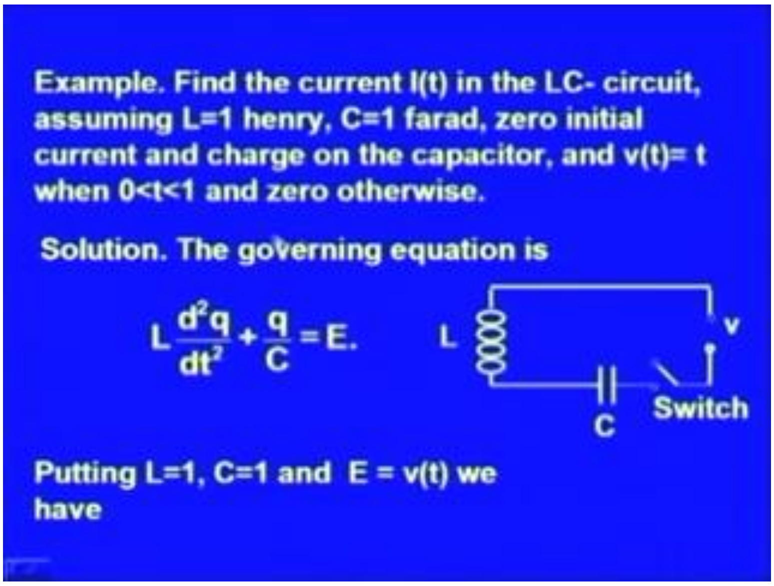
Mathematics III
Prof. P. N. Agrawal
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture - 11
Applications of Laplace Transformation (Contd.)

Dear viewers, today we shall talk about some more Applications of the Laplace Transformation. In my last lecture, we had discussed some applications of Laplace transformation like application of Laplace transformation to the problems in dynamics, then the application of Laplace transformation to a simple electrical circuit. We shall today discuss the application of Laplace transformation to bending of beams and then the application of Laplace transformation to problems in mechanics. We shall also discuss the application of Laplace transformation to boundary values problems like, how to find the solution of a heat conduction equation and how to find the application of a wave equation by using Laplace transformation.

(Refer Slide Time: 01:27)

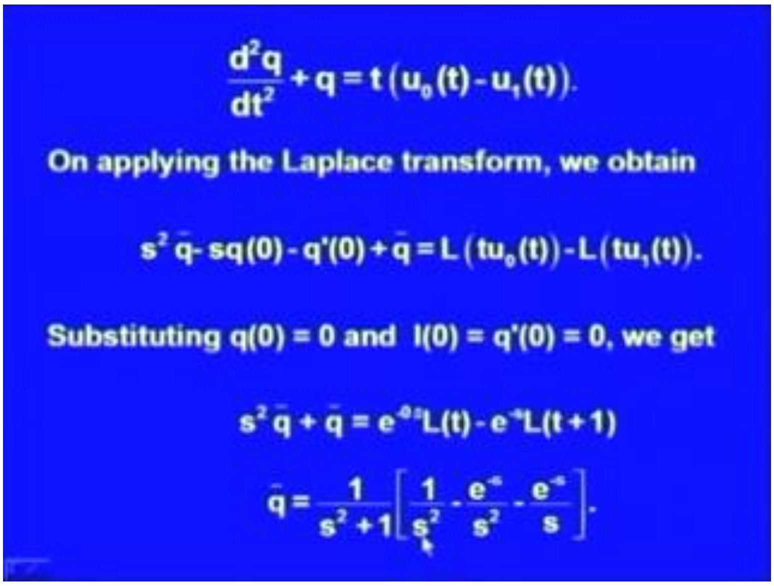


So, let us begin with a problem on electrical circuit, let us find the current I t in the L C circuit, here we are considering only I t, I c, L C circuit, r is not there that is resistance is not there. We are assuming L to be of 1 Henry, C to be of 1 farad and with initial current 0 and initial charge on the capacitor also 0. And we are given that v t is equal to t, when

0 is less than t less than 1 and 0, otherwise v t means the electromotive source of voltage v t.

Now, in this case of the given problem the governing equation will be L d i by d t plus q by C equal to E, but I we know is d q by d t. So, we get L d square q by d t square plus q by C equal to E, E is the E here, will be replaced by v that is v t, so and we are given that L is equal to 1 and C also equal to 1. So, L is equal to 1 and C equal to 1, let us put and we put v E equal to v t, but v t is equal to t when 0 less than t less than 1 and 0, otherwise so we can replace v t, we can write v t in terms of unit step functions as t times u naught t minus u 1 t.

(Refer Slide Time: 02:46)



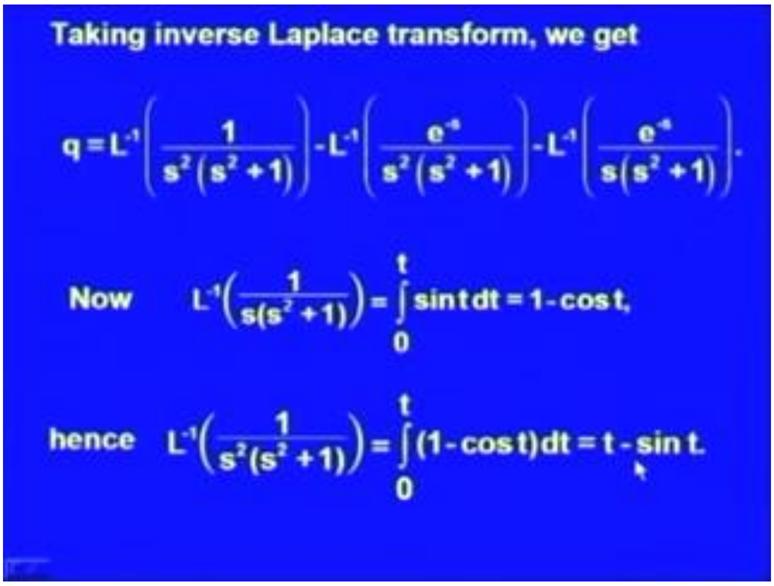
So if we do that we will have the differential equation like this d square q by d t square plus q equal to t times u naught t minus u 1 t. Now, let us apply Laplace transform to this equation L of d square q by d t square, we know will be given by s square q bar minus sq 0 minus q dash 0, then L of q is q bar and then we have Laplace transform of t into u naught minus Laplace transform of t into u 1 t.

Now, let us substitute q 0 equal to 0 and then I 0 that is q dash 0 equal to 0, we will have s square q bar plus q bar equal to. Now, Laplace transform of t into u naught t is equal to e to the power minus 0 s into Laplace transform of t, because we know that Laplace transform of ft into u a t is e to the power minus s into Laplace transform of f of t plus a where a is equal to 0, so we get e to the power minus 0 s into Laplace transform of t plus

0 that is t. And then, Laplace transform of t into u 1 t similarly will be e to the power minus s into Laplace transform of t plus 1 because here a is equal to 1.

And, now simplifying this we have, then q bar equal to 1 over s square plus 1 Laplace transform of t is, we know is 1 by s square minus e to the power minus s multiplied to Laplace transform of t plus Laplace transform of 1 Laplace transform of t is 1 by s square and Laplace transform of 1 is 1 by s. So, we get q bar s 1 by s square plus 1, multiplied to 1 by s square, minus e to the power minus s by s square minus e to the power minus s by s.

(Refer Slide Time: 04:48)

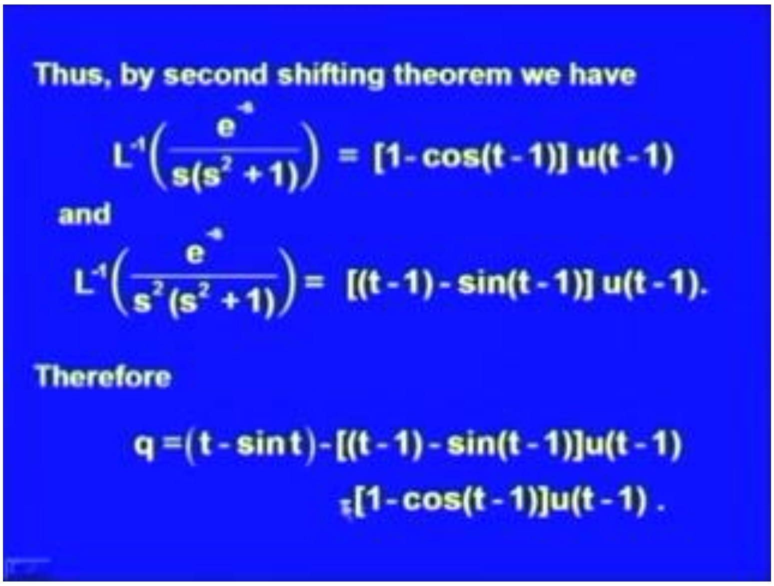


Now, let us take the inverse Laplace transform of this equation, left hand side will give us q and the right hand side using linearity, we will have q equal to L inverse of 1 over s square into s square plus 1 minus inverse transform of e to the power minus s over s square into s square plus 1 minus inverse transform of e to the power minus s over s into s square plus 1.

Now, we know that the inverse transform of 1 over s square plus 1 is sin t, so inverse transform of 1 over s into s square plus 1, we can write as integral over 0 to t sin t d t, because we know that Laplace transform of integral over 0 to t ft d t is f s by s, where f s is the Laplace transform of f t. So, making use of that theorem we have inverse transform of 1 over s into s square plus 1 equal to integral 0 to t sin t d t and which is equal to 1 minus cos t, now let us find the inverse transform of 1 over s square into s square plus 1.

So, again making use of that theorem, where we have set that Laplace trans transform of integral 0 to t f tau d tau is equal to f s over s making use of that theorem again. We now have integral of how many in inverse Laplace transform of 1 over s square into s square plus 1 as integral 0 to t and then inverse Laplace transform of 1 over s into s square plus 1, which we have found as 1 minus cos t. So, this will when we integrate this and put the limits we get the inverse Laplace transform as t minus sin t.

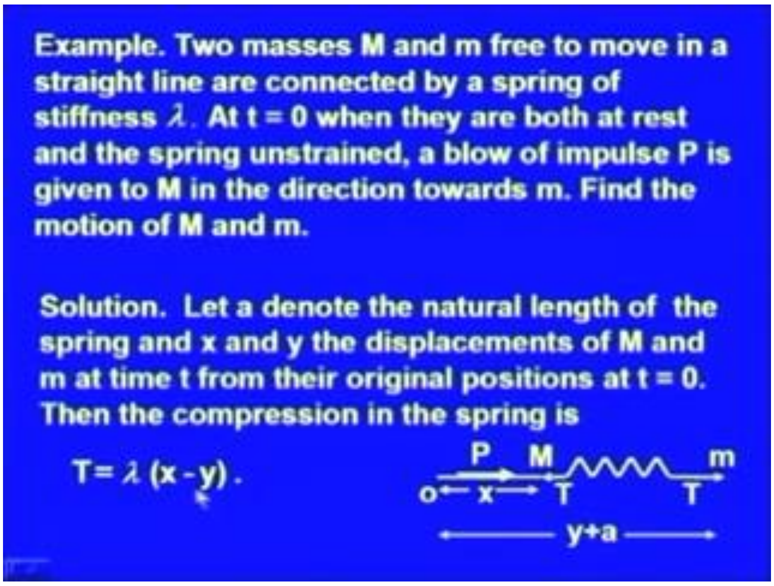
(Refer Slide Time: 06:37)



And then, let us apply second shifting theorem inverse Laplace transform of e to the power minus s over s into s square plus 1 will be equal to, then 1 minus cos t minus 1 into u t minus 1, because we know that inverse Laplace transform of e to the power minus s into f s is equal to f t minus a into u t minus a. So, here a is equal to 1 and f s is 1 over s into s square plus 1 whose inverse Laplace transform we have seen comes out to be 1 minus cos t, so by second shifting theorem we have inverse Laplace transform like this.

And then, similarly in inverse Laplace transform of e to the power minus s over s square into s square plus 1 may be written it will be equal to t minus 1 minus sin t minus 1 into u t minus 1. And therefore, we will have q equal to t minus sin t minus t minus 1 minus sin t minus 1 into u t minus 1 minus 1 minus cos t minus 1 into u t minus 1.

