**Mathematics - III**
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**Indian Institute of Technology, Roorkee**

**Lecture - 1**
# Solution of Ode of First Order and First Degree

Dear viewers, I am Dr. P. N. Agrawal professor in Department of Mathematics, IIT, Roorkee. In my lecture, I am going to discuss the various methods of obtaining a general solution of ordinary differential equation of first order and first degree.

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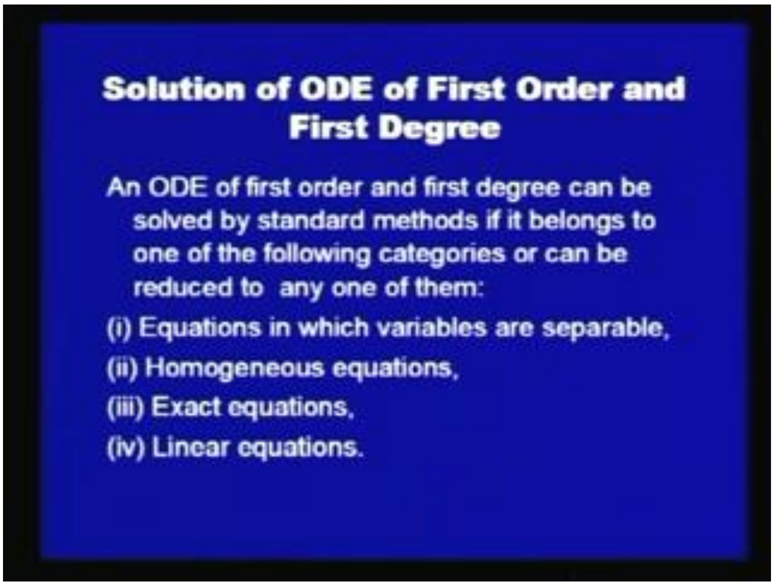


In ordinary differential equation of first order and first degree can be solved by standard methods, if it belongs to one of the following categories or can be reduced to any one of them. First equations in which variables are separable, second in homogeneous equations, third exact equations, fourth is linear equations. All other differential equations are first order and first degree, which do not belong to any one of those categories can be solved using numerical methods, which give us an approximate solution of the given differential equation. We shall talk about approximate methods, which gives us solutions of those differential equations later on. First we shall discuss those differential equations where, which can be solved analytically, so we begin with equations in which variables are separable.

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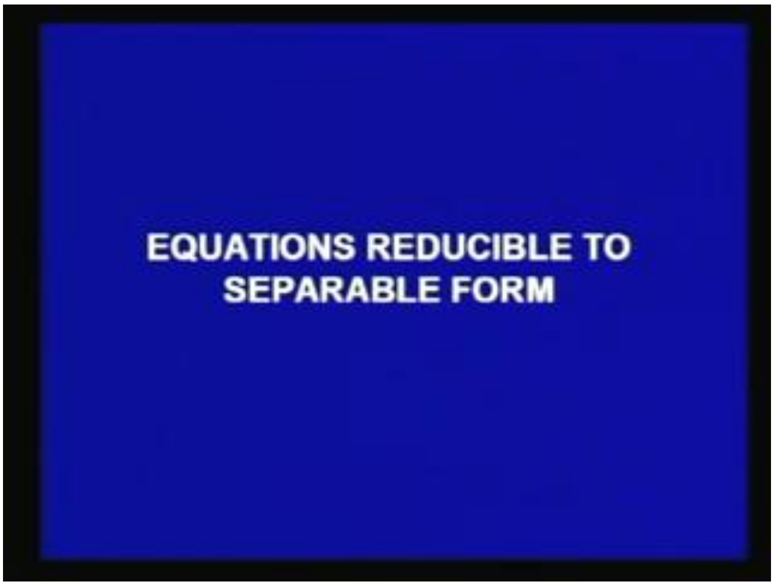


Equations reducible to separable form.

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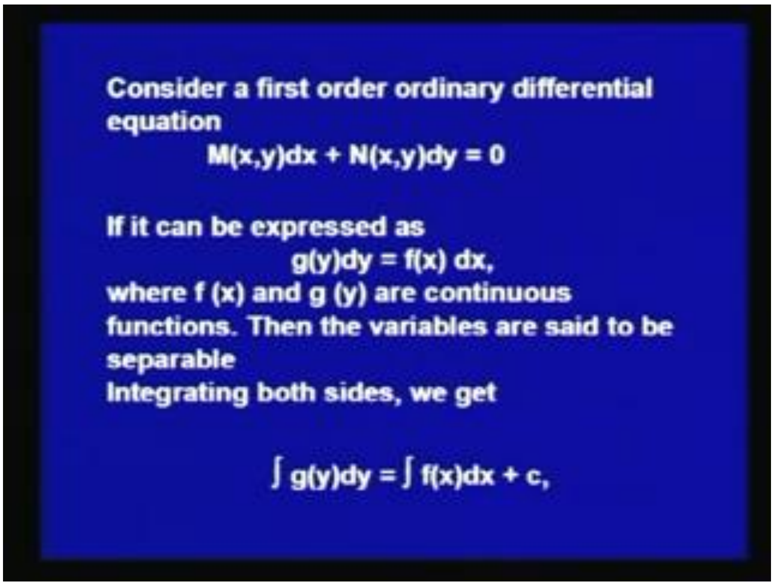


Let us begin with a ordinary differential equation of first order in first degree, it can be expressed as M d x plus M d y equal to 0 where M and N are functions of x and y. Now, if we are able to express it in the form g y d y equal to f x d x, where g y is the function of y only and f x is the function of x only and f x and g y are continuous functions. Then we say that the given differential equation belongs to the category of those differential equations where variables are separable.

General solution of such a differential equation can be obtained very easily, we just have to integrate both sides of the given differential equation and we have integral g y d y equal to integral f x d x plus c.

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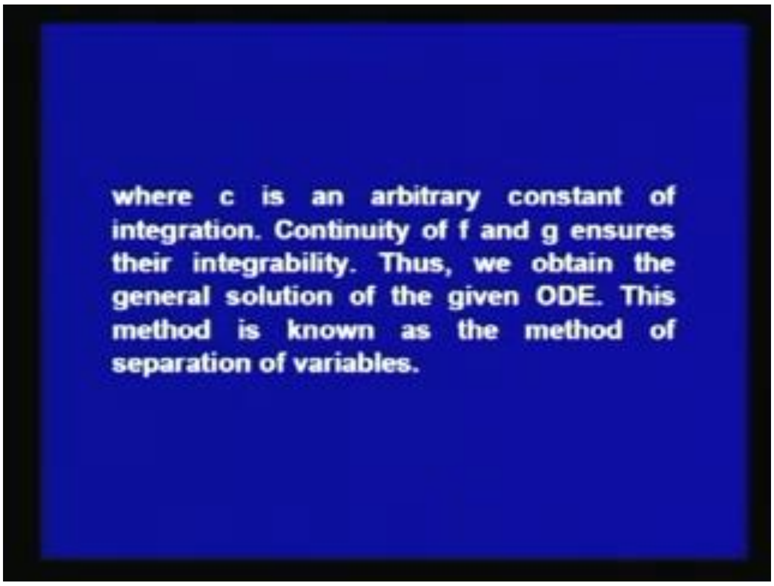


Where c is an arbitrary constant of integration, since f and g are continuous functions they are integrable. So, the integral of f and integral of g axis can be obtained the general solution of the given ordinary differential equation, this method is known as the method of separation of variables.

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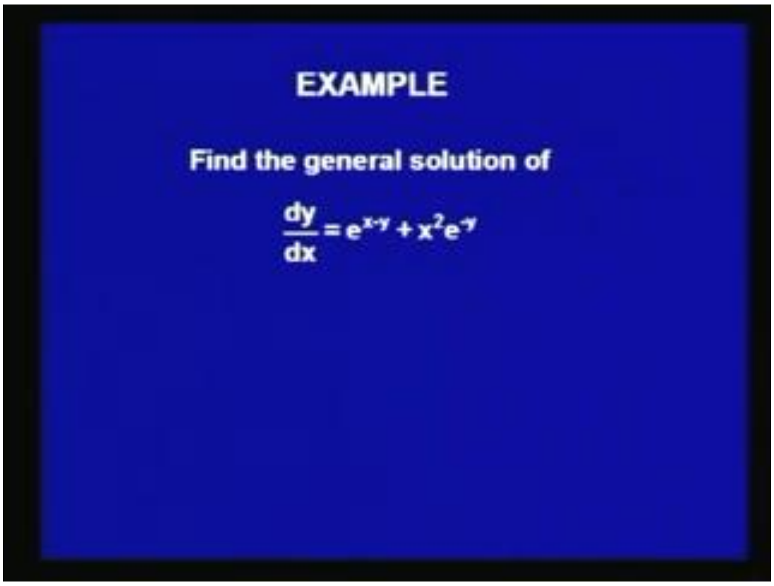

Let us consider an example of a first order ordinary differential equation of a degree, which belongs to this category of variable separable form. Let us take d y by d x equal to e to the power x minus y plus x square into e to the power minus y.

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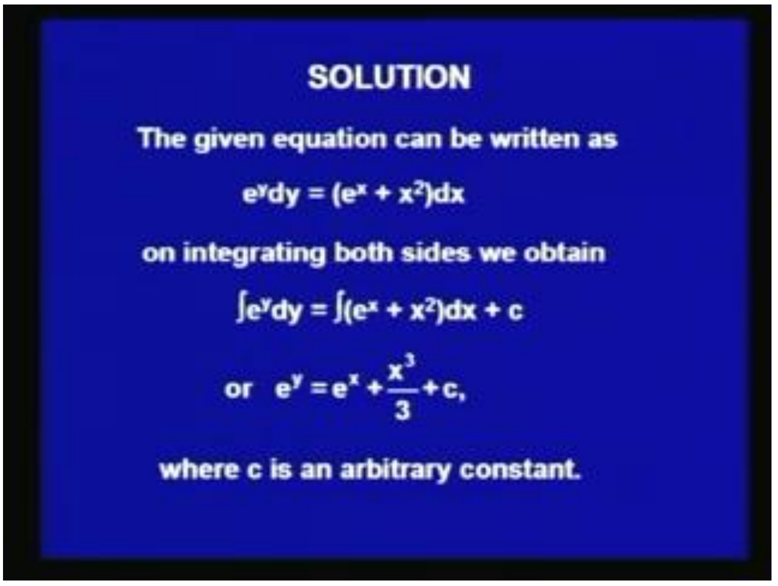


Now, this differential equation can be written as e to the power y d y equal to e to the power x plus x square into d x, when do you multiply the given differential equation by e to the power y and then by d x be arrive at this equation. And we can know that on the left side we have a function of y only, while on the right side we have a function of x only. And therefore, when we integrate both sides of this differential equation, we have integral e to the power y d y equal to integral of e to the power x plus x square into d x plus; or we will have e to the power y equal to e to the power x plus x cube over 3 plus c, where c is an arbitrary constant.

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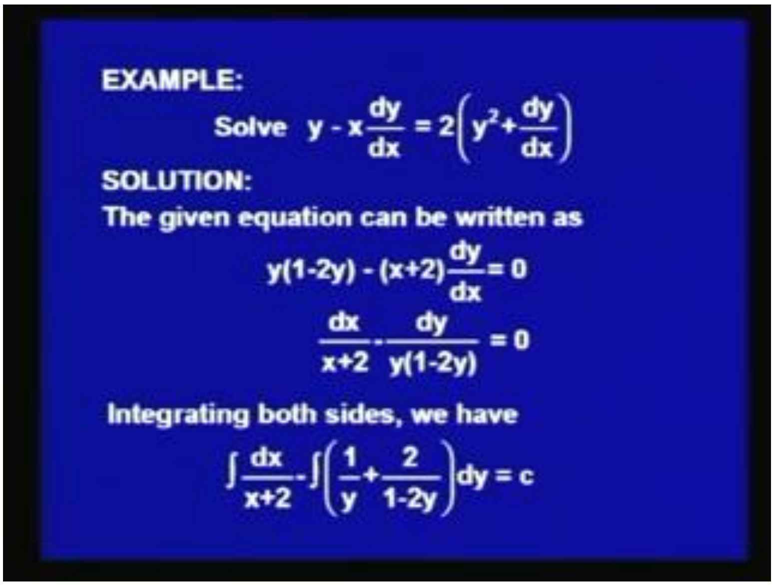


Let us take up another problem, which is y minus x into d y by d x equal to 2 times y square plus d y by d x, now this equation can be rewritten as y into 1 minus 2 y minus x plus 2 into d y by d x equal to 0. Or we may write it as d x over x plus 2 minus d y over y into 1 minus 2 y equal to 0, now you can see that the variables x and y have again being separated. First term is a function of x only, while the second term is a function of y only and with this equation again belongs to the case of differential equations where the variables are separable. We integrate both sides and get integral d x over x plus 2 minus integral of 1 over y plus 2 over 1 minus 2 y d y equal to c. Here we have the broken we term 1 over y into 1 minus 2 y into partial fractions, the partial fractions are 1 over y plus 2 over 1 minus 2 y.

