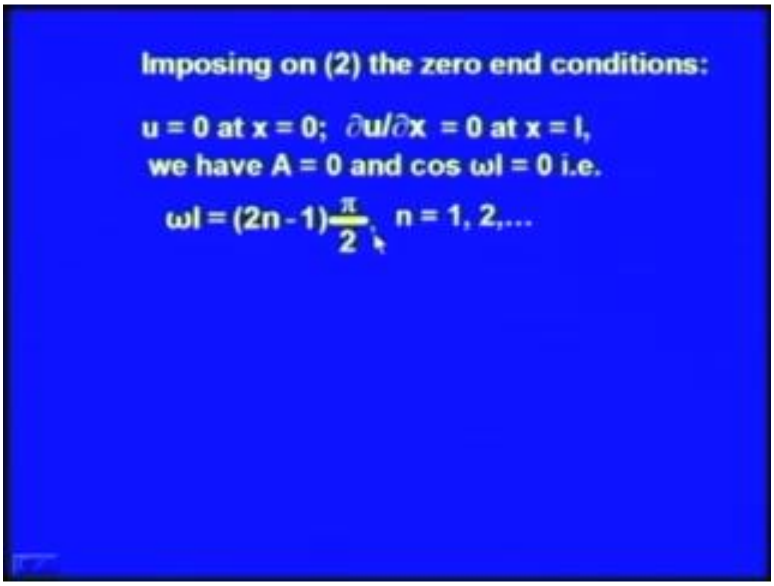

So, what we will do as we have done in the case of finding temperature for a bar with non 0 end temperatures. We hear what are all will be do is that we start with 0 and conditions, we assume that at x equal to 0 u is 0, which is given to us at the end x equal to 0 the temperature gradient is 0. We first at assume that, the temperature gradient at the end x equal to 0 is also 0 and then we shall take the case where the temperature at the end x equal to l will be taken as A.

So, u x t equal to A cos omega x plus B sin omega x into e to the power minus c square omega square into t, implies that A is equal to 0, which follows, when you take a u to be 0 at x equal to 0. And also cos omega l equal to 0, which follows, when you take the partial derivative of u with respect to l and u is the condition that delta u over delta x is 0 at x equal to l at for all the time t.

Now, cos omega l equal to 0 gives you omega l equal to 2 n minus 1 into pi by 2, where n takes value 1, 2, 3 and so on, here again as we have done in the previous case as the bar with 0 end temperatures, that here we have to take the only positive integral values of n. When, you take the negative integral value of n, you only get the solutions which or the negative of the solutions for the positive integral values of n.

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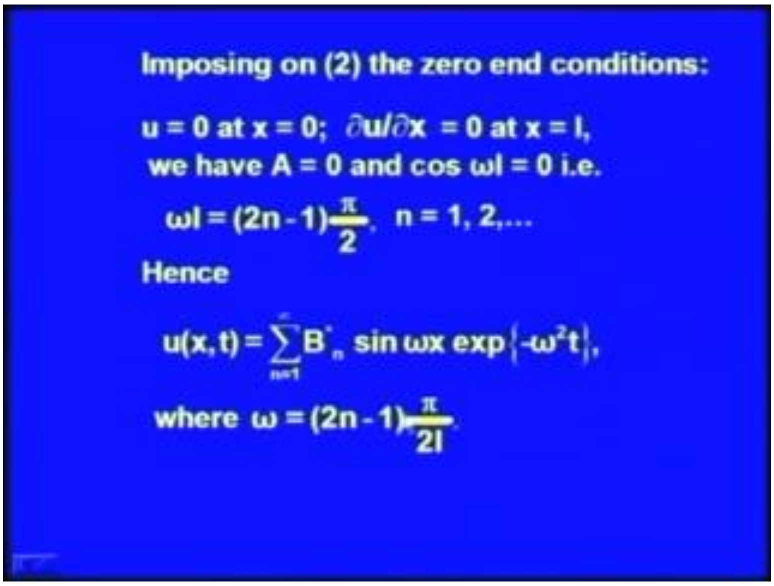

So, we will get u x t equal to sigma n equal to 1 to infinity, B n star sin omega x into exponential minus omega square into t, where omega is equal to 2 n minus 1 into pi over 2 l.