(Refer Slide Time: 07:46)

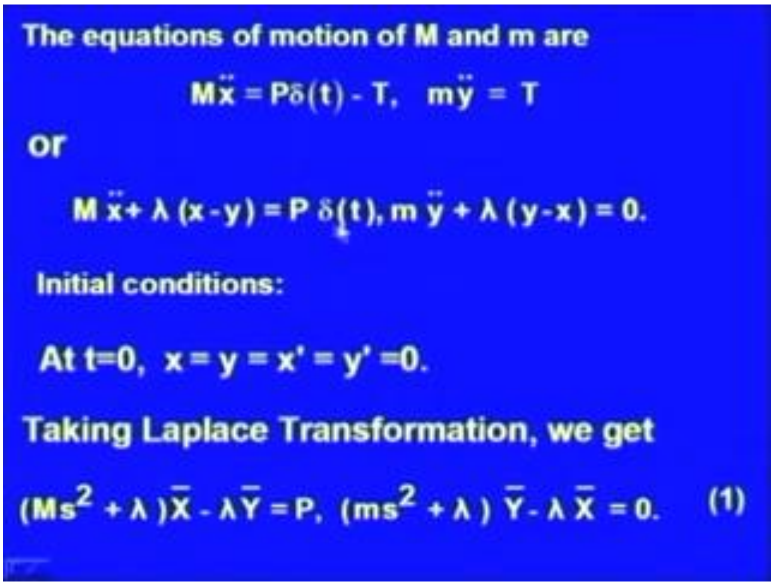


Now, let us discuss the application of Laplace transform to a problem in mechanics, two masses M and small m free to move in a straight line are connected by a spring of stiffness lambda. At t equal to 0, when they are both at rest and the spring unstrained, a blow of impulse P is given to M in the direction towards small m. We have to find the motion of the mass capital M and the small m.

So, let us let us say that let us say this is our spring a its natural length and x and y are the displacements of the mass capital M and these small m from their original positions at time t equal to 0 that is from their from the equilibrium position. So, at time t equal to 0 the mass M is at distance x from the origin and this mass m is at the distance y plus a, where a is the natural length of the spring and y is the displacement of the mass m from its original position.

Now, then by the hoops law it follows that the compression in the spring will be given by T equal to lambda times x minus y, because lambda is the spring when stiffness of the spring and x minus y gives us the compression. So, this will be the compression in the spring.

(Refer Slide Time: 09:24)



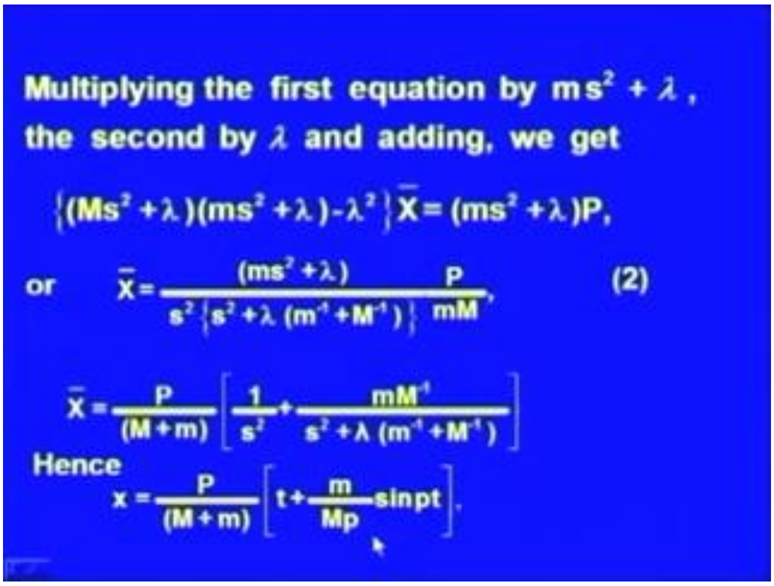
The equations of motion of M and a small m, will then the given by M x double dot will be equal to P delta t minus T. Because P the blow of impulse P is given to the cap bigger mass m to the capital to the mass capital M and so m; as the equation of motion for the cap mass m capital M will be M x double dot equal to p delta t minus T, while for the smaller mass m it will be m y double dot equal to T.

And, now let us put the value of T equal to lambda times x minus y in these two equations, we will have M x double dot plus lambda times x minus y equal to p delta t, and m y double dot equal to lambda times x minus y, which we can write as m y double dot plus lambda times y minus x equal to 0. The initial conditions are at t equal to 0, x is 0 y is 0 and both the masses were at rest, so x dash and y dash are also 0 x dash is the derivative of x that is d x by d t and y dash is d y by d t.

Now, let us take the Laplace transform of this equation, so Laplace transform of this will be m times s square x bar minus s x 0 minus x dash 0 making use of the initial conditions x 0 equal to 0 x dash 0 equal to 0. We will have the Laplace after lap taking Laplace transform of this equation we shall have Ms square plus lambda times X bar minus lambda times Y bar equal to P times lambda P times l of delta t Laplace transform of delta t.

Now, we know that Laplace transform of delta t minus a is equal to e to the power minus s, so taking a equal to 0, we get the Laplace transform of delta t as e to the power 0 as that is 1. So, we have the right hand side Laplace transform of the right hand side as P and the when we take the Laplace transform here, we get m times s square y bar minus s by 0 minus y dash 0, again y 0 is equal to 0 and y dash equal to 0 implies as implies that the lapla after taking Laplace transform of this equation 1 will have m s square plus lambda into Y bar minus lambda X bar equal to 0, where X bar denotes the Laplace transform of X and Y bar denotes the Laplace transform of Y.

(Refer Slide Time: 12:01)

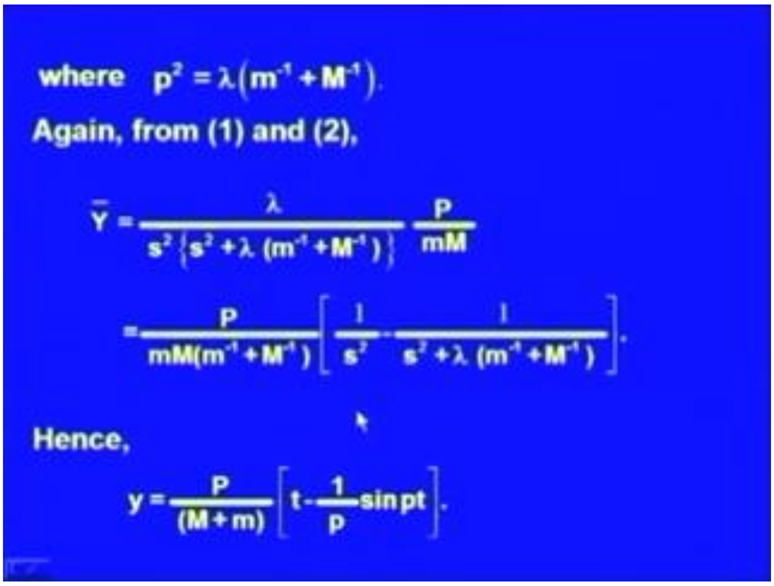


Now, we multiplication by m s square plus lambda the second by lambda and then add this by doing this, we will be eliminating 1 variable that is Y bar and we shall have this equation in X bar. So, M s square plus lambda into m s square plus lambda minus lambda square into x bar equal to m s square plus lambda into P.

We had a system of simultaneous equations in X bar and Y bar by eliminating 1 variable Y bar we got this equation in the variable X bar, which after simplification gives as X bar equal to m s square plus lambda over s square times s square plus lambda into m to the power minus 1 plus capital M to the power minus 1 into P by m M, which we call as equation number 2.

And, we can write this further as X bar equal to P over M plus m into 1 over s square plus m into M to the power minus 1 over s square plus lambda times m to the power minus 1 plus capital M to the power minus 1, that is we break it into its partial fractions. And then, we take the inverse Laplace transform of this equation. So, inverse Laplace transform X bar gives x P over M plus m is a constant, inverse Laplace transform 1 over s square is t and then m into M inverse is a constant let us call lambda into m to the power minus 1 plus capital M to the power minus 1 as p square. So, that we have inverse Laplace transform of 1 over s square plus p square, which we know is sin pt over p. So, when we take the inverse Laplace transform of this equation, we get this where we have assumed that p square is equal to m to the power minus 1 plus capital M to the power minus 1.

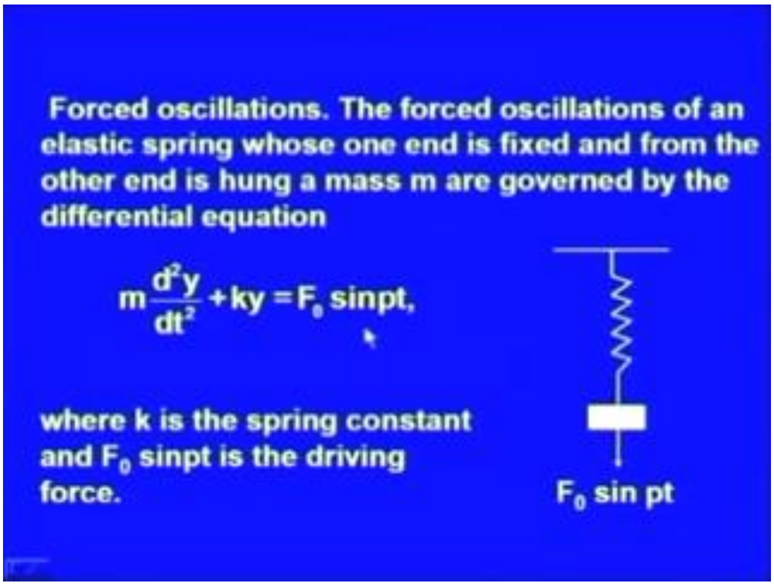
(Refer Slide Time: 14:07)



Now, from equations 1 and 2 we then can also find Y bar, Y bar equal to lambda over s square in to s square plus lambda times m to the power minus 1 plus capital M to the power minus 1 into P over m into capital M, after breaking it into partial fractions. We will have P over m into capital M into m to the power minus 1 plus capital m to the power minus 1 into 1 over s square minus 1 over s square plus lambda times m to the power minus 1 plus capital M to the power minus 1.

And, when we take the inverse Laplace transform of this equation, we get y equal to P over M plus m inverse Laplace transform of 1 over s square is t minus again here, we take p square equal to lambda times m to the power minus 1 plus capital M to the power lam minus 1. So, then inverse Laplace transform of this is sin pt over p.

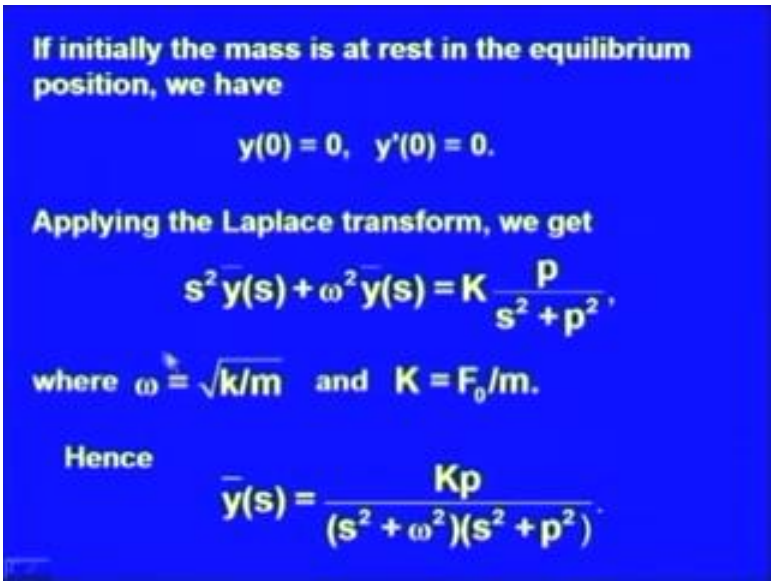
(Refer Slide Time: 15:09)



Now, let us discuss the case of an elastic spring whose one end is fixed and from the other end is hung a mass m. So, such a problem is governed by this differential equation m d square y over d t square plus k y equal to F naught sin pt, where k is the spring constant and F naught sin pt is the driving force, because of this force at y is the displacement in the mass m at time t.

And by hoops law, we know that the spring force will be k y, because k is the spring constant and y is the different displacement in the spring from the equilibrium position, so k y will be the spring force. So, resultant force will be F naught sin pt will be acting downwards while k y force will be acting upwards, so resultant force will be F naught sin pt minus k y and that will be equal to m into d square y by d t square.

