Mathematics III

Prof. P. N. Agrawal
Department of Mathematics
Indian Institute of Technology, Roorkee

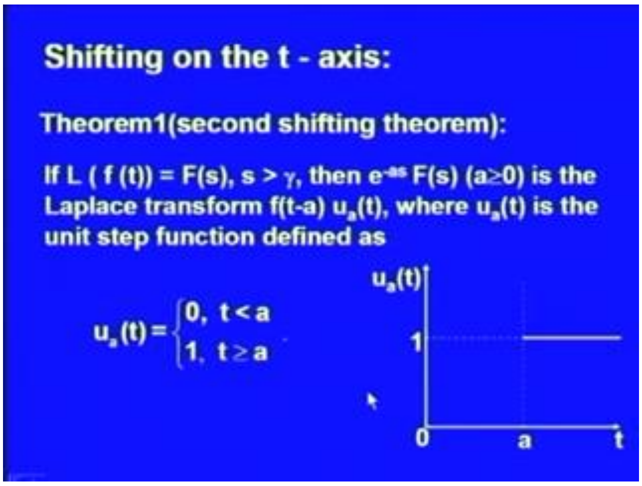
Lecture - 10
Applications of Laplace Transformation

Dear viewers in the title of my lecture is Applications of Laplace Transformation. In order to show the real power of Laplace transformation in it is applications to various problems of Engineering, Mathematics, we shall derive some more properties of the Laplace transformation. Two very important properties of the Laplace transformation, concern the shifting on the s axis and the shifting on the t axis.

The shifting on the s axis we had covered in our previous lectures on Laplace transformation, where we had shown that, if the replacement of s minus a in F s, which is the Laplace transform of the function f (t) corresponds to the multiplication of the original function f (t) by the exponential function e to the power a t. In the second shifting theorem which concerns the shifting on the t axis, we shall show that the shifting if we replace t by t minus a in the function f (t).

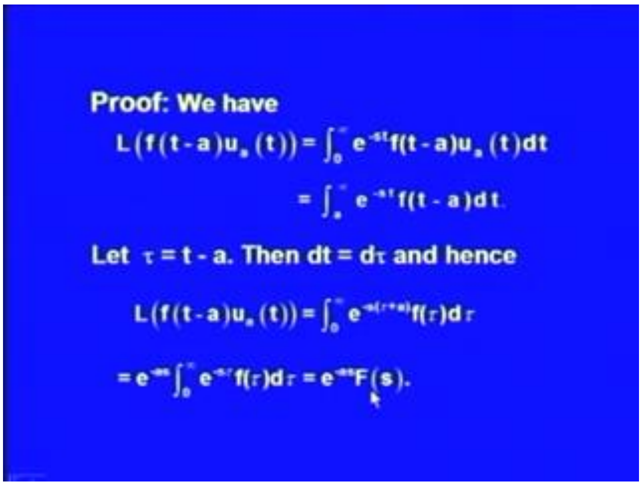
Then, it corresponds roughly to the multiplication of the Laplace transform F s of f (t) by e to the power minus a s, the precise formulation of the second shifting theorem is as follows.

(Refer Slide Time: 01:57)



It says that, if L f (t) is equal to F s, where s is greater than gamma, then e to the power minus a s into F s a greater than or equal to 0 is the Laplace transform of f (t) minus a into u a t, where u a (t) is the unit step function defined as u a (t) is equal to 0 when t is less than a. And u a (t) is equal to 1, when t is greater than or equal to a this is the graph of the function u a t, in the interval 0 to a that is when t lies between 0 and a, u a (t) is defined as 0 and when t is greater than or equal to a, u a (t) is defined as equal to 1.

(Refer Slide Time: 02:39)

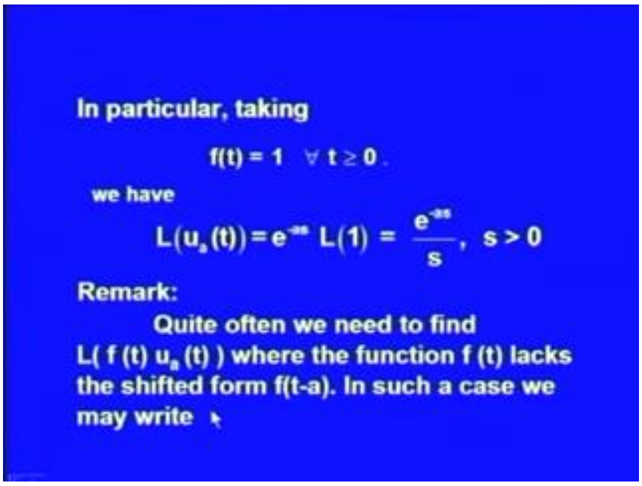


Let us look at the proof of the second shifting theorem, the Laplace transform of f (t) minus a into u a (t) by definition can be written as integral over 0 to infinity e to the power minus s t into f (t) minus a into u a (t) d t, which is equal to integral over a to infinity e to the power minus s t into f (t) minus a d t. Because, u a (t) is equal to 0 over the interval 0 to a and over the interval a to infinity it is defined as 1. So, the integral over 0 to infinity reduces to the integral over a to infinity and u a (t) becomes 1.

Now, let us make a substitution here, let us put tau is equal to t minus a, so when we put tau equal to t minus a the limits of integration change from a to infinity to 0 to infinity and d t becomes equal to d tau. And hence, we get L of f (t) minus a u a (t) equal to integral over 0 to infinity e to the power minus tau plus a into f tau d tau, which can be detected as e to the power minus a s integral over 0 to infinity e to the power minus s tau into f tau d tau.

Now, integral over 0 to infinity e to the power minus s tau into f tau d tau is the Laplace transform of the function f tau, so we can say it is equal to F s. And thus we get the Laplace transform of f (t) minus a u a (t) equal to e to the power minus a s in to F s.

(Refer Slide Time: 04:15)

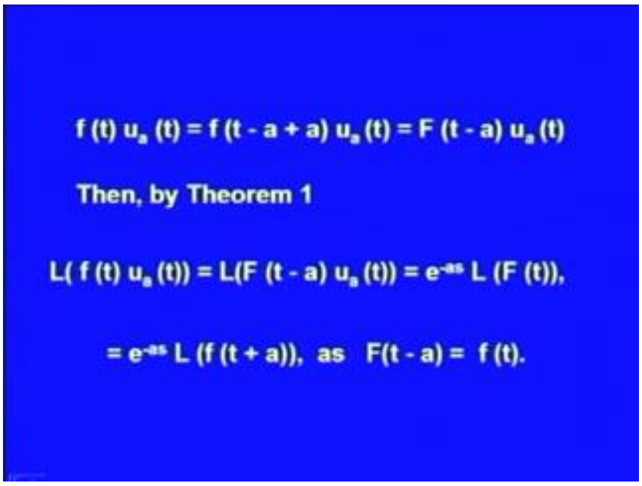


Now, if we take a particular case here, let us assume that f (t) is equal to 1 for all t greater than or equal to 0, then we will have the Laplace transform of f (t) minus a into u a (t) equal to Laplace transform of u a t, because f (t) is equal to 1 for all t greater than or equal to 0. So, we get Laplace transform for u a (t) equal to e to the power minus a s into Laplace transform of f t, that is Laplace transform of 1.

We know that Laplace transform of 1 is 1 by s whenever s is greater than 0, this we have shown earlier in our previous lecture on Laplace transformation; so the Laplace transform of u a (t) will be equal to e to the power minus a s over s, whenever s is greater than 0. Now, let us study a remark, quite often we have to find the Laplace transform of f (t) into u a t, where the function f (t) lacks the shifted form f (t) minus a.

If we have f (t) minus a here, in place of f (t) we can directly apply the second shifting theorem, but quite often it is not so, instead of f (t) minus a, we have f t. And we have to find the Laplace transform of f (t) into u a t, so in such a case we do the following.

(Refer Slide Time: 05:34)



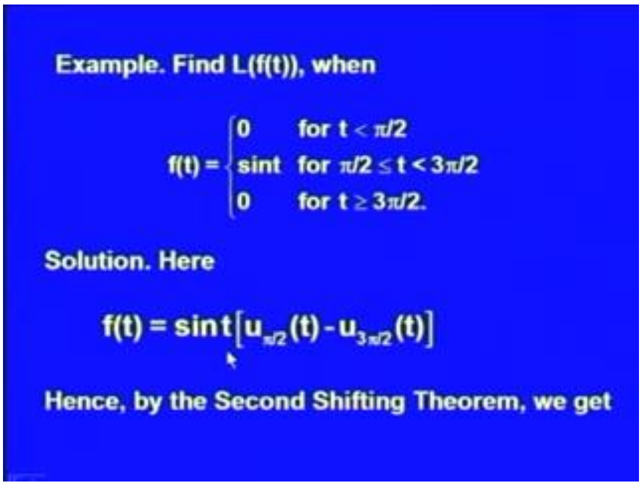
F (t) into u a (t) here we written as, f (t) minus a plus a into u a t, now f (t) minus a plus a is a function of t minus a, which we have denoted by capital f (t) minus a, so we get f (t) into u a (t) equal to f (t) minus a into u a t. And then, by the second shifting theorem that is theorem number 1, Laplace transform of f (t) into u a (t) will be equal to Laplace transform of f (t) minus a into u a t.

Now, Laplace transform of f (t) minus a into u a (t) by shift by theorem 1 is e to the power minus as into Laplace transform of the function capital F t, which is equal to e to the power minus a s into Laplace transform of F t, F t is equal to f of t plus a. Because, we have assumed that capital F of t minus a is equal to a small f t, so f of F t is equal to f of t plus a, so we get the Laplace transform of f (t) into u a (t) equal to e to the power minus a s into Laplace transform of f (t) plus a.

