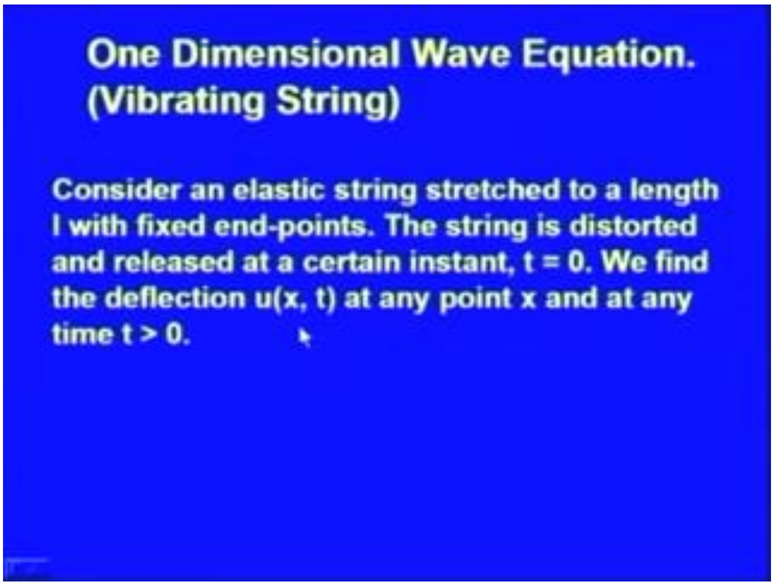
Mathematics III

Prof. P. N. Agrawal
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture - 12
One Dimensional Wave Equation

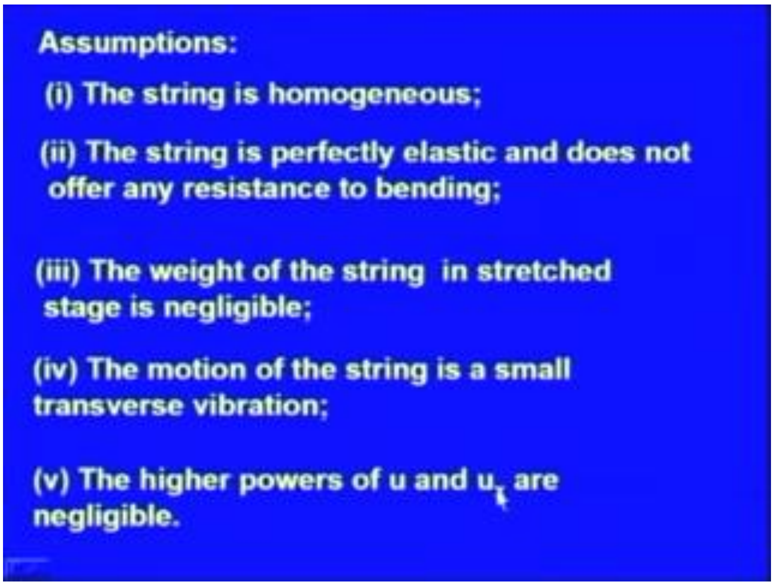
Dear viewers, in my lecture today, we shall discuss the One Dimensional Wave Equation and it is solution by the product method and D'Alembert's method. We shall take a uniform elastic string tightly stretch between two fix points and consider it is transfers vibrations unrespectable assumptions. We shall see that it gives us the equation as delta s square u over delta t square equal to c square times delta square u over delta x square, where u x t denotes the displacement of the string at a distance x and at time t. This equation as we know from classification partial differential equations is a hyperbolic partial differential equation. So, we shall discuss its solution by taking the product method and then by the classification by the D'Alembert's method.

(Refer Slide Time: 01:39)



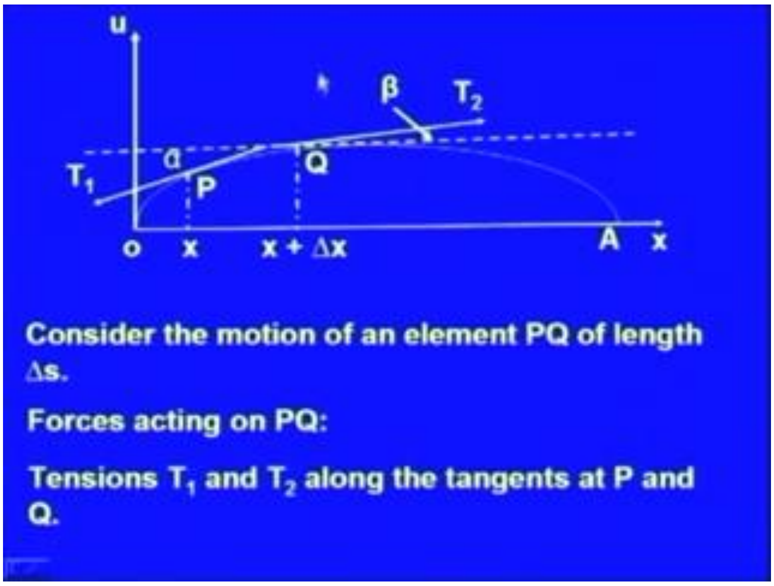
Let us consider an elastic string stretch to a length l with fixed end points, the string is distorted and released at a certain instant we state that instant as t equal to 0 and then we will discuss the deflection, which we denote by u x t at any point x and at a time t greater than 0 of the instant.

(Refer Slide Time: 02:01)



The assumptions are the string is homogeneous it is perfectly elastic and does not offer any resistance to bending. The weight of the string in the stretched stage is negligible the motion of the string is a small transverse vibration the higher powers of u and delta u over delta x which we have denoted by u x are negligible that is the higher powers of u and its slope delta u over delta x are negligible.

(Refer Slide Time: 02:43)

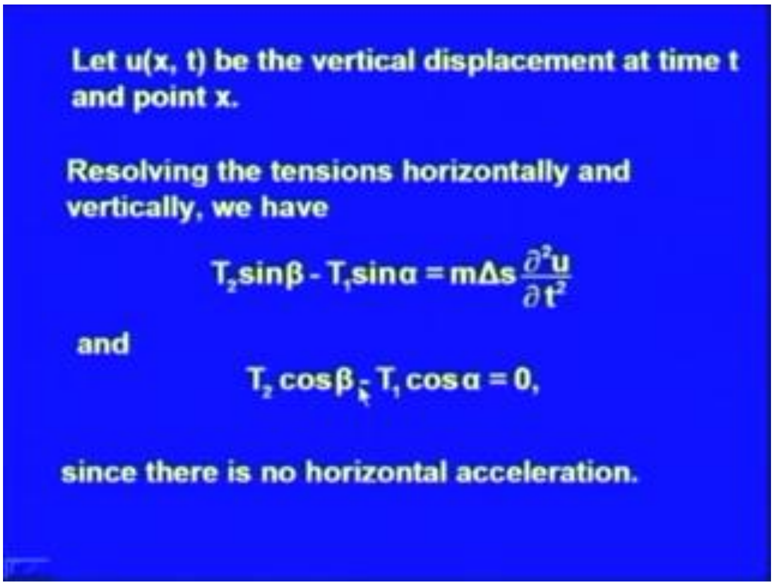


Let us consider the motion of an element PQ of length delta s the forces that are obtain on this element or the tensions at the points P and Q, which at along the tangents T 1 is the tension at the point P and T 2 is the tension at the point Q. Since, the string is perfectly elastic it does not offer any resistance to bending.

Now, when we resolve these forces T 1 and T 2 horizontally and vertically, then the horizontal force horizontal component of the force T 1 will be T 1 cos of alpha horizontal component of the force T 2 will be T 2 cos of beta. So, we shall have T 1 cos of alpha equal to T 2 cos of beta, because there is no horizontal a long horizontal direction, then they are equal and then in the vertical direction there is a movement only in the transverse direction of the elastic string.

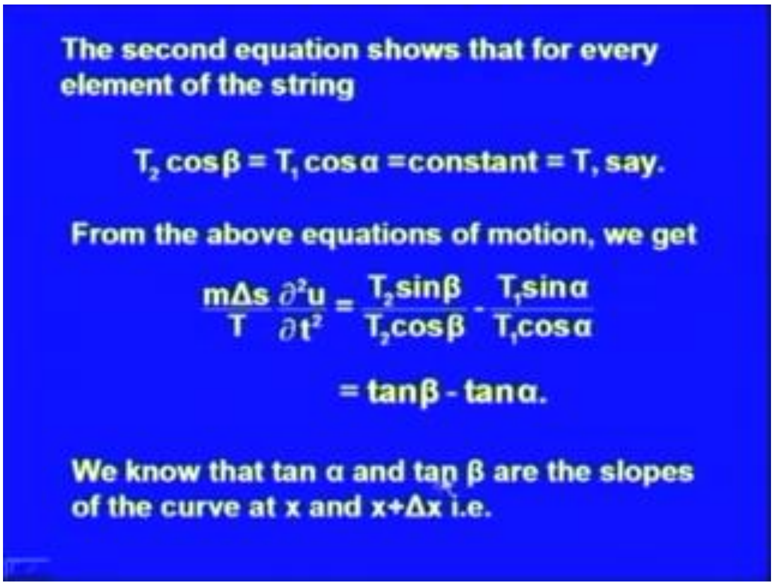
So, T 1 cos of alpha mew s t be equal to T 2 cos of beta and in the vertical direction, that is along the by axis. We have m delta s is the mass of the element PQ m delta s into delta square u over delta t square equal to this is a and here, we will have vertical component of T 2 will be t 2 sin of beta where the vertical component of T 1 will be T 1 sin of alpha, so T2 sin of beta minus T 1 sin of alpha.

(Refer Slide Time: 04:42)



So, we shall have the equations of motion as if u xt denotes the vertical displacement at time t and at a distance point x, then after resolving the tensions horizontally and vertically we have T 2 sin beta minus T 1 sin of alpha equal to m delta s into delta square u over delta t square and T 2 cos beta equal to T 1 cos of alpha because there is no horizontal acceleration.

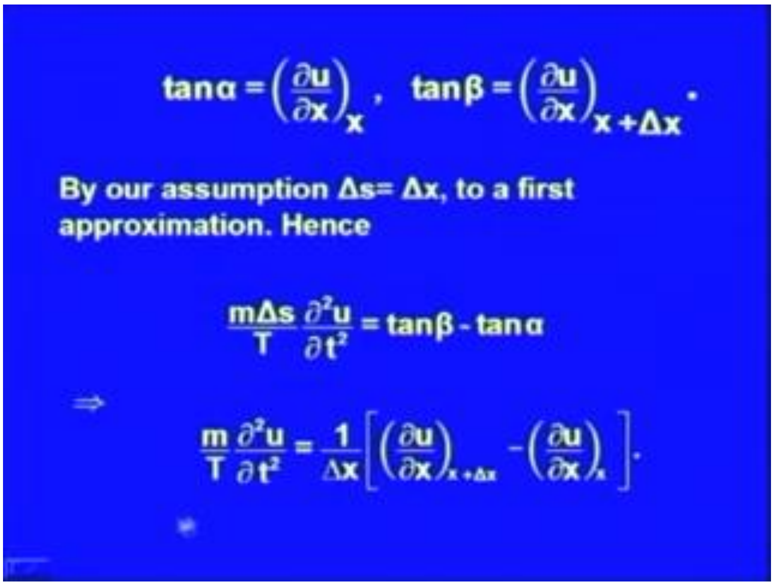
(Refer Slide Time: 05:06)



Now, the second equation T 2 cos beta minus T 1 cos of alpha equal to 0 show that for every element of these string T 2 cos beta is equal to T 1 cos of alpha and so we can take it as a constant say T. Now, from the above equations of motion we have then m delta T mm delta s over T, let us divide the equation first equation by T, then m delta s by T into delta square u by delta t square equal to T 2 sin beta over T that is T we can take us T 2 cos beta.