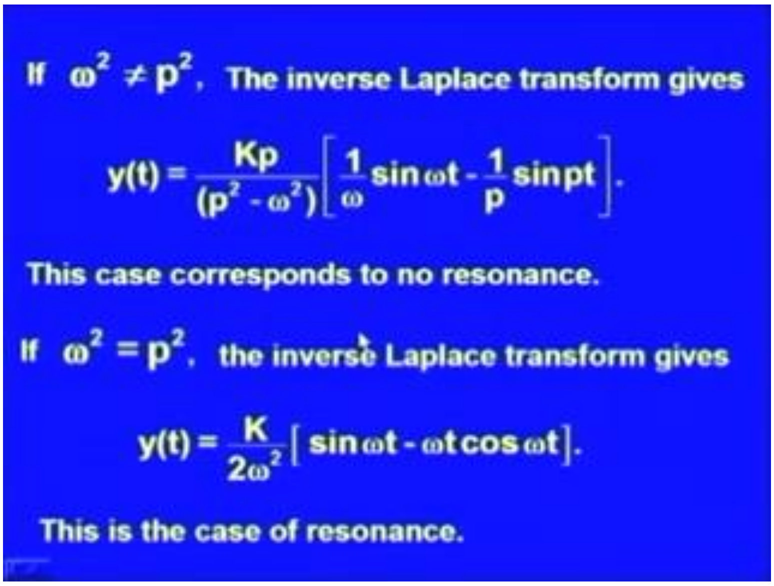
(Refer Slide Time: 16:27)



Now, we are assuming that initially the mass is at rest in the equilibrium position, so we will have y 0 equal to 0 and y dash 0 equal to 0. And, when we take the inverse lap when we take the Laplace transform of the governing equation of motion of the mass m we have s square y bar s plus omega square into y bar s equal to K into p over s square plus p square, where we have assume that omega is equal to square root K by m and capital K denotes F naught over small m.

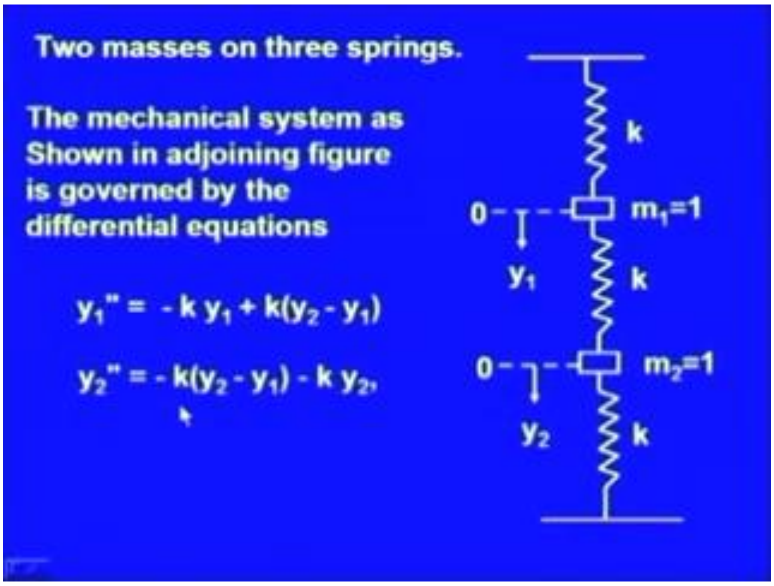
So, when we solve this equation for y bar we will have y bar s equal to K into p over s square plus omega square into s square plus p square. Now, will bracket into partial fractions and then take the inverse Laplace transform in order to find the displacement of the mass m at time t.

(Refer Slide Time: 17:24)



So, let there are two cases one is if omega square is not equal to p square, then when we take the inverse Laplace transform after breaking into partial fractions, we get y t equal to K p over p square minus omega square into sin omega t over omega minus sin pt over p this case corresponds to no resonance. If, omega square is equal to p square, then the inverse Laplace transform will give us y t equal to K over 2 omega square into sin omega t minus omega t into cos omega t this is the case of resonance.

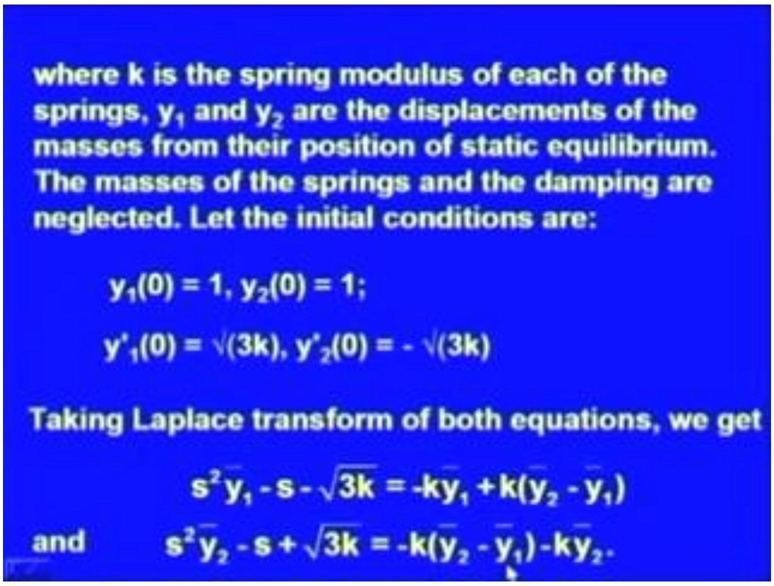
(Refer Slide Time: 18:16)



Now, let us study another problem of mechanics where we have two masses connected to three springs, we have and all the three springs have the same spring constant k the stiffness. So, here the governing equations of motion for the mass m, m 1 and the for mass m 2 are these two equations for the mass m 1 we have y 1 double dash minus k y 1 plus k times y 2 minus y 1, where we have made use of the given value of m 1 that is m 1 equal to 1 and for the mass m 2, which is again given to be equal to 1 we have the equation of motion as y 2 double dash minus k times y 2 minus y 1 minus k y 2.

Now, when the because of the mass m 1, let us say at time t the displacement in the spring this first spring is y 1. So, I mean, so then by the k y 1 force that is the spring force will act upwards and will this mass m 1 will compress this lower spring, so then the and then this there is a displacement y 2 in this mass at time t, so the net resultant displacement will be in the mass m 2 will be y 2 minus y 1. So, for this lower spring k times y 2 minus y 1 will act upwards for the mass m 1, while for the mass m 2 k times y 2 minus y 1 will act upwards and also k y 2 will act upwards. So, we have for the mass m 1 the resultant force will be k times y 2 minus y 1 minus k y 1, while for the mass m 2 it will be k times y 2 minus y 1 minus k y 2 plus k y 2, which is acting upwards.

(Refer Slide Time: 20:17)

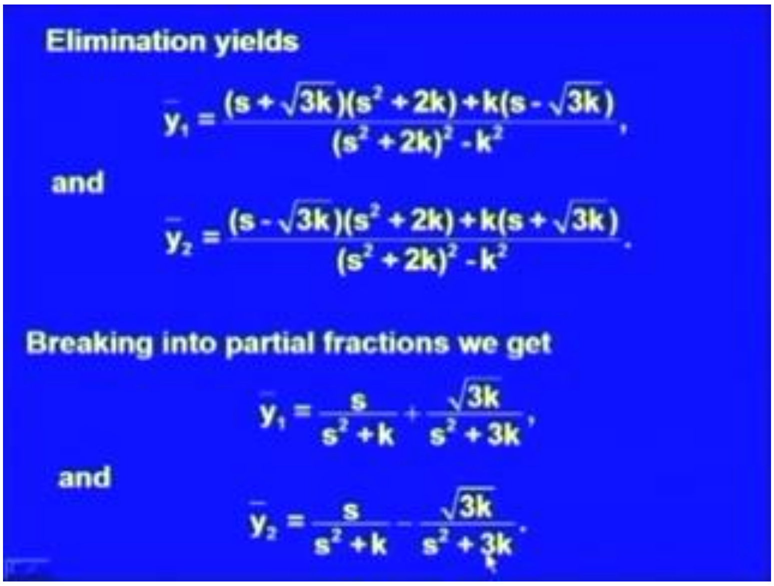


so we have put negative sign here, where k is the spring modulus of each of these springs y 1, y 2 are the displacements of the masses from their positions of static equilibrium, we are assuming that the masses of the springs and the damping is negligible. So, the initial conditions here are at t equal to 0 y 1 equal to 1 and at t equal to 0 y 2 is equal to 1, at t equal to 0 d y 1 by d y 1 y d t is equal to root 3 k and d y 2 y d t is equal to minus root 3 k.

Let us take now, the Laplace transform of both the equations of motion for the mass m 1 and m 2. Then, we shall have for the mass m 1 we shall have s square y 1 bar minus s minus root 3 k minus k y 1 bar plus k times y 2 bar minus y 1 bar in the left hand side there we had y 1 double dot by d square y 1 over d t square. So, when we take the Laplace transform of that we have s square y 1 bar minus s y 1 0, y 1 0 is equal to 1 then minus y 1 dash 0 and y 1 dash 0 is equal to root 3 k.

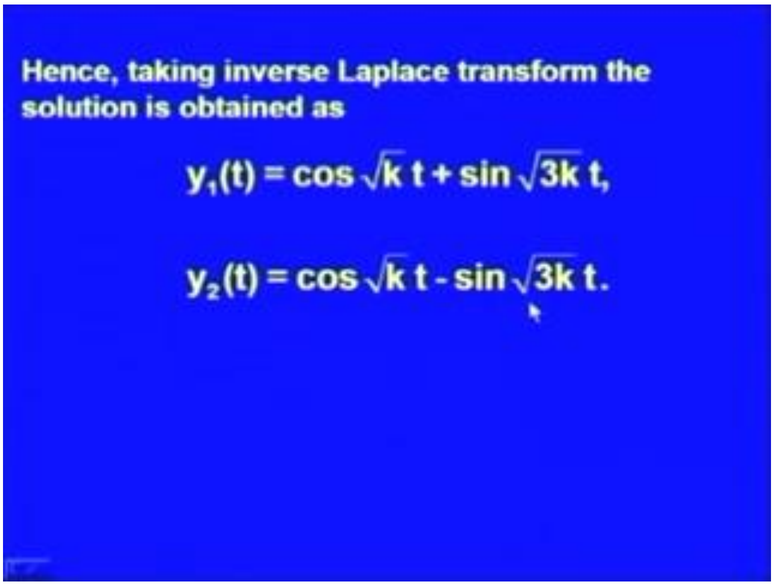
So, we have the left hand side after taking the Laplace transform like this right hand side we had k y minus k y 1 plus k times y 2 minus y 1, so when we take Laplace transform of the right hand side, we get minus k y 1 bar plus k times y 2 bar minus y 1 bar. And, when we take the Laplace transform of the second equation of motion that is the motion of the mass m 2, we have s square y 2 bar minus s y 2 0, y 2 0 is 1 plus minus y 2 dash 0, but the y 2 dash 0 is negative root 3 k. So, we have plus root 3 k here, equal to minus k times Laplace transform of y 2 minus y 1, which gives us y 2 bar minus y 1 bar minus Laplace transform of k times y 2 that is minus k y 2 bar.