So, whenever we want to find the Laplace transform of the function f (t) into u a t, what we will do is, we will write it as e to the power minus a s into Laplace transform of f (t) plus a. In the second shifting theorem we had f of t minus a here and we had f (t) here, now the change is that we have f (t) here and here we have f of t plus a, so we get the Laplace transform of the function f of t plus a.

In the function f (t) we replace t by t plus a and then, find it is Laplace transform, in order to get the Laplace transform of the function f (t) into u a t.

(Refer Slide Time: 07:26)

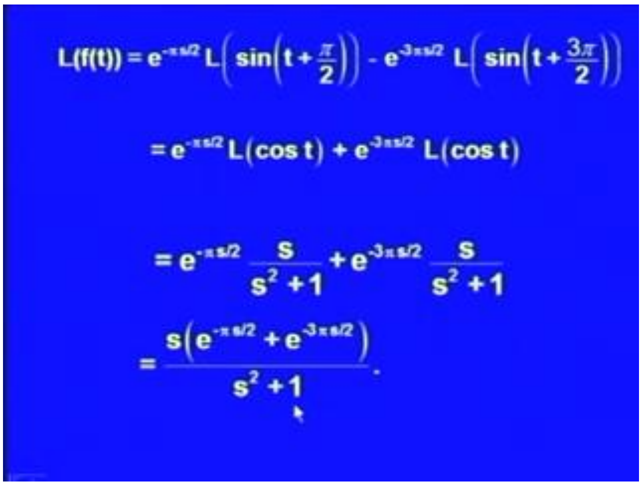


Now, let us study an example based on the second shifting theorem, let us find the Laplace transform of the function f t. Where f (t) is defined as 0 for t less than pi by 2, f (t) is defined as sin t for pi by 2 less than or equal to t less than 3 pi by 2 and f (t) is defined as 0 for t greater than or equal to 3 pi by 2. So, we can write the function f t, in terms of unit step functions as f (t) equal to sin t into u pi by 2 t minus u 3 pi by 2 t, because when t is less than pi by 2 u pi by 2 is t is 0, u 3 pi by 2 t is also 0, so f (t) is equal to 0.

And when t is equal to pi by 2 or more than pi by 2, but less than 3 pi by 2, then u pi by 2 t will be equal to 1 while u 3 pi by 2 t will be 0, so f (t) will be equal to sin t and when t is equal to 3 pi by 2 or more than 3 pi by 2 u pi by 2 t will be equal to 1 u 3 pi by 2 t will also be equal to 1, so f (t) will be equal to 0. Hence f t, the given function f (t) can be described in terms of unit step functions u pi by 2 t and u 3 pi by 2 t as f (t) equal to sin t times u pi by 2 t minus u 3 pi by 2 t.

Now, let us apply the second shifting theorem, so when you take the Laplace transform of this by linearity property of the Laplace transform, L f (t) will be equal to L of sin t into u pi by 2 t minus L of sin t into u 3 pi by 2 t. Now, we can see here that this is L of sin t into the u pi by 2 t is not actually in the form of the second shifting theorem, rather it is in the form of the remark, which follows the second shifting theorem. Where we are discuss that how to find the Laplace transform of f (t) into u a t, so using that remark we will have the following.

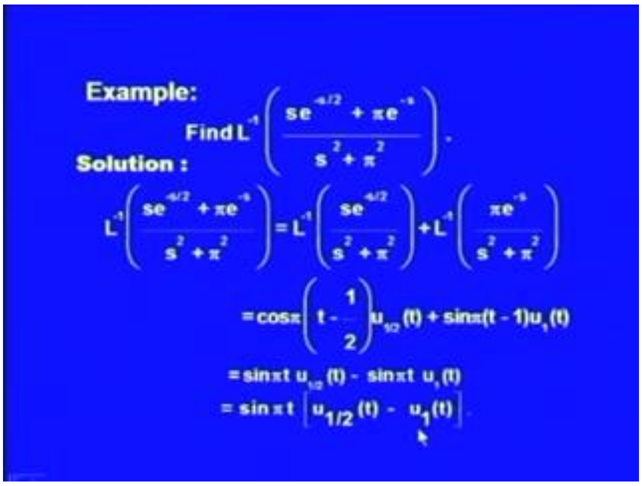
(Refer Slide Time: 09:27)



Laplace transform of the function f (t) will be e to the power minus s a is pi by 2 here, so e to the power minus pi s by 2 into Laplace transform f of t plus a f (t) sin t a is pi by 2. So, Laplace transform of sin t plus pi by 2 we have minus e to the power minus s a is 3 pi by 2 here, so we get e to the power minus 3 pi s by 2 into Laplace transform of sin t plus 3 pi by 2; now sin t plus pi by 2 is cos t, so Laplace transform of cos t we have here.

And then, Laplace transform of sin t plus 3 pi sin t plus 3 pi by 2 is equal to minus cos t, so we have minus minus plus here, e to the power minus 3 pi s by 2 into Laplace transform of cos t. And Laplace transform of cos t we know, it is s over s square plus 1, so we get the right hand side as e power minus pi s by 2 into s over s square plus 1 plus e to the power minus 3 pi s by 2 into s over s square plus 1, which is further equal to s into e to the power minus pi s by 2 plus e to the power minus 3 pi s by 2 over s square plus 1.

(Refer Slide Time: 10:40)



Let us take another example based on second shifting theorem, here we have to find the inverse Laplace transform of s into e to the power minus s by 2 plus pi into e to the power minus s over s square plus pi square. We can express the inverse Laplace transform of s e to the power minus s by 2 plus pi e to the power minus s over s square plus pi square is the sum of the inverse Laplace transforms of s into e to the power minus s by 2 over s square plus pi square, and pi into e to the power minus s over s square plus pi square.

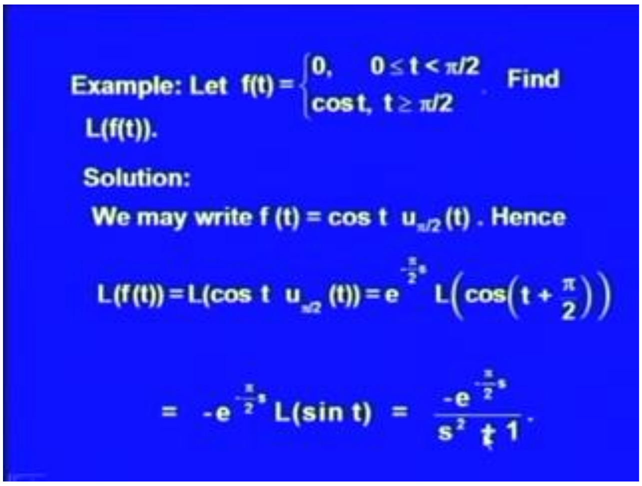
And we know that s over s square plus pi square is the Laplace transform of cos pi t and here, we have if you identify with e to the power minus a s here, s into e to the power minus s by 2 tells us that a is equal to half here. So, we have to find inverse Laplace transform of e to the power minus a s into f s, where a is half and f s is s over s square plus pi square, whose inverse Laplace transform is cos pi t.

And so by second shifting theorem, inverse Laplace transform of s into e to the power minus s by 2 over s square plus pi square will be u a t, u a (t) becomes u half t into f (t) minus a, f (t) minus a means, f (t) is cos t cos pi t. So, f (t) minus a will be f (t) minus a will be f (t) minus half that will be given as cos pi t minus half and then, we have inverse Laplace transform here of pi e to the power minus s over s square plus pi square.

Pi over s square plus pi square is the Laplace transform of sin pi t function and e to the power minus s tells us that, a is equal to 1 here, so using second shifting theorem we get u 1 (t) that is u a (t) becomes u 1 (t) and then, is f of t minus 1. So, f (t) is sin pi t, so f (t) minus 1 becomes sin pi t minus 1, so we get the inverse Laplace transform as cos pi into t minus half into u half t plus sin pi t minus 1 into u 1 (t), which we can write in the simplified form as cos pi t minus half becomes sin pi t into u half t.

And sin pi t minus 1 becomes minus sin pi t and so we get minus sin pi t into u 1 (t) and we can simplify it further, we can write it as sin pi t times u half t minus u 1 (t).

(Refer Slide Time: 13:17)

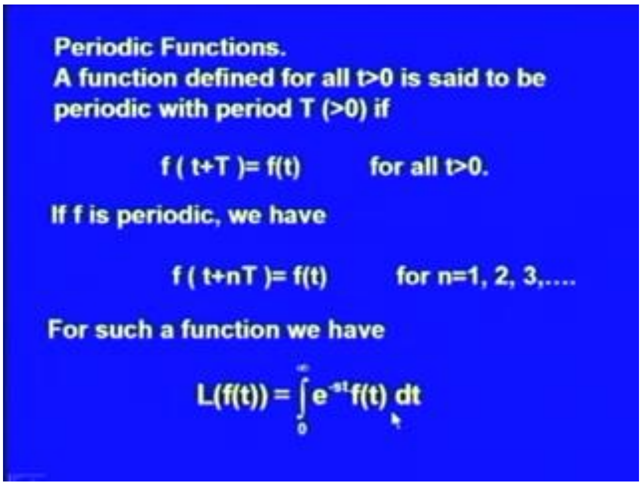


Now, let us take another function f t, which is defined as 0, when t is greater than or equal to 0, but less than pi by 2 and equal to cos t when t is greater than or equal to pi by 2; and let us find the Laplace transform of this function. So, again we can express this function as in terms of unit step functions, we can write f (t) as cos t into u pi by 2 t, when t is less than pi by 2, u pi by 2 is 0, u pi by 2 t is 0, so f (t) becomes 0.

And when t is equal to pi by 2 or more than that, then u pi by 2 t is equal to 1, so we get f (t) equal to cos t and then, Laplace transform of cos t into u pi by 2 t will be equal to e to the power minus a s, a i is equal to pi by 2 here, so e to the power minus pi by 2 into s. And then Laplace transform of f (t) plus a, f (t) is cos t here, a is pi by 2, so we get Laplace transform of cos t plus pi by 2 and cos pi by 2 plus theta we know this equal to minus sin theta, so we get L of sin t here.