So, T 2 sin beta over T 2 cos beta minus T 1 sin alpha over T here we take the value of T as T 1 cos of alpha. So, we have T 1 sin alpha over T 1 cos of alpha and then the right hand side becomes right hand side of this equation becomes tan beta minus tan of alpha and from the figure it is clear that tan alpha and tan beta are the slopes of the curve at the point x and at the point x plus delta x.

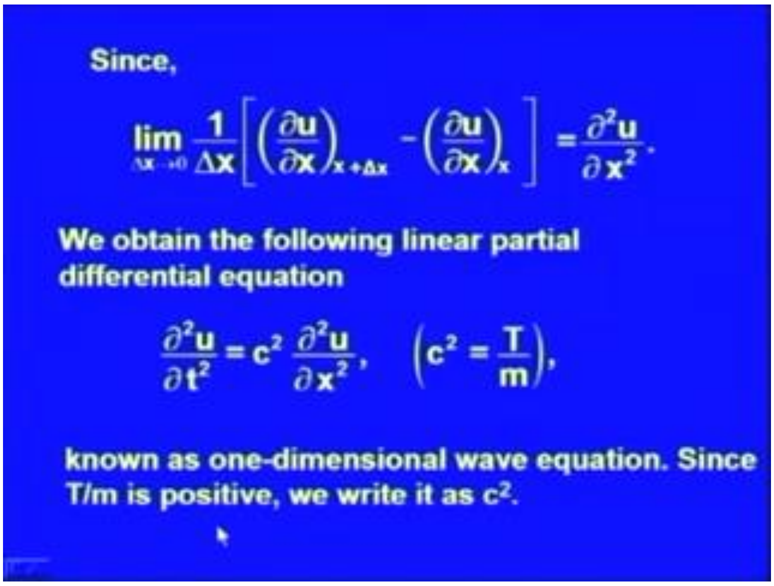
(Refer Slide Time: 06:14)



And therefore, tan alpha is equal to the partial derivative of u with respect to x at x and tan beta will be the partial derivative of u with respect to x at x plus delta x that is this slopes at x and at x plus delta x and by our assumption delta s is equal to delta x to a first order approximation. Hence m delta s over delta m delta s over T delta square u over delta t square is equal to tan beta minus tan of alpha, which implies that m delta square u over delta t square m over T delta square u over delta t square is equal to tan beta minus tan alpha that is delta u over delta x at x plus tan alpha minus delta u over delta x at x divided by delta x.

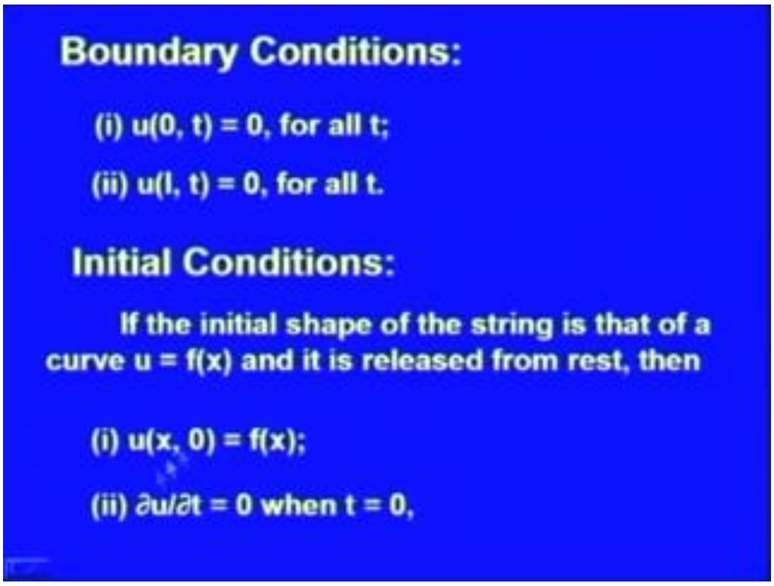
Now, as delta x goes to 0 the right hand side of this equation the right hand side of this equation that is 1 over delta x delta u over delta x at x plus delta x minus delta u over delta x at x will tend to delta x square u over delta x square and m over T are we will take it to the right side, so it will become T over m, T over m is the physical constant T over m is positive. So, we will write we can write it as c square and that will lead as to the partial differential equation delta square u over delta t square equal to c square delta u square u over delta x square.

(Refer Slide Time: 07:48)



So, since this expression as delta x goes to 0 tends to delta square u over delta x square, we obtain the following linear partial differential equation delta square u over delta t square equal to c square delta square u over delta x square, which is known as given dimensional wave equation further, because c T over m the physical constant T over m is positive, we may take it this as the square of c. So, we have the differential equation as u t t equal to c square into u xx.

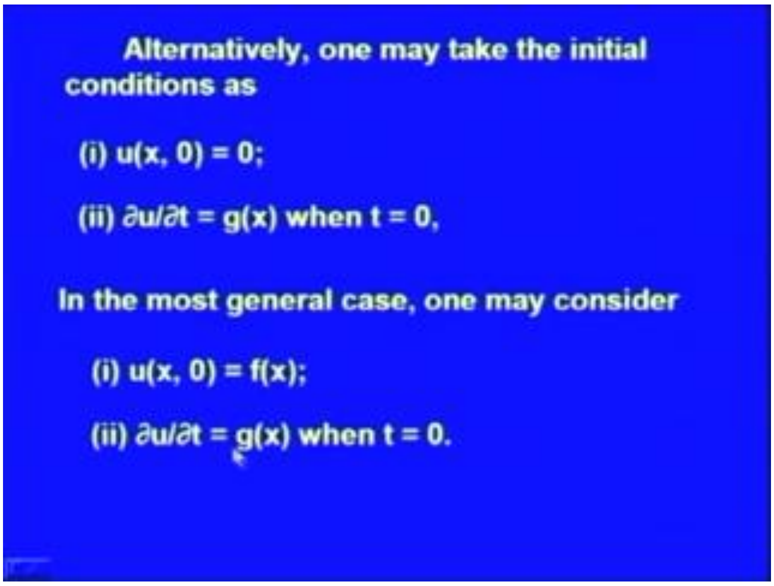
(Refer Slide Time: 08:30)



Now, the boundary conditions in every for the in this case are u 0 t equal to 0, because at the ends they are the ends of the string are fixed. So, there is no displacement at the ends of the boundary string that is at x equal to 0 for all the time t there is no displacement. So, u 0 t is equal to 0 for their at the other end that is at x is equal to l for all the time t there is no displacement, so u l t is this is equal to 0.

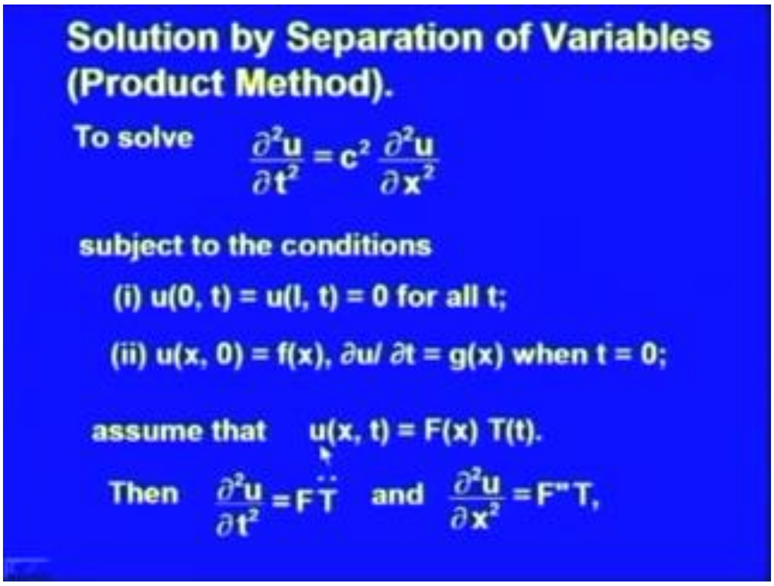
The initial conditions are the if the initial shape of the string is that of a curve u equal to f x and it is released from rest, then we have u x 0 equal to f x and delta u over delta t equal to 0 when t equal to 0 here. We are assuming that the initial shape of the string is given by a curve say u equal to f x and it is released from rest of there is the velocity delta u over delta t is 0 at t equal to 0 and at t equal to 0 u x t is equal to f x

(Refer Slide Time: 09:46)



Now, another set of initial conditions may be taken as u x 0 is equal to 0 and delta u over delta t equal to g x, when t is equal to 0 that is initially there is no displacement in the string at time t equal to 0, u x 0 is u x t is 0, but the initial velocity of the string is given by a function of x, so that is g x. So, when at t equal to 0 the initial velocity is given by this g x in the most general con case one may take the initial conditions as the initial shape of the string is given by a curve that is the function f x that is, so u x 0 is equal to f x and the initial velocity at the time t equal to 0 is also given by a function of x that is g x.

(Refer Slide Time: 10:42)

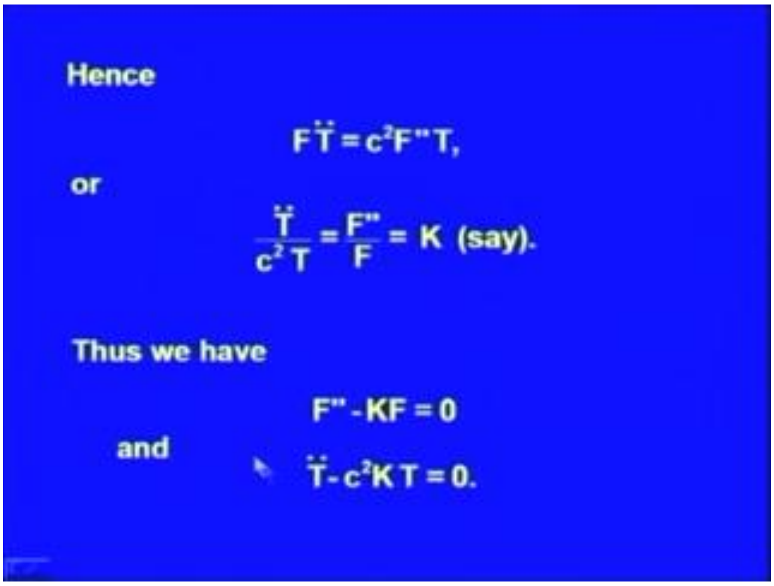


Now, let us solve this differential equation using the separation of variables method, which is also known as product method. So, we shall be solving this wave equation one dimensional wave equation subject to the conditions the u 0 t equal to u l t equal to 0, which are the two boundary conditions and the initial conditions we shall take as the most general case that is at the time t equal to 0 u is g is given by f x and at time t equal to 0 the velocity is given by g x.

So, let us assume that the function u, then one function u can be written as the product of two functions one is a function of x another one is a function of t and. So, u is assume to be a f of a particular form it is a product of two functions one is a function of x and another one is a function of t that is why we call it a productive method or separation of variables method.

So, assume that u x u is equal to F into T, then if you differentiate u partially with respect to t twice you will get F into T double dot. So, we had their dots represent the derivatives with respect to T and if you differentiated partially with respect to x twice, then you get delta square u over delta x square as F double dash x into T. So, F double dash here represents the represent the second derivative of F with respect to x. So, primes denote the derivatives with respect to x where the dots denotes derivative with respect to T.