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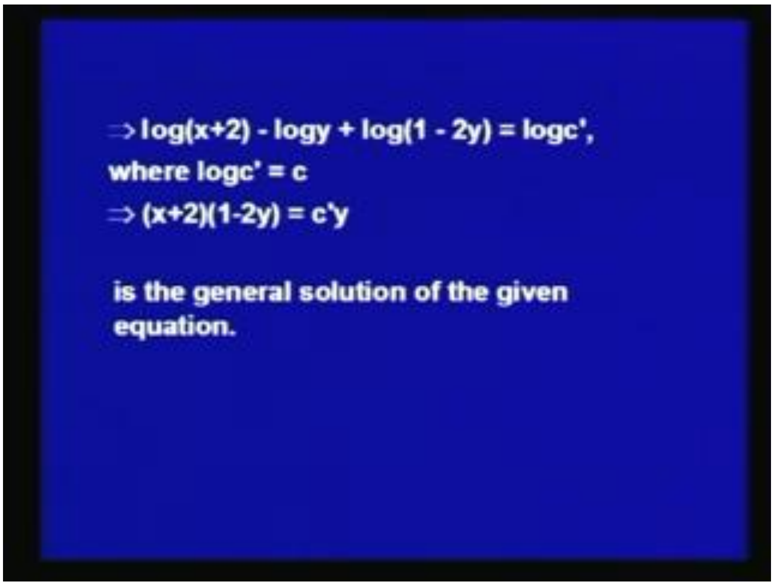


Now, after integration we get log of x plus 2 minus log y plus log 1 minus 2 y equal to log c dash, where we define log c dash equal to c. This will give us x plus 2 into 1 minus 2 y equal to c dash y, which is then the general solution of the given differential equation.

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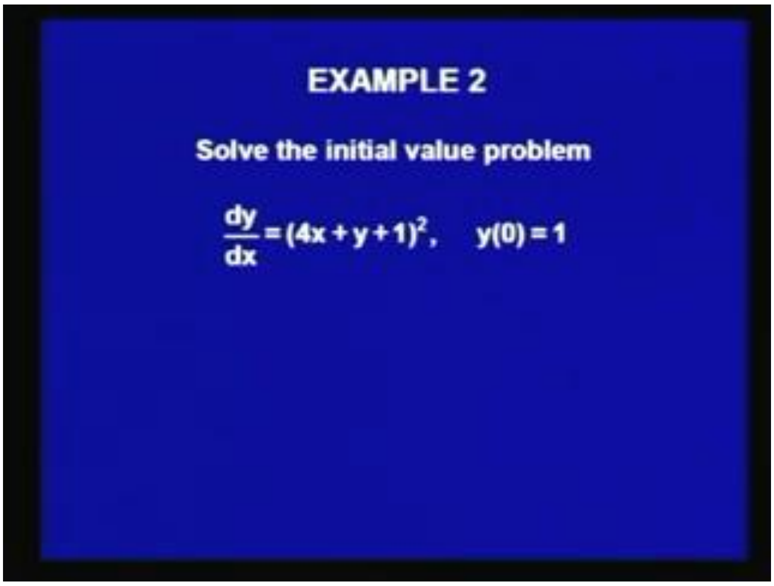


Now, let us consider the initial value problem d y over d x equal to 4 x plus y plus 1 whole square where y 0 equal to 1. And initial value problem is 1, where the value of the given depended variable y is given at an initial value of x, that is x equals to 0.

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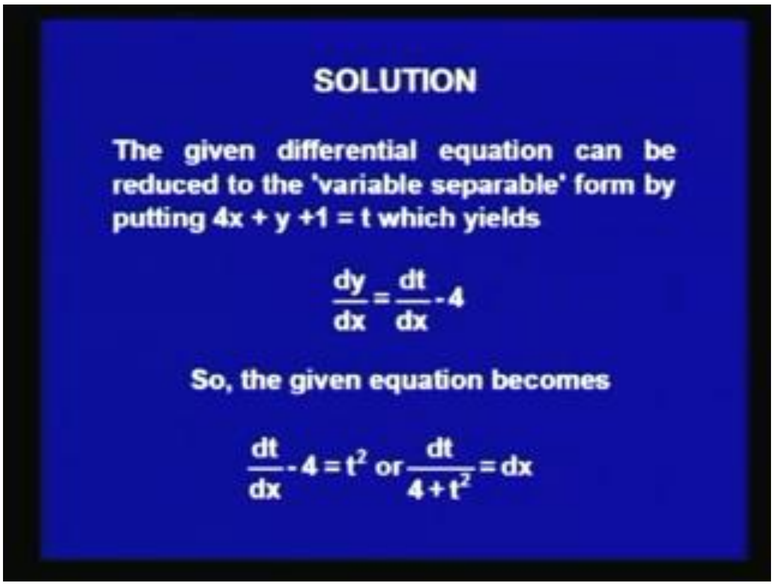


In the case of this differential equation, we substitute 4 x plus y plus 1 equal to t, in order to bring it to the variable separable form, because then d y by d x will be equal to d t by d x minus 4. And so we will be getting the given equation in the form d t by d x minus 4 equal to t square or d t by 4 plus t square equal to d x. You can see now the variables x and t have been separated on the left hand side we have a function of t only, and on the right hand side we have a function of x.

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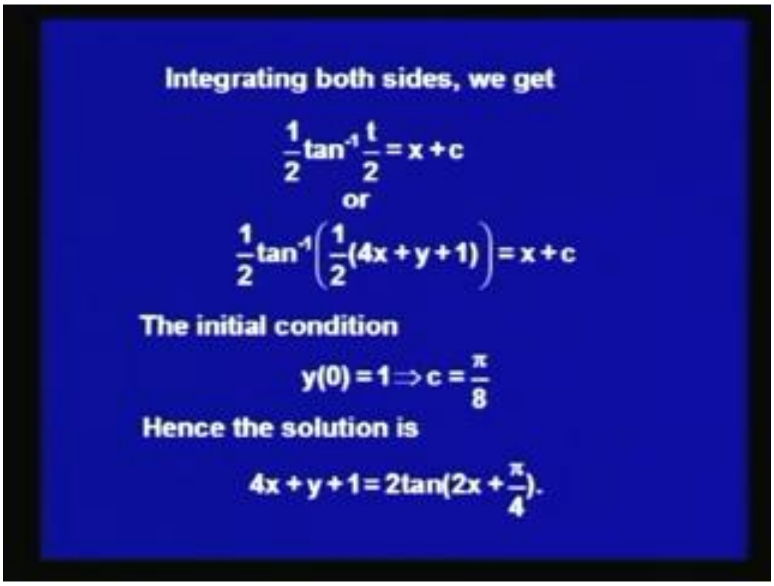

So, after integrating those sides we get half of tan inverse t by 2 equal to x plus c, which is equal to half tan inverse half 4 x plus y plus 1 equal to x plus c. Now, when we make use of the initial condition y 0 equal to 1, we get c equal to phi by 8 and thus the solution is 4 x plus y plus 1 equal to 2 tan 2 x plus phi by 4.

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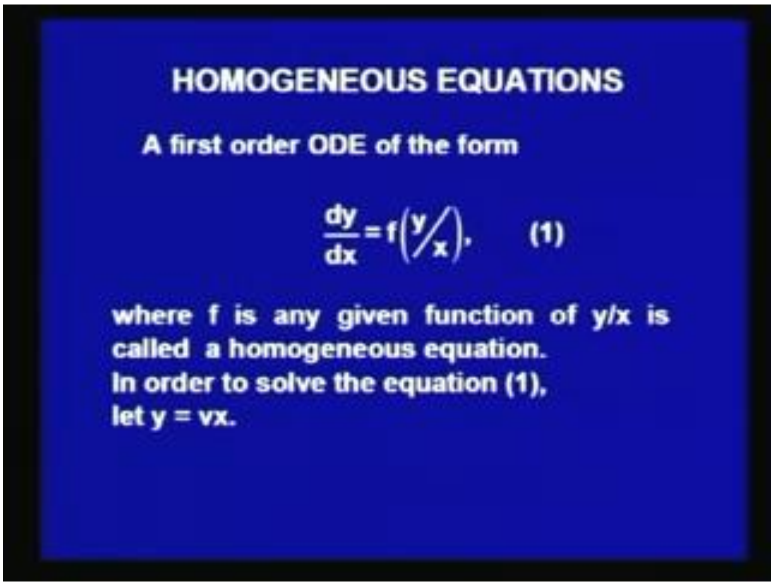


Let us now take up another case of differential equations, ordinary differential equations of first order first degree, that is homogeneous equations. If a first order ordinary differential equation of first degree is of the form d y by d x equal to f of y over x, where f is any given function of y over x. We shall call such a differential equation as a homogeneous equation, in order to solve such a differential equation we put y equal to v into x.

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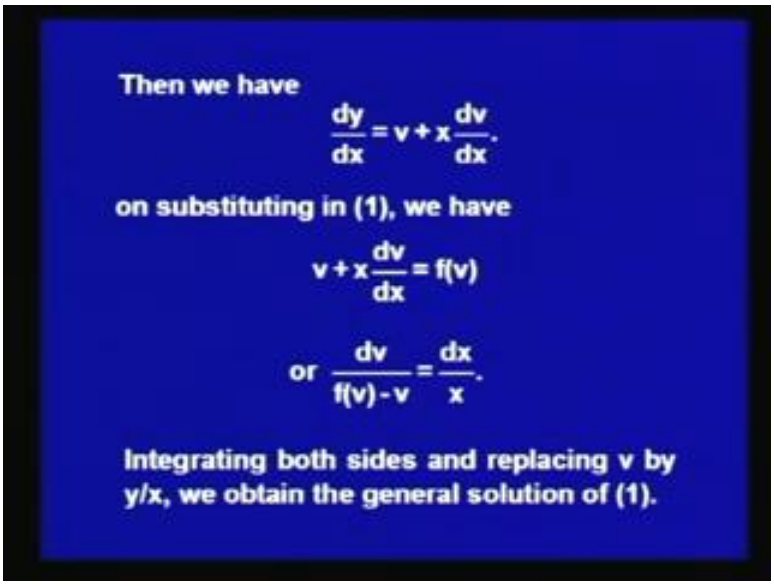


Then d y by d x will be equal to v plus x d v by d x and when we substitute it in the equation 1, we shall have v plus x d v by d x equal to f v or we can separate the variables x and v and get d v over f v minus v equal to d x over x. Now, we can integrate both sides of this differential equations and then replace v by y over x to obtain the general solution of the given differential equation.

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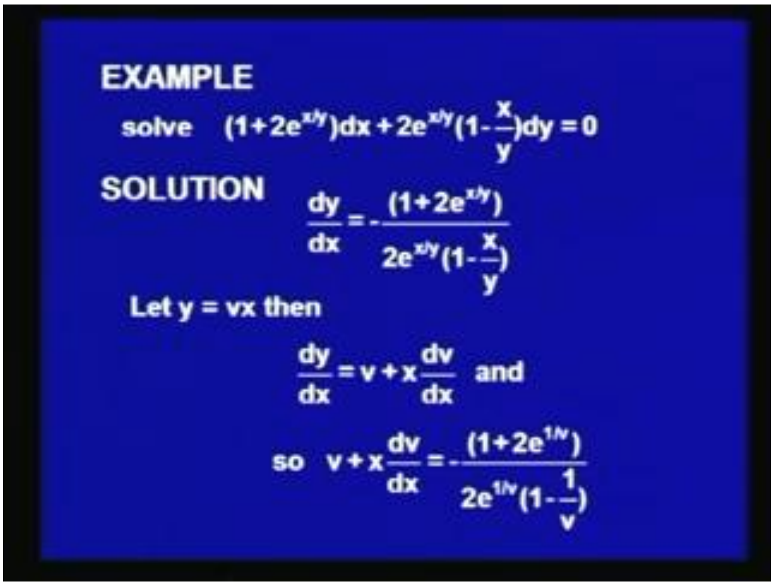


Let us take an example of a differential equation belonging to this type, let us consider 1 plus 2 times e to the power x over y d x plus 2 time e to the power x over y into 1 minus x over y d y equal to 0. You can see here, that this differential equation may be written as d y by d x equal to minus 1 plus 2 times e to the power x over y divided by 2 times e to the power x over y into 1 minus x over y. The right hand side of this equation is the function of y over x, so let us put y equal to v x, then we shall have v plus x d v by d x equal to minus 1 plus 2 times e to the power 1 over v divided by 2 times e to the power 1 over v into 1 minus 1 over v.