(Refer Slide Time: 22:22)



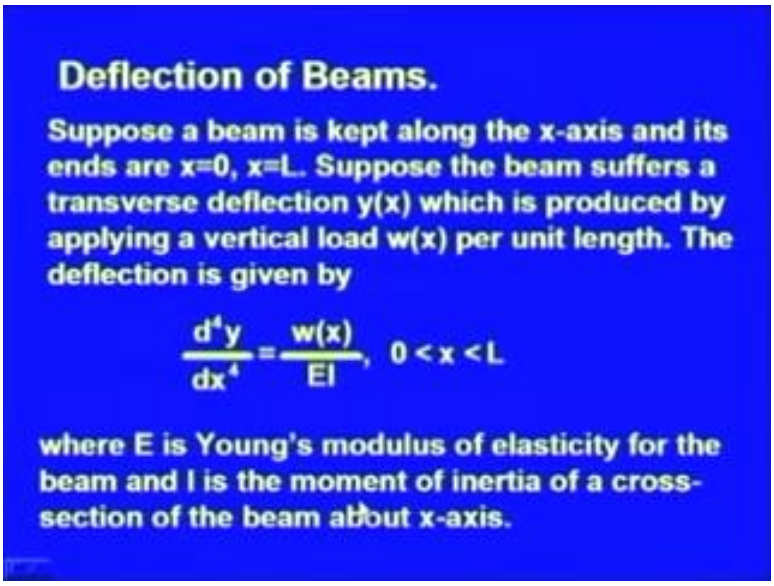
And, when we eliminate the variable y 2 bar here, we get y 1 bar equal to s plus root 3 k into s square plus 2 k plus k times s minus root 3 k over s plus 2 k whole square minus k square and y 2 bar comes out to be s minus root 3 into k plus into s square plus 2 k plus k times s plus root 3 k over s square plus 2 k whole square minus k square. After breaking into partial fractions, we get y 1 bar equal to s over s square plus k plus root 3 k over s square plus 3 k and y 2 bar becomes s over s square plus k minus root 3 k over s square plus 3 k.

(Refer Slide Time: 23:16)



Now, let us take inverse Laplace transform of these two equations, they will give us the displacements of the masses m 1 and m 2 at a time t. So, y 1 t comes out to be cos root k into t plus sin root 3 k into t and y 2 comes out to be cos root k into t minus sin root 3 k into t.

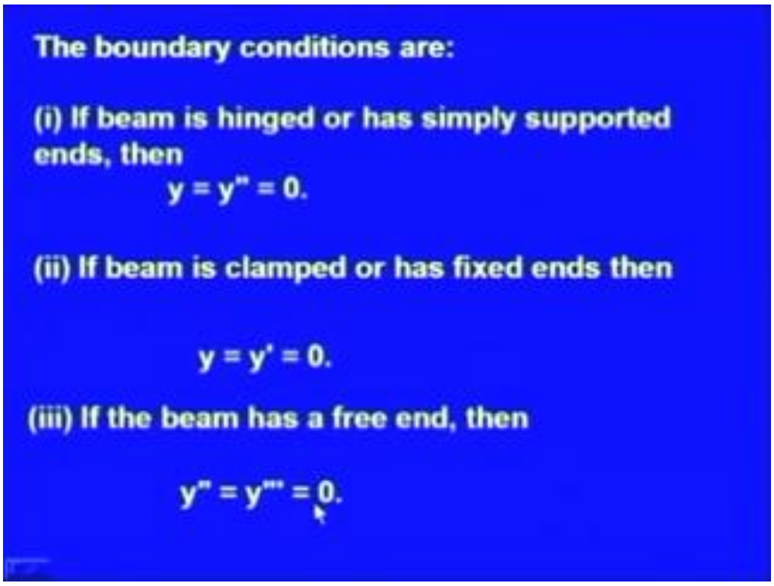
(Refer Slide Time: 23:40)



Now, let us study application of Laplace transformation to deflection of beams, so let us say we are given a beam, which is kept along the x axis and it is of length L its 1 end is at x equal to 0 the other end is at x equal to L. And, let us suppose that the beams suffers a transverse deflection y x, which is produced by applying a vertical load to the beam say w x per unit length.

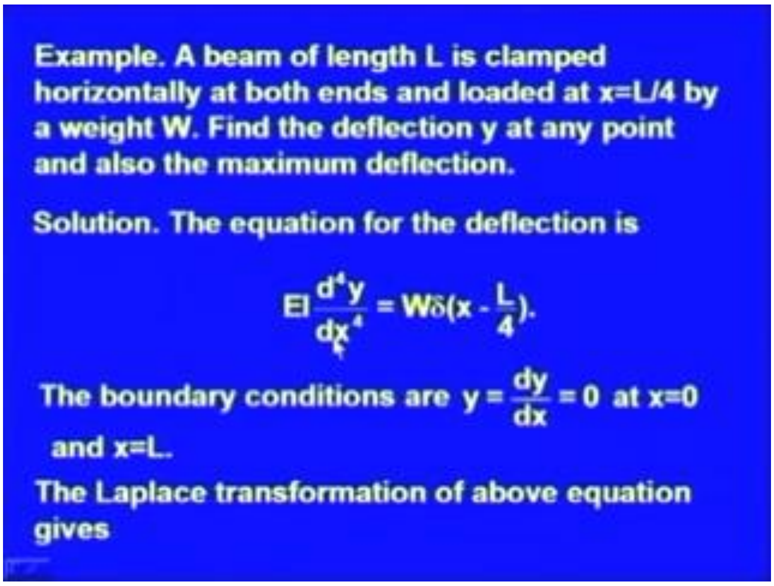
Then, the deflection is given by these differential equation d 4 y over d x 4 equal to w x over E I for x varying from 0 to L, 0 less than x less than L. Here E is young’s modulus of elasticity for the beam and I is the moment of inertia of a cross section of the beam about the x axis.

(Refer Slide Time: 24:40)



The boundary conditions are if the beam is hinged or has sup simply supported ends, then at those ends y and y double dash are 0. If, the beam is clamped at both the ends or it has fixed ends then at those ends y and y dash are 0, now if the beam has a free end then at that end y double dash and y triple dash are 0.

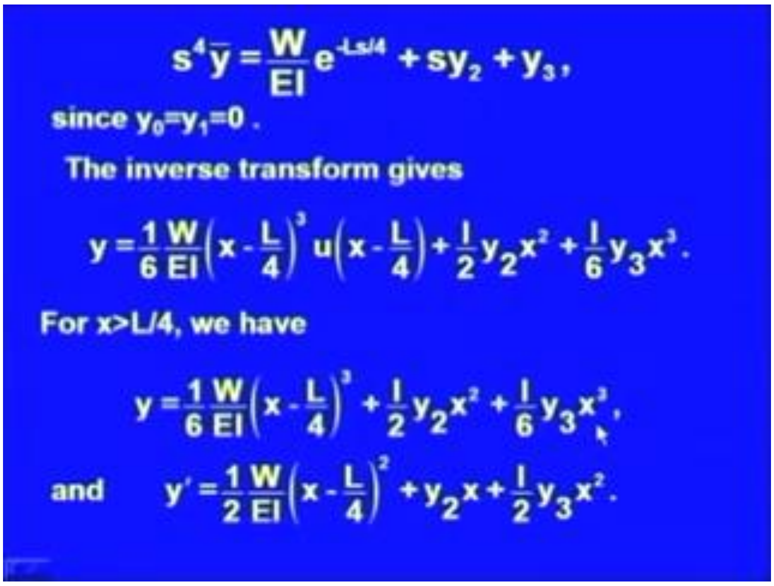
(Refer Slide Time: 25:14)



Let us now study, an example of a beam of length L, which is clamped horizontally at both at both its ends and loaded at x equal to L by 4 by a weight capital W. We have to find the deflection y at any point and also the maximum deflection in the beam, so as we have seen the equation for the deflection of the beam is given by E I d 4 y over d x 4 and the weight this is the point load here W.

So, we write the right hand side as W times delta x minus L by 4 it is applied at the point l by 4, so we write W in to delta x minus L by 4. The boundary conditions are because the ends of the beam are clamped, so the boundary conditions are y and y d y by d x both are 0 at the ends x equal to 0 and x equal to L.

(Refer Slide Time: 26:26)



When we take the Laplace transform of this equation for the deflection of the beam we will have, S 4 y bar equal to W over E I into e to the power minus L s by 4, because Laplace transform of delta x minus L by 4 gives you e to the power minus L by 4 into s and then plus s y 2 plus y 3, where we have made use of y naught and y 1 equal to 0 y 1 denotes d y by d x at x equal to 0 y naught is y at x equal to 0, y 2 is the y double dash at x equal to 0 and y 3 is y triple dash at x equal to 0.

Now, let us take inverse Laplace transform of this equation, so we will divide this equation by s to the power 4 and then take the inverse Laplace transform y bar is equal to W over E I to the power minus Ls by 4 over s 4. Then, the second term will become y 2 over s cube, third term will become y 2 y 3 over s to the power 4.

So, inverse Laplace transform of y bar is y, then w over E I is a constant, so we will write it like that and then inverse Laplace transform of e power L by 4 over s to the power 4.

Now, we know that inverse Laplace transform of 1 over s to the power 4 is t to the power 3 over 3 factorial that is 6. So, t cube over 6 is the inverse Laplace transform of 1 by s to the power 4, but here we will have to make use of the second shifting theorem, because we have to find the inverse Laplace transform of e to the power minus Ls by 4 over s to the power 4.

So, we will get a here is L by 4, so we will get the inverse Laplace transform as x minus L by 4 raise to the power 3 over 6 into u times into u of x minus L by 4 unit step function of x minus L by 4. Then, we will have y 2 over s to the power the inverse Laplace transform of that will be x square over 2 factorial that is x square over 2 and here, we will have to find the inverse Laplace transform of L over s to the power 4, which is s cube by 6.

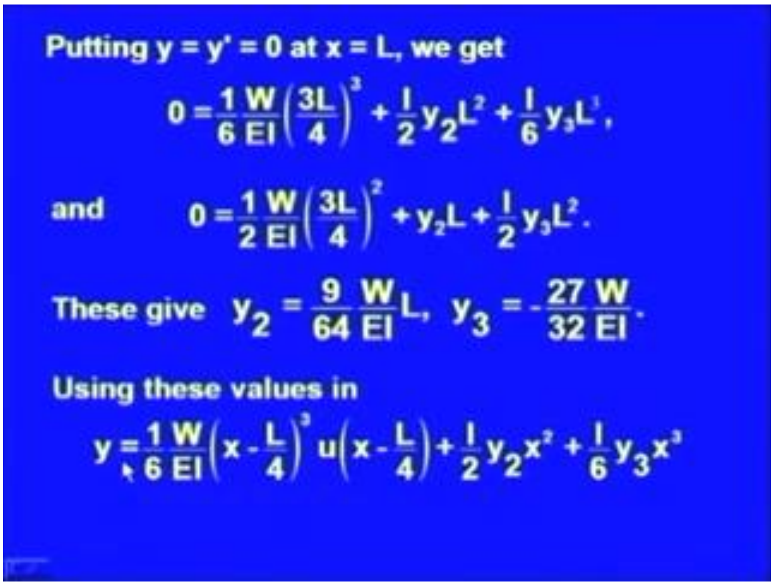
Now, let us in order to find this is the deflection at any time t and at a distance x, where we have to still find the values of the unknown constants y 2 and y 3. So, for that we will have to make use of the boundary conditions at the end x equal to l we have, so for made use of the boundary conditions at the end x equal to 0 only.

So, now let us, but we will have to somehow get rid of this unit step function, in order to find the values of y 2 and y 3. So, what we do is let us see what happens when we take x equal to L by 4, because the end x equal to L satisfies x greater than L by 4, so with if you take x to be greater than L by 4 will, you will be able to get rid of u of x minus L by 4.

And then, you can take the derivative of y with respect to x, so for x greater than l by 4 we have y equal to 1 by 6 into W by EI into x minus L by 4 raise to the power 3, because when x is greater than L by 4 unit step function gives us value 1 and then 1 by 2 into y 2 x square 1 by 6 into y 3 into x cube.

Now, let us take the derivative of this. So, d y by d x of this equation we will gives us 1 by 6 into W over EI in to 3 times x minus L by 4 whole square. So, we will after simplification we get the first term like this and then 1 by 2 into y 2 into 2 x, so second term after simplification gives y 2 x, third term after simplification will give us 1 by 2 into y 3 into x square.

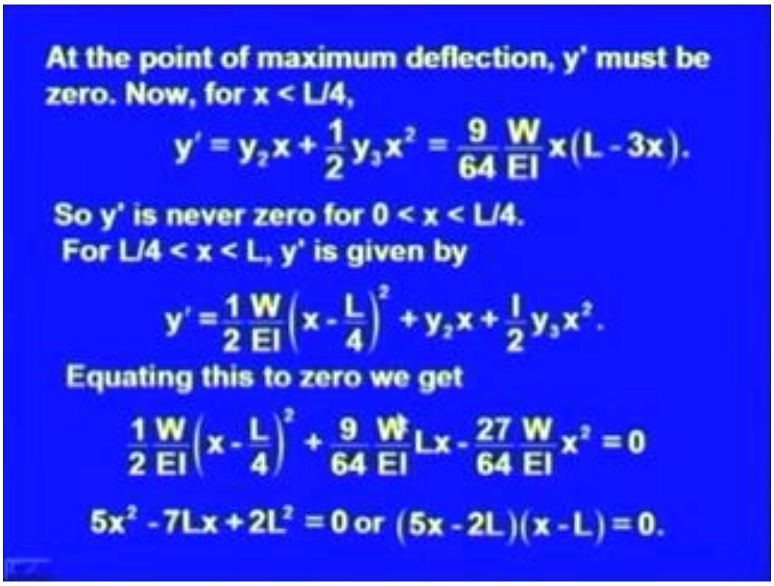
(Refer Slide Time: 30:26)



Now, let us put the in boundary conditions at the other end that is at x equal to L the boundary conditions are y and y dash are zeros at x equal to L, then we will get from the equations for y and y dash, will have these two equations where we have put x equal to L.