So, we get minus e to the power minus pi by 2 into s L of sin t and L of sin t is 1 over s square plus 1, so we get minus e to the power minus pi by 2 s over s square plus 1.

(Refer Slide Time: 14:40)

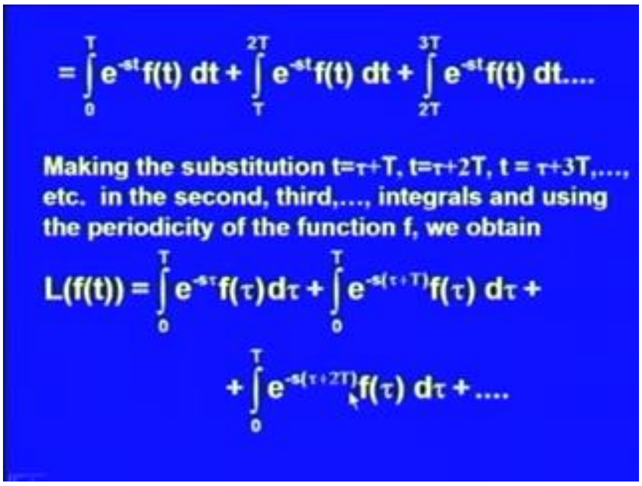


Now, we are going to study periodic functions, periodic functions occur in many practical problems and in most of the cases they are more complicated, then the single sin or cosine functions. We are going to show that if the function f (t) is periodic with period t and it is piecewise continuous over a length t, then the Laplace transform of the function L exist, so let us define first periodic function.

A periodic function defined for all t greater than 0 is said to be periodic with period t greater than 0, if f of t plus T is equal to f (t) for all t greater than 0. And if f is periodic with period t, then we can easily see by induction on n that f of t plus n T is equal to f t, where n takes values 1, 2, 3 and so on. Now, for such a function we have L of f (t) equal to 0 to infinity e to the power minus s t into f (t) d t, we shall see that for the existence of the integral of a periodic function with period t, we just need the piecewise continuity of the function.

So, let us see when this Laplace transform the function f (t) exist, so for the periodic function L of f (t) can be written as integral 0 to infinity e to the power minus s t into f (t) d t.

(Refer Slide Time: 16:11)

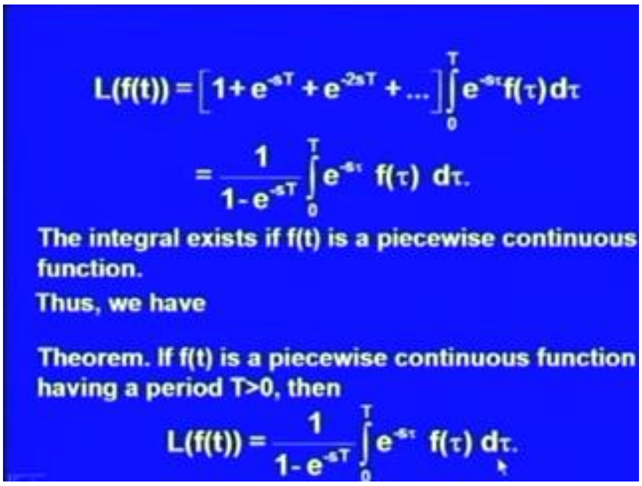


And now using periodicity what we will do, we will get the interval 0 to infinity into parts, 0 to T, T to 2 T, 2 T to 3 T and so on and make use of the periodicity of the function f t. So, what we will do in the second integral onwards on the right side, that is integral T to 2 T, integral 2 T to 3 T and so on, in the we shall make substitutions t equal to tau plus T, t equal to tau plus 2 T, t equal to tau plus 3 T and so on.

And use periodicity of the function f to obtain L f (t) is equal to integral 0 to T e to the power minus s t, s tau f tau d tau plus integral over 0 to T e to the power minus s tau plus T into f tau d tau. Here when you substitute T equal to tau plus T using periodicity of the function f tau plus t will be f tau, e to the power minus s t will become e to the power minus s tau plus T.

And d t will become d tau, the limits of integration which are T and 2 T will change to 0 to T, here they will become 0 and T and in the integrant we will have e to the power minus s tau plus 2 T into f tau d tau, again using periodicity here.

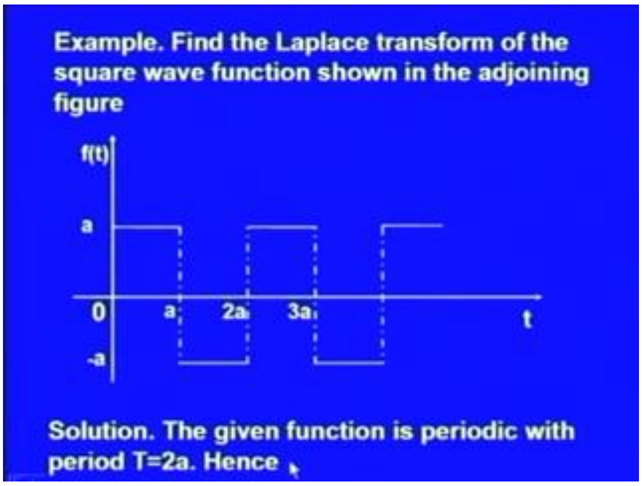
(Refer Slide Time: 17:36)



This way we shall then have L f (t) equal to 1 plus e to the power minus s T plus e to the power minus 2 s T and so on, multiplied by integral 0 to T e to the power minus s tau into f tau d tau. Now, these are geometric series, there I show is e to the power minus s T, so we can write the sum of the series as 1 over 1 minus e to the power minus s T and then, multiplied by integral 0 to T e to the power minus s tau f tau d tau.

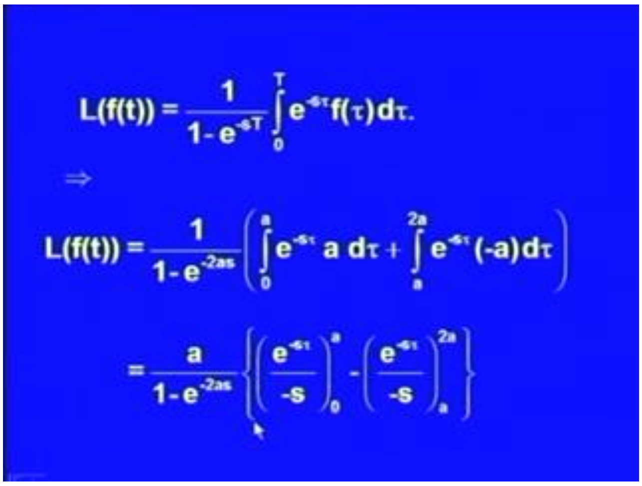
This integral which occurs here, exist if f (t) is a piecewise continuous function and of course periodic with period T, so thus we have the following theorem for a periodic function. If f (t) is a piecewise continuous function having a period T greater than 0, then the Laplace transform of the function f (t) exist and is given by 1 over 1 minus e to the power minus s T into integral 0 to T e to the power minus s tau f tau d tau.

(Refer Slide Time: 18:48)



Let us take an example on this article, let us find the Laplace transform of the square wave function, which is shown here in this figure. You can see here that the function f given by this figure is periodic with period 2 a here, you can see the graph of the function over the interval 0 to a it takes the value a and then, over the interval a to 2 a it takes the value minus a. And then, the graph of the function f over the length 2 a is repeated over the from 2 a to 4 a and so on, so the it is a periodic function with period 2 a; the given function is periodic with period, we have denoted by T so T is equal to 2 a here.

(Refer Slide Time: 19:37)



And f (t) is clearly a piecewise continuous function on the length t equal to 2 a, so Laplace transform f (t) of the given function exist and will be obtained from these result 1 over 1 minus e to the power minus s T integral over 0 to T e to the power minus s tau f tau d tau. So, t is equal to 2 a gives us Laplace transform of the given function f (t) as 1 over 1 minus e to the power minus 2 a s and then, integral over 0 to 2 a is broken up into two parts integral over 0 to a and then, integral over a to 2 a.

Integral over 0 to a we have, in the integral over 0 to a we use the definition of f, f is equal to a in this part of the interval and in the interval a to 2 a function f takes the value minus a, so we have put those values of f. And then, we integrate e to the power minus s tau integral of e to the power minus s tau with respect to tau gives us into e to the power minus s tau over minus s and so when you put the limits 0 a here, and 2 a here what we get is the following.

(Refer Slide Time: 20:56)

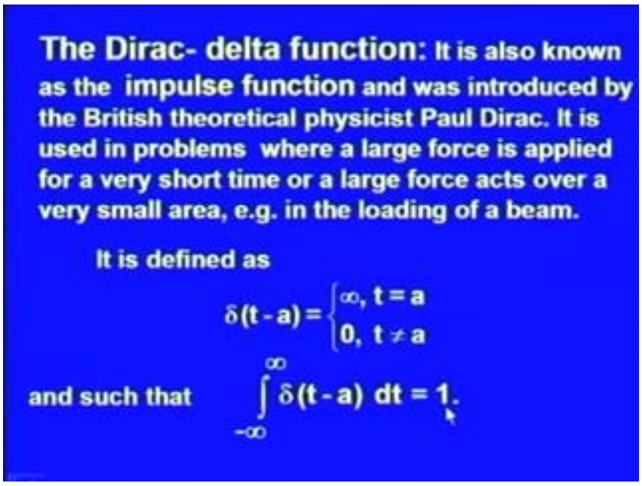
$$= \frac{a(1-2e^{-as}+e^{-2as})}{s(1-e^{-2as})}$$

$$= \frac{a(1-e^{-as})^2}{s(1-e^{-2as})} = \frac{a(1-e^{-as})}{s(1+e^{-as})} = \frac{a}{s}\tanh\left(\frac{as}{2}\right).$$

A times 1 minus 2 times e to the power minus a s plus e to the power minus 2 a s over s into 1 minus e to the power minus 2 s, now 1 minus 2 e to the power minus a s plus e to the power minus 2 a s is the square of 1 minus e to the power minus a s. So, we replace this numerator by a times 1 minus e to the power minus a s whole square and the denominator we have s into 1 minus e to the power minus 2 s 1 minus e to the power minus 2 a s can be factorized as 1 minus e to the power minus s into 1 plus e to the power minus a s.