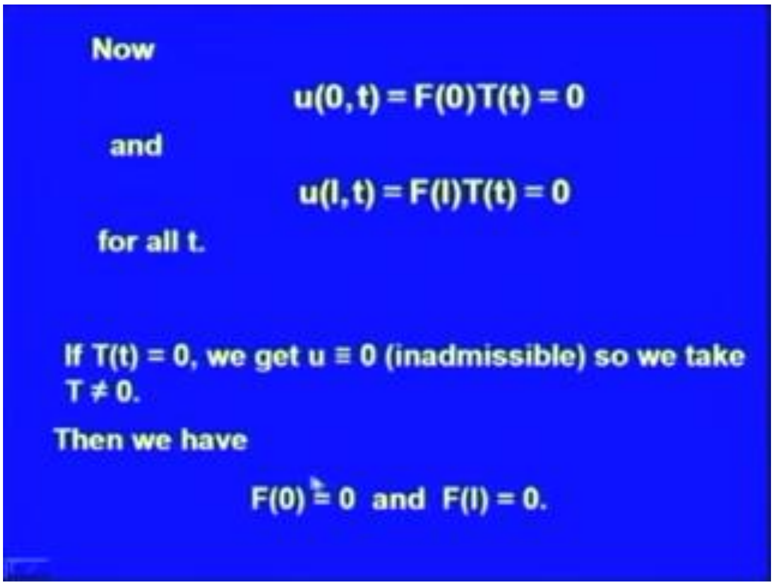
(Refer Slide Time: 12:23)



And, now let us put the values of these u xx and u t t in the given one dimensional wave equation we will have FT double dot equal to c square F double dash into T. We can separate the functions of x and T, so we shall have T double dot over c square into T equal to F double dash over F. Now, if you look at this equation T double dash T double dot over c square T equal to F double dash over F, then T double dot over c square T is a function of T alone, while F double dash over F is a function of x alone and they are equal such an equation is possible only when both are equal to a constant.

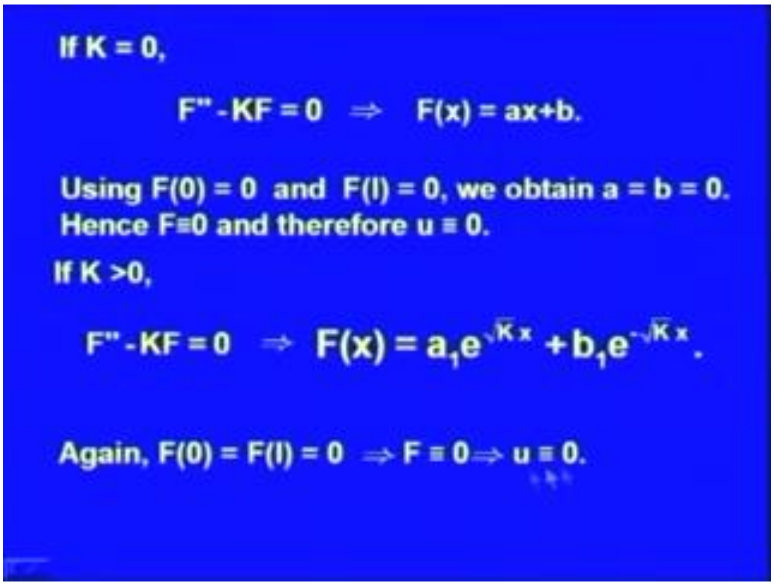
So, T double dot over c square T equal to F double dash over F mewst be equal to a constant, which we can take as say K. Then, we will have this will lead us to two ordinary differential equations of second order F double dot F double dash if minus K of equal to 0 and T double dot minus c square into KT equal to 0.

(Refer Slide Time: 13:37)



Now, we are having the boundary condition that u 0 t is equal to 0 and we have assume that u x t equal to F x into T t. So, let us put x equal to 0 there, then what will have is what will happen we will get u 0 t equal to F 0 into T t, which is equal to 0. And, also we have u l t equal to 0, so u l t is equal to F l into T t equal to 0 for all the time T, now if T t is equal to 0, then u x t equal to F x into T t will give you u as identically 0 that is at for all the time t and the for all x u is 0. So, this is an inadmissible case and so we will take u t to be not equal to 0, and then one will have f 0 equal to 0 and F l equal to 0.

(Refer Slide Time: 14:34)



Now, let us discuss the various cases, which occur due to the constant K the constant K may be 0 it may be positive it may be negative. So, we first discuss the case when K is 0, if K is 0 f double dash minus KF equal to 0 will implies at F double dash is 0 . So, that will give us after integration twice with respect to x it will lead as to F x equal to ax plus b.

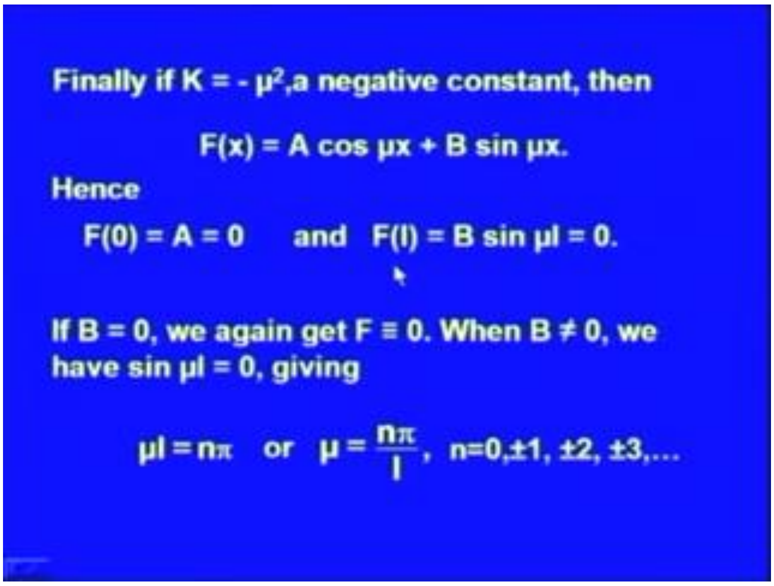
Now, we already have from the boundary conditions that F 0 is 0 and F l equal to 0. So, if we make use of these two conditions, then we shall have a and b both 0, and since a and b both are 0, F x will be 0 for all values of x that is F is identically 0 and therefore, again u will be u being product of F and T will be identically 0.

So, this again inadmissible case now if k is taken to be positive then f double dash minus KF equal to 0 will be giving us this homogeneous second order ordinary second order differential equation. So, we can write it is auxiliary equation as m square minus K equal to 0, which will give us two values of m as m equal to plus minus root K they are both real and distinct.

And therefore, the complementary function will be equal to some constant that is a 1 times e to the power root k into x plus b 1 times e to the power minus root k into x here the particular integral is 0, because the right hand side is 0. So, the general solution may be written as the complementary function plus particular integral that is F x equal to a 1 e to the power root K into x plus b 1 e to the power minus root K into x.

Now, again in order to determine a 1 and b 1 let us make use of F 0 equal to 0 and F l equal to 0. So, F l F 0 equal to F l equal to 0, then implies that F x is 0 for all values of x that is F is identically 0, which will imply that u is identically 0, so K greater than 0 is also not suitable.

(Refer Slide Time: 16:56)

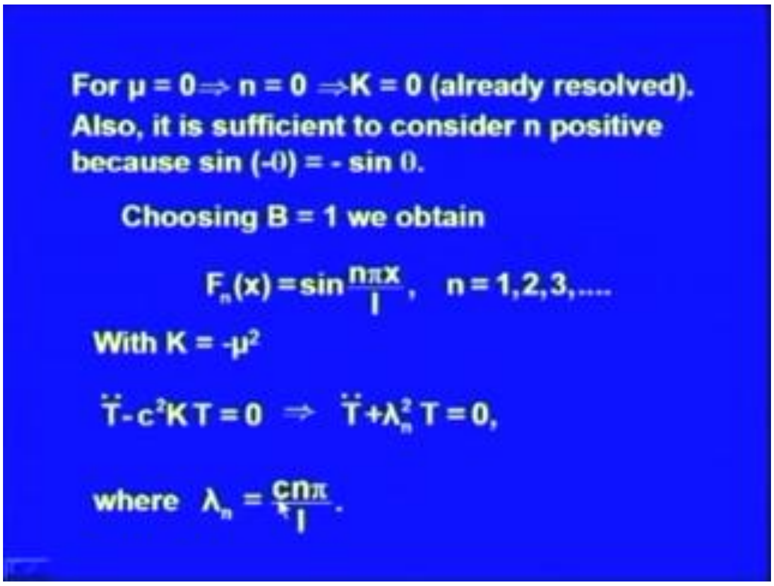


Let us now, consider K to be negative and since K is negative constant we can take it as negative of mu square. Then, the differential equation the second order differential equation for F leads us those to this solution F x equal to A cos mu x plus B sin mu x. Now, making use of F 0 equal to 0 it follows that A is equal to 0 and when you take Fl equal to 0 we get B sin mu l equal to 0. Now, if you take here B equal to 0, then A and B both 0, will give you F x equal to 0 for all values of x, which will give you u equal to 0 for all x and t, so B cannot be taken as 0, so the other possibility is that sin mu l is equal to 0.

And, sin mu l equal to 0 then gives us mu l equal to n pi or mu equal to n pi by l, where n can take all integral values 0 plus minus 1 plus minus 2 plus minus 3 and so on. If, you take n equal to 0 here, then mu is equal to 0 and mu equal to 0 will give you K equal to 0 and we have seen that when K is equal to 0, we get the displacement function u as identically 0.

So, that case was not in a was not admissible and therefore, n is equal to 0 is ruled out here when, you take negative values of n, n equal to minus 1 minus 2 minus 3 and so on that is, then we know that sin of minus theta is minus sin theta. So, what we will get they will get these solutions are negative they will get a same set of solutions except that will have a negative sign associated with it. So, we get nothing new, so that is why the negative values of n can also be discarded, so we I will say that mu is equal to n pi by l where n is a positive integer that is n is taking values 1, 2, 3, and so on.

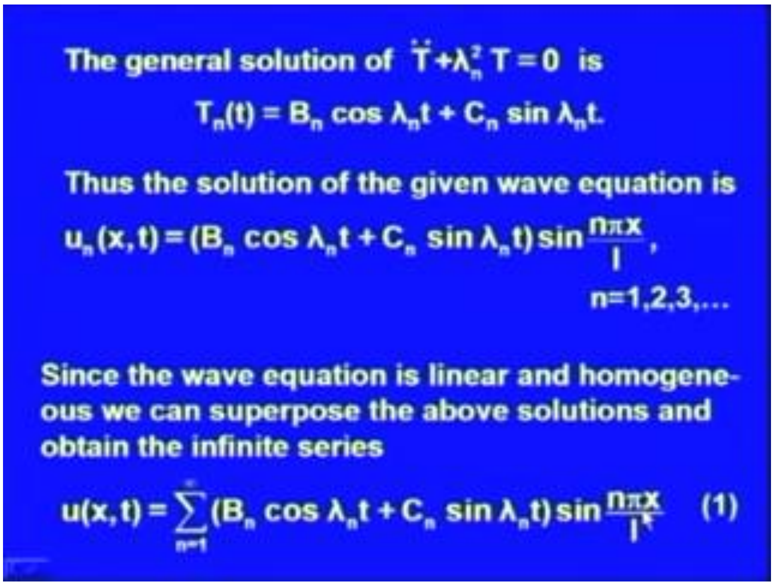
(Refer Slide Time: 19:02)



So, now, will it is sufficient to consider n as positive integer, because sin minus theta is equal to minus sin theta. So, now, we can also take for simple convenience we can also take b equal to 1 and then we shall get these solutions as sin n pi x over l where n takes values 1 2 3 and so on. So, each value of n is giving as a solution of the second order ordinary differential equation for F and so we denote them by F n x, so fn x is equal to sin n pi x over l when n takes values 1 2 3 and so on.

Where now, with k equal to minus mu square let us see, but do we get from the second differential equation that is the second order differential equation for T. So, T double dot minus c square KT equal to 0, becomes T double dot plus lambda is n square T equal to 0. Where lambda n is equal to c n pi over l after putting K equal to minus mu square and we have T double dot mi plus c square mu square into T equal to 0 and that will give us and mu is we have seen and pi over l, so we will get lambda n equal to c n pi over l.