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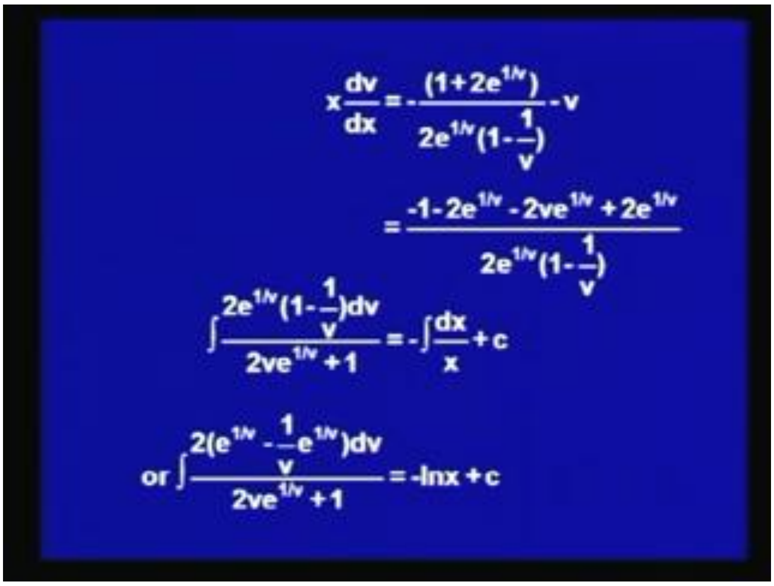


We can simplify this equation and get x into d v by d x equal to minus 1 plus 2 times e to the power 1 over v over 2 times e to the power 1 over v minus into 1 minus 1 over v minus v; which will give us after simplification minus 1 minus 2 e to the power 1 over v minus 2 times v into e the power 1 over v plus 2 times e to the power 1 over v divided by 2 times e to the power 1 over v into 1 minus 1 over v.

Now, we can bring the variables v on 1 both side and the variable x on the other side and then integrate both sides. So, we shall have integral of 2 e to the power 1 over v into 1 minus 1 over v d v divided by 2 v into e to the power 1 over v plus 1 equal to minus integral of d x over x plus c; now the denominator 2 v e to the power 1 over v plus 1 when we put equal to t.

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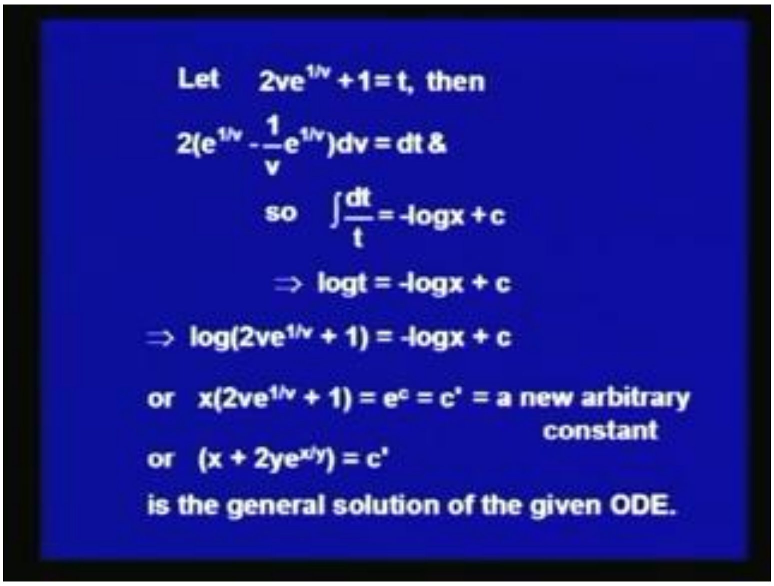


We note that we get the derivative as 2 times e to the power 1 over v minus 1 over v into e to the power 1 over v into d v equal to d t and so the equation becomes integral d t by t equal to minus log x plus c. The integration gives us log t equal to minus log x plus c or we may say log of 2 v into e to the power 1 over v plus 1 equal to minus log x plus c, which will further give x into 2 v e to the power 1 over v plus 1 equal to e to the power c. E to the power c can be written as a new arbitrary constant c dash and we shall then have x plus 2 y e to the power x over y equal to c dash when we put the value of v equal to y over x. So, this is the general solution of the given ordinary differential equation.

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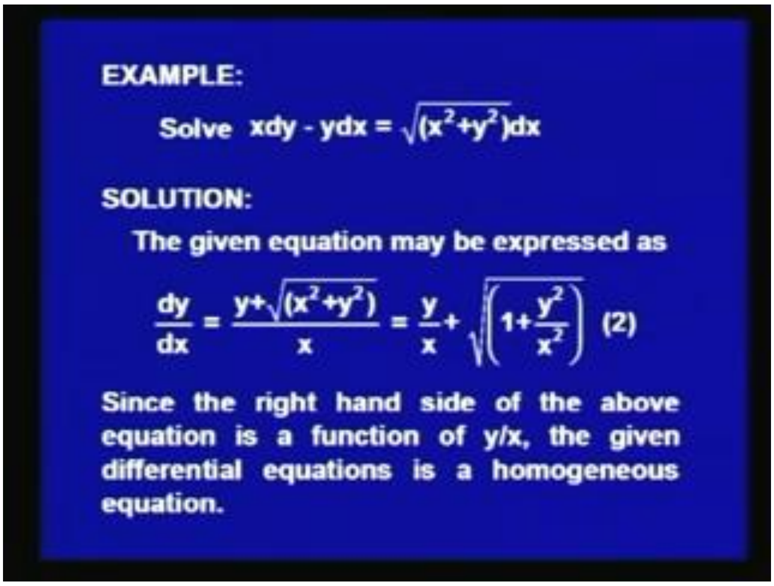

Now, let us consider the case of the differential equation x d y minus y d x equal to under root x square plus y square d x, this equation may be written as d y by d x equal to y over x plus under root 1 plus y square over x square. So, again we see that this differential equation is by d y by d x equal to a function of y over x and hence, this differential equation is a homogeneous differential equation.

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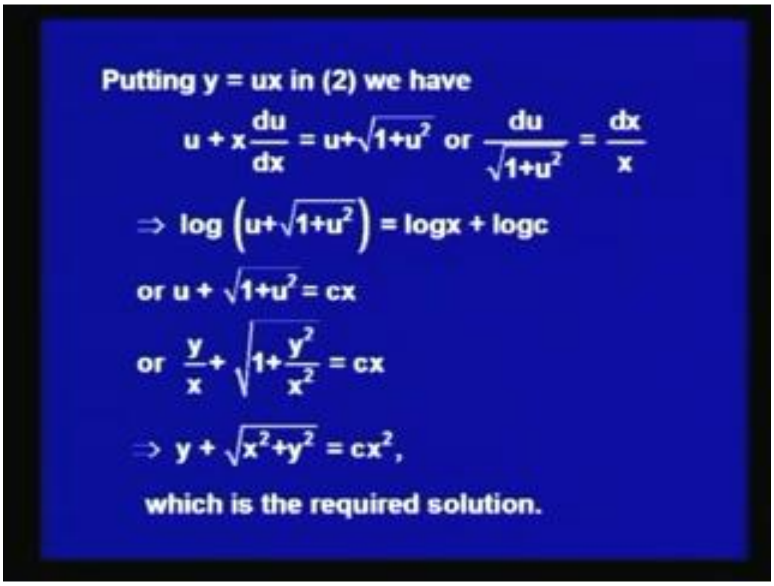


Now, so we will put y equal to u x here and that will give us u plus x d u by d x equal to u plus under root 1 plus u square, we can separate the variables u and x and get d u by under root 1 plus u square equal to d x by x. When we integrate both sides, we get log of u plus under root 1 plus u square equal to log x plus the constant of integration will be taking here as log c, so this will give us u plus under root 1 plus u square equal to c x. Let us put the value of u and we get y over x plus under root 1 plus y square over x square equal to c x, which then gives us the required solution as y plus under root x square plus y square equal to c x square.

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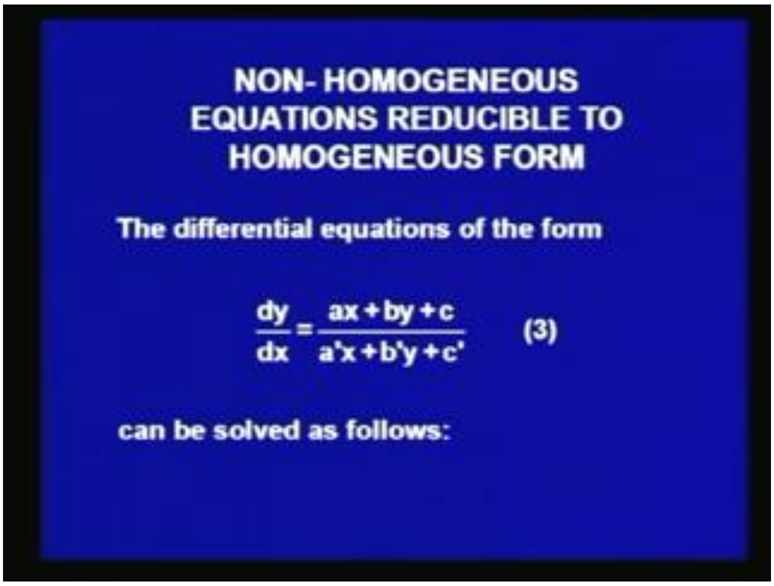


Now, we consider the case of non homogeneous equations, which are reducible to homogeneous form, the differential equations of the form d y by d x equal to a x plus b y plus c over a dash x plus b dash y plus c dash can be solved as follows.

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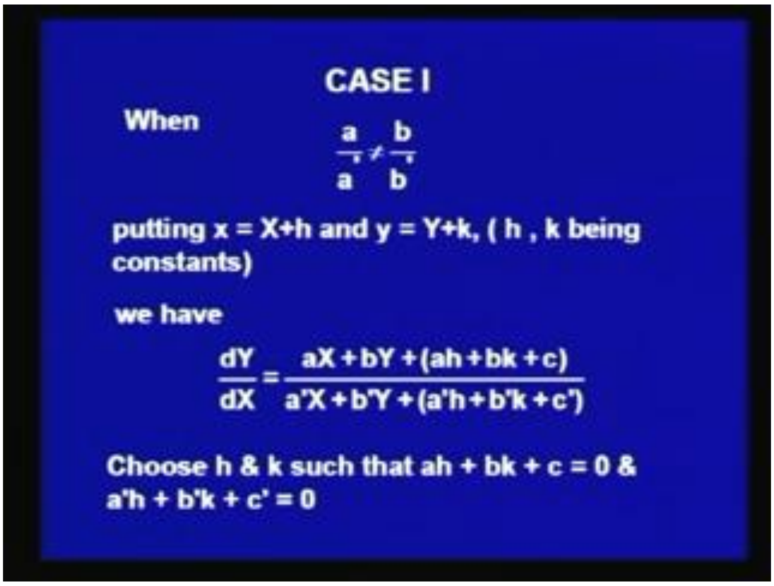


First we take up the case where a over a dash is not equal to b over b dash, so then we will put x equal to capital X plus h and y equal to capital Y plus k where h and k are some constants. And this substitution will give us d Y by d X equal to a X plus b Y plus a h plus b k plus c over a dash x plus b dash y plus a dash h plus b dash k plus c dash.

Now, we will choose the constants h and k in such a way that a h plus b k plus c equal to 0 and a dash h plus b dash k plus c dash equal to 0 if we do this, then the differential equation will reduce to homogeneous form.

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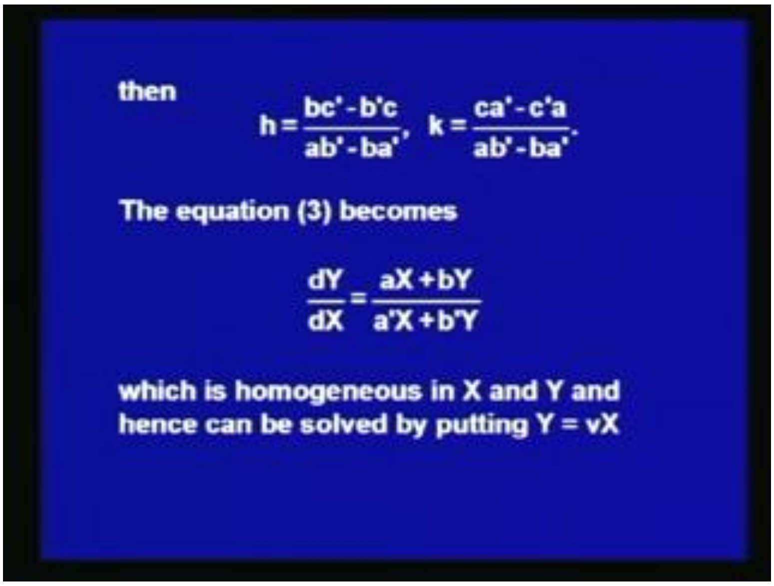


The values of h and k can be obtained from the two linear equations involving h and k as b c dash minus b dash c over a b dash minus b a dash and k equal to c a dash minus c dash a over a b dash minus b a dash. Now, since we have assumed that a over a dash is not equal to b over b dash a b dash minus b a dash is not equal to 0 and therefore, h and k adjust. The equation 3 now becomes d Y by d X equal to a X plus b Y over a dash X plus b dash Y, which is a homogeneous equation in the variables capital X and capital Y and can then we solved by putting capital Y equal to v into capital X.