And then, 1 minus e to the power minus a s can be cancelled from the numerator and denominator and we will be getting a times 1 minus e to the power minus s over s times 1 plus e to the power minus s. Now, if you multiply in the numerator and denominator by e to the power a s by 2, then what we will get a over s times e to the power a s by 2 minus e to the power minus a s by 2 divided by e to the power a s by 2 plus e to the power minus a s by 2, which we know that uh denotes the tan hyperbolic of a s by 2, so we have the Laplace transform of the given square wave function as a over s into tan hyperbolic a s by 2.

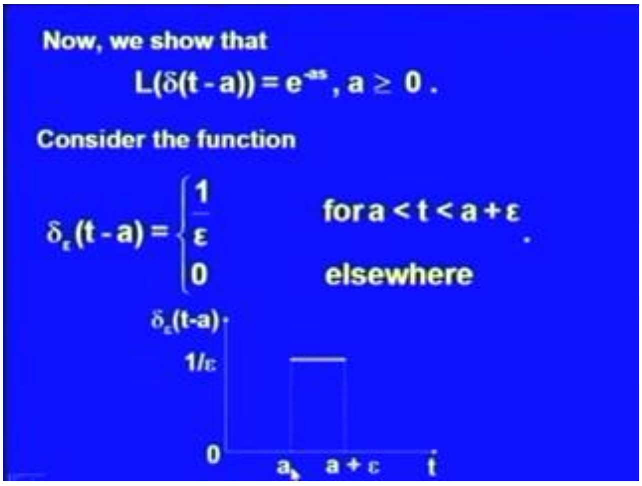
(Refer Slide Time: 22:23)



Now, we are going to discuss the Dirac delta function, the Dirac delta function is also called the impulse function and has applications in these problems where a large force occurs is applied a very, very small, for a large force is applied for a very short time. Say for example in the case of bending of beams, when we use the pipe loads, the pipe load means, we are applying a very large pressure over a very small area. So, in such cases we use the Dirac delta function, with Dirac delta function is also called as unit impulse function or impulse function and it was introduced by British theoretical physicist Paul Dirac.

So, it is applied or used in problems where a large force is applied for a very short time or a large force acts over a very small area, for example in the loading of a beam. And because of it is nature it is defined as delta t minus a is equal to infinity at t equal to a and 0 when t is not equal to a. So and further that integral of delta t minus a over the interval minus infinity to infinity is defined as equal to 1.

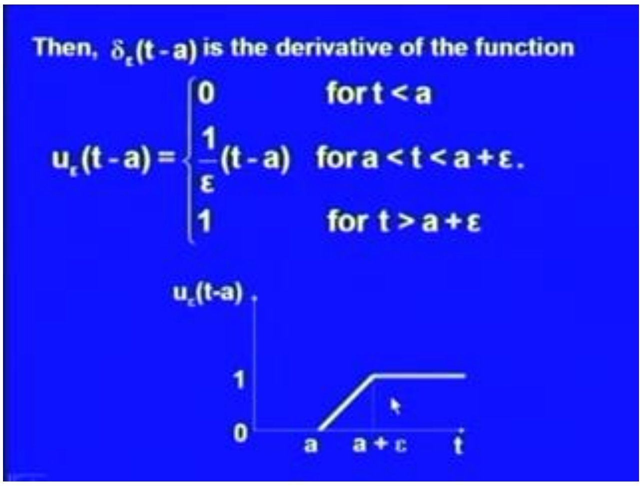
(Refer Slide Time: 23:48)



We are going to show that the Laplace transform of delta t minus a exist and is equal to e to the power minus a s for all a greater than or equal to 0. So, let us consider the function delta epsilon t minus a, which is defined as 1 over epsilon for t bigger than a and less than a plus epsilon and 0 elsewhere. Let us look at the graph of this delta epsilon t minus a function, delta epsilon t minus a function by it is definition can be drawn like this, this is t axis, this delta epsilon t minus a representing that vertical axis.

And then, we have over the interval a to a plus epsilon delta epsilon t minus a taking the value 1 by epsilon and 0 elsewhere.

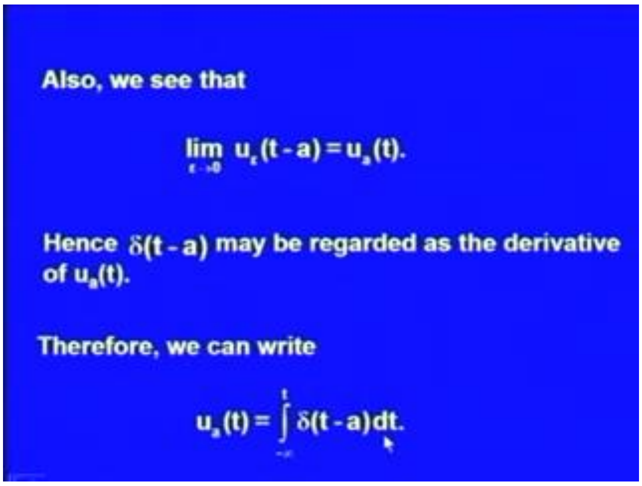
(Refer Slide Time: 24:37)



So, this delta epsilon t minus a is then the derivative of the function u epsilon t minus a, which is defined as 0 for t less than a and 1 by epsilon times t minus a for a less than t less than a plus epsilon. And 1 for t greater than a plus are all, clearly you can see that the derivative of u epsilon t minus a gives us delta epsilon t minus a, so this the graph of u epsilon t minus a is like this.

Over the interval whole for values of t uh less than a, it is taking values 0, over the interval a to a plus epsilon it is given by this line segment 1 by epsilon t minus a having slope 1 by epsilon and then, over the uh values of t greater than a plus epsilon it is taking value 1.

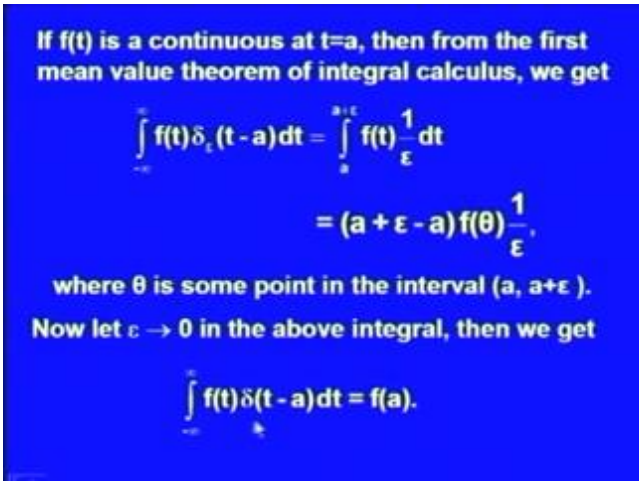
(Refer Slide Time: 25:31)



Further we note that, s epsilon tends to 0, u epsilon t minus a tends to u a (t) when that is the unit step function, which is 0 for t less than a and equals 1 for t bigger than or equal to a. And that is why we can regard delta t minus a as the derivative of u a t, delta t minus a was the limit of u epsilon, delta t minus a was the limit of delta epsilon t minus a, delta epsilon t minus a was the derivative of u epsilon t minus a.

So and the limit of u epsilon t minus a as epsilon tends to 0 is u a t, so we say that delta t minus a is the derivative of u a t. And since it is the derivative of u a t, we can write it as u a (t) equal to integral over minus infinity to t delta t minus a d t.

(Refer Slide Time: 26:29)



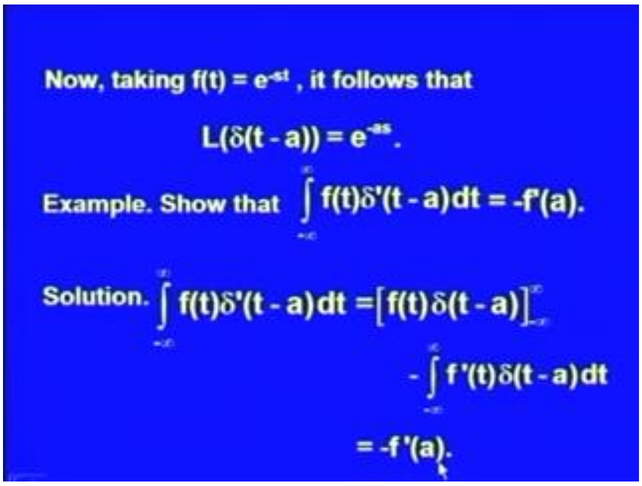
Now, if f is assume to be continuous at t equal to a, then from the mean value theorem of integral calculus, we will be getting integral over minus infinity to infinity f (t) into delta epsilon t minus a d t equal to f (t) into 1 by epsilon d t. Because, delta epsilon t minus a assumes value 1 by epsilon over the interval a to a plus epsilon and elsewhere it is defined as 0, so this integral is equal to integral over a to a plus epsilon f (t) into 1 by epsilon into d t.

Now, let us use the mean value theorem here, if we assume that f is continuous at t equal to a, we can apply the mean value theorem of integral calculus and then, this will be equal to when my epsilon is a constant, we have put the constant out here. And then, a plus epsilon minus a, a plus epsilon minus a into the value of the function at some intermediate point, theta is an intermediate point of the interval a to a plus epsilon.