So, we get these two equations and these two are linear equations in y 2 and y 3 one can solve them. They will give us the values of y 2 as 9 by 64 into W by EI into L, y 3 as minus 27 by 32 into W over EI. Using these values in the equation for y, we will have the deflection at any time t and at a distance x given by this equation, this gives us deflection at any point of the beam, now in order to find the maximum deflection of the beam. at the point of maximum deflection Y dash must be 0.

(Refer Slide Time: 31:28)

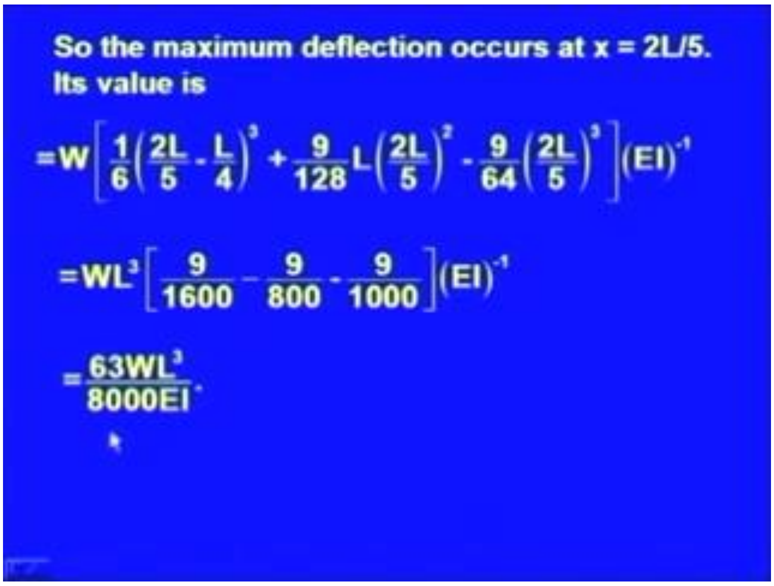


And now let us note that for x less than L by 4 here, we have for x less than L by 4 in the expression for y dash from the proof from y dash we shall have x is less than L by 4 the expression for y in the expression for y u x minus l by 4 will be equal to 0. So, for x less than l by 4 will differentiate that equation and get y dash as y 2 x plus 1 by 2 into y 3 in to x square, where after putting the values of y 2 and y 3 we get y dash as nine by 64 into W by EI into x in to L minus 3 x. And, from here we can see that when x is less than L by 4, L minus 3 x will never be equal to 0.

So, y dash is never 0 for less than x less than L by 4 and therefore, maximum deflection cannot occur in this interval. Now, let us note check for 1 by L by 4 less than x less than L, here y dash will be given by 1 by 2 into W over EI x minus L by 4 whole square plus y 2 x plus 1 by 2 y 3 x square, because u x minus L by 4 will be equal to 1 for this interval.

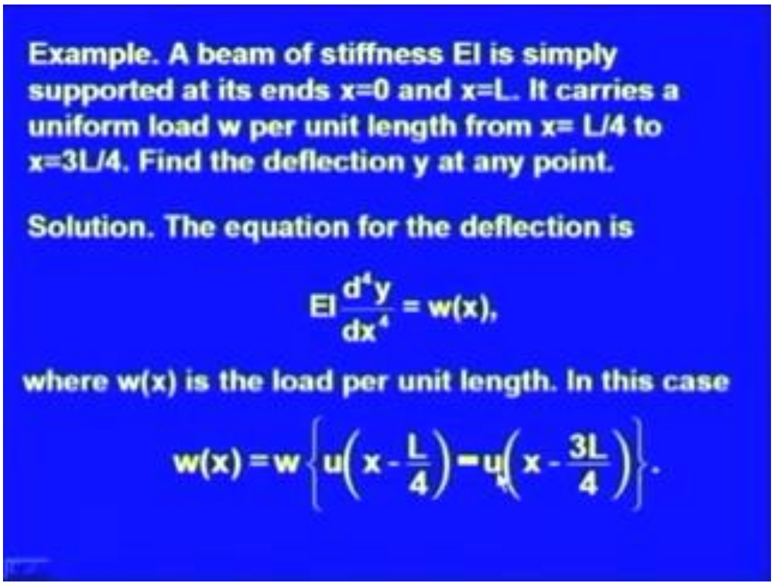
And after we put it equal to 0, we get and put the values of y 2 and after putting the values of y 2 and y 3 and equating y dash to 0 we get this equation from this equation. After simplification 1 will have 5 x square minus 7 L x plus 2 L square equal to 0, which can be factorized in to 2 factors like 5 x minus 2 L into x minus L and since x is not equal to L it is less than L this gives, you the value of x as 2 L by 5.

(Refer Slide Time: 33:41)



So, the maximum deflection of the beam occurs at x equal to 2 L by 5 and the deflection, then that is the maximum deflection y at x equal to 2 L by 5 is then given by W into this 1 by 6 2 L by 5 minus l by 4 whole to the power 3 plus 9 by 1 28 into L 2 L by 5 whole square minus 9 by 64 2 L by 5 whole cube into 1 over EI or EI to the power minus 1, after simplification the value of this expression comes out to be 63 WL cube over 8000 into EI.

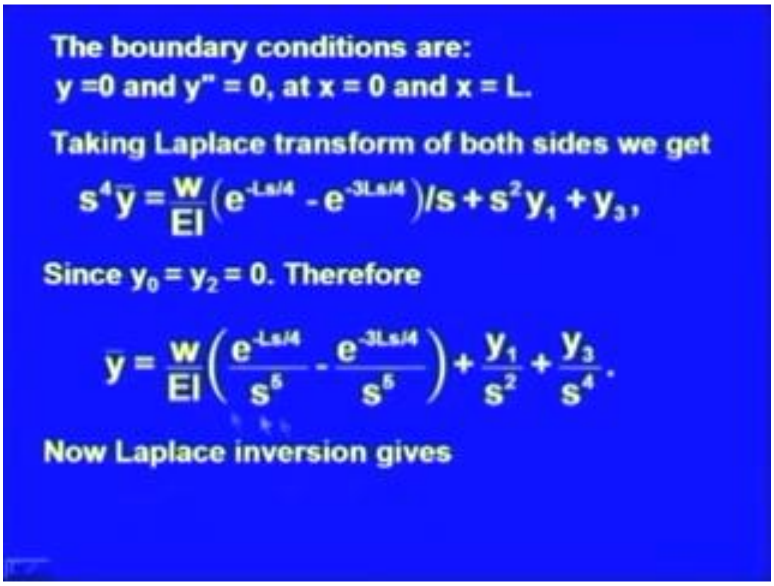
(Refer Slide Time: 34:14)



Now, let us study 1 more example on deflection of the beam a beam of stiffness EI is simply supported at its ends x equal to 0 or at and x equal to L it carries a uniform load W per unit length from x equal to L by 4 to x equal to 3 L by 4 find the deflection y at any point.

So, the we know that the deflection equation for the deflection of the beam is EI d 4 y over d x 4 equal to W x by W x is the load per unit length, in this case we are given that the beam carries uniform load W per unit length from x equal to L by 4 to 3 L by 4. So, W x the right hand side of this equation will be uh will be equal to w times u x minus L by 4 minus u x minus 3 L by 4

(Refer Slide Time: 35:08)



And therefore, and the boundary conditions for, because the beam is simply supported at both its ends, so y equal to 0 and y double dash equal to 0 at x equal to 0 and x equal to L. After taking the Laplace transform of the deflection of the beam, where we have seen that the right hand side is W times u x minus L by 4 minus u x minus 3 L by 4, so we have and make ma making use of these boundary conditions at x equal to 0, we get the Laplace transform of the deflection of beam as s 4 y bar equal to W over EI into e to the power minus Ls by 4 minus e to the power minus 3 s Ls by 4 over s plus s square y 1 plus y 3.

Since y naught and y 2 are 0, at x equal to 0 and therefore, after simplification y bar will be equal to W over EI, e to the power minus Ls by 4 over s to the power 5, e to the power minus 3 Ls by 4 over s to the power 5 y 1 over s square plus y 3 over s to the power 4.

(Refer Slide Time: 36:21)

$$y = \frac{1}{24}\frac{w}{EI}\left\{\left(x - \frac{L}{4}\right)^4 u\left(x - \frac{L}{4}\right) - \left(x - \frac{3L}{4}\right)^4 u\left(x - \frac{3L}{4}\right)\right\} + y_1 x + \frac{1}{6} y_3 x^3.$$

For $x > 3L/4$,

$$y = \frac{1}{24}\frac{w}{EI}\left\{\left(x - \frac{L}{4}\right)^4 - \left(x - \frac{3L}{4}\right)^4\right\} + y_1 x + \frac{1}{6} y_3 x^3,$$

and

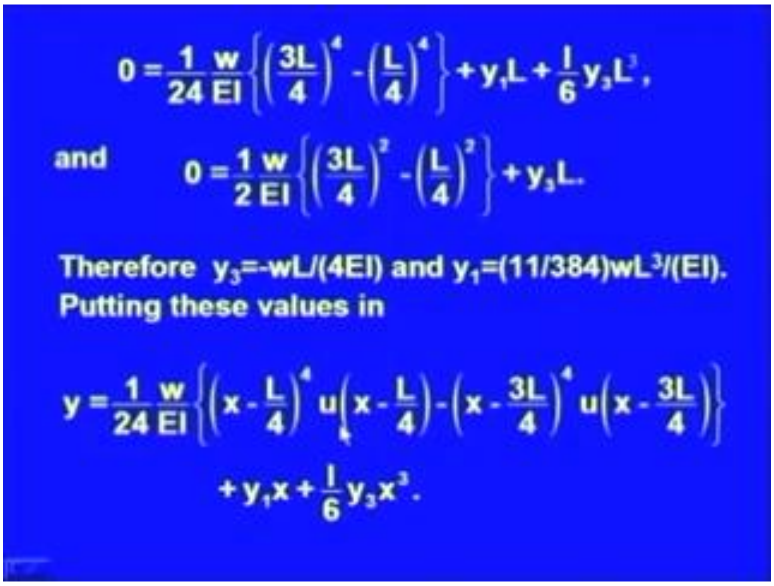
$$y'' = \frac{1}{2}\frac{w}{EI}\left\{\left(x - \frac{L}{4}\right)^2 - \left(x - \frac{3L}{4}\right)^2\right\} + y_3 x.$$

Now, let us take inverse Laplace transform of this when we take inverse Laplace transform this equation we get y equal to 1 by 24, W over EI x minus L by 4 raise to the power 4 into u x minus L by 4 minus x minus 3 L by 4 raise to the power 4 into u x minus 3 L by 4 plus y 1 x plus 1 by 6 y 3 x cube, where we have made use of the second shifting theorem.

Now, for x greater than 3 L by 4 we have y equal to 1 by 24 into W over EI, because u x minus 3 L by 4 will be equal to 1 while this will be equal to 0, this will be equal to 1 and all this will also be equal to 1. So, will have x minus L by 4 raise to the power 4 minus x minus 3 L by 4 raise to the power 4 plus y 1 x plus 1 by 6 y 3 x cube, we are going to find the values of y 1 and y 3.