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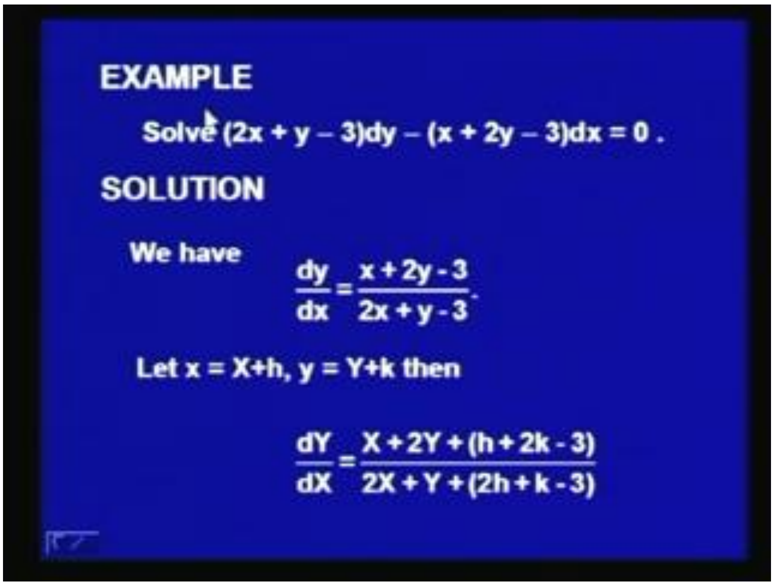


Let us take an example of a differential equation of this category, suppose we consider the differential equation as 2 x plus y minus 3 into d y minus x plus 2 y minus 3 d x equal to 0. We can write the given differential equation as d y by d x equal to x plus 2 y minus 3 over 2 x plus y minus 3. And let us note that here a over a dash is half, while b over b dash is 2, so a over a dash is not equal to b over b dash. And therefore, we can put a small x equal to capital X plus h and small y equal to capital Y plus k to reduce it to homogeneous form. These substitutions will give us d Y by d X equal to X plus 2 Y plus h plus 2 k minus 3 divided by 2 X plus Y plus 2 h plus k minus 3.

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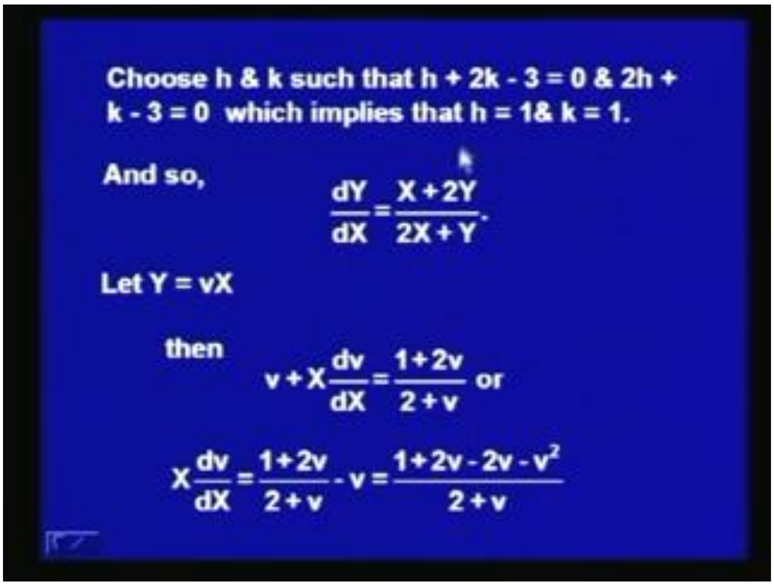

Now, you will have to find h and k in such a way that, h plus 2 k minus 3 is equal to 0 and 2 h plus k minus 3 equal to 0, on solving these two linear equations in h and k we obtain h equal to 1 and k equal to 1. And so the equation now reduces to d Y by d X equal to X plus 2 Y over 2 X plus Y, now let us put Y equal to v X; and then we shall have v plus X d v over d X equal to 1 plus 2 v over 2 plus v.

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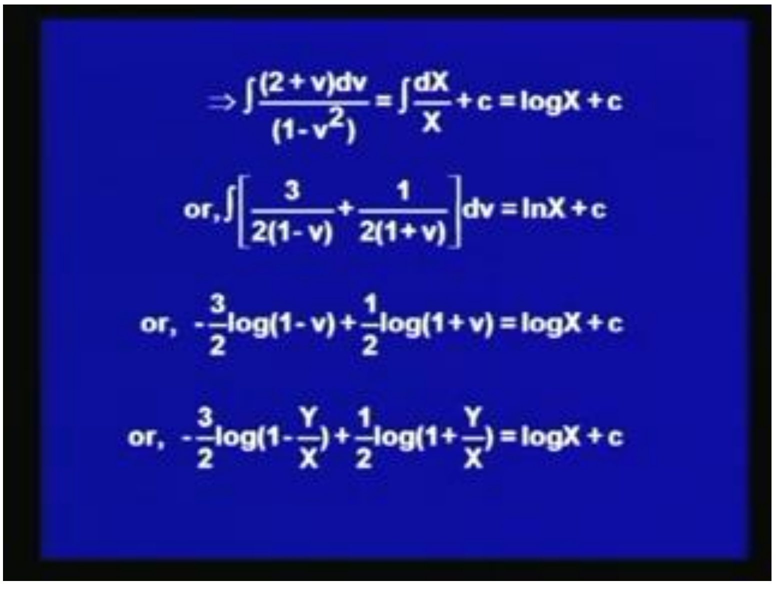


Now, we can separate the variables X and v and then integrate we shall have integral 2 plus v d v over 1 minus v square equal to integral of d X over X plus c, which will be equal to log X plus c. In the left hand side, we break it into partial fractions, the partial fractions are 3 by 2 into 1 over 1 minus v plus 1 over 2 into 1 over 1 plus v, so after integration the left hand side will become minus 3 by 2 log 1 minus v plus half log 1 plus v. And we get the general solution of the given differential equation after substituting the value of v, that is minus 3 by 2 log 1 minus Y over X plus half log 1 plus Y over X equal to log X plus c.

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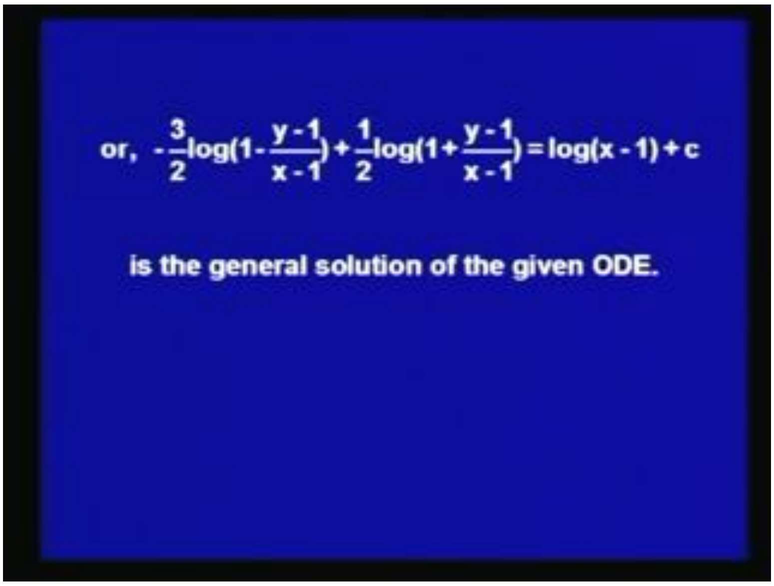


Where capital Y is equal to y minus 1 and capital X is x minus 1, so let us put their values and get the general solution of the given differential equation.

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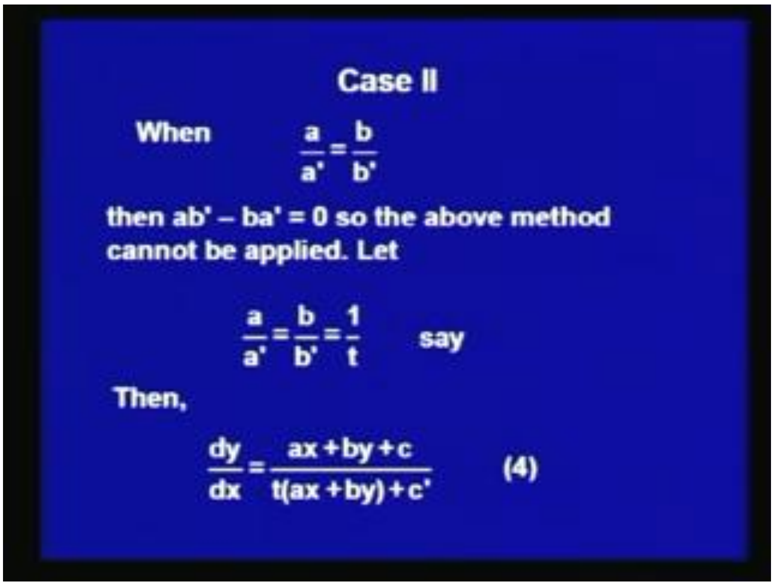


Now, let us consider the case of those differential equations, where a over a dash becomes equal to b over b dash, now in this case you can see that a b dash minus b a dash is equal to 0, so the previous case cannot be applied. Let us take a over a dash equal to b over b dash equal to 1 by t, then we will be getting d y by d x equal to a x plus b y plus c over t times a x plus b y plus c dash.

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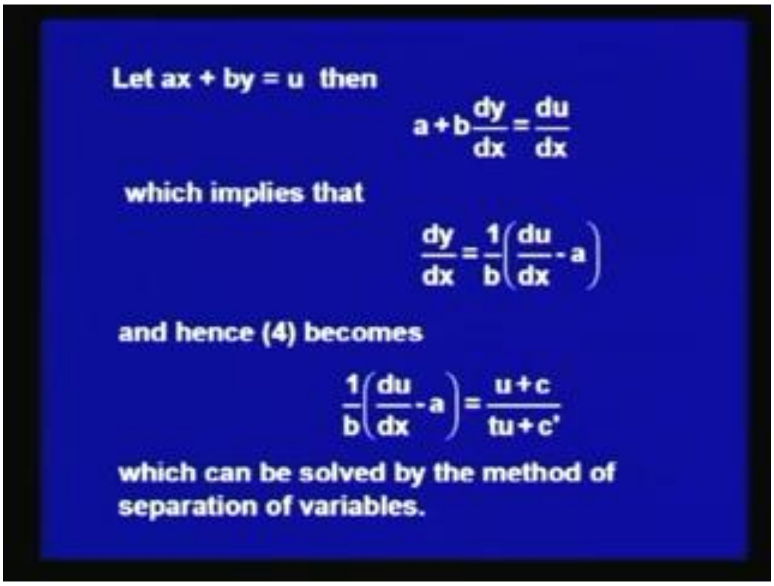


Let us put a x plus b y equal to u, then we will have after differentiation with respect to x a plus b d y by d x equal to d u by d x, which will give us the value of d y by d x as 1 over b into d u by d x minus a and so the equation 4 will become 1 over b into d u by d x minus a equal to u plus c over t u plus c dash. And then we can separate the variables u and x and u is the method of separation of variables to solve this differential equation, we can put the value of u as a x plus b y to obtain the general solution of the given differential equation.

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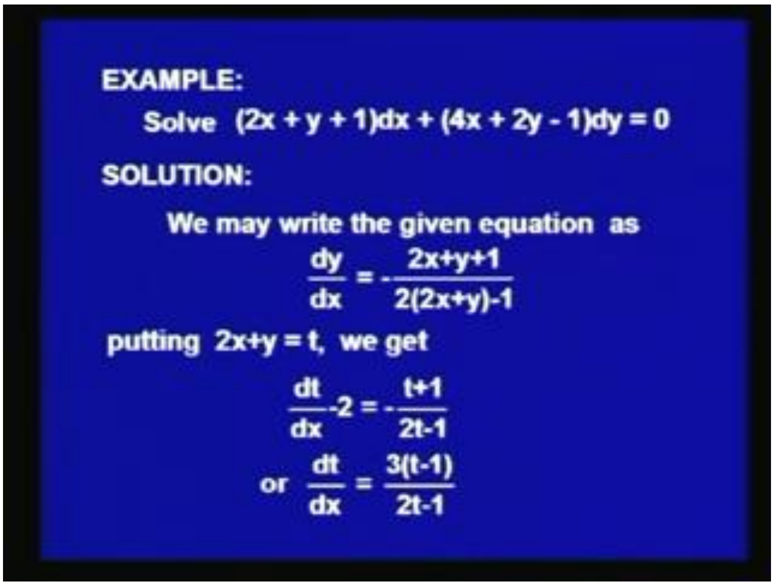

Let us take an example of a differential equation of this type, consider 2 x plus y plus 1 into d x plus 4 x plus 2 y minus 1 into d y equal to 0. You can see here that a over a dash is equal to b over b dash equal to half and so this differential equation belongs to the case of a over a dash equal to b over b dash. We can write the given differential equation as d y by d x equal to minus 2 x plus y plus 1 over 2 times 2 x plus y minus 1. So, let us put 2 x plus y equal to t, which will give us d t by d x minus 2 equal to minus t plus 1 over 2 t minus 1 or we may say d t by d x equals 3 times t minus 1 over 2 t minus 1.