So, where theta is some point in the interval a to a plus epsilon and then, let us now take let epsilon go to 0, when epsilon goes to 0 we shall have delta epsilon t minus a tending to delta t minus a, so we will have the left hand side tends to integral over minus infinity to infinity f (t) into delta t minus a d t. While the right hand side epsilon will cancel with this epsilon f theta will tend to f a as epsilon tends to 0.

So, the value of the integral of f (t) into delta t minus a over the interval minus infinity to infinity gives us f a, the value of the function f at the point a.

(Refer Slide Time: 28:25)



Now, let us take a particular case, let us assume f (t) to be equal to e to the power minus s t, then it follows that L of delta t minus a is equal to e to the power minus a s. So, if you take f (t) equal to e to the power minus a s, then Laplace transform of delta t minus a, Laplace transform of delta t minus a will be integral 0 to infinity e to the power minus s t into delta t minus a d t, so that will give us e to the power minus s.

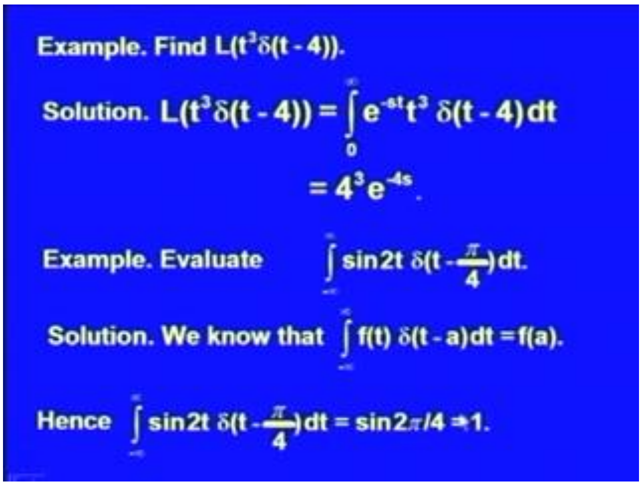
Because, we have shown that integral over minus infinity to infinity f (t) delta t minus a d t is equal to f a, so from that f (t) equal to e to the power minus s t will transform to f a, that is e to the power minus s. Laplace transform of delta t minus a is e to the power minus s, now if you take a equal to 0 here, then we get that Laplace transform of delta t is e to the power 0 that is 1.

So, Laplace transform of delta t function is equal to 1 or we can say that inverse Laplace transform of 1 is equal to delta t, now which is a very useful result. Let us take an example on this derived delta function, let us find the integral of f (t) into delta dash t minus a over the interval minus infinity to infinity, we shall see that it is equal to minus f dash a. Now, making use of the integration by parts, integral over minus infinity to infinity f (t) delta dash t minus a d t can be written as f (t) into delta t minus a valuated at minus infinity and plus infinity.

And then, minus integral over minus infinity to infinity f dash t into delta t minus a d t, now delta t minus a is defined as 0 everywhere except at t equal to a varied text value infinity. So, this will change go to 0, as t goes to infinity or minus infinity and the right hand side will then become... So, we get the integral over minus infinity to infinity f (t) delta dash t minus a d t equal to minus f dash a, because this becomes 0 as t goes to plus infinity and minus infinity.

And this we have earlier seen that, if f (t) is continuous at t equal to a, then integral over minus infinity to infinity f (t) delta t minus a d t is equal to f a, so from that we get the value of this integral as f dash a, so we get the value of the desired integral as minus f dash a.

(Refer Slide Time: 30:59)

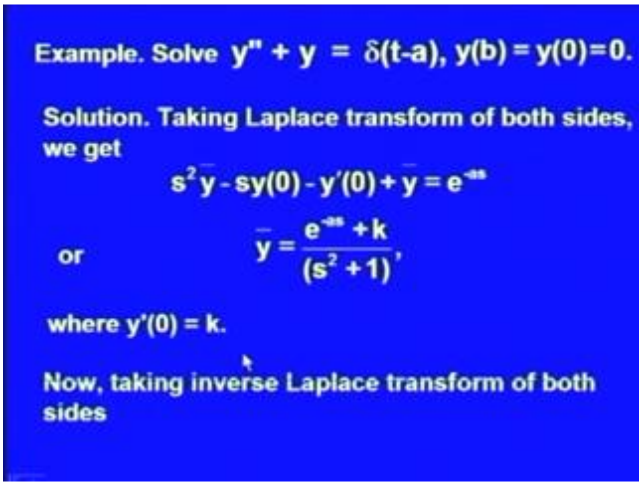


Now, let us find the Laplace transform of t cube into delta t minus 4, so Laplace transform of t cube into delta t minus 4 is equal to integral 0 to infinity e to the power minus s t into t cube into delta t minus 4 d t. Now, so what we have now, we have this can be identified with the integral over minus infinity to infinity f (t) delta t minus a d t, so here a is equal to 4 and with that integral over minus infinity to infinity f (t) delta t minus a d t will then reduce to 0 to infinity.

Because, over the interval minus infinity to 0 delta t minus a will be equal to 0, so that integral if you identify this integral with that, this will be equal to the value of this integral will be equal to e to the power minus 4 s into 4 cube. If this is we can regard this as f t, then it will become f a, a is equal to 4 here, so the value of the integral will be 4 cube e to the power minus a s.

A similar case is here, if you take integral over minus infinity to infinity sin 2 t delta t minus pi by 4 d t, then the value of this integral, again by this property of the Dirac delta function that integral over minus infinity to infinity f (t) delta t minus a d t is f a, will get the value of the given integral. That is integral over minus infinity to infinity sin 2 t delta t minus pi by 4 d t as sin 2 times pi by 4, with this is sin 2 a, a is equal to pi by 4, so sin 2 pi by 4 and which is sin pi by 2 and we know that sin pi by 2 is equal to 1.

(Refer Slide Time: 32:53)

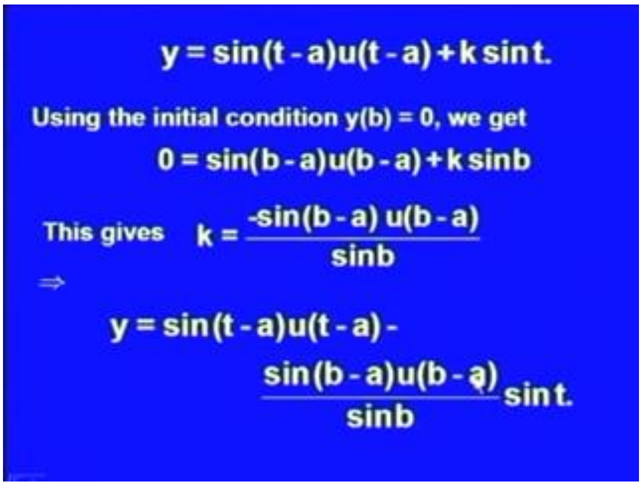


Now, we shall study a example of a differential equation where the derived delta function is used, let us solve the differential equation y double dash plus y equal to delta t minus a where we are given that y b equal to y 0 equal to 0, a and b are some real constants. So, let us take the Laplace transform of the given differential equation, we will get Laplace transform of using linearity of the Laplace transformation, we get Laplace transform of the left hand side as Laplace transform of y double dash plus Laplace transform of y.

So, Laplace transform of y double dash, we know is given by s square y bar minus s y 0 minus y dash 0 and Laplace transform of y is y bar Laplace transform of the right hand side, that is Laplace transform of delta t minus a we have seen, it is given by e to the power minus a s. So, because we are given no value of y dash 0, so let us assume that y dash 0 is equal to k, then we can find the value of y bar solve this equation for y bar.

And we get y bar equal to e to the power minus a s plus y dash 0, that is e to the power minus a s plus k divided by y 0 is given equal to 0, so we get y bar equal to e to the power minus a s plus k over s square plus 1.

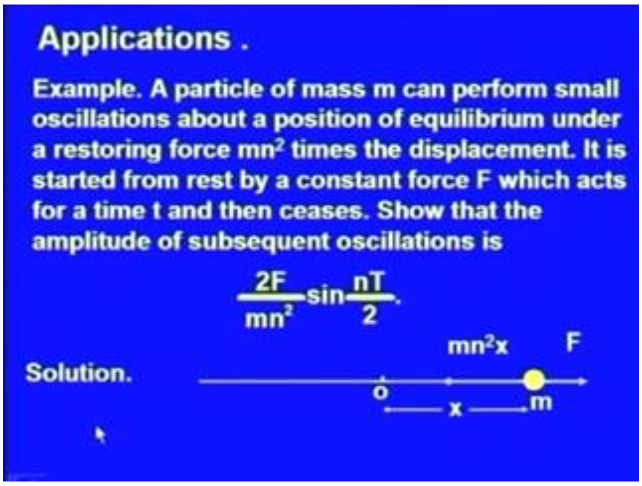
(Refer Slide Time: 34:23)



Now, let us take the inverse Laplace transform of the Laplace transform of y bar, we shall have y equal to sin t minus a into u t minus a plus k sin t, we can then use the initial condition that is y at b is equal to 0. So, when you put t equal to b here, we get 0 equal to sin b minus a into u b minus a plus k sin b and therefore, the value of k is equal to minus sin b minus a into u b minus a over sin b.

And hence, y is equal to sin t minus a into u t minus a minus k, value of k is minus sin b minus u b minus a over sin b, so this value we are substituting here, so y is equal to sin t minus a into u t minus a minus sin b minus a into u b minus a over sin b into sin t.