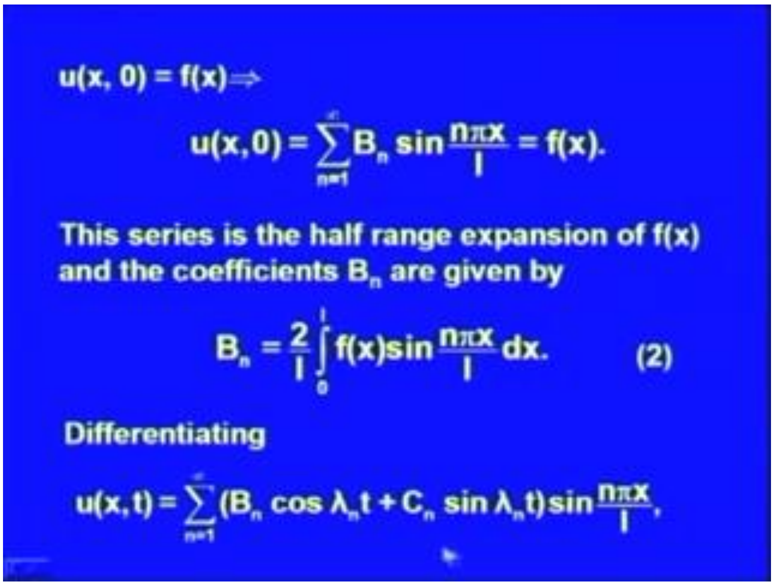
(Refer Slide Time: 20:36)



So, this will give us this as the solution T double dot plus lambda n square T equal to 0 will give us the solution general solution as T n t equal to B n cos lambda n t plus C n sin lambda n t for each value of n this differential equation will give as a solution. So, we can call this a solution for the nth equation as the equation for n is T n t, so T n t is B n cos lambda n t plus C n sin lambda n t thus the solution of the given wave equation is un x t equal to the this is T n t B n cos lambda n t plus C n sin lambda n t and this is F n x sin n pi x over l we had assume that u x t equal to F into t, so from that we are getting this and n is taking values 1 2 3 and so on.

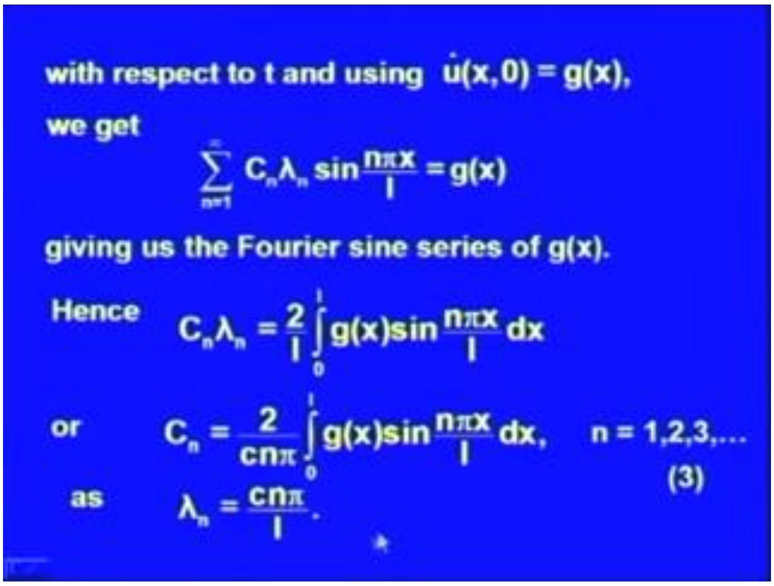
Now, since the wave equation is the linear or a partial differential equation of second order and it is homogeneous. So, we can superpose the above solutions a obtain the infinite series u x t equal to sigma n equal to 1 to infinity B n cos lambda n t plus C n sin lambda n t into sin n pi x over l let us call it as equation number 1.

(Refer Slide Time: 21:56)



Now, we have assume that the initial shape of these string elastic string is given by the corrupt u equal to f x. So, u x 0 equal to f x implies if you put t equal to 0 in the equation number 1 what you get is u x 0 equal to sigma n equal to 1 to infinity B n sin n pi x over l which is equal to f x. Now, this series is the half range expansion of the function f x this we know from the forever study of the Fourier series. So, this is the half range expansion of f x and if you recall the in the case of the half range expansion the coefficients B n is are given by 2 over l integral 0 to l f x sin n pi x over l d x this equation we call as equation 2.

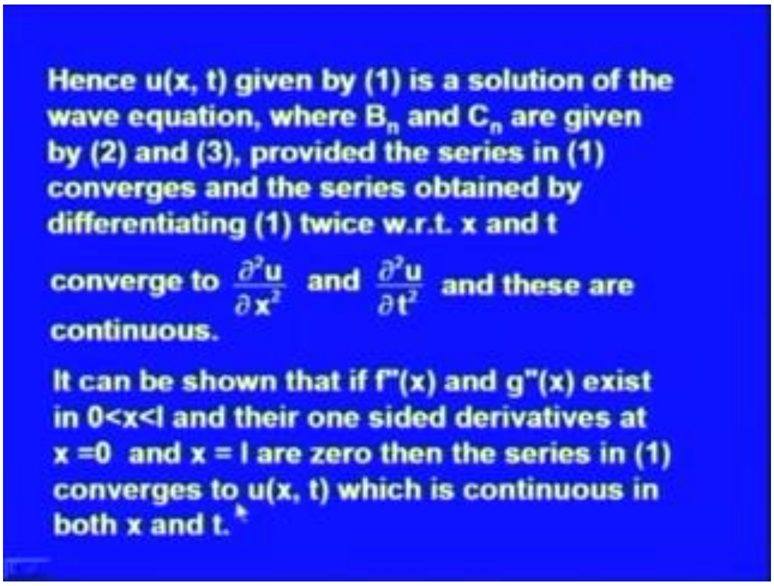
(Refer Slide Time: 23:03)



Now, if you differentiate this equation 1 that is the equation for u x t with respect to t partially, then what you get this is the equation 1 we differentiate we are going to differentiate it partially with respect to t in order to use the second condition delta u over delta t at t equal to 0. So, with respect to t when you differentiate it what you get and u will that delta u over delta t at t equal to 0 is g x that is the initial velocity of the string is given by the function g x we will have sigma n equal to 1 to infinities C n lambda n sin n pi x over l equal to g x again giving us the Fourier sin series of the function g x.

And therefore, these C n lambda n will be equal to 2 over l integral 0 to l g x sin n pi x over l d x. Now, let us put the value of lambda n here as C n pi over l then we will get C n as 2 over C n pi integral 0 to l g x sin n pi x over l d x, where n takes values 1 2 3 and. so on thus call it as equation number 3.

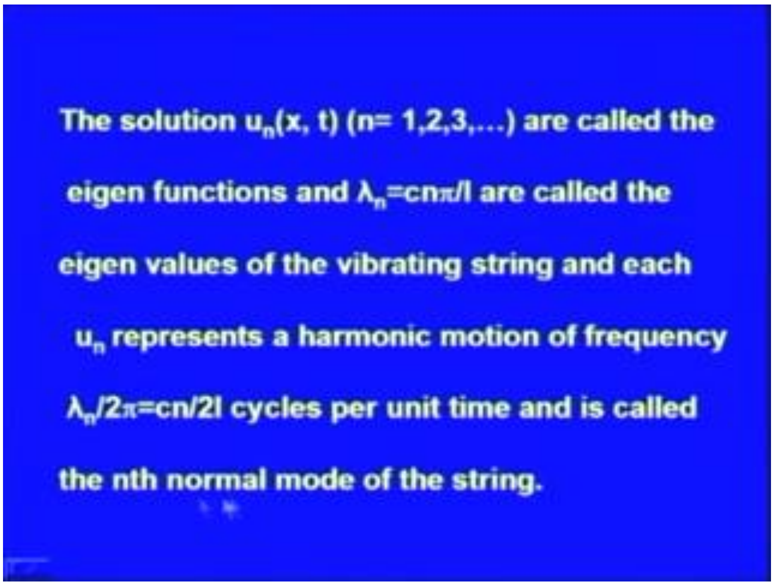
(Refer Slide Time: 24:06)



Hence u x t given by the equation 1 is a solution of the wave equation, where B n and C n is are given by equations 2 and 3, provided the series in one converges and the series obtained by differentiating one twice with respect to x and t converge to the sums delta square u over delta x square and delta square u over delta t square and these sums are continuous.

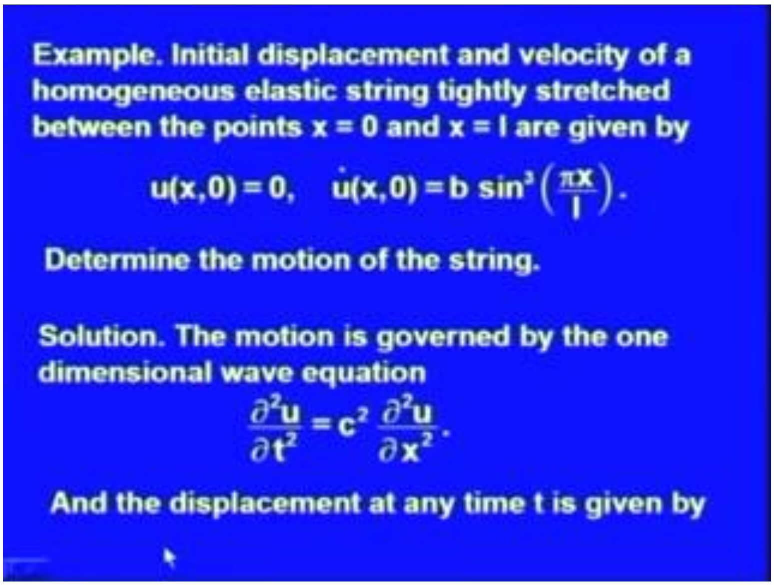
Now, it can be shown that if f double dash x and g double dash x exist in the interval 0 to l and their one sided derivatives at x equal to 0 that is f prime 0 plus and g prime 0 plus f prime l minus and g prime l minus are 0, then the series in one converges to u x t, which is continuous in both x and t.

(Refer Slide Time: 24:59)



The solution un x t and taking values 1, 2, 3, and so on are called the Eigen functions and lambda n is equal to c n pi over l are called the Eigen values of the vibrating string each un represents a harmonic motion of frequency lambda n over 2 pi lambda n is c n pi over l. So, lambda n over 2 pi is equal to c n over 2 l cycles per unit time and is called the nth normal mode of the string.

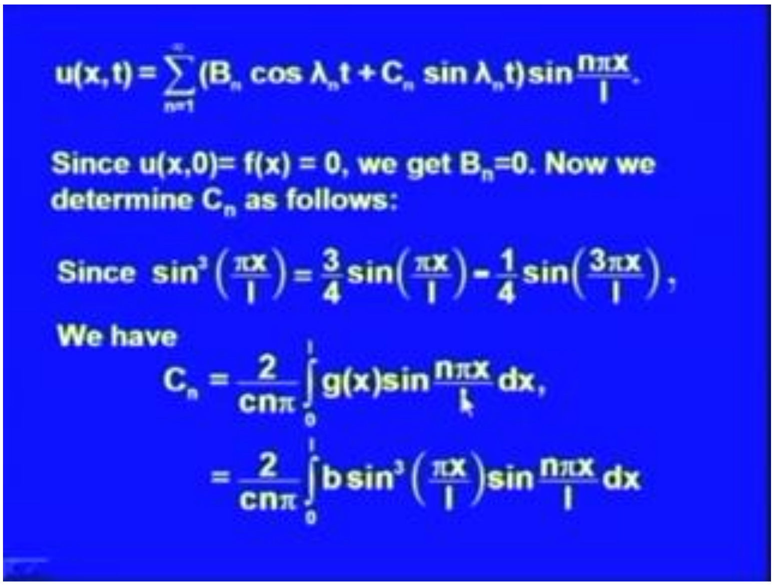
(Refer Slide Time: 25:30)



Now, let us take an example based on the one dimensional wave equation, so initial displacement and velocity of a homogeneous elastic string tightly stretch between the points x equal to 0 and x equal to are l are given by u x 0 equal to 0 u dot x 0 equal to b sin cube pi x over l.