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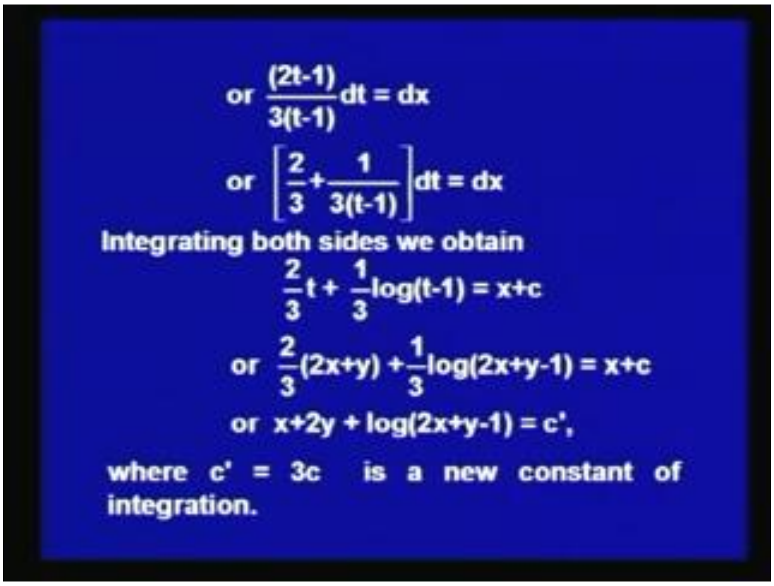


We can separate the variables t and x, this will give us 2 t minus 1 over 3 times t minus 1 d t equal to d x after breaking the left hand side into partial fractions, we shall have 2 over 3 plus 1 over 3 into t minus 1 d t equal to d x. And when we integrate both sides of this equation, we shall have 2 over 3 into t plus 1 by 3 log t minus 1 equal to x plus c; and substituting the value of t, we shall have 2 over 3 into 2 x plus y plus 1 over 3 log 2 x plus y minus 1 equal to x plus c. Or we will have x plus 2 y plus log 2 x plus y minus 1 equal to c dash, where c dash is a new arbitrary constant and is equal to 3 c.

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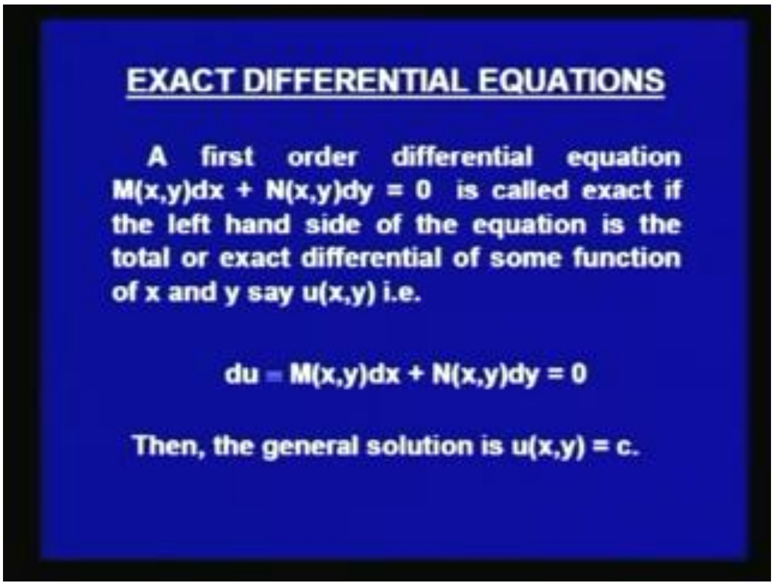


Now, let us consider the case of exact differential equations, a first order differential equation M d x plus N d y equal to 0 is said to be exact if the left hand side of the differential equation that is M d x plus N d y is the total or exact differential of some function of x and y. Say u (x, y), that is to say that d u is equal to M d x plus N d y equal to 0 and if this is the case, then we can write the general solution of the given differential equation as u (x, y) equal to c after integration.

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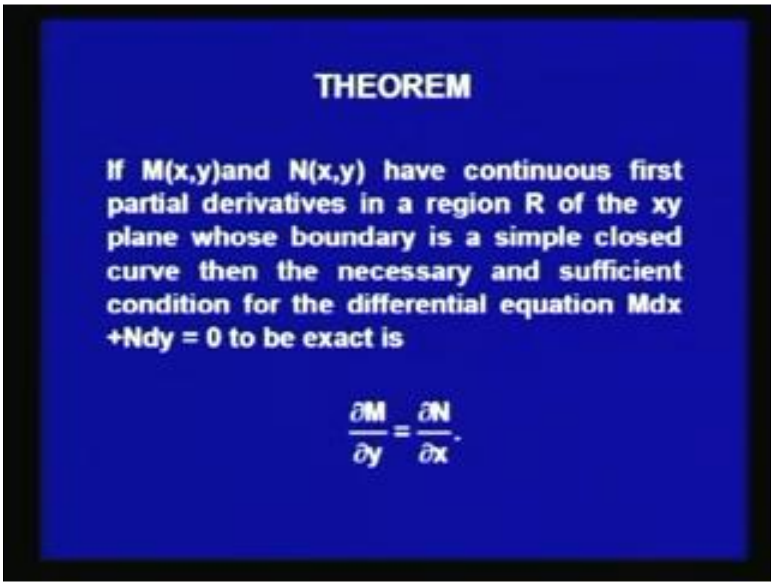

Now, we have an important theorem here, which gives us necessary and sufficient condition for the differential equation M d x plus N d y equal to 0 to be exact. The theorem is the following, if M (x, y) and N (x, y) have continuous first partial derivatives in a region R of the x y plane, whose boundary is a simple closed curve. Then the necessary and sufficient condition for the differential equation M d x plus N d y equal to 0 to be exact is delta M over delta y is equal to delta N over delta x.

Now, a simple closed curve is a closed curve, which does not intersect itself for example, you can take a circle or you can take an ellipse or you can take a square or you may take a rectangle, but if you take the number aid it is the closed curve, but since it intersects itself it cannot be called as a simple closed curve, so let us prove this result.

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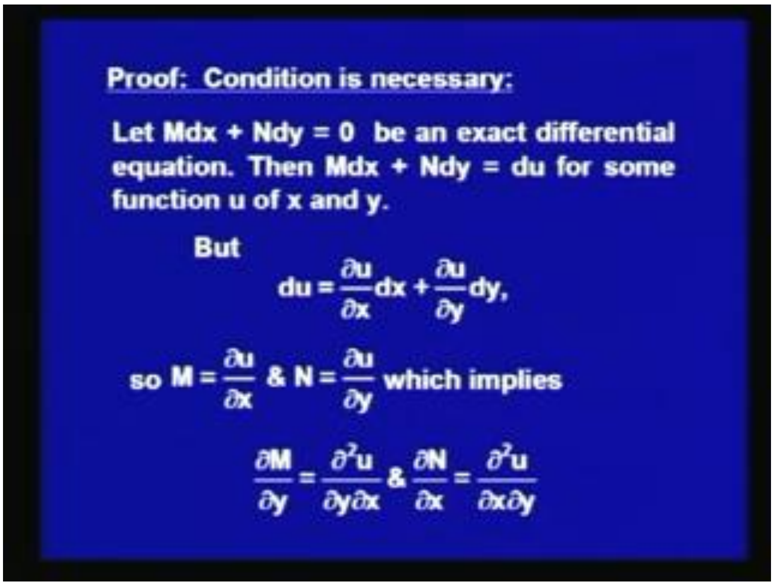


First we show that the condition is necessary that is we shall assume that, M d x plus n d y equal to 0 is an exact differential equation and then we shall show that the first order partial derivatives of M and N that is delta M over delta y and delta N over delta x are equal to each other. So, let us assume that M d x plus N d y is an exact differential equation, then we may write M d x plus N d y equal to d u for some function u of x and y.

Since u is a function of x and y, we shall have the differential d u is equal to delta u by delta x into d x plus delta u by delta y into d y on comparing it with M d x plus N d y. We shall then have M equal to delta u by delta x and N equal to delta u over delta y, which will gives us delta M over delta y equal to delta square u over delta y delta x and delta N over delta x equal to delta square u over delta x delta y. Now, since we have assumed that the first order partial derivatives of M and N are continuous the second order partial derivatives of u delta square u over delta y delta x and delta square u over delta x delta y are continuous and are therefore equal.

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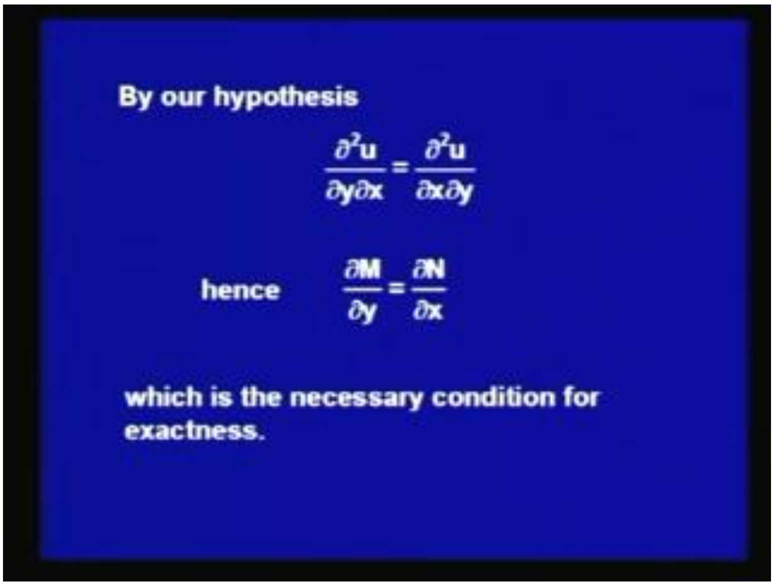


So, this will give us delta M over delta y equal to delta N over delta x, which is the necessary condition for exactness.

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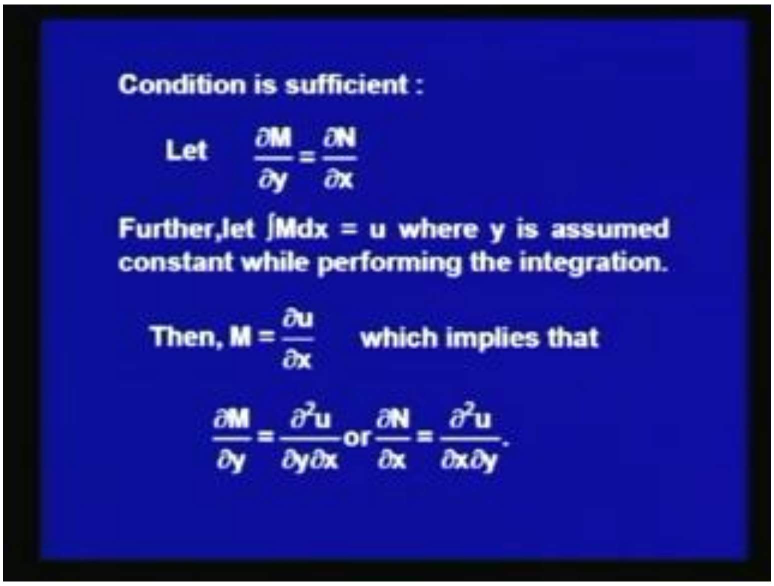

Now, let us show that the condition is sufficient, so let us assume that delta M over delta Y is equal to delta N over delta X, further we assume that integral of M with respect to x is say u. Where while integrating we assume y as a constant, this will give us M equal to delta u over delta x on differentiating partially with respect to M x on both sides, so this will imply that delta M over delta y is equal to delta square u over delta y delta x. Or delta N over delta x equal to delta square u over delta x delta y, since delta M over delta y and delta N over delta x are equal and the first order partial derivatives are continuous.