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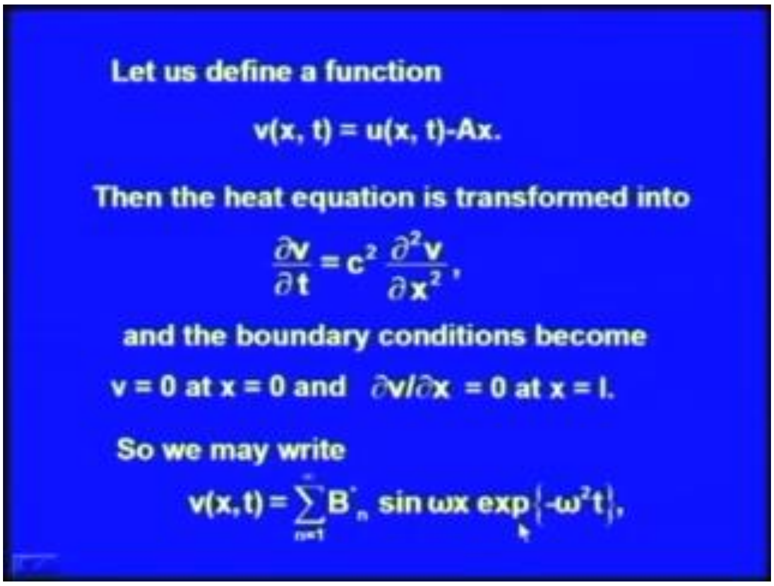

Now, let us define a function v x t equal to u x t minus A x, in a similar way as we had define for the case, where we are taken bar with non 0 end temperatures to reduce the problem to a bar with 0 end temperatures. So, here we again define a function v x t equal to u x t minus A x and then the heat equation as we had done in the case of the bar with 0 and non 0 end temperatures, the heat equation is transformed into delta v over delta t equal to c square delta square v over delta x square.

So, the boundary conditions reduce or you can say the boundary conditions change into when we have assume that at t equal to 0, u x 0 is equal to 0, so here also, when you put t equal to 0, v x 0 comes out to be 0 minus 0, so v is equal to 0 at x equal to 0. We have assume that at x equal to l, delta u over delta x is equal to A, it is given to us, so delta v over delta x, if you write delta v over delta x, will be delta u over delta x minus A. And when you put delta u over delta x as A, then you get delta v over delta x as equal to 0 at x equal to l.

So, the problem now reduces to the problem, which we have are already just now done that, where we had to set that at x equal to 0, the temperature is 0 and at x equal to l, the temp the temperature gradient is 0. So, while defining this function v x t, we reduce the problem, where at the end is equal to 0, the temperature gradient is given to be a constant A to the problem, where the temperature gradient at the end x equal to l is also 0.

So, with this at a transformation we will have the solution of the heat equation, delta v over delta t or this equation delta v over delta t equal to c square, delta square v over delta x square as v x t equal to sigma n equal to 1 to infinity, B n star sin omega x into exponential minus omega square into t.

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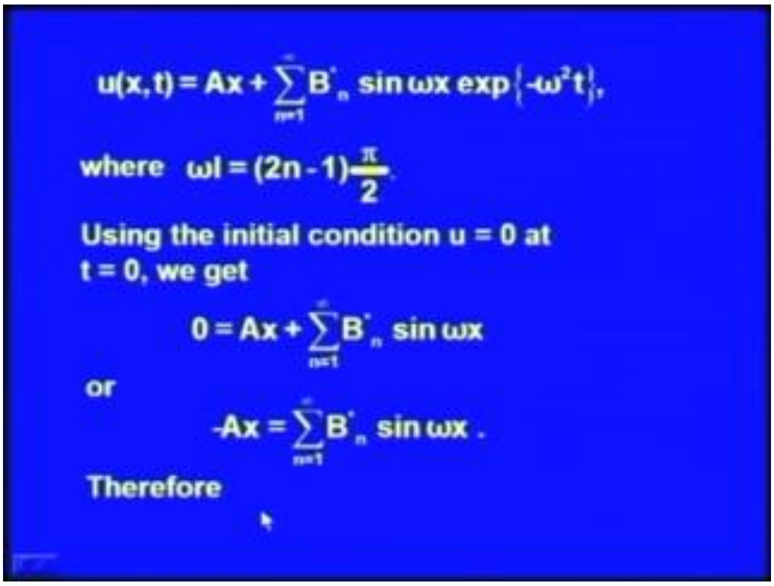

Let us replace the value of v x t, v x t is equal to u x t minus A x, so u x t will be equal to Ax plus sigma n equal to 1 to infinity B n star sin omega x into exponential of minus omega square t; where omega l, we know is equal to 2 n minus 1 into pi by 2 or we can say omega is equal to 2 n minus 1 into pi over 2 l and taking positive integral values.

Now, let us make use of the initial condition that is at t equal to 0, the initial temperature distribution in the bar is 0. So, when you make use of that, we will get 0 equal to A x plus sigma n equal to 1 to infinity, B n star sin omega x or we can say that sigma n equal to 1 to infinity, B n star sin omega x will be equal to minus b minus x. So, we shall get half range expansion of the function minus A x.