So, we are given that the initial shape of the homogeneous velocity string is given by u x 0 equal to 0 that is we are given f x equal to 0 and we are given the initial velocity of the elastic string are the function of x given by b sin cube by x over l and by our notation the initial velocity is g x. So, g x is equal to b sin cube pi x over l here, we are to determine the motion of the given velocity the string, now we know that the motion is governed by the one dimensional wave equation delta square u over delta t square equal to c square delta square u over delta x square.

(Refer Slide Time: 26:43)

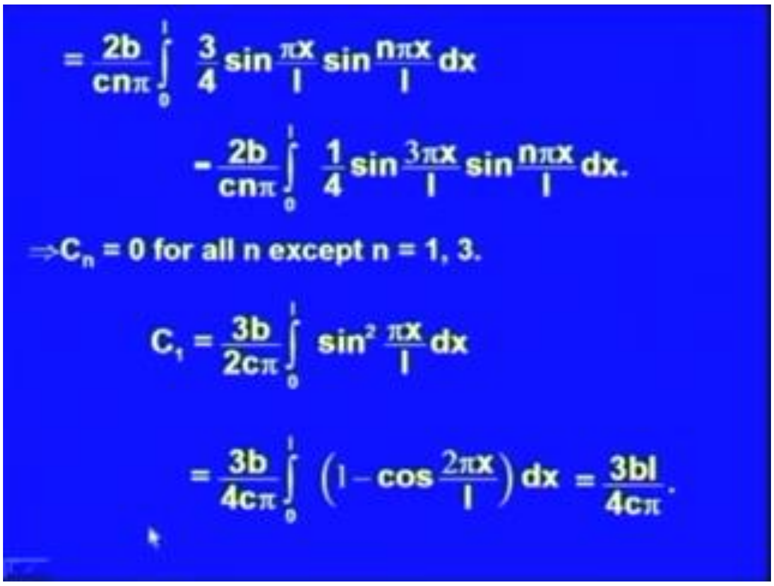


And, the displacement at any time t is given by u x t equal to sigma n equal to 1 to infinity B n cos lambda n t plus C n sin lambda n t into sin n pi x over l. Now, we are given that u x 0 is equal to f x equal to 0, so when you put t equal to 0 here, what you get sigma n u x 0 as sigma n equal to 1 to infinity B n sin n pi x over l, because when you put t equal to 0 this will vanish. So, and sigma n equal to 1 to infinity B n sin n pi x over l is equal to 0 implies that B n is equal to 0 for all values of n, now we have to determine the values of C n.

So, for that let us write sin cube pi x over l as 3 over 4 sin pi x over l minus 1 over 4 sin 3 pi x over l, because we know that sin 3 theta is 3 sin theta minus 4 sin cube theta. So, from that sin cube pi x over l may be written in this form, now this form will be very convenient to us in the evaluation of C n, so we know that C n is given by 2 over C n pi integral 0 to l g x sin n pi x over l d x let us put the value of g x we are given that g x is equal to b sin cube pi x over l.

So, C n is equal to 2 over C n pi integral 0 to l b sin cube pi x over l into sin n pi x over l d x. Now, here when you put the value of sin cube pi x over l as 3 by 4 sin pi x over l minus 1 by 4 sin 3 pi x over l then using the orthogonality of the sin functions sin n pi x over l over the interval 0 to l, will be able to determine the value of the constant constant C n is very easily.

(Refer Slide Time: 28:40)

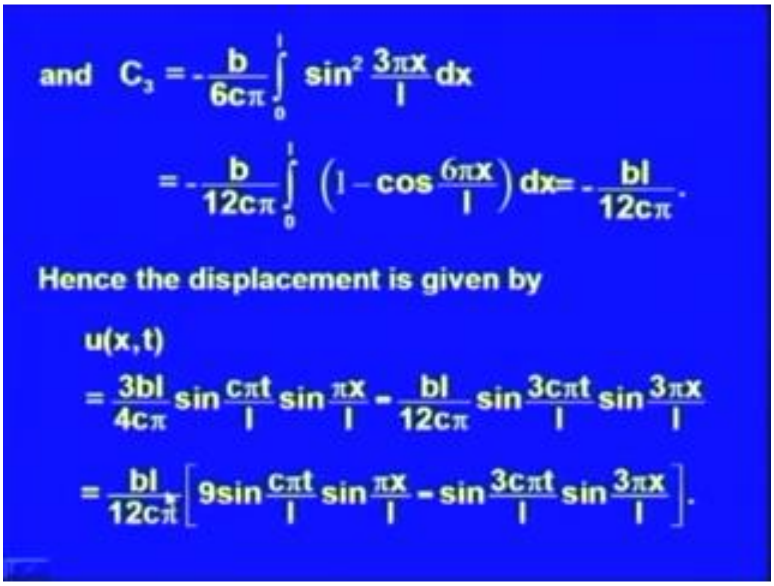


As we shall see now, so c n is equal to 2 b over c n pi integral 0 to l 3 over 4 sin pi x over l sin n pi x over l d x, we have broken the right hand side into two parts corresponding to the two terms in the expression for sin cube pi x over l. So, we have the right hand side as this and minus 2 b over c n pi integral 0 to l, 1 by 4 sin 3 pi x over l into sin n pi x over l.

Now, let us use the orthogonality of sin n pi x over l functions over the interval 0 to l, then 3 over 4 is a constant. So, integral 0 to l sin pi x over l into sin n pi x over l d x will always be 0 for all values of n except n equal to 1 and here, integral 0 to l sin 3 pi x over l into sin n pi x over l d x will be 0 for all values of n except n is equal to 3. So, if n is taking a value other than 1 or 3, then this integral will be 0 this integral will also be 0, so c n will be equal to 0.

And therefore, C n is 0 for all values of n except when n is taking values one or 3, so let us determine the value of C n for n equal to 1 and the value of C n for n equal to 3. So, first we take n equal to 1 to determine C 1 when we take n equal to 1 in the in here, then what will happen this will integral will vanish integral 0 to l sin cube 3 pi by pi x by l into sin pi x by l d x, so c 1 will be equal to then 3 2 b over C n pi into 3 by 4 will give us 3 b over 2 c pi integral 0 to l sin square pi x by l d x, which can be written as 3 b 3 b over 4 c pi integral 0 to l 1 minus cos 2 pi x over l d x and when you evaluate the value of this integral and simplify it you get 3 b l over 4 c pi as the value of.

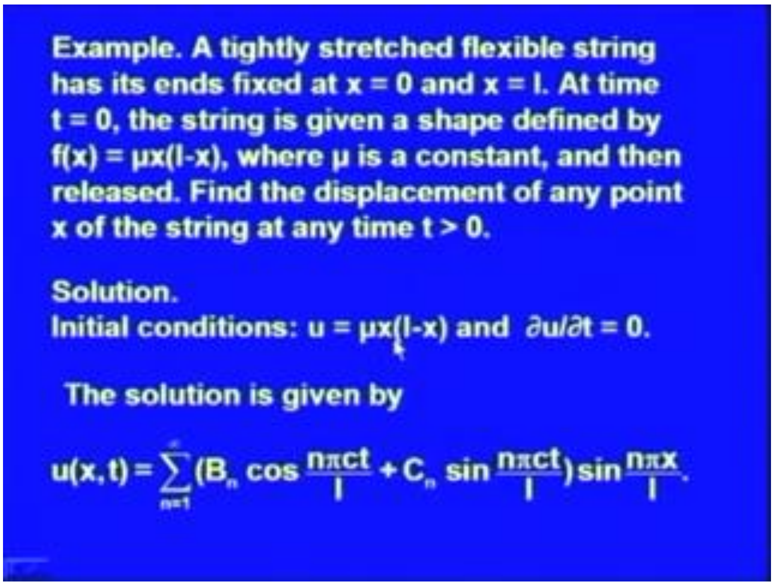
(Refer Slide Time: 30:54)



Similarly, we may compute the value of C 3, which is given by minus b over 6 c pi integral 0 to l sin square 3 pi x over l into d x, it is equal to minus b over 12 c pi integral 0 to l 1 minus cos 6 pi x over l d x, when you evaluate the value of this integral and simplify the value of C 3 comes out to be minus b l over 12 c pi.

And, hence the displacement function u x t is given by 3 b l over 4 c pi sin c pi over l into sin pi x over l this is the term corresponding to c corresponding to c 1 and this the term corresponding to C 3. So, we have minus b l over 12 c pi sin 3 pi c t over l into sin 3 pi x over l, now which can further be written as b l over 12 c pi into 9 sin c pi t over l into sin pi x over l minus sin 3 pi c t over l into sin 3 pi x over l.

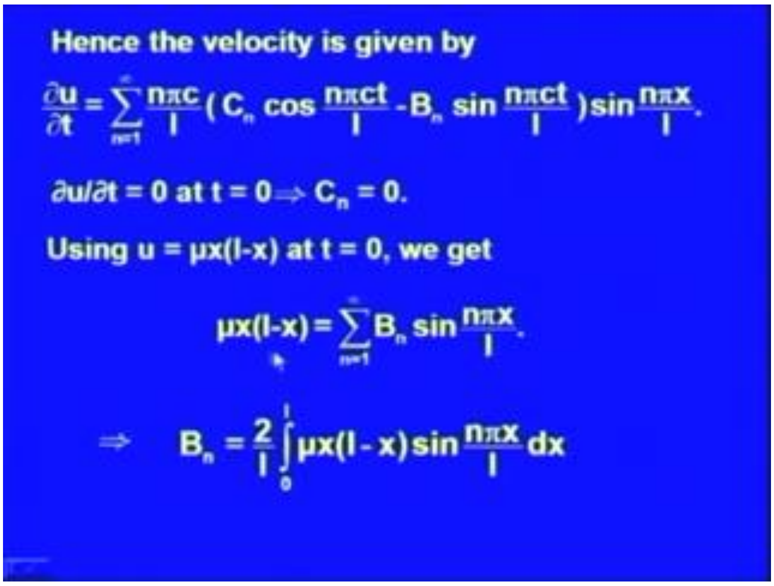
(Refer Slide Time: 32:06)



Let us look at another example of a tightly stretched flexible string, which has it is ends fixed at x equal to 0 and x equal to l. And, at time t equal to 0 the string is given a shape defined by the function f x equal to mu into x into l minus x, where mu is a constant and then released find the displacement of any point x of the string and at any time t greater than 0.

So, the initial conditions here are mu x 0 is equal to mu x into l minus x and the initial velocity that is delta u over delta t at t equal to 0 is given to be 0. The solution is then given by u x t equal to sigma n equal to 1 to infinity B n cos n pi c t over l plus C n sin n pi c t over l into sin n pi x over l. Now, here we are given the function g x as 0, so that will give as C n equal to 0 for all values of n and we are given the function f x as mu x into l minus x. So, we will determine the value of B n using the function f x as mu x into l minus x.

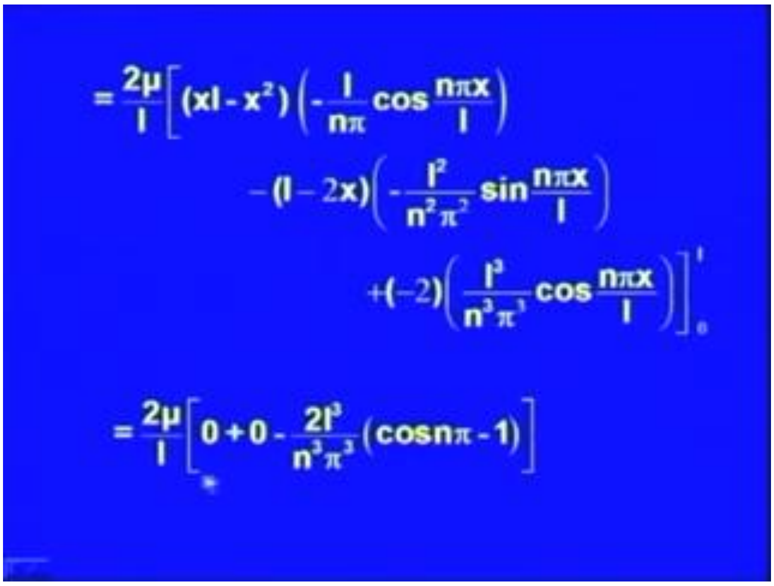
(Refer Slide Time: 33:33)



So, if you differentiate that expression for u x t delta u over delta t sigma n equal to 1 to infinity n pi c over l into C n cos n pi c t over l minus B n sin n pi c t over l sin n pi x over l. So, we are given that delta u over delta t is equal to 0 at t equal to 0, if you put t equal to 0 here, this terms vanish and what we have is delta u over delta t at t equal to 0 sigma n equal to 1 to infinity n pi c over l into C n sin n pi x over l.