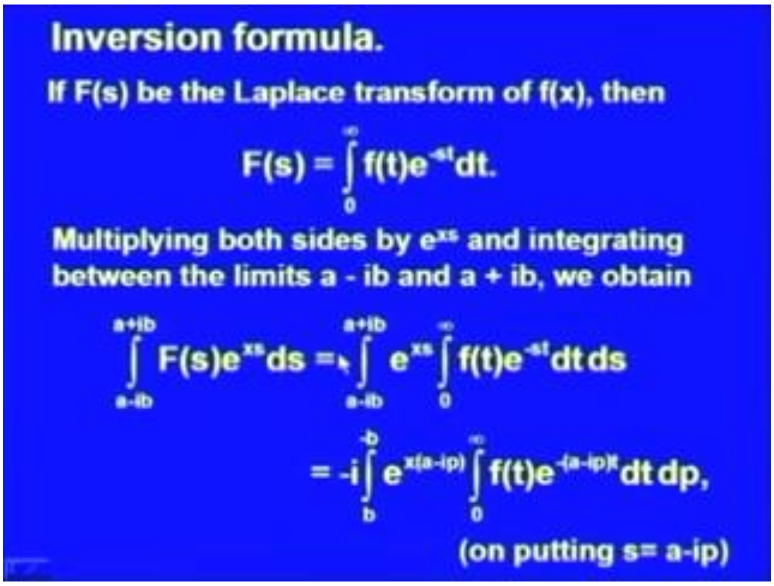
So, for that we will make use of the boundary conditions at the end x equal to L and that is why we have taken x to be greater than 3 L by 4 with y with this we can the replace the unit step functions by their values that is 1. And now, we can take the second derivative of this y double dash gives us 1 over 2 times W over EI x minus L by 4 to the power raise to power 2 minus x minus 3 L by power raise 3 L by 4 raise to power 2 plus y 3 x.

(Refer Slide Time: 37:59)



Putting the boundary conditions at the end x equal to L that is y equal to 0 and y double dash equal to 0, we get these two conditions equations which are again linear in y 1 and y 3. So, we can solve them for the values of y 1 and y 3, y 3 comes out to be minus w L by 4 EI while y 1 comes out to be eleven by 3 84 into w L cube by EI. Putting these values in the expression for y we get the deflection at any point x of the beam as this 1 by 24 W over EI x minus l by 4 raise to the power 4 u in into u x minus l by 4 minus x minus 3 L by 4 raise to the power 4 into u x minus 3 L by 4 plus y 1 x plus 1 by 6 y 3 x cube.

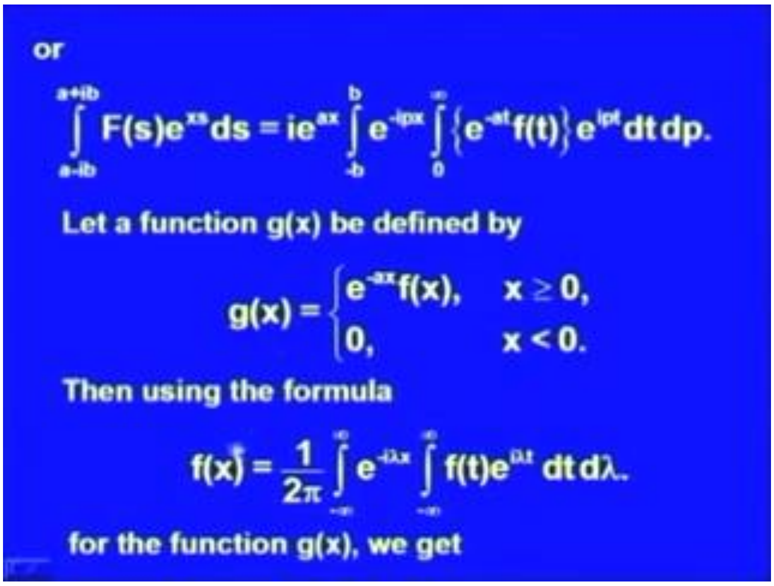
(Refer Slide Time: 38:44)



Now, we are going to study the inversion formula for the Laplace transform this we shall make use of why when we study the application of Laplace transform to the boundary value problems. So, let us say F s denotes the Laplace transform the function f x then F s will be equal to integral 0 to infinity ft e to the power minus s t d t.

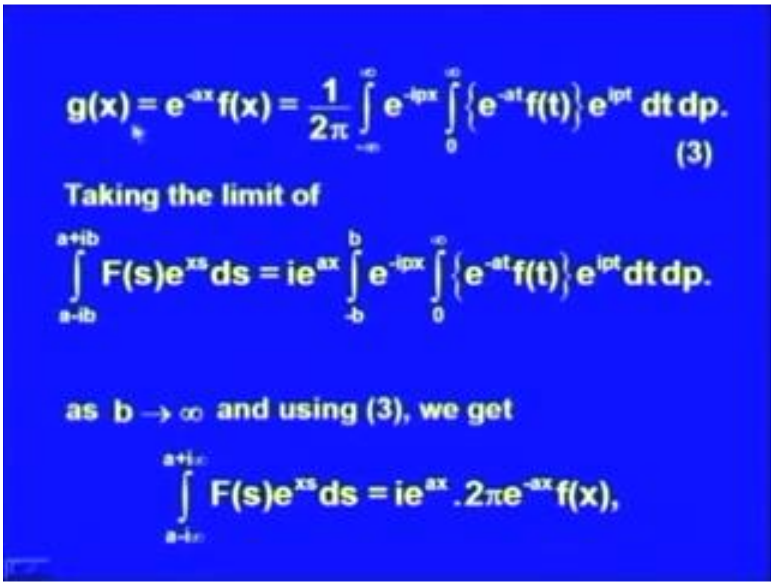
Now, let us multiply both sides of this equation by e to the power x s and integrate between the limits a minus i b and a plus i b we will have integral over a minus i b to a plus i b into f s I mean F s e to the power x s d s equal to a minus i b to a plus i b e to the power x s integral 0 to infinity ft e to the power minus s t d t d s. And when, you put in the right hand side s equal to a minus i b then d s becomes minus i d p the limits of integration for s change from a minus i b to a plus i b to b and minus b and we get this expression on the right hand side.

(Refer Slide Time: 39:53)



Or we will have integral over a minus i b to a plus i b F s e to the power x s d s as i e to the power ix minus integral over minus b to b e to the power minus i p x integral over 0 to infinity e to the power minus at into ft e to the power i pt d t d p. Now, let us define a function g x as e to the power minus a x into f x where x is greater than or equal to 0 and 0 when x is less than 0. Then, let us use the formula for the Fourier integral of f, let us recall that the formula for the Fourier integral of f at each point of continuity is f x equal to 1 over 2 pi integral over minus infinity to infinity e to the power minus psi lambda x integral over minus infinity to infinity ft e to the power lambda t d t d lambda, so let us apply this formula for the function g x.

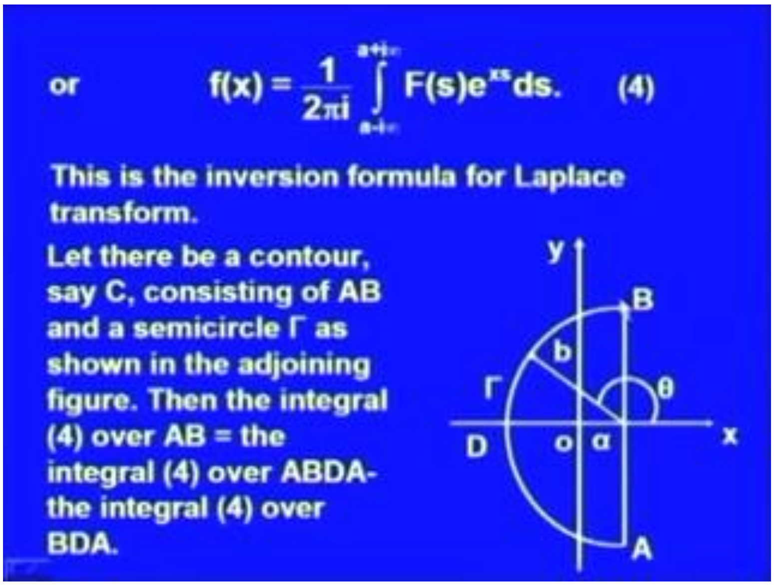
(Refer Slide Time: 40:59)



So, then g x will be equal to integral g x will be equal to e to power minus x into f x equal to 1 over 2 pi integral over minus infinity to infinity e to the power minus i p x 0 to infinity and then e to the power minus a t into ft here the limits of integration from minus infinity to infinity are reduced to 0 to infinity, because g x is defined as 0 from minus infinity to 0.

So, e to the power minus a t a t into ft in to e to the power i pt d t d p taking the limit of a minus i b to a plus i b F s e to the power x s d s that is this, so that is as b tends to infinity and using the equation 3, we get integral over a minus i infinity to a plus i infinity Fs e to the power x s d s as i e to the power i a x into 2 pi e to the power minus a x into f x.

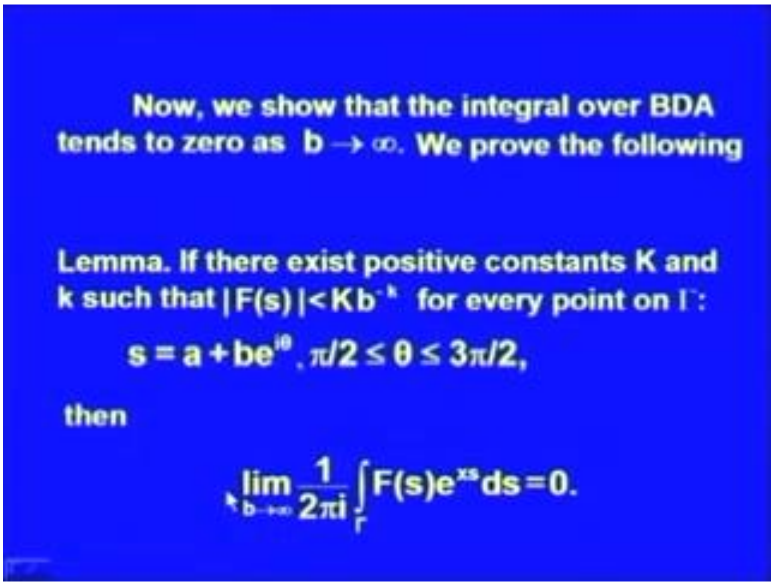
(Refer Slide Time: 41:54)



Or we get f x equal to 1 over 2 pi i integral over a minus i infinity to a plus i infinity F s e to the power x s d s this is the inversion formula for the Laplace transform. Here the integral is being taken over the line a b, which is parallel to the y axis and is to the right of all the similarities in the s plane all the similarities of the fun a function F s into e to the power x s lie to the left of the line a b and r enclosed by this contour a b, a b d this contour, which consist of a b and the semi circle gamma.

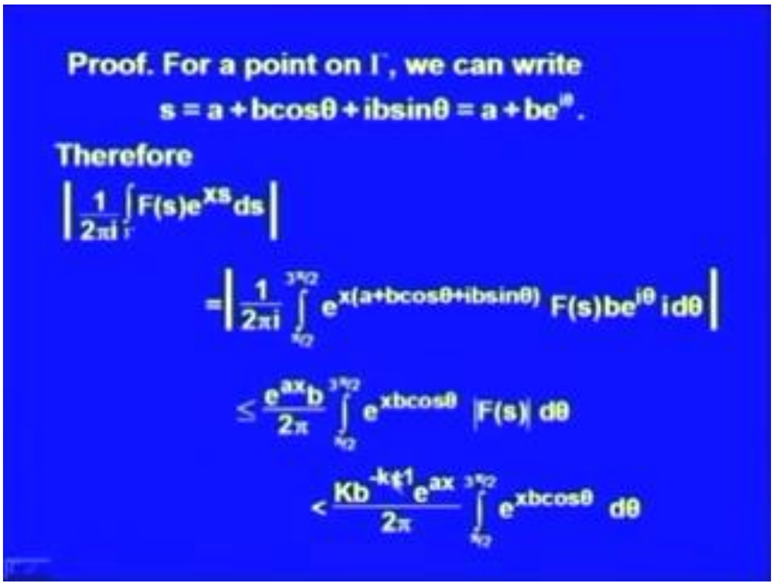
So, let us assume that there is a contour C, which consist of AB the line segment AB and a semi circle gamma as shown in this figure, then the integral 4 this integral 4 along AB is equal to integral over ABDA minus integral over BDA that is integral o a over ABDA from that, we subtract the integral over BDA in order to get the integral over AB.