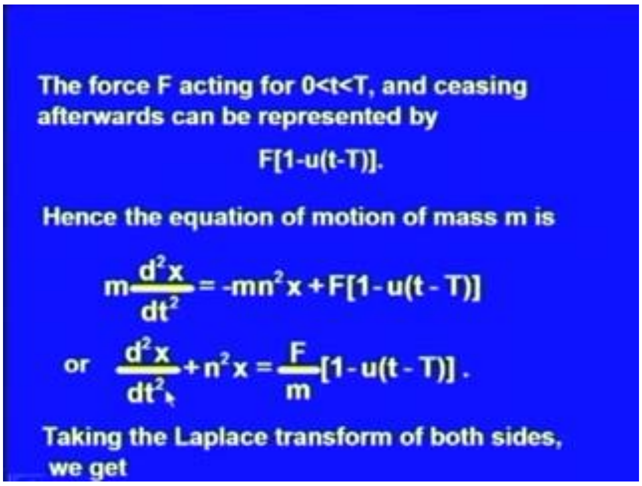
(Refer Slide Time: 35:25)



Now, let us study some applications of the Laplace transformation, first we study the applications of the Laplace transformation to the problems in dynamics. Let us consider the case of a particle of mass m, which can perform a small oscillations about a position of equilibrium under a restoring force m n square times the displacement. It is started from rest by a constant force F which acts for a time t and then ceases, show that the amplitude of subsequent oscillations is given by 2 F over m n square into sin n T by 2.

So, let us say this is the position of equilibrium at o the particle was here, at time t the particle is here the mass of the particle is m and at time t it is at a distance x from the position of the equilibrium that is o, acted upon by forces, this ((Refer Time: 36:27)) forces m n square x and this is the force F, which is acting for a time t.

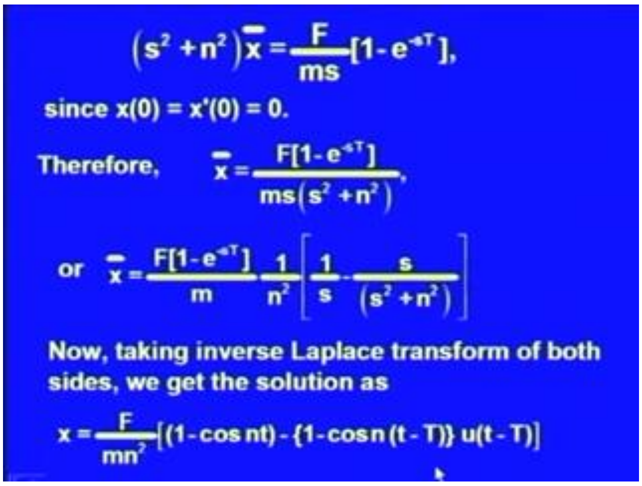
(Refer Slide Time: 36:33)



So, the force F which is acting for the time t only can be represented as and then, it is ceases afterwards can be represented in terms of unit step functions as F into 1 minus u t minus T. And therefore, the equation of the motion of the mass m will be equal to m x double dot, that is m into d square x by d t square equal to resultant of the a force F, which is acting for the time t only and then, the restoring force.

So, we get m d square x by d t square equal to minus m n square into x plus F times 1 minus u t minus T, now dividing by m we get d square x by d t square plus n square x equal to F by m into 1 minus u t minus T. Let us now take the Laplace transform of both sides, Laplace transform of x double dot or x double dash will give us s square x bar minus s x 0 minus x dash 0. X 0 is equal to 0, because at the time t equal to 0 the particle was at the point o and it was at rest, so x dash 0 is also 0.

(Refer Slide Time: 37:57)



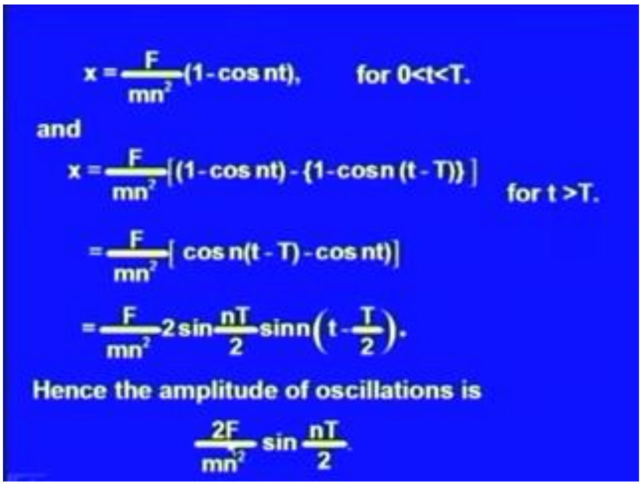
So, making use of those initial conditions we get the Laplace transform of the equation of motion as s square plus n square into x bar equal to F over m s into 1 minus e to the power minus s T. The Laplace transform on the right hand side we get, the Laplace transform of 1 over s and Laplace transform of u t minus T s e to the power minus s T over s, so the Laplace transform of the right hand side gives us F over m s into 1 minus e to the power minus s T.

After we make use of the given initial conditions that is x at t equal to 0 is 0 and x dash at t equal to 0 is 0, now solving for x bar,, we get x bar equal to F into 1 minus e to the power minus s T over m s into s square plus n square. Let us break it now into it is partial fractions, we can write x bar further as F into 1 minus e to the power minus s T over m and then, 1 over s into s square plus n square when we break into partial fractions, we get 1 over n square times 1 over s minus s over s square plus n square.

Now, in order to find the displacement of the particle m from the position of equilibrium, in order to find x let us take the inverse Laplace transform of this equation, so then we will get x equal to... Now, F over m n square, F over m n square is a constant, so we have kept it like that only F over m n square, then this 1 multiplied by 1 over s minus s over s square plus n square. When we take the inverse Laplace transform of that, we get inverse Laplace transform of 1 over s as 1, inverse Laplace transform of s over s square plus n square as cos n t.

So, we get 1 minus cos n t the inverse Laplace transform of this expression, so 1 into this whole thing, when we take the inverse Laplace transform of that we get 1 minus cos n t and then, minus this minus is here then e to the power minus s T multiplied by this. Then we take inverse Laplace transform of e to the power minus s T into this expression by second shifting theorem will be equal to 1 minus cos n t minus T into u t minus T.

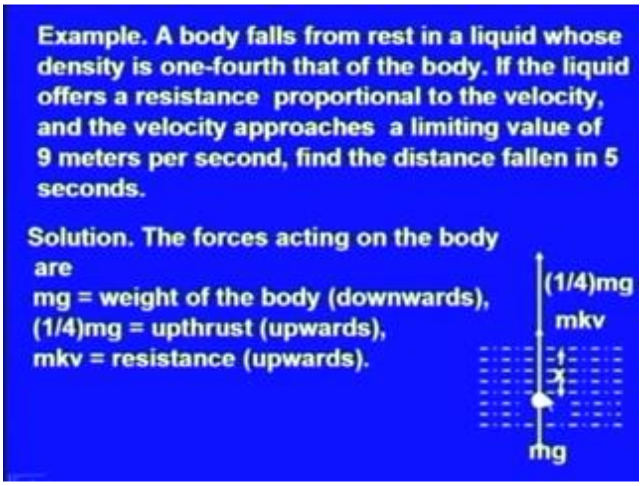
(Refer Slide Time: 40:17)



Now, we can study the various cases here, when t will be more than 0, but less than capital T, u t minus T the unit step function u t minus T will take value of 0, so x will be equal to F over m n square into 1 minus cos n t. And when t will take value more than capital T u t minus T will be equal to 1, so the displacement x will be equal to F over m n square into 1 minus cos n t minus 1 minus cos n t minus T.

And this can be simplified further as F over m n square into cos n t minus T minus cos n t, now making use of the formula cos a minus cos b as 2 sin a plus b by 2 into sin b minus a by 2, we get this expression further equal to F over m n square into 2 sin n T by 2 into sin n t minus T by 2. And thus the amplitude of the oscillations of the mass m about the position of equilibrium will be equal to 2 F by m n square into sin n T by 2, now let us study another example on dynamics, where a body falls from rest in a liquid.

(Refer Slide Time: 41:35)

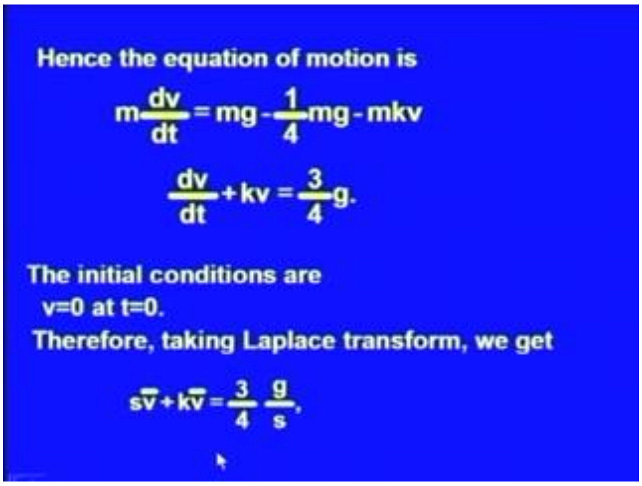


The density of the liquid is one fourth of that of the body, if the liquid offers a resistance proportional to the velocity of the body, and the velocity approaches the limiting value of 9 meters per second, then let us find the distance which the body falls in 5 seconds. So, this is the liquid and this is the body of mass m, after it has fallen in the liquid mg is the weight of the body acting downwards.

1 by 4 times mg, 1 by 4 times mg the body falls whose density of the liquid is one fourth of the density of the body, so we get 1 by 4 mg acting upwards and then, m k v 1 by 4 mg is the up thrust, which is acting upwards. And then, m k v we are taking the resistance is proportional to the velocity for convenience, we are writing resistance as m k into v, v is the velocity, the constants of proportionality we are writing as m into k.

So, that we may cancel this m from the equation, we can divide that equation by m, so by for convenience we are writing the resistance equal to m k v, so m k v is acting upwards 1 by 4 mg is acting upwards, which is the up thrust. And then, mg which is we get of the wave body is acting downwards, so these are the forces that are acting on the body at time t after it has fallen a distance x from the surface of the water, the x is measured from the surface of the liquid, so this is the situation after and a at an instant t.