So, since delta u over delta t is 0 at t equal to 0 it follows that C n is are 0, for all values of n 1 2 3 and so on. And, using u equal to mu x into l minus x at t equal to 0 we get from the equation for u x t it follows that mu into x into l minus x is equal to sigma n equal to 1 to infinity B n sin n pi x over l and which gives as B n as 2 over l integral 0 to l mu x into l minus x into sin n pi x over l d x using the half range expansion of the function mu x into l minus x.

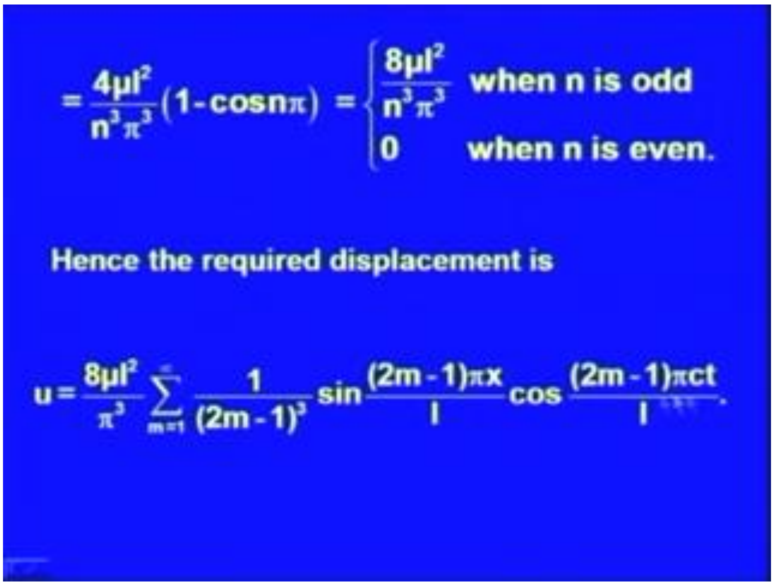
(Refer Slide Time: 34:57)



So, that last integral then gives us one integration by parts 2 mu by l into lx minus x square into minus l over n pi cos n pi x over l minus l minus 2 x into minus l square over n square pi square into sin n pi x over l plus minus 2 into l cube over n cube pi cube cos n pi x over l the limits of integration are 0 and l.

So, here when you put the limit is l and 0, but you get on simplification you get B n as 2 mu over l into this becomes 0, when you put x equal to l and you put this 0. So, for both the upper and lower limit is this expression this expression gives as 0, we have 0 plus 0. And, when you put x equal to l here, in the third term on the right side, but you get is minus 2 l cube over n cube pi cube into cos n pi and then you get when you put the lower limit you get plus 2 l cube over n cube pi cube into 1.

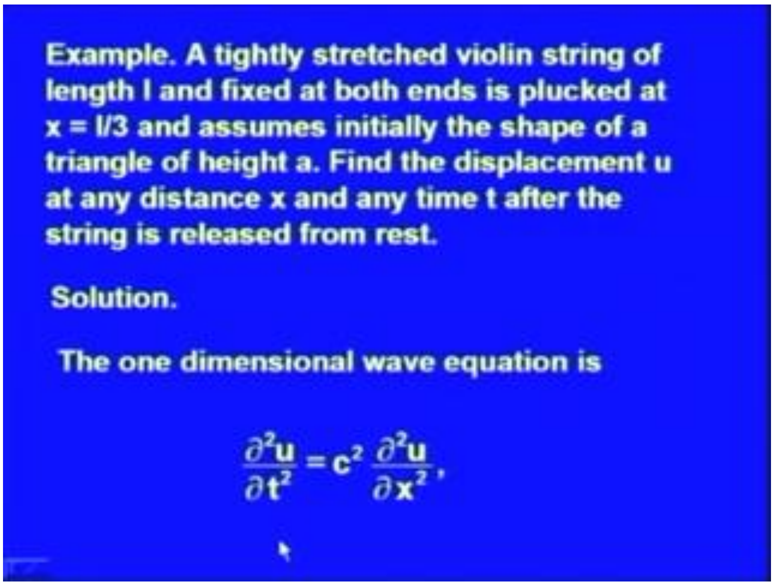
(Refer Slide Time: 36:20)



So, we have the right hand side like this which is equal to 4 mu l square over n cube pi cube into 1 minus cos n pi. Now, let us look at the case when n is an odd positive integer cos n pi will be equal to minus 1. So, 1 minus cos n pi will be equal to 2 and therefore, we get the value of B n as 8 mu l square over n cube pi cube if n is an odd integer, but if you take n to be an even integer, then cos n pi is equal to 1 so will get B n as 0, so the value of B n is 0 when is given when n is an even positive integer.

Hence, we get the required displacement function u x t as 8 mu l square over pi cube, now in the we have seen that B n is 0 when n is even and it is 8 mu l square over n cube pi cube, when n is odd. So, when n is an odd integer let us take n to be equal to 2 mu 2 m minus 1, so when where m takes values 1 2 3 and so on, so then u will be equal to 8 mu l square over pi cube sigma m takes values from 1 to infinity n is replaced by 2 m minus 1. So, we will get 1 over this n is 2 m minus 1, so 1 over 2 m minus 1 whole cube into sin n pi x over l becomes sin 2 m minus 1 into pi x over l and cos n pi ct over l becomes cos 2 m minus 1 pi c t over l.

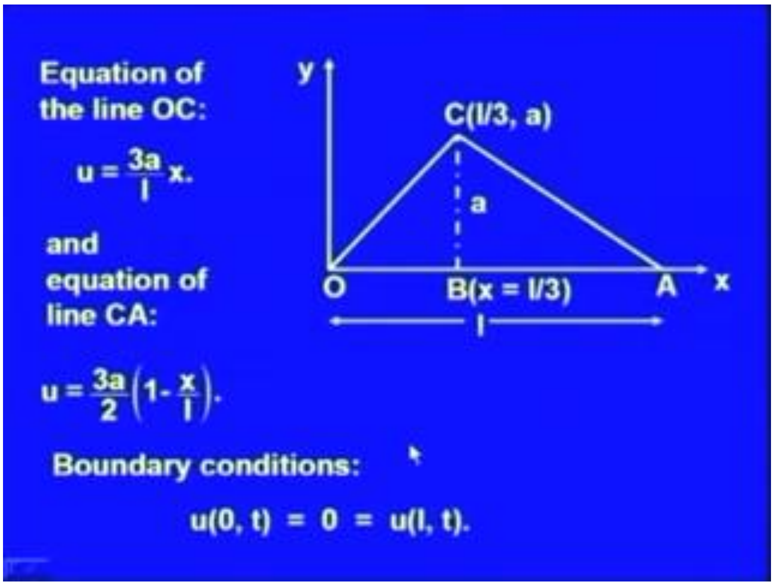
(Refer Slide Time: 37:53)



Now, let us discuss one more example before, we discuss the dl numbers a solution for the one dimensional wave equation. Here we are considering a tightly stretched violin string of length l, which is fixed at both its ends and is plucked at x equal to l by 3 it assumes initially the shape of a triangle of height a, at x equal to l by 3 we have to find the displacement function u at any distance x and at a time t after the string is released from rest.

So, we are given that the initial velocity of the string is 0 delta u over delta t at t equal to 0 and we are given that u x 0 is given by the triangle of height a at x equal to l by 3. So, let us let us again discuss the solution of the one dimensional wave equation, which is governing the emotion of the stretched violin string the equation is delta square u over delta t square equal to c square delta square u over delta x square.

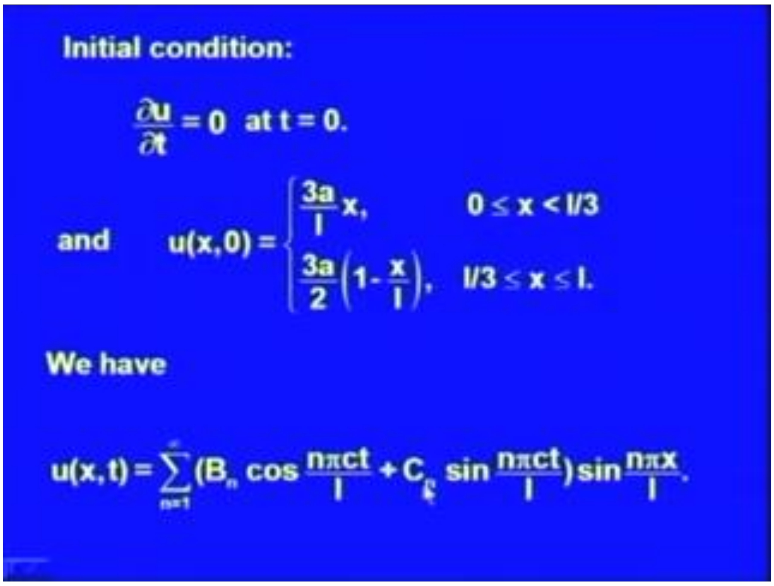
(Refer Slide Time: 39:00)



So, this is the shape of the triangle at x equal to l by 3, we have assumed that this thing is plugged and it is it assumes height A. So, now, let us find the equation of the line segment OC equation of the line segment OC will be u equal to 3 a by l into x at x equal to 0, we know that u is equal to 0 and at x equal to l by 3 they are given that the height of the triangle is A.

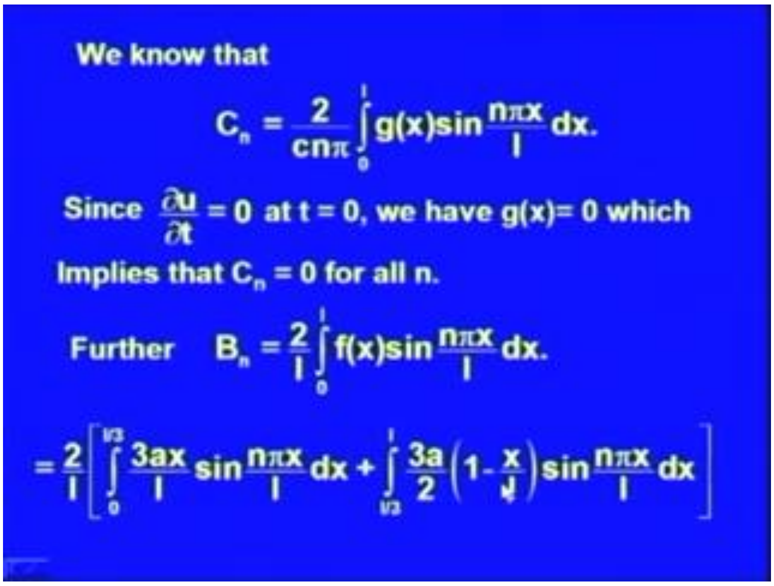
So, u is equal to a at x equal to l by 3 and therefore, the equation of OC is u equal to 3 a by l into x; similarly, the equation of the line segment ca will be u equal to 3 a by 2 into 1 minus x by l, when x is equal to l we know that u is equal to 0 there is no displacement at the end x equal to l that is A and at x equal to l by 3 u is equal to a. So, when you put x is equal to l by 3 here, we get 1 minus 1 by 3 that is 2 by 3 multiplied with 3 a by 2 gives you u as a. So, the u equal to 3 a by 2 into 1 minus x by l is the equation of the line segment CA the boundary conditions are as usual u 0 t equal to 0 and u l t equal to 0.