(Refer Slide Time: 43:27)



We show that the integral over BDA that is the curved path that is the over semi circle tends to 0 as b goes to infinity the semi circle is of radius b and has centre at a. Let us show that if there exist positive constants capital K and small k such that mod of F s is less than k times b to the power minus small k for every point on gamma, which is described by s equal to a plus b e i theta pi by 2 less than or equal to theta less than or equal to 3 pi by 2, then limit of 1 over 2 pi i integral over gamma F s e to the power x s goes to 0 as b goes to infinity.

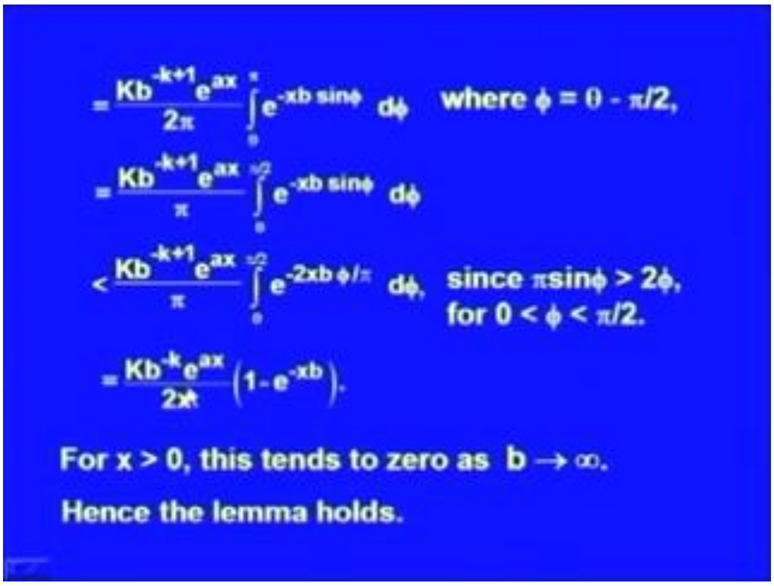
(Refer Slide Time: 44:07)



Now, a point on gamma for a point on gamma we can write s equal to a plus bi b e i theta or a plus b cos theta plus i b sin theta, because the gamma has center at the point a and its radius is b and theta here varies from pi by 2 to 3 pi by 2. Then, the modulus of 1 over 2 pi i integral over gamma F s e power x s d s is equal to modulus of 1 over 2 pi i integral over pi by 2 to 3 pi by 2 e to the power x s that is a plus b cos theta plus i b sin theta into F s and then d s will give you i into d s will give you i into b into e to the power i theta d theta.

So, now this is further less than or equal to this is less than or equal to e to the power a x into b this b we can write here, and then mod of i is 1. So, 2 pi integral over pi by 2 to 3 pi by 2 integral then e to the power b x sin theta we have and mod of e to the power i b x sin theta is equal to 1. And then we have mod of F s into d theta and this is further less than K b to the power minus k plus 1 here we are making use of the condition that mod of F s is less than k times b to the power minus k. So, k times b to the power minus k plus 1, because of this b and then we have e to the power a x over 2 pi by 2 to 3 pi by 2 e to the power b x cos theta d theta.

(Refer Slide Time: 45:54)

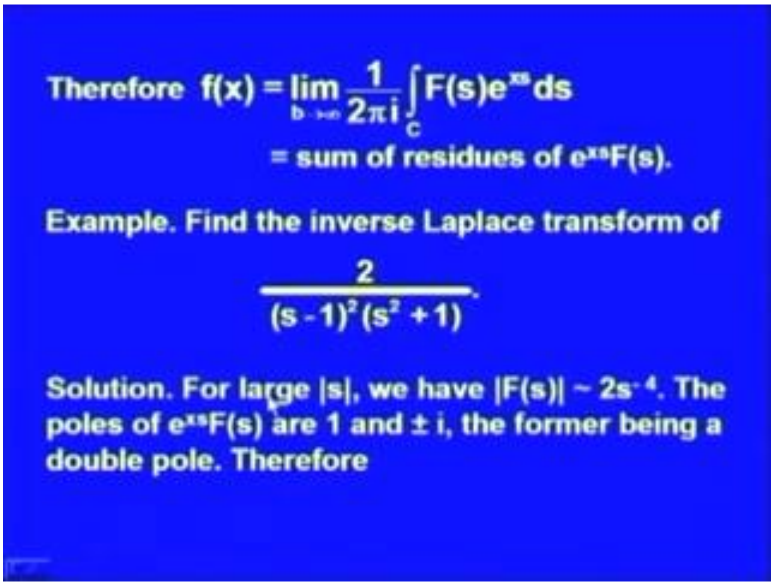


And, let us now put this phi equal to theta minus pi by 2, then the previous integral replace is replaced by integral 0 to pi e to the power minus b x sin phi d phi and will be multiplied by Kb to the power minus k plus 1 e to the power a x over 2 pi. And here now we make use of a property of the definite integral because sin pi minus phi is sin phi, so we can write it as 2 times 0 to pi by 2 e to the power minus b x sin phi d phi.

And, now let us make use of an equality which is well known we know that when 0 is phi 0 is less than phi less than pi by 2 sin phi over phi is greater than 2 over pi. And, so let us make use of that here, then this will be further less than K times b to power minus k plus 1 into e to power a x over pi integral over 0 to pi by 2 e to the power minus 2 b x phi over pi d phi.

And when you evaluate this integral and substitute the limits it comes out to be equal to K times b to the power minus k e to the power a x over 2 x into 1 minus e to the power minus x b for x greater than 0 it clearly goes to 0 as b goes to infinity. Because, e to the power minus x b as b goes to infinity goes to 0 and b to the power minus k goes to 0 as b goes to infinity.

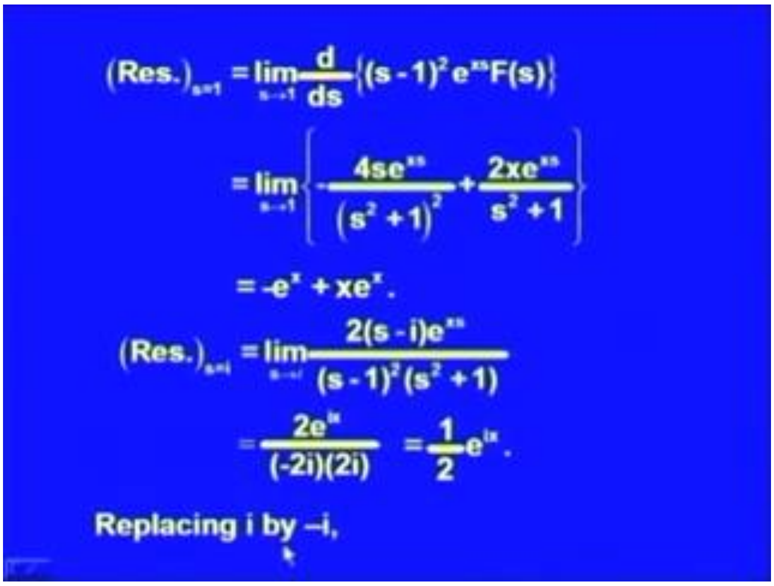
(Refer Slide Time: 47:28)



And thus f x is equal to limit of 1 over 2 pi i integral over c F s e to the power x s d s as b goes to infinity and from a residue theorem in complex analysis it follows that the value of this integral as b goes to infinity is some of residues of e to the power x s into F s at the similarities, which lie at the similarities of e to the power x s F s, which lie in the s plane and we have taken the line a b in such a way that all the similarities of e to the power x s F s lie to the left of it and are inside the contour.

Let us find the inverse Laplace transform of this function of s, 2 over s minus 1 whole square into s square plus 1. Now, we can see that for large values of modulus of s this F s is asymptotic to 2 times s to the power minus 4, so and the poles of e to the power x s into F s poles are positioning the similarities of this function F s occur at s equal to 1 and s equal to plus minus i, at s equal to 1 we have a pole of order 2 and at s, s equal to plus minus i, we have pole of order 1 these concepts follow from the complex analysis.

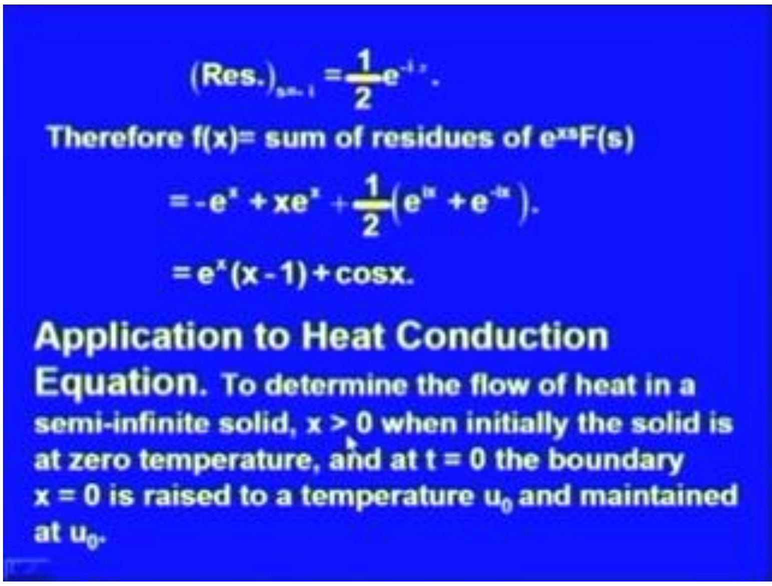
(Refer Slide Time: 48:58



So, therefore, the residue at this a pole double pole at s equal to 1 will be given by limit s tends to 1 d over d s of s minus 1 whole square into e to the power x s over F s, which after differentiating this we and we can find we can see that the differentiation comes out to be this. Now, let us take the limit of this as s tends to 1, so i just tends to 1 the limit of this is minus e to the power x plus x e to the power x.

Now, let us find the residue at the simple pole s equal to i it is again by a formula from complex analysis is it limit s tends to i 2 times s minus i into e to the power x s over s minus 1 whole square into s square plus 1, which will be equal to 2 times e to the power ix over minus 2 i into 2 i if you put if you let i over s go to i the denominator becomes minus 2 i into 2 i, which is after simplification half of e to the power ix. Now, at s equal to minus i we again have a simple pole, so replacing i by minus i here we get the residue at the other pole that that is at s equal to minus i it comes out half of e to the power minus ix.

(Refer Slide Time: 50:18)

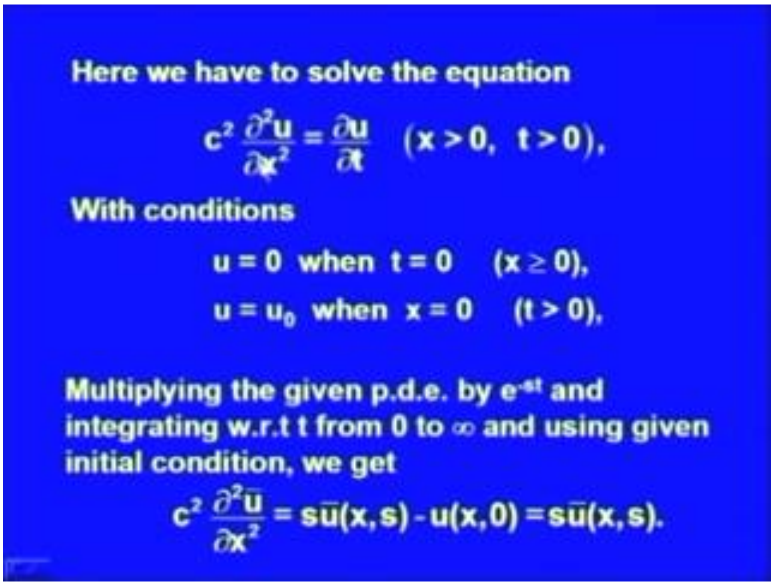


v

So now, f x is sum of residues of e to the power x s into F s, so we have the f x equal to minus e to the power x plus x e to the power x plus half of e to the power ix plus e to the power minus ix and we know that half of e to power ix plus e to power minus ix is cos x. So, the inversion formula for the Laplace transform gives us f x equal to e power x into x minus 1 plus cos x 1, can verify that f x is having this value directly by breaking F s the Laplace transform of f x the given function F s into its partial fractions and then using the known results that is the Laplace transforms of elementary functions, which we earlier done. So, from there also 1 can see that f x comes out to be this.