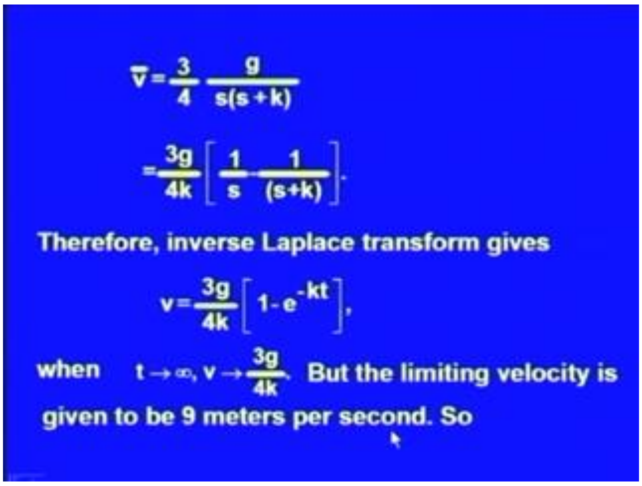
(Refer Slide Time: 43:32)



And thus the equation of motion will be given by m into d v by d t equal to mg minus 1 by 4 mg minus m k v, this is the resultant force acting on the body and the m into d v by d t is therefore equal to the resultant force, which is gives us the equation as d v by d t plus k v equal to 3 by 4 g, after we divide the equation by m and simplify it. The initial condition is that at t equal to 0, the body was at rest that is v was equal to 0.

And let us now take the Laplace transform of this equation, when we take the Laplace transform here, the Laplace transform of d v by d t will give us s into v bar minus v 0, v 0 is equal to 0. And then k times v bar and then, 3 by 4 into g is a constant then L of 1, L of 1 is 1 by s, so this is what we get after taking the Laplace transform of this equation and we can then find v bar from here.

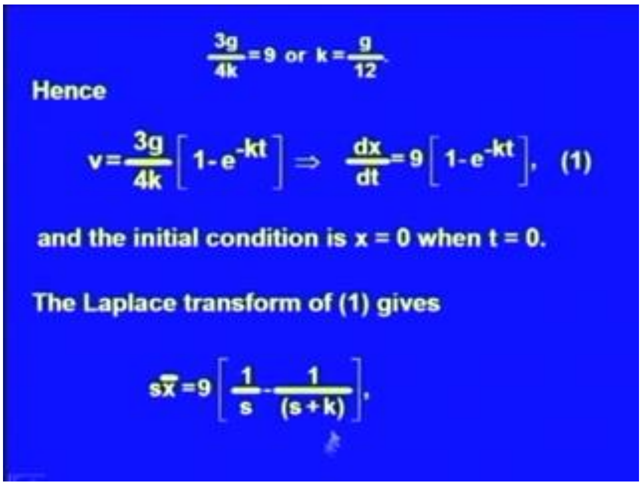
(Refer Slide Time: 44:42)



V bar give will give us 3 by 4 into g by s into s plus k, let us break it into it is partial fractions v bar will be 3 g by 4 k into 1 by s minus 1 by s plus k. And when we take the inverse Laplace transform of this equation, we shall have the value of v, which is 3 g by 4 k into 1 minus e to the power minus k t; inverse Laplace transform of 1 by s is 1 an inverse Laplace transform of 1 over s plus k is e to the power minus k t.

Now, when t tends to infinity e to the power minus k t will go to 0, so v will tend to 3 g by 4 k, the limiting velocity of the body is therefore 3 g by 4 k, but we are given that the limiting velocity is 9 meters per second.

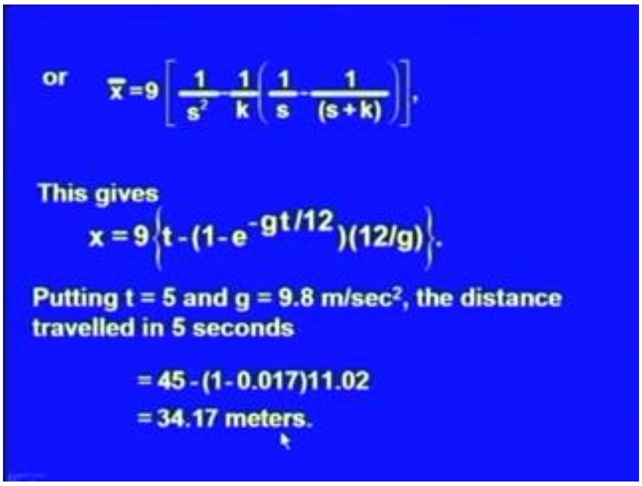
(Refer Slide Time: 45:33)



So, thus we have 3 g by 4 k equal to 9 or we have k equal to g by 12 and hence, v equal to 3 g by 4 k into 1 minus e to the power minus k t implies that, d x by d t is equal to 9 into 1 minus e to the power minus k t, we know v is equal to d x by d t and 3 g by 4 k is equal to 9. So, we get the differential equation again, d x by d t equal to 9 times 1 minus e to the power minus k t, let us call this as equation number 1.

And the initial condition is that where t equal to 0, the body was at the surface of the liquid that is x is equal to 0. So, in our Laplace transform of, if you take the Laplace transform of equation 1, we are going to solve it for x, so then we will have s x bar minus x 0, x 0 is equal to 0. And the right hand side will give as 9 times Laplace transform of 1 as 1 over s minus Laplace transform e to the power minus k t as 1 over s plus k.

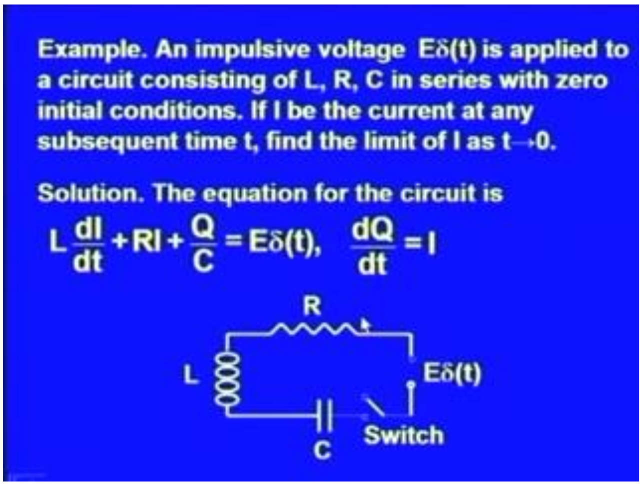
(Refer Slide Time: 46:45)



Or we can say that x bar is equal to 9 into 1 over s square minus 1 by s into 1 by s minus 1 over s plus k, after we break it into partial fractions. So, the inverse Laplace transform of this equation, then gives as x equal to 9 times, inverse Laplace transform of 1 over s square is t and then, we have minus 1 over k. 1 over k is equal to twelve over g this is the value of 1 over k and then 1 over s, inverse Laplace transform 1 over s is 1, inverse Laplace transform of 1 over s plus k is e to the power minus k t and k is equal to g by 12; so we get e to the power minus k t as e to the power minus g t by 12.

Now, let us put t equal to 5 and g equal to 9.8 meters per second square in this expression to find the value of x, which will give us the distance travelled by the body in 5 seconds, so we get x equal to 9 into 5 that is 45. Then we calculate the value of this, 1 minus e to the power minus g is 9.8 t is 5 divided by 12, the value of this expression comes out to be 1 minus 0.017 and then, 9 times 12 by g gives us 11.02, so we get the value of x as 34.17 meters.

(Refer Slide Time: 48:20)



Now, let us study some problems on the simple electric circuit, a simple electric circuit consists of a resistance given by R, inductance L and capacitance C that which are connected in series. And a switch is provided to connect or disconnect the circuit and then, a source of electric motive power is also there, so then we know that the if Q is the charge on the condenser at time t and I denotes the current in the circuit at time t.

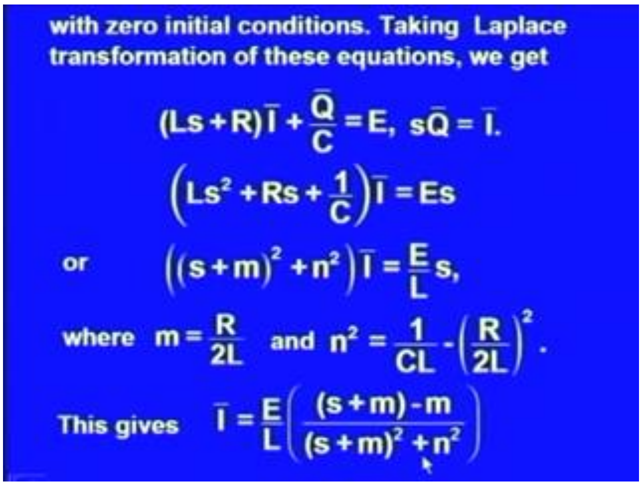
Then d Q by d t is equal to I and the voltages developed across the resistance inductance and capacitance are given by L d I by d t is the voltage developed across the inductance, R I is the voltage developed across the resistance. And Q by C is the voltage developed across the capacitance, which is equal to the electric motive source of which is provided in the given circuit.

Here in this problem it is E delta t, so by Kirchhoff's law the L d I by d t plus R I plus Q by C will be equal to E delta t and d Q by d t will be equal to I, now we are further given that, here it is also assume that the connective wires I have negligible resistance. So, if I is the current at any times subsequent time t we have to find the limit of I as t tends to 0, now let us solve these equations using the Laplace transformation method.

So, when we take the Laplace transform of this equation will get this L is a constant, so L times Laplace transform of d I by d t and Laplace transform of d I by d t will give us s into I bar minus I 0. I 0 is equal to 0, because we have 0 initial conditions, so at t equal to 0 the charge on the condenser is 0 and at t equal to 0, the current in the circuit is also 0, so we get L into s into I bar plus R times I bar plus Q bar over C.