(Refer Slide Time: 40:35)



Now, the initial condition that is given to assist that this thing is released from rest, so delta u over delta t is 0 at t equal to 0 and it is initially in the shape of the triangle, we have found the equation of the equations of the triangle, the two sides of it that is OC and CA this is OC. So, the u x 0 equal to 3 a by l into x when 0 is less than or equal to x and x is less than l by 3 and it is 3 a by 2 into 1 minus x by l when, then l by 3 is less than or equal to x and x is less than or equal to l. Now, we have the solution of the one dimensional wave equation as u x t equal to sigma n equal to 1 to infinity B n cos n pi c t over l plus C n sin n pi c t over l into sin n pi x over l.

(Refer Slide Time: 41:29)



Since delta u over delta t is 0 at t equal to 0 C n and c n is given by two over c n pi integral 0 to l g x sin n pi x over l d x it follows that c n is equal to 0 for all values of n. Because, the very initial velocity at t equal to 0 we have represented by g x and we are given the initial velocity as 0, so g x is 0 and so C n is 0 for all values of n the value of B n is given by 2 over l integral 0 to l f x sin n pi x over l d x we have the expression for f x over the interval 0 to l f x is 3 a x by l over the interval l by 3 to l f x is 3 a by 2 into 1 minus x by l. So, we have breaken this integral into two parts 0 to l by 3 and l by 3 to l and substituted the values of f x over these intervals.

(Refer Slide Time: 42:28)

$$= \frac{3a}{l^2}\left[\int_0^{l/3} 2x \sin\frac{n\pi x}{l}\,dx + \int_{l/3}^{l}(l-x)\sin\frac{n\pi x}{l}\,dx\right]$$

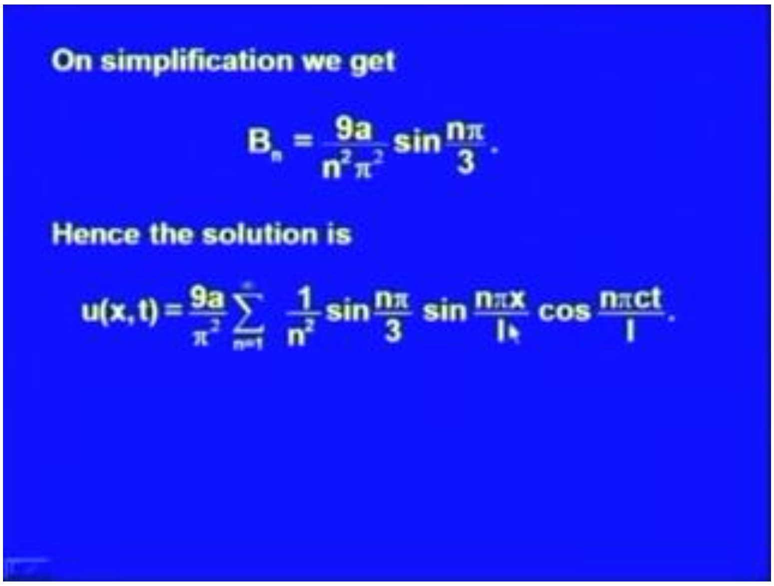
$$= \frac{3a}{l^2}\left\{\left(\frac{-2l}{n\pi}x\cos\frac{n\pi x}{l} + \frac{2l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)_0^{l/3}\right.$$

$$+\left(-\frac{l^2}{n\pi}\cos\frac{n\pi x}{l}\right.$$

$$\left.\left. -\frac{l}{n\pi}\left(-x\cos\frac{n\pi x}{l} + \frac{l}{n\pi}\sin\frac{n\pi x}{l}\right)\right)_{l/3}^{l}\right\}$$

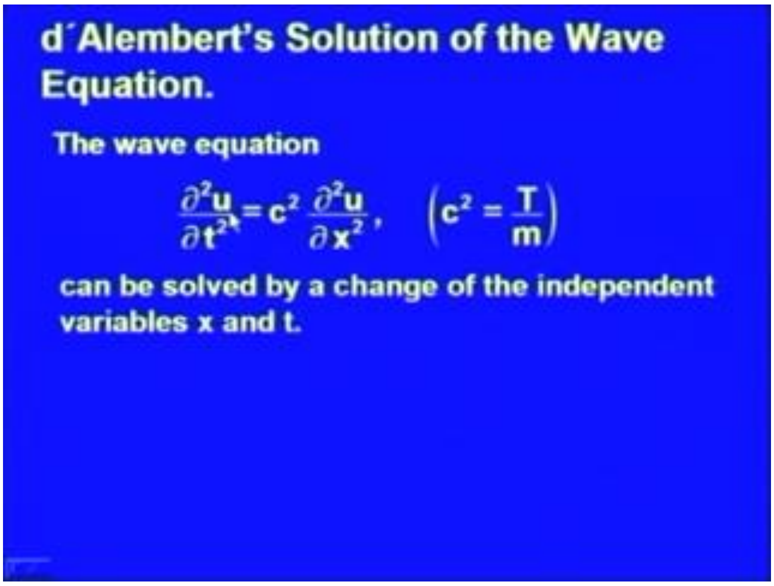
Now, the right hand side of B n then gives us 3 a by l square integral 0 to l by 3 2 x sin n pi x over l d x plus l by 3 to l, l minus x into sin n pi x over l d x. When we integrate it by parts the first integral what we get is minus 2 l over n pi x cos n pi x over l plus 2 l square over n square pi square sin n pi x over l these are the integrates of integration 0 and l by 3. And then we have after integrating the second integral by parts, we have minus n square over n pi cos n pi x over l minus l over n pi minus x cos n pi x over l plus l over n pi sin n pi x over l l by 3 and l are the limits of integration.

(Refer Slide Time: 43:19)



When you simplify this, you get the value of B n as 9 a over n square pi square sin n pi by 3. And hence, the solution of the given a problem is u x t equal to 9 a by pi square sigma n equal to 1 to infinity 1 by n square sin n pi by 3 into sin n pi x by l into cos n pi c t over .l

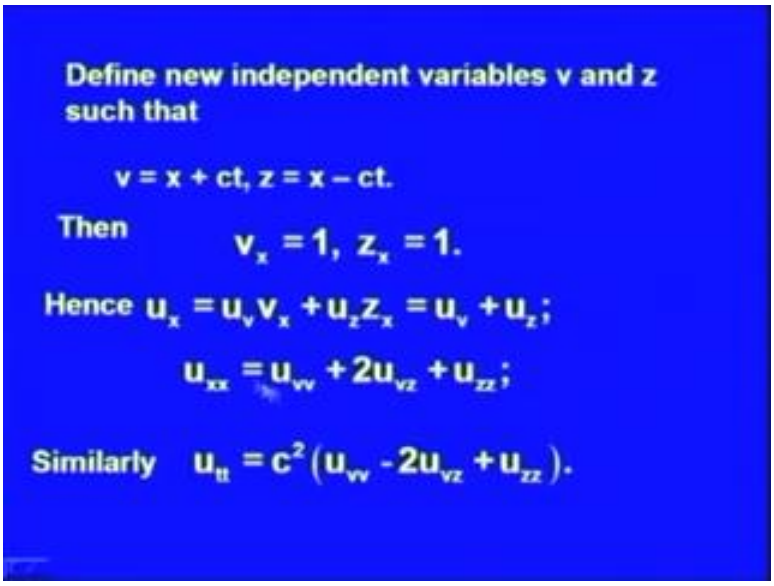
(Refer Slide Time: 43:44)



Now, we are going to discuss the d'Alembert's solution of the wave equation, so as let us we call the wave equation it is delta square u over delta t square equal to delta square u over delta x square, where we have by c square we have where c square is equal to t by

m. Now, we are going to show that it can be solved by a change of the independent variables and the independent variables as we know are x and t here.

(Refer Slide Time: 44:15)

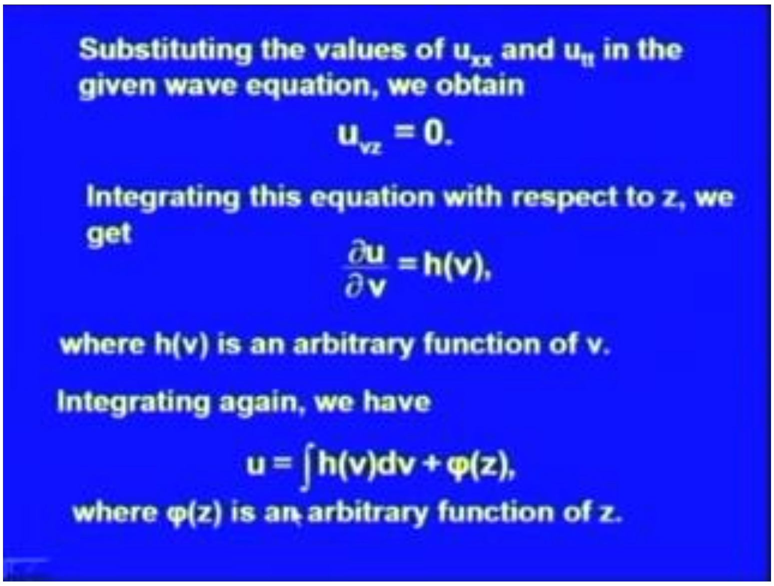


So, let us define a new set of independent variables v and z such that v is equal to x plus ct and z is equal to x minus ct. Now, from these relations we it follows that when you differentiate we partially with respect to x, you get delta v over delta x as 1 and delta v over delta x we have denoted by v x. So, v x is equal to 1 similarly delta z over delta z delta x that is z x is equal to 1.

Now, using the chain rule of partial differentiation we can, then write u x that is delta u over delta x, since u is a function of x and t and x and t are functions of v and z u is a function of v and z. So, delta u over delta x by chain rule will be delta u over delta v into delta v by delta x plus delta u by delta z into delta z by delta x, now making use of v x is equal to 1 and z x is equal to 1 it follows that the partial derivative of u with respect to x that is u x is equal to delta u over delta v that is u v plus delta u over delta z that is u z.

Now, when we apply this operator delta by delta x and u again and u x we get u xx is equal to u vv plus 2 u v z plus u z z. And similarly, we can show that u t t comes out to be equal to c square times u vv minus 2 u v z plus u z z, because for this u t t we will meet the values of v t and z t and v t is that is delta v over delta t comes out to be c delta z over delta c t comes out to be minus c. So, we can follow the same procedure as we have done for finding u xx if you follow the same procedure for u t t it comes out to be this.

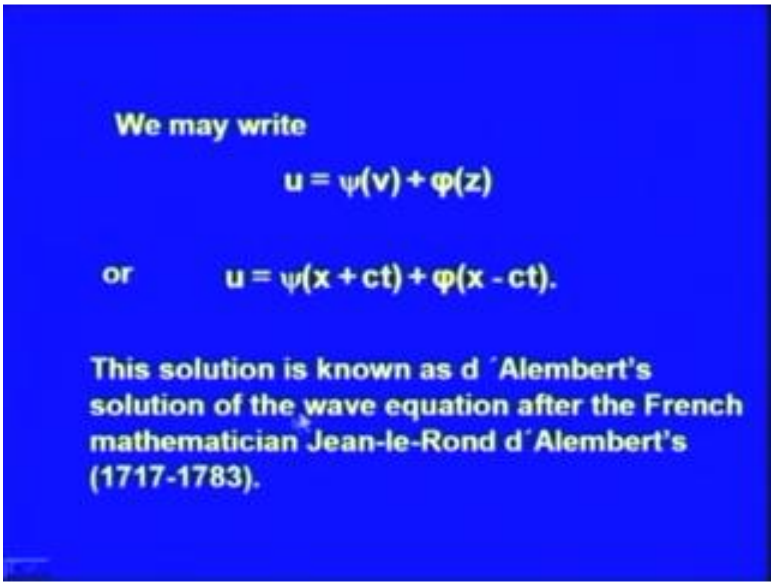
(Refer Slide Time: 46:16)



And, substituting these values of u xx and u t t in the given wave equation that is u t t equal to c square u xx it tells out that u v z is equal to 0 that is delta square u over delta v delta z is equal to 0, when we integrate this equation with respect to z what we will have delta u over delta v is equal to h v where h v is an arbitrary function of v alone.

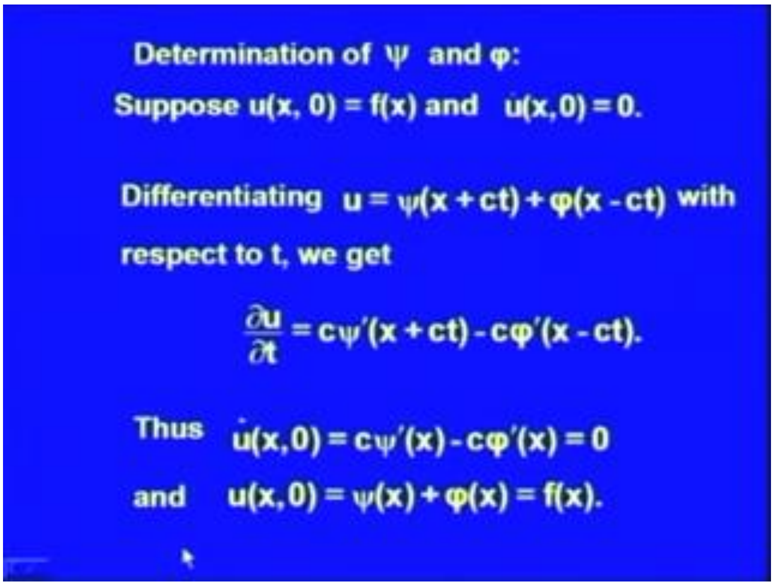
Now, when we integrate this equation again with respect to v, we will have u that is u equal to integral h v d v plus a function of z, which we have denoted by phi z phi z is an arbitrary function of z, so here constant of integration is the function of z.

(Refer Slide Time: 47:01)



And, thus u may be written as psi v plus phi z psi is integral of h v d v here, or we may write u as psi of x plus ct, because we have written v for x plus ct and we have written z for x minus ct, so u is equal to psi of x plus ct and phi of x minus ct. Now, this is this solution of the wave equation is known as the d'alembert's solution of the wave equation after the French mathematician Jean-le-Rond d'Alembert's.

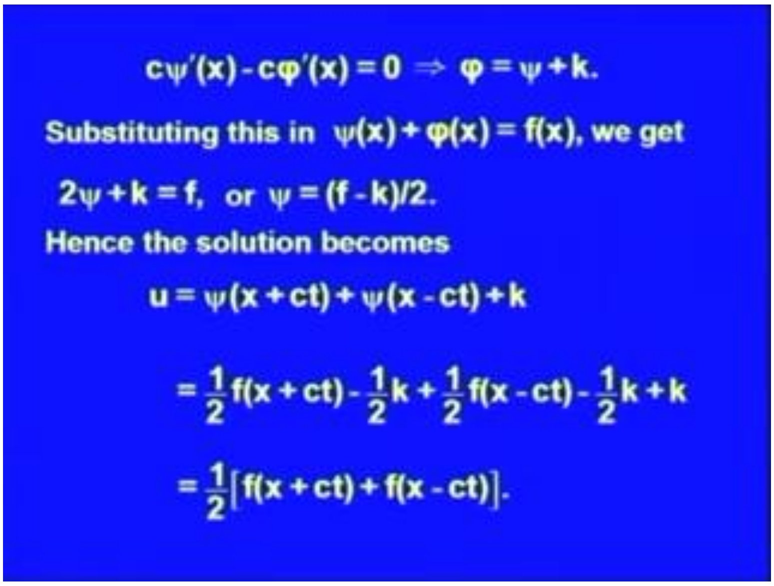
(Refer Slide Time: 47:47)



Now, let us find the values of the arbitrary function psi and phi, let us assume that u x 0 is equal to f x and delta u over delta t at t equal to 0 is 0. So, initially the string is released from rest and it is a it is shaped is initially given by the function f x, let us differentiate u equal to psi x plus ct plus phi x minus ct with respect to t we will get delta u over delta t equal to c into psi dash x plus ct minus c into phi dash x minus ct, where the prime c are denote the derivative with respect to x plus ct and the derivative with respect to x minus ct.

Thus u dot x 0 equal to, so when you put here, t equal to 0 what you get delta u over delta t at t equal to 0 gives u dot x 0 u dot x 0, will be equal to c times psi dash x minus c times phi dash x and we are given that u dot x 0 is equal to 0 this condition. So, u dot x 0 equal to 0 gives you c times psi dash x minus c times phi dash x equal to 0, and when you put here t equal to 0 in this what you get u x 0 equal to psi x plus phi x and we are given u x 0 equal to f x, so psi x plus phi x is equal to f x. Now, from these two equations we are now going to determine the known functions arbitrary functions phi x and psi x.

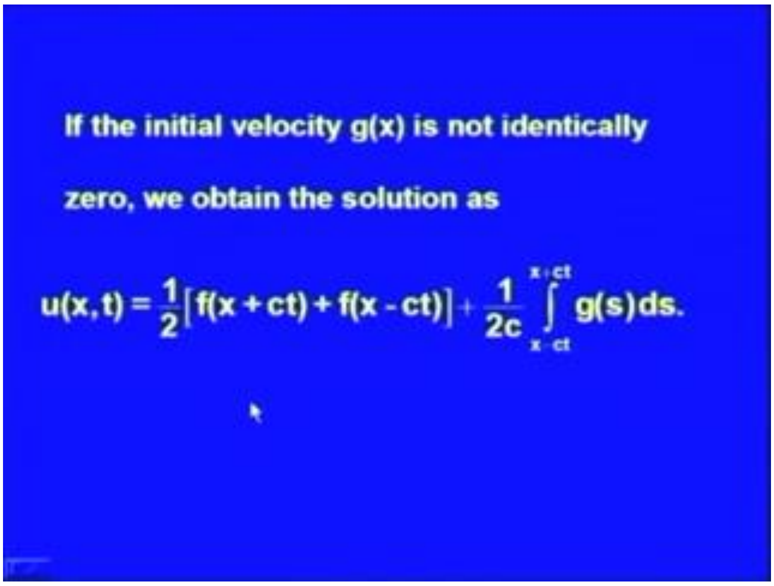
(Refer Slide Time: 49:24)



So, c times psi dash x minus c times phi dash x equal to 0 gives you phi equal to psi plus k. This give we can divide by c and then you have phi x phi dash x equal to psi dash x integrate with respect to x we will have phi x equal to psi x plus k this k is an arbitrary constant. Now, let us substitute this value of phi in the other equation psi x plus phi x equal to f x what we will 2 psi plus k equal to f or we may say that psi is equal to f minus k by 2.

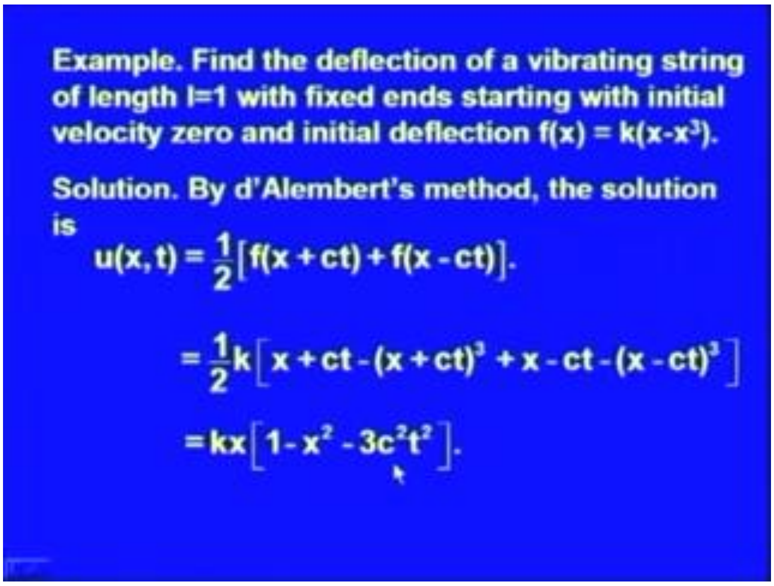
And hence, the solution becomes u equal to psi x plus ct plus phi x minus ct becomes psi x minus ct plus k from this equation and let us put the value of psi here now. So, psi x plus ct gives you f x plus ct minus k by two that is half of f x plus ct minus half k and psi x minus ct when you find from here, what you get is half of f x minus ct minus half of k and then plus k here, so this gives you half of f x plus ct plus f x minus ct.

(Refer Slide Time: 50:45)



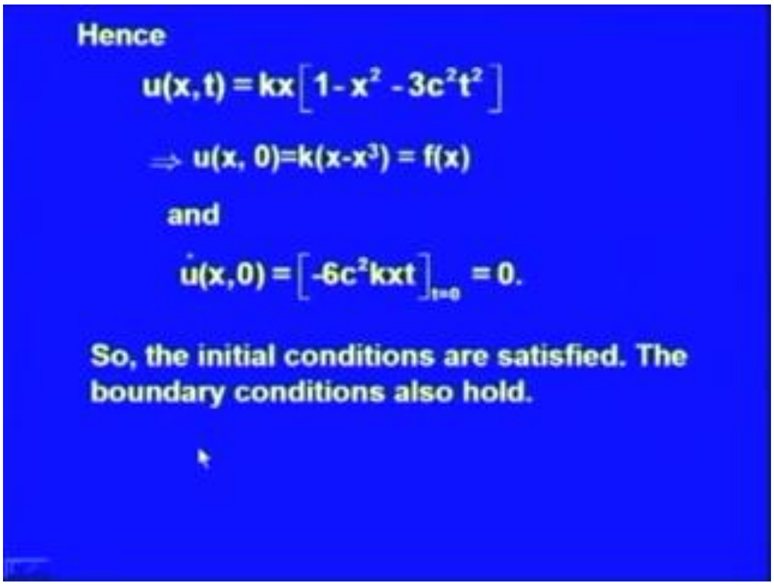
Now, if we assume that the initial velocity g x is not identically 0, then we shall obtain the solution of the wave equation as u x t equal to half of f x plus ct plus f x minus ct plus 1 by 2 c integral over x minus ct to x plus ct g s d s this can be easily shown.

(Refer Slide Time: 51:10)



Now, let us take an example on the d'Alembert's method, let us find the deflection of a vibrating string of length l equal to 1 with fixed ends starting with initial velocity 0 and initial deflection f x equal to k times x minus x cube. Now, by at d'alembert's method, we know that the solution is given by u x t equal to half of f x plus ct plus f x minus ct, so let us put the value of f here f is equal to k times x minus x cube. So, we will get u x t as half of k into x plus ct minus x plus ct whole cube plus x minus ct minus x minus ct whole cube, and if you simplify this expression you get the value of u x t as k into x into 1 minus x square minus 3 c square t square.

(Refer Slide Time: 52:09)



So, u x t comes out to be k x into 1 minus x square minus 3 c square t square, which implies that when you put t equal to 0 here you get u x 0 as k times x minus x cube and which is nothing but the function f x. So, this solution of the wave equation satisfies the a initial condition u x 0 equal to f x and if you differentiated with respect to t and then put t equal to 0 what you get u dot x 0 as minus 6 c square k x t at t equal to 0, which gives as the value 0, so it satisfies the other initial condition also.

And, so the boundary conditions also hold for the solution u x t equal to k x into 1 minus x square minus 3 c square t square and so it gives us the a solution of the given problem. Now, in our lecture in our next lecture, we shall discuss one dimensional heat equation, which is delta u over delta t equal to c square delta square u over delta x square that equation we shall again solve by using the separation of variables method that is the product method, which we have discussed here, will there also we shall see that we I will have to use the half range expansion of the Fourier series for finding the constants b n is and c n is that be this all will done in our next lecture on one dimensional heat equation.

Thank you.