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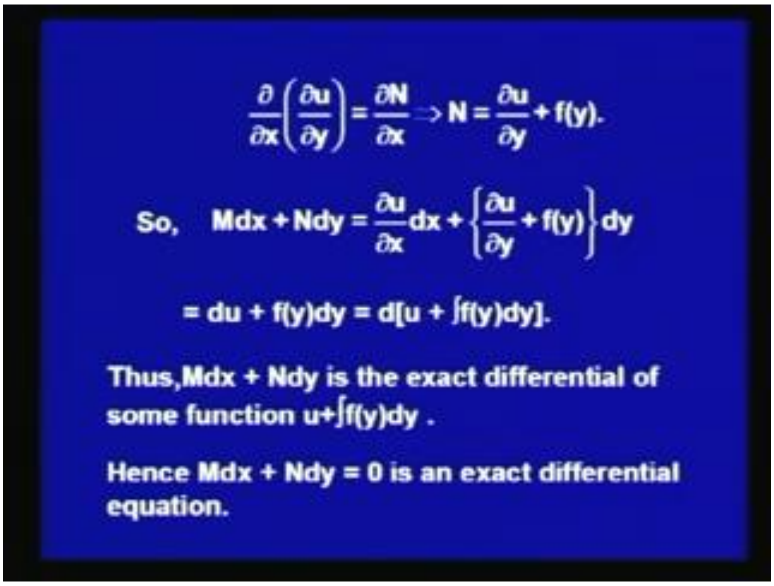


Then we shall have delta over delta x of delta u over delta y equal to delta N over delta x implying that, N is equal to delta u over delta y plus f y, on integrating with respect to x keeping y as a constant. So, here the constant of integration in general will be a function of y, which we have denoted by f y and this will give us M d x plus N d y equal to delta u over delta x into d x plus delta u over delta y plus f y multiplied by d y.

Now, delta u over delta x d x plus delta u over delta y d y can be written as u and we have d u plus f y d y as the right hand side, which is equal to differential of u plus integral f y d y. And thus we have shown that M d x plus N d y is the exact differential of some function u plus integral f y d y and hence, M d x plus N d y is an exact differential equation, so this is the proof of the theorem.

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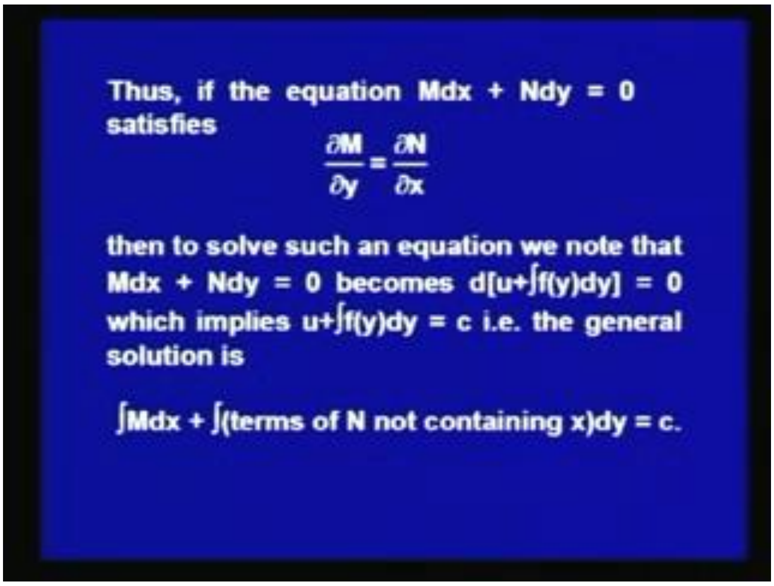


Thus if the equation M d x plus N d y equal to 0 satisfies delta M over delta y equal to delta N over delta x, then the equation is exact and in order to solve such an equation. Let us note that M d x plus N d y equal to 0 becomes the differential of u plus integral f y d y equal to 0, which implies that u plus integral f y d y equal to c. That is the general solution of the exact equation M d x plus N d y equal to 0 is nothing but, integral M d x which is u plus integral f y, f y is the terms of N which do not contain x into d y equal to c. So, while finding the solution of the given differential equation M d x plus N d y equal to 0, we shall simply have to find the integral of M with respect to x and the integral of those terms of n which do not contain x. And then take the sum of the two and put it equal to c which is an arbitrary constant.

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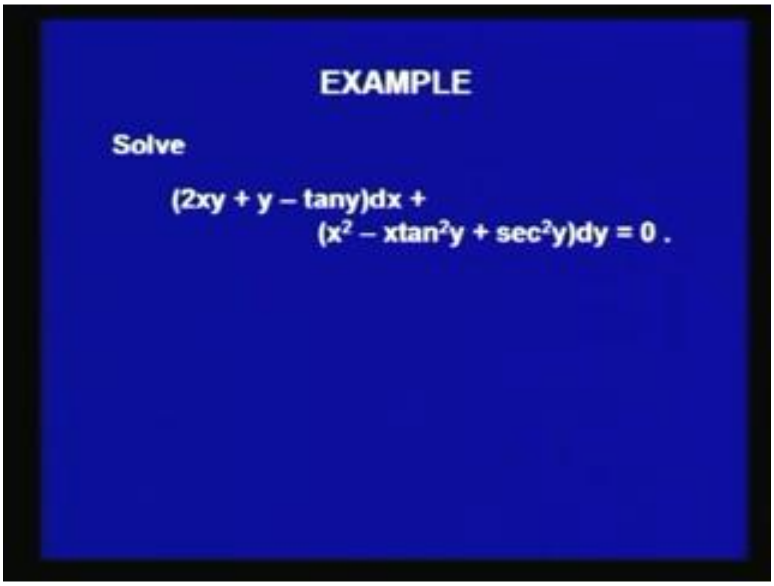


Now, let us take an example of an equation of this type, 2 x y plus y minus tan y into d x plus x square minus x tan square y plus sec square y into d y equal to 0.

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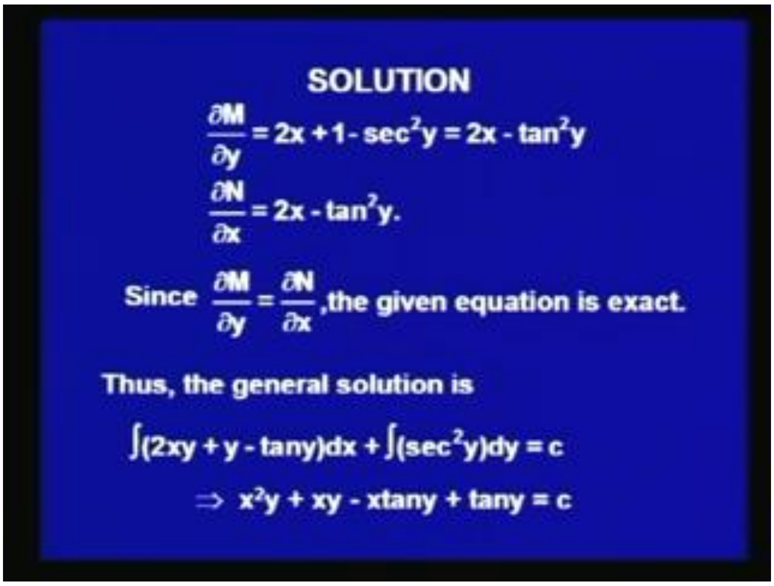


So, here M is equal to 2 x y plus y minus tan y and N is equal to x square minus x tan square y plus sec square y, when we find the partial order derivatives of M and N with respect to y and x, we shall have delta M over delta y equal to 2 x minus tan square y and delta N over delta x equal to 2 x minus tan square y, so they are both equal and therefore, the given equation is an exact differential equation. And hence, the general solution of the differential equation will be integral of M, that is integral of 2 x y plus y minus tan y d x plus integral of those terms of N, which do not contain x that is integral of sec square y d y equal to c. In the first integral we integrate with respect to x keeping y as a constant, which will give us x square y plus x y minus x tan y and then the second integral, integral sec square y d y gives us tan y. So, we get the general solution as x square y plus x y minus x tan y plus tan y equal to c.

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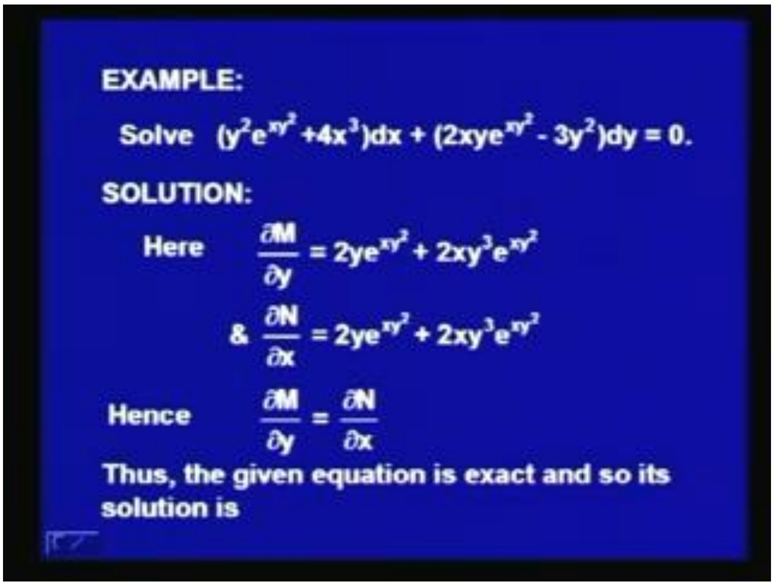


Let us take one more example of N exact differential equation y square into e to the power x y square plus 4 x cube into d x plus 2 x y into e to the power x y square minus 3 y square into d y equal to 0. So, here you can see that delta M over delta y is equal to 2 y into e to the power x y square plus 2 x y cube into e to the power x y square, we have differentiated M partially with respect to y keeping x as the constant.

And when we differentiate N with respect to x keeping y as a constant that is we find delta n over delta x, we get 2 y e to the power x y square plus 2 x y cube e to the x y square. So, delta over delta y and delta N over delta x are both equal and therefore, the given equation is an exact differential equation.

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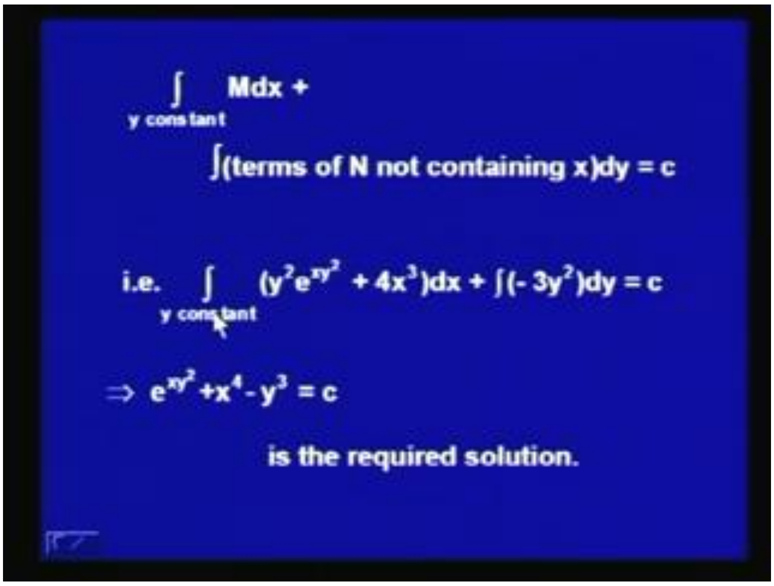


Here the solution of the differential equation is integral M d x, where y is constant plus integral of terms of N not containing x d y equal to c, which will give us integral of y square e to the power x y square plus 4 x cube d x, where we will take y as a constant during integration process. And then integral of minus 3 y square d y equal to c and we can see that, the integral is e to the power x y square plus x 4 minus y cube equal to c which is the required solution.

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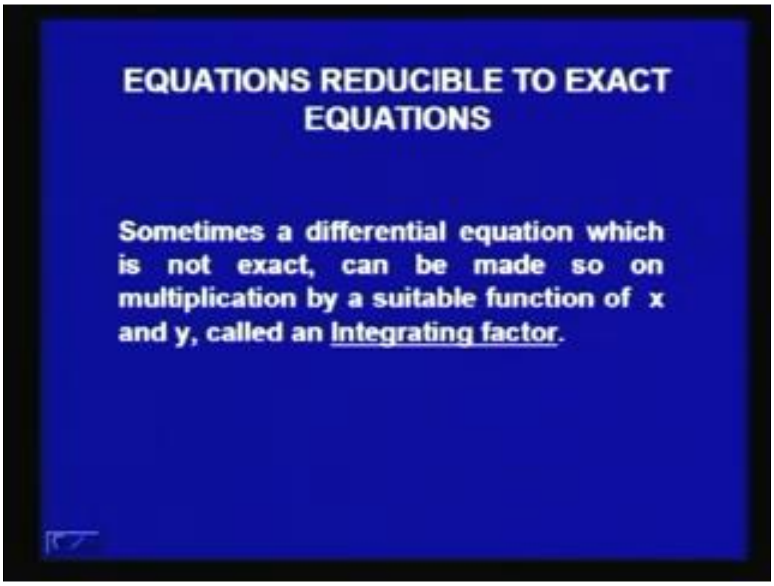

Now, we shall consider equations that are reducible to exact form, sometimes differential equation which is not exact can be made so on multiplication by a suitable function of x and y, we call this function as an integrating factor.

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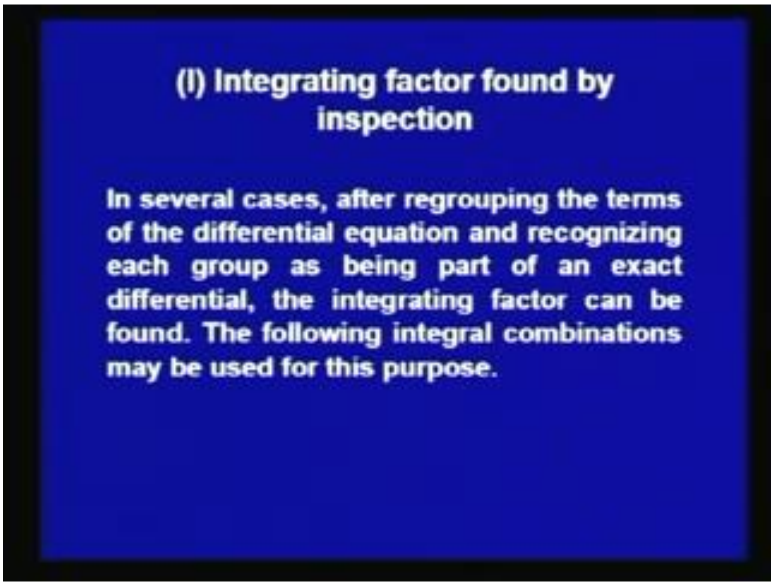


In several cases after regrouping the terms of the differential equation and recognizing each group as being part of an exact differential, the integrating factor can be found the following integral combinations may be used for this purpose.

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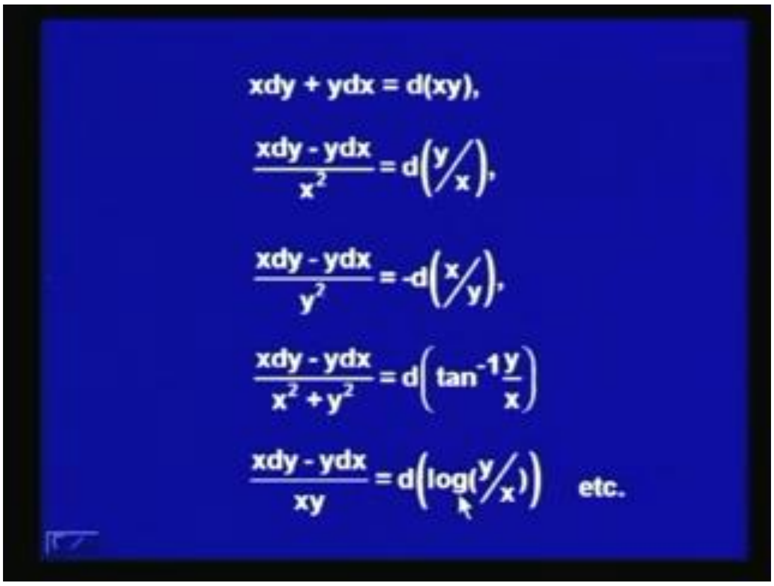

X d y plus y d x is differential of x y, x d y minus y d x over x square is the differential of y over x and x d y minus y d x over y square is equal to minus d of x over y. X d y minus y d x over x square plus y square is the differential of tan inverse y over x and x d y minus y d x over x y is the differential of log y over x, and so on.

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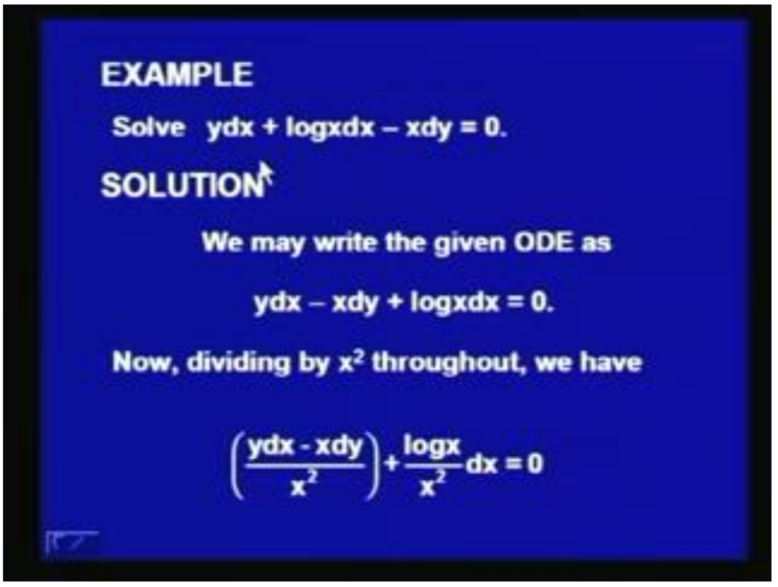


Now, let us consider the differential equation y d x plus log x d x minus x d y equal to 0, we may write the given ordinary differential equation as y d x minus x d y plus log x d x equal to 0. We can regroup the terms y d x and minus x d y together and write y d x minus x d y plus log x d x equal to 0, now let us divide the differential equation by x square. If we divide it by x square, we can note that y d x minus x d y over x square is an exact differential of a function of x y and log x over x square is integrable with respect to x; so by dividing this differential equation by x square, it becomes an exact differential equation.

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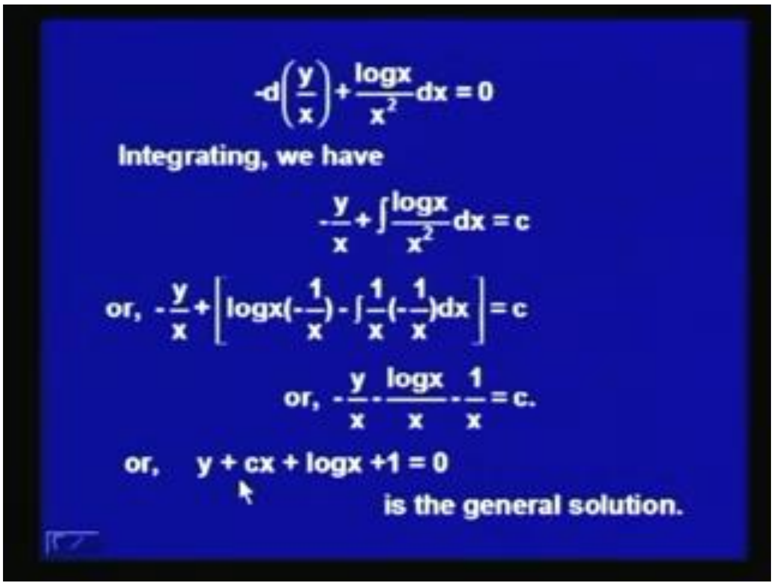


And we note that, the first term is equal to minus d of y over x plus log x over x square d x equal to 0, now we can integrate with this both sides and we have minus y over x plus integral of log x over x square d x. Log x over x square can be integrated by parts, we will have the solution as minus y over x plus log x into minus 1 over x, here we are taking log x as the first function and 1 over x square as the second function.

So, we have the first function log x, the integral of the second function 1 over x square as minus 1 by x, minus integral of the derivative of log x that is 1 over x in to the integral of 1 over x square, which is minus 1 over x. And then we shall after simplification we shall have the solution as minus y over x minus log x over x minus 1 over x equal to c, which can be written as y plus c x plus log x plus 1 equal to 0 which gives us the general solution of the given differential equation.

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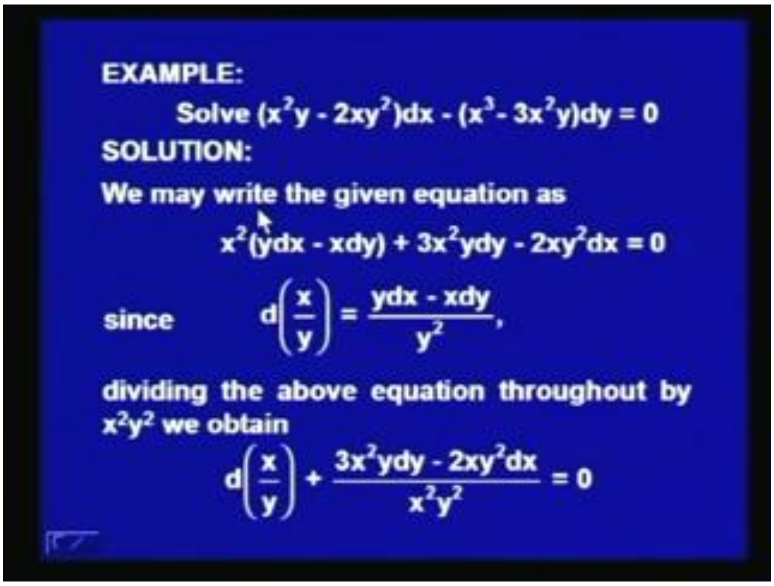


Next, let us consider the case of the differential equation x square y minus 2 x y square d x minus x cube minus 3 x square y d y equal to 0, we regroup the terms here again and write it as x square in to y d x minus x d y plus 3 x square y d y minus 2 x y square d x equal to 0. We note that, since d of x over y is equal to y d x minus x d y over y square, we divide this equation as by x square y square, then the first term will become y d x minus x d y over y square, which will be differential of x over y. And the second term and the third terms 3 x square y d y minus 2 x y square d x will be turned into 3 x square y d y minus 2 x y square d x over x square y square.

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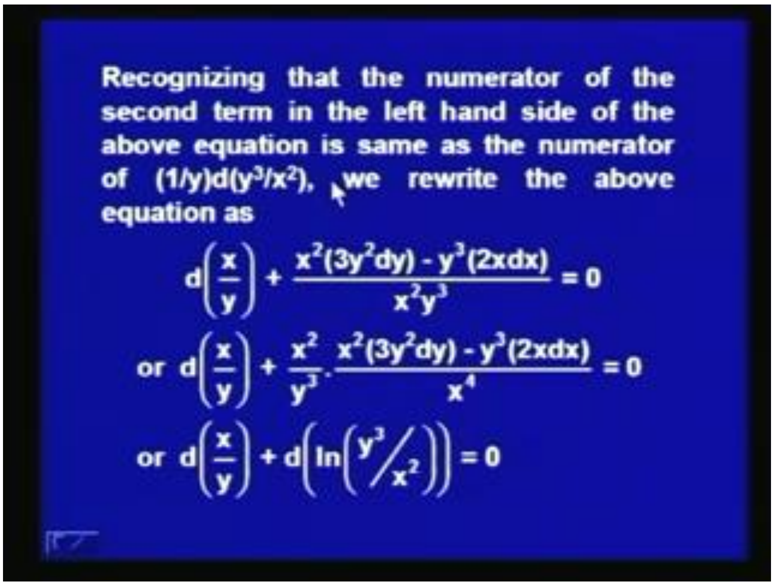

Now, let us note that the numerator of the second term, in the left hand side of the above equation is same as the numerator of 1 over y into differential of y cube over x square. But, we do is we rewrite the given differential, we get the above equation as differential of x over y plus x square into 3 y square d y minus y cube into 2 x d x over x square y cube equal to 0. And then we multiply by x square in the numerator and denominator, we shall have d of x over y plus x square over y cube in to x square in to 3 y square d y minus y cube in to 2 x d x over x to the power 4 equal to 0, which will give us differential of x over y plus differential of ln over x square equal to 0. And so we can now obtain the solution of the given differential equation easily.

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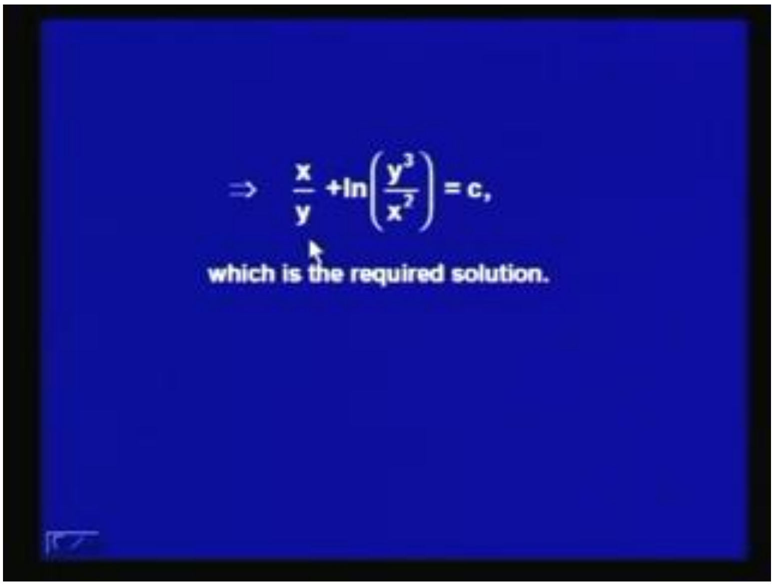


As x over y plus log of y cube over x square equal to c, which is the required solution.

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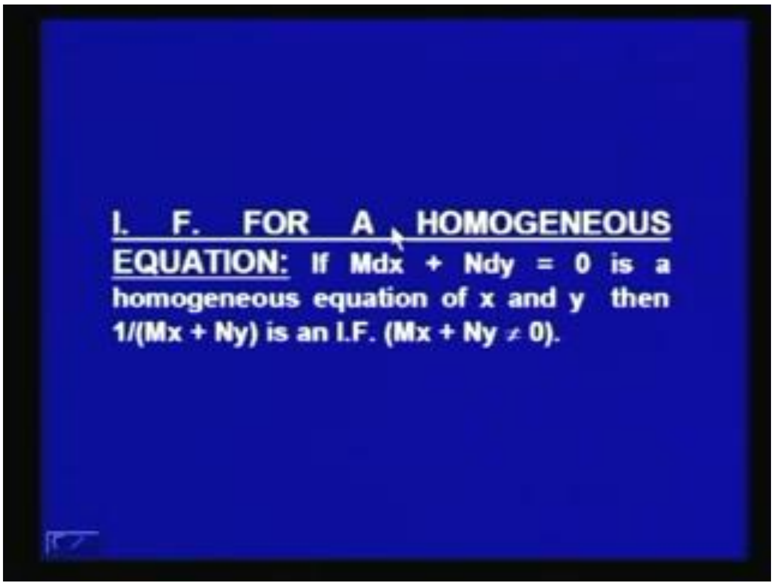


Now, let us find the integrating factor for a homogeneous equation, let us recall that homogeneous equation is 1 where d y by d x is equal to a function of y over x, so if M d x plus N d y equal to 0 is a homogeneous equation of x and y. Then it follows that 1 over M x plus N y is an integrating factor, that is we multiply the equation M d x plus N d y equal to 0 by 1 over M x plus N y provided M x plus N y is not equal to 0, it will become an exact differential equation.

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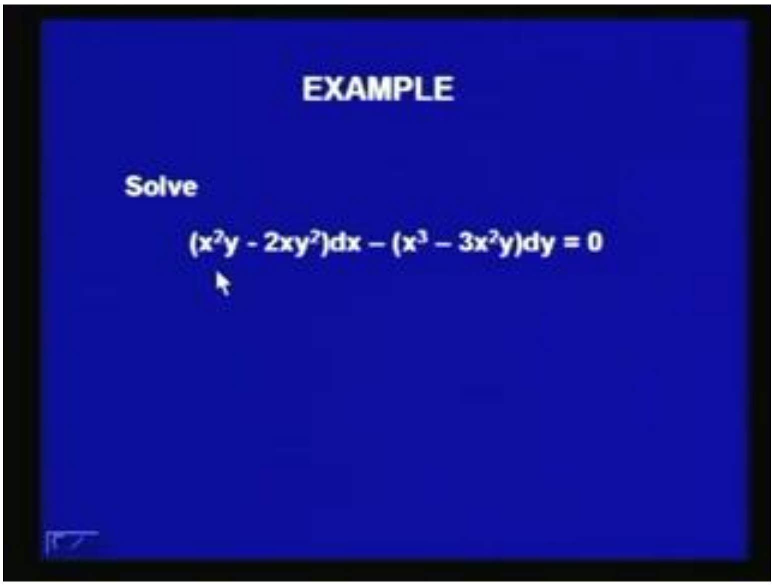

And we can easily find it is solution, so let us look at the differential equation x square y minus 2 x y square d x minus x cube minus 3 x square y d y equal to 0, you can see here that d y over d x is equal to a function of y over x. And so it is a homogeneous equation, let us we can solve this differential equation by the method of homogeneous equations also, where we put y equal to v in to x.

Here we shall find the integrating factor an integrating factor for this differential equation, that is 1 over M x plus N y. And then we will multiply this differential equation by 1 over M x plus N y and make it an exact differential equation; and from there we shall find the general solution of this differential equation.

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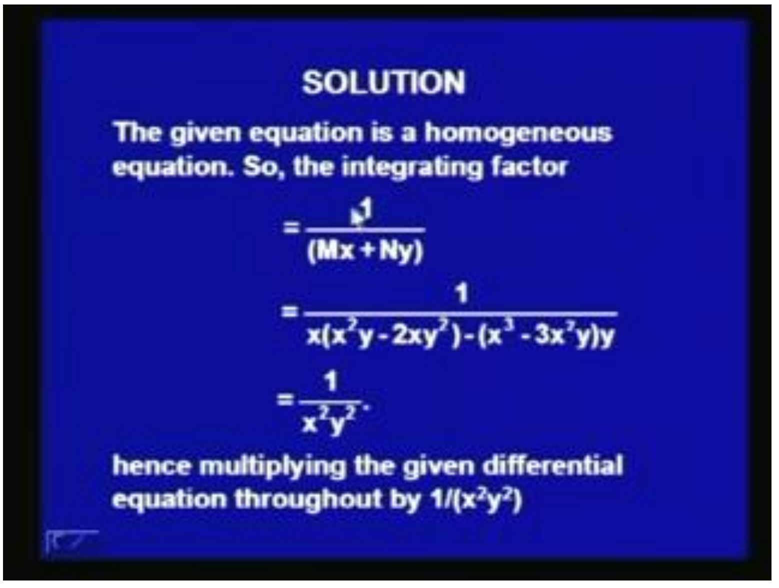


So, let us see this equation, for this equation the integrating factor 1 over M x plus N y comes out to be 1 over x square plus y square. So, obviously, M x plus N y is not equal to 0 and therefore, multiplying the given differential equation throughout by 1 over x square y square.

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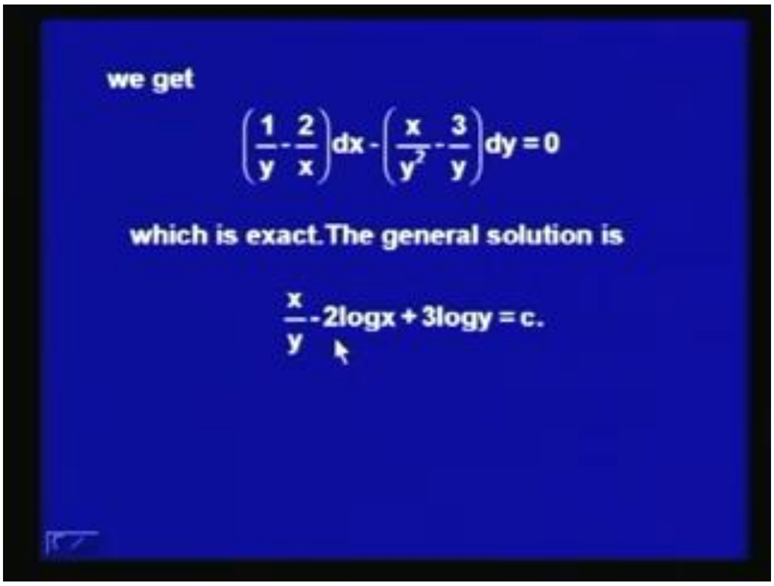


We shall have 1 over y minus 2 over x d x minus x over y square minus 3 over y d y equal to 0, now we shall group the terms 1 over y d x with minus x over y square that will be differential of x over y. And the integral of minus 2 over x will give us minus 2 log x, the integral of 3 over y will give us 3 log y and so we will get the general solution as x over y minus 2 log x plus 3 log y equal to c.

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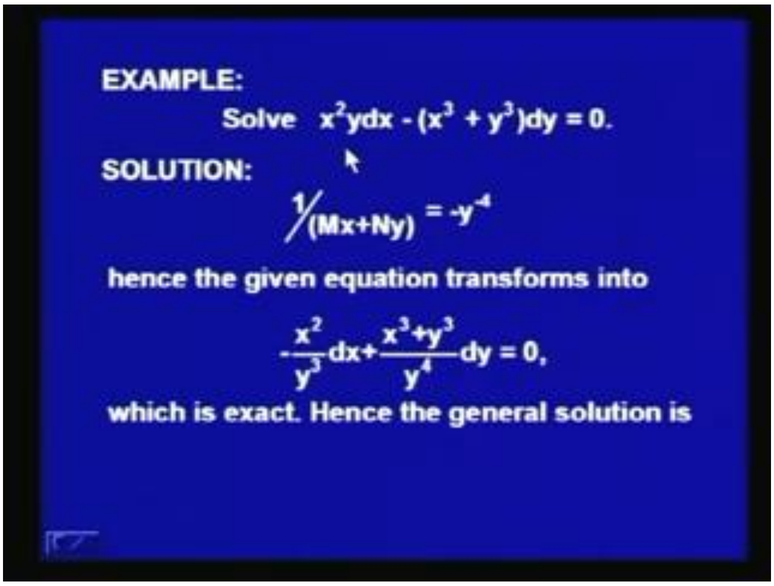


Let us take one more example of homogeneous equation x square y d x minus x cube plus y cube d y equal to 0, so in this case 1 over M x plus N y is equal to minus y to the power minus 4. Multiplying the given equation by the integrating factor, it will transform in to minus x square over y cube d x plus x cube plus y cube over y to the power 4 d y equal to 0, which is an exact differential equation; and therefore, the general solution is...

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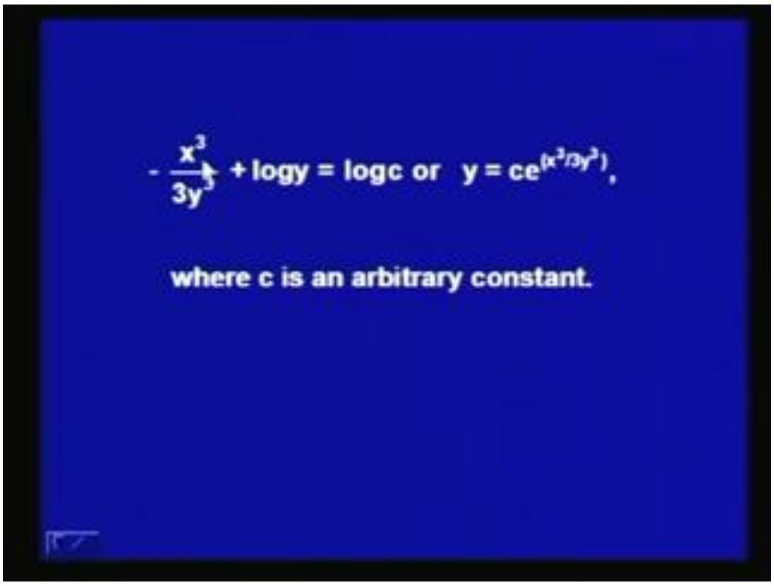


So, we will integrate M with respect to x assuming y as a constant, that will give us minus x cube over 3 y cube and then we integrate those terms of n which do not contain x with respect to y that will give us log y. And we shall have minus x cube over 3 y cube plus log y equal to log c or we shall y equal to c e to e to the power x cube over 3 y cube as the general solution of the given differential equation, where c is an arbitrary constant.

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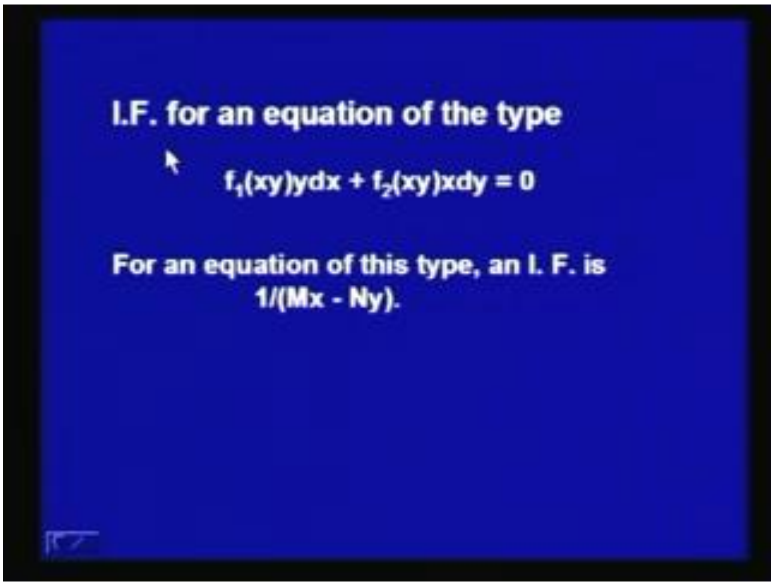


Now, we take up the case of those ordinary differential equations, which are of the type f 1 x y into y d x plus f 2 x y x d y equal to 0, f 1 x y is the function of x y f 2 x y is also a function of x y. So, the differential equation, in the case of such differential equation we shall see that integrating factor is 1 over M x minus N y that is when we multiply this equation by 1 over M x minus N y, it will become an exact differential equation.

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