So, this is what we get, 0 equal to A x plus this and then or we may write this equation as minus A x plus sigma equal to sigma n equal to 1 to infinity, B n star sin omega x.

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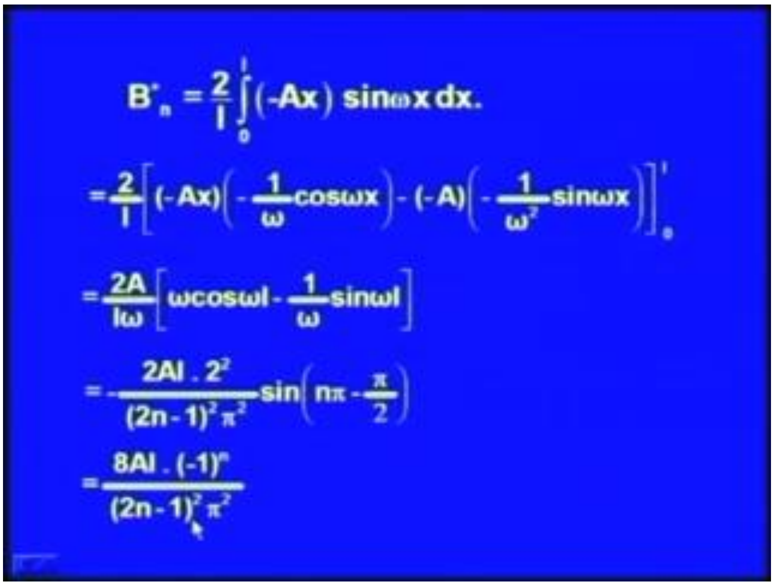

Therefore, B n star will be given by 2 over l integral 0 to l minus A x into sin omega x d x and we can then evaluate the value of this integral by using integration by parts. It will come out to be 2 over l into minus A x minus 1 over omega sin cos omega x, which is the integral of sin omega x minus derivative of minus x, which is minus A into integral of this function, that is minus 1 over 1 by omega square sin omega x evaluated at 0 and l..

So, this when you put the lower and upper limits 0 and l, we will get 2 A over l omega cos omega l minus 1 over omega, sin omega l and we simplified further use omega equal to 2 n minus 1 into pi by 2 l. So, we get it as minus 2 A l into 2 squares over 2 n minus 1 square into pi square sin n minus 1 as sin n pi minus pi by 2. And when you evaluate the value of sin n pi minus pi by 2, you gets this value of B n star as eight A l into minus 1 to the power n over 2 n minus 1 whole square into pi square.

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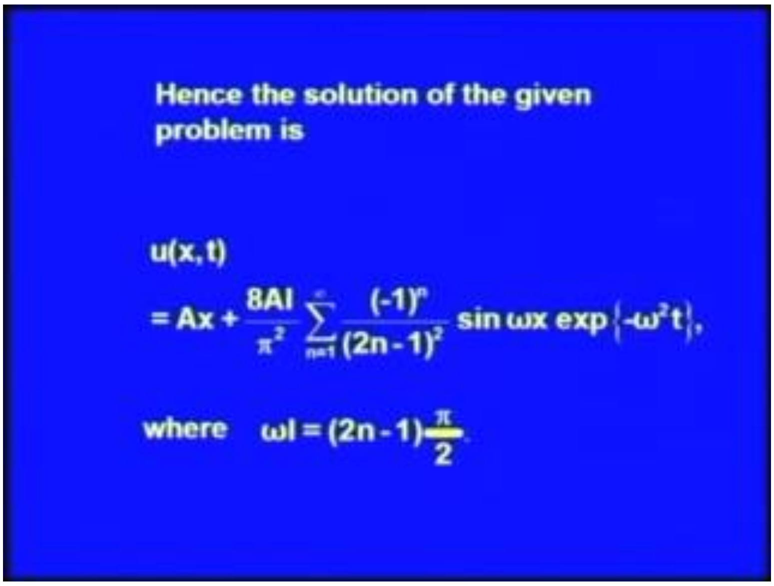

And thus, we can write the solution of the given problem u x t equal to A x plus 8 A l over pi square into sigma n equal to 1 to infinity minus 1 to the power n over 2 n minus 1 whole square into sin omega x into exponential minus omega square t; where omega is equal to 2 n minus 1 into pi over 2 l.

Thus, we have solved the one dimensional wave equation and the one dimensional heat equation using the method of separation of variables. We can also discuss two dimensional wave equation and the two dimensional heat equation or even three dimensional heat equation using the method of separation of variables. So, the two dimensional cases will be covered in the next in the other lectures, where we will be discussing the rectangular region. In the case of a two dimensional wave equation and a rectangular plate for two dimensional heat equation and we shall solve them using the product method, that is the method of separation of variables in a similar fashion.

Thank you.