So now, let us apply the Laplace inversion formula to the heat conduction equation, let us now determine the flow of heat in a semi infinite bar x greater than 0, when initially the bar we are writing it as a solid here, but we are considering a semi infinite bar here. So, when initially the bar is at 0 temperature and at t equal to 0 the boundary x equal to 0 is raised to a temperature u naught and maintained at u naught. So, here we are given the initial condition that at t equal to 0 the temperature of the bar is 0 and we are given 1 boundary condition that at x equal to 0 for all the time t uxt is equal to u naught.

(Refer Slide Time: 52:10)

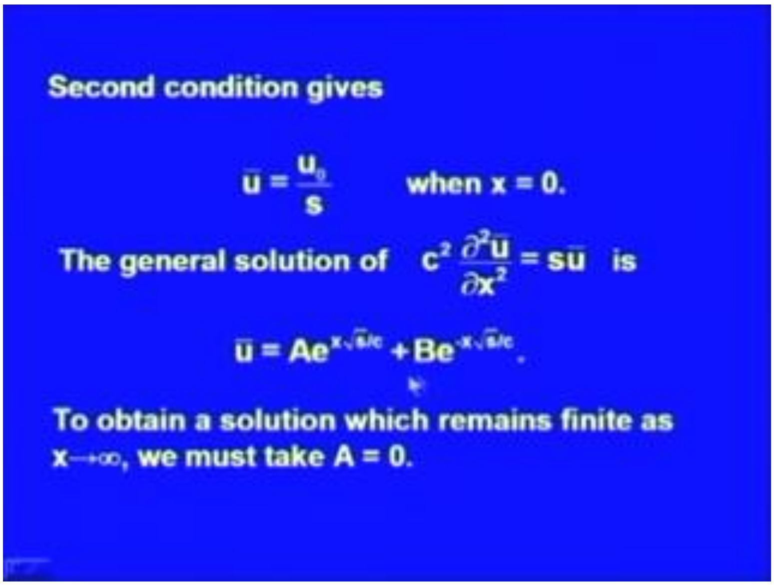


And we know that, the heat conduction equation in one dimension is delta u over delta t equal to c square delta square u over delta x square, where x is greater than 0 and t is greater than 0. We are given the boundary condition and initial conditions s follows the initial condition is that t equal to 0 u is equal to 0 u denotes the temperature it raise the function of 2 variables x and t and we are given the boundary condition that at x equal to 0 u is equal to u naught for all the time t.

Now, let us multiply the partial differential equation by e to the power minus s t and integrate with respect to t from 0 to infinity and use the given initial condition that is at t equal to 0 u is equal to 0. Now, when you take the Laplace transform of the left hand side c square is a constants it will remain as it is delta square over delta x square x is independent of t. So, we will get after multiplying by e to the power minus s t and taking the integral from with respect to t from 0 to infinity, we will get delta square over delta x square of u bar.

And then the right hand side will be Laplace transform of delta u over delta t, so that is s u bar x s t is replaced by s because we are integrating with respect to t and then minus u x 0. So, this follows from the Laplace transform for derivatives, which we have earlier done. So, s u bar x s minus u x 0 and u x 0 is given to be 0 at t equal to 0 u is 0. So, right hand side becomes s u bar x s.

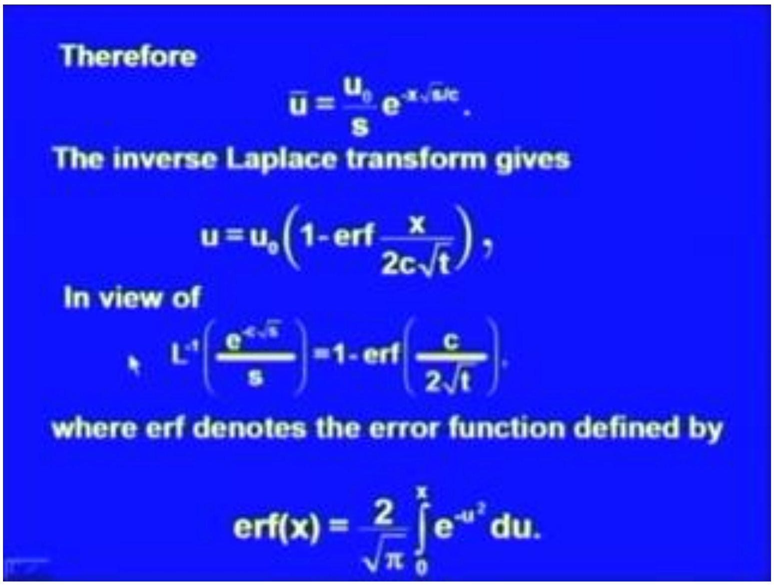
(Refer Slide Time: 54:00)



The second condition is that at x equal to 0 u is equal to u naught, so when you take the Laplace transform of that that is you multiply by e to the power minus s t and integrate with respect to t over 0 to infinity what you get is u bar equal to u naught by s where, when x is equal to 0. And, now the general solution of c square delta u square u bar over delta x square equal to s u bar this u bar is u bar axis is u bar equal to A into e to the power x root s by c plus B times e to the power minus x root s by c where A and B are functions of x.

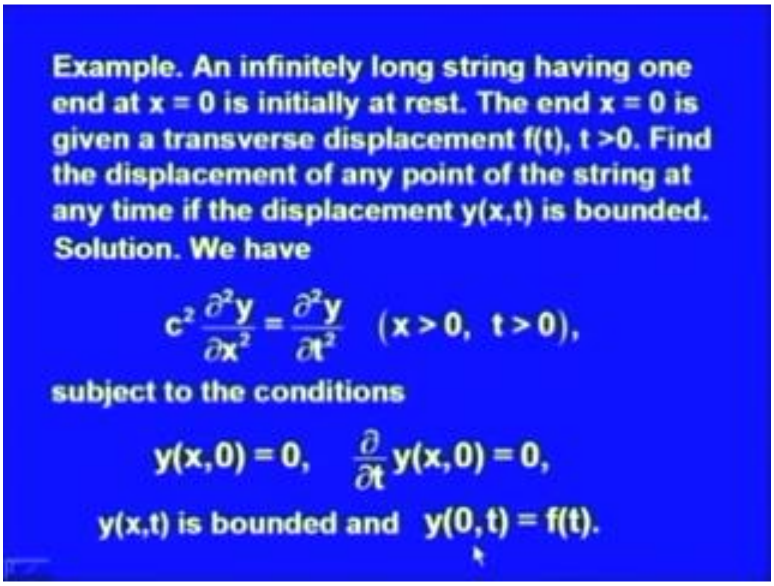
Now, let us we have to find a solution which remains finite as x tends to infinity and if that is the case, then we must have a equal to 0 otherwise e to the power x root s will tend to infinity as x goes to infinity. So, in order to find a finite solution or a solution, which remains finite as x may goes to infinity we must have A equal to 0. So, u bar reduces to u bar equal to b into e to the power minus x root s by c, now let us use the bound the condition that at x equal to 0 u bar is equal to u by s u naught by s this will give you B equal to u naught by s.

(Refer Slide Time: 55:15)



And so, u bar will be equal to u naught by s into e to the power minus x root s by c, when we take the inverse Laplace transform of this then inverse Laplace transform of u bar will be u, u naught is a constants it will remain as it is, then inverse Laplace transform of e to the power minus x root s by c over s is 1 minus erf x over 2 root 2 c root t in view of the result that is L inverse of e to the power minus c root s over s equal to 1 minus erf c over 2 root t. Now, where erf denotes the error function defined by erf x equal to 2 over root pi integral 0 to x e to the power minus u square d u. So, the solution of the given problem is u equal to u naught 1 minus erf x over 2 c root t.

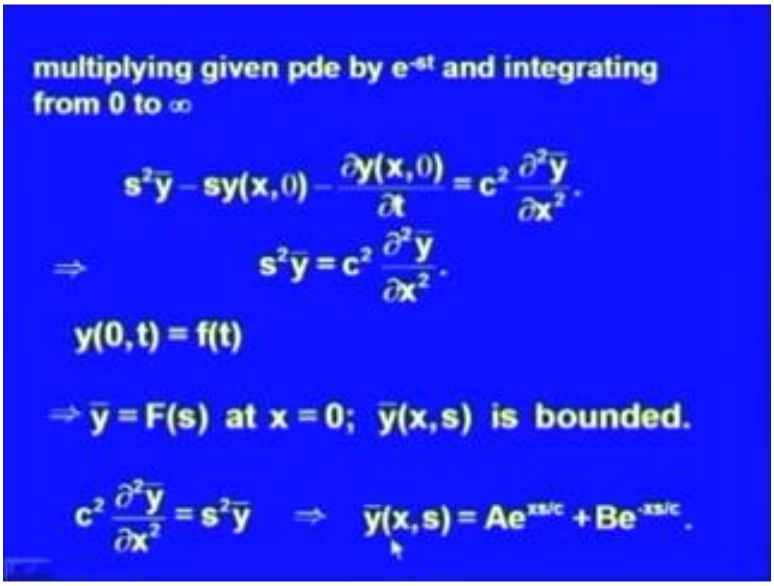
(Refer Slide Time: 56:21)



Let us now, consider the case of an infinitely long string, which is semi infinitely long string at the end x equal to 0 of this string is at rest initially at t equal to 0 it is at rest the end x equal to 0 is given a transverse displacement ft here to find the displacement of any point of the string at any time t if the displacement y x t is bounded.

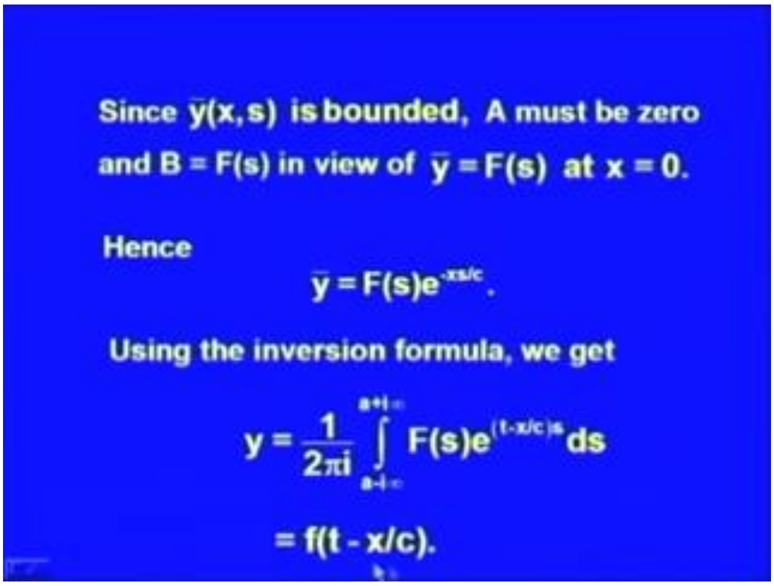
So, we take we are giving an infinite this thing transverse displacement to these spring to the string at the end x equal to 0 and this is the differential equation, which governs the transverse displacement of a string that is delta square y over delta t square equal to c square delta square y over delta x square, where x is greater than 0 t is greater than 0 we are given that at x equal to 0 the displacement is 0, so y x 0 is equal to 0 and the string is at rest at the at t equal to 0. So, delta over delta t y x 0 equal to 0 and we are also given that y x t is bounded and at x equal to 0 y x t is equal to ft.

(Refer Slide Time: 57:36)



Now, let us take the Laplace transform of the given partial differential equation will have this equation after using the given a conditions it reduces to s square y bar equal to c square delta square y bar over delta x square y 0 t equal to ft gives you y bar equal to F s at x equal to 0 and y x t is bounded. So, y bar x s is also bounded these equation gives the solution uh as y bar x s equal to A e to the power x s by c plus B x e to the power minus x s by c.

(Refer Slide Time: 58:16)



Since y bar x s is bounded a must be 0 and b equal to F s in view of y bar equal to F s at x equal to 0 hence y bar is equal to F s into e to the power minus x s over c. Now, let us use the inversion formula for the Laplace transform, we have y equal to 1 over 2 pi i integral over a minus i infinity to a plus i infinity F s e to the power t minus x by c into s d s which is equal to f of t minus x by c. Now, in our next lecture we shall be doing the application of Fourier series to the solutions of heat and to find the solutions of heat conduction equation and wave equations that is we will be applying Fourier series method to the boundary value problems in one dimension case.

Thank you.