C is the capacitance which is constant, the Laplace transform of Q will be Q bar equal to E times Laplace transform of delta t we have seen earlier, that it is equal to 1, so we get Laplace transform of E delta t as E. And Laplace transform of d Q by d t will be s into Q bar minus Q 0, Q 0 is equal to 0 and which will be equal to I bar; so this equation, when we take the Laplace transform of this equation will get s Q bar equal to I bar.

(Refer Slide Time: 51:34)



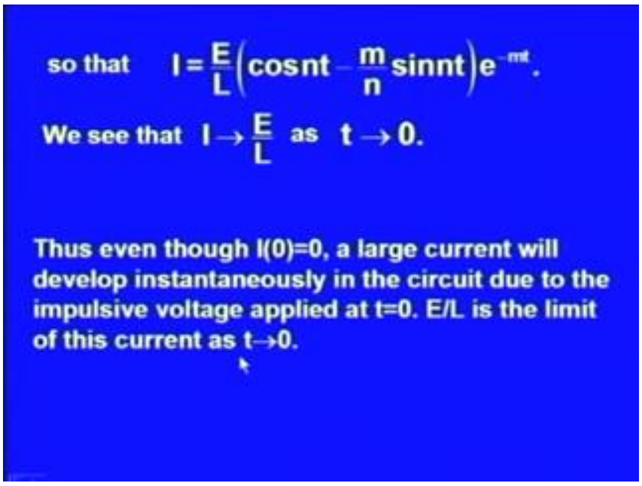
And hence, we get the equations, following equations L s plus R into I bar plus Q bar by C equal to E and s Q bar equal to I bar. Now, substituting the value of Q bar from here, as I bar over s in this equation, we will have the following L s square plus R s plus 1 by C into I bar equal to E s, which can be expressed as s plus m whole square plus n square into I bar equal to E by L into s, after dividing by L.

Where m is equal to R by 2 L and n square is equal to 1 by C L minus R by 2 L whole square, and we can then is express this equation as I bar equal to E by L, s we can write as s plus m minus m over s plus m whole square plus n square. We have added and subtracted m here, in order to take the inverse Laplace transform of this equation, so when you take the inverse Laplace transform of this equation, inverse Laplace transform of I bar will give us I, E by L is a constant.

And then, inverse Laplace transform of this expression will be inverse Laplace transform of s plus m over s plus m whole square plus n square minus inverse Laplace transform of m over s plus m whole square plus n square. Inverse Laplace transform of s plus m over s plus m whole square plus n square will be e to the power minus m t into cos n t, because Laplace transform of cos n t is s over s square plus n square.

And here s is being represent by s plus m, so by first shifting theorem inverse Laplace transform of this will be e to the power minus m t into cos n t. And similarly, inverse Laplace transform of m over s plus m whole square plus n square will be e to the power minus m t into sin n t.

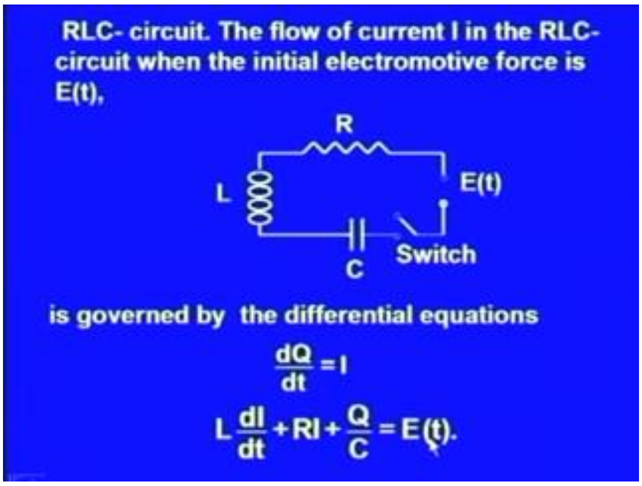
(Refer Slide Time: 53:35)



So, after taking inverse Laplace transform, we get I equal to E by L into cos n t minus m by n into sin n t into e to the power minus m t and then, have s t tends to 0, the limit of I is E by L. Thus we have the following remark that even though at initially the current was 0 in the circuit I 0 was equal to 0, a large current develops instantaneously in the circuit due to the impulsive voltage, which is applied at t equal to 0.

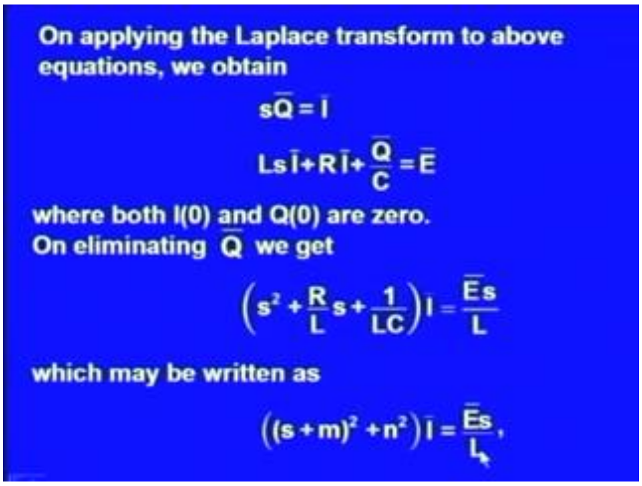
And we are finding the limit that is E by L, the E by L is the limit of this current which develops instantaneously in the circuit due to the impulsive voltage, so E by L is the limit of this current as t tends to 0.

(Refer Slide Time: 54:28)



Now, let us take one more example on this RLC circuit, the flow of current I in the RLC circuit, when the initial electromotive force is E t is governed by the differential equations d I by d Q by d t equal to I and L d I by d t plus R I plus Q by C equal to E t.

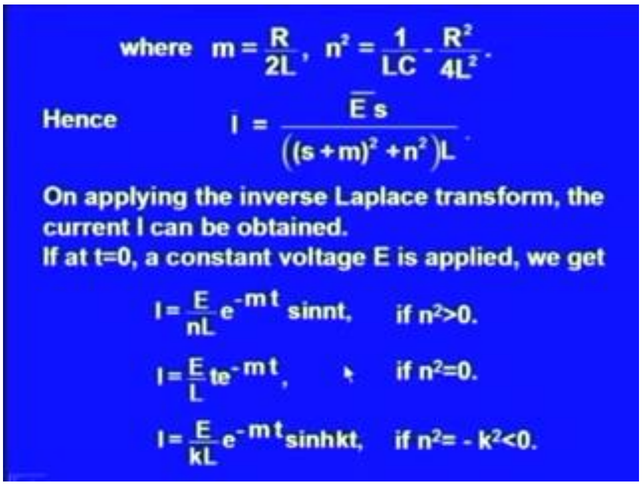
(Refer Slide Time: 54:50)



And when we take the Laplace transforms of the above equations we get s Q bar equal to I bar plus R I bar plus Q bar by C equal to E bar. So, here we are then studying a general case, where E t denotes in a electromotive source of voltage, I mean above E t denotes the voltage of the electromotive source.

So, here we again we are assuming that I and Q both are 0 at t equal to 0, and when we eliminate Q bar from these two equations, or we put the value of Q bar as I bar over s from this equation into this equation; we will get s square plus R by L into s plus 1 by L C into I equal to E bar s over L. So, E bar is the Laplace transform of E t, so this can be written as s plus m whole square plus n square into I bar equal to E bar s by L.

(Refer Slide Time: 55:51)



And where m is R by 2 L, n square is 1 by LC minus R square by 4 L square as we had seen earlier and therefore, I bar is equal to E bar s over s plus m whole square plus n square into L, after solving for I bar. And when we take the inverse Laplace transform of this equation, we can then find the value of the current I, if at t equal to 0, now let us study some particular cases of the electromotive source of voltage E t.

If we assume that at t equal to 0 a constant voltage E is applied, that is E t is equal to a constant E, then we will get I as after taking inverse Laplace transform E t is equal to E, so E bar will become E by s and therefore, E bar s will be equal to E, so we will have E here. And then, inverse Laplace transform of, so 1 over s plus m whole square plus n square will be e to the power minus m t into sin n t by n and so I will be equal to E upon n L into e to the power minus m t into sin n t.

Now, if 1 by L C minus R square by 4 L square is equal to 0, that is if R, L and C have value such that 1 by L C becomes equal to R square by 4 L square, we will have the inverse Laplace transform of this equation as... Because, in that case n square will be 0, so we will have I bar equal to E bar s over s plus m whole square into L, so when we will take the inverse Laplace transform of I bar will be I.

And we will get E bar s, E bar s is equal to E so we will get E here, L is a constant will come like that and then, 1 over s plus m whole square, inverse Laplace transform of 1 over s square is equal to t, so by first shifting theorem inverse Laplace transform of 1 over s plus m whole square will be equal to t e to the power minus m t. And if it is so happens that R, L and C have values such that, 1 by LC becomes less than R square by 4 L square.

Then, the n square which has occurring here will be replaced by minus k square and will then, have I bar equal to E bar s over s plus m whole square minus k square into L. So, then we will take the inverse Laplace transform, we shall get I equal to E upon K L into e to the power minus m t into sin hyperbolic k t. Because, we know that the Laplace transform of sin hyperbolic k t is k over s square minus k square, so when we multiply by e to the power minus m t and take the Laplace transform will get k over s plus m whole square minus k square.

And so inverse Laplace transform here, we will give us I equal to E by k L into e to the power minus m t into sin hyperbolic k t, if n square is equal to minus k square and where we are assuming that 1 by L C minus R square by 4 L square is less than 0. Now, in our next lecture we shall discuss some more applications of the Laplace transformation, like the application of Laplace transformation to bending of beams.

And the application of Laplace transform to the boundary value problems, which are occur in the Engineering Mathematics will find the solution of the heat conduction equation by using the Laplace transformation method.

~~,~~ T~~t~~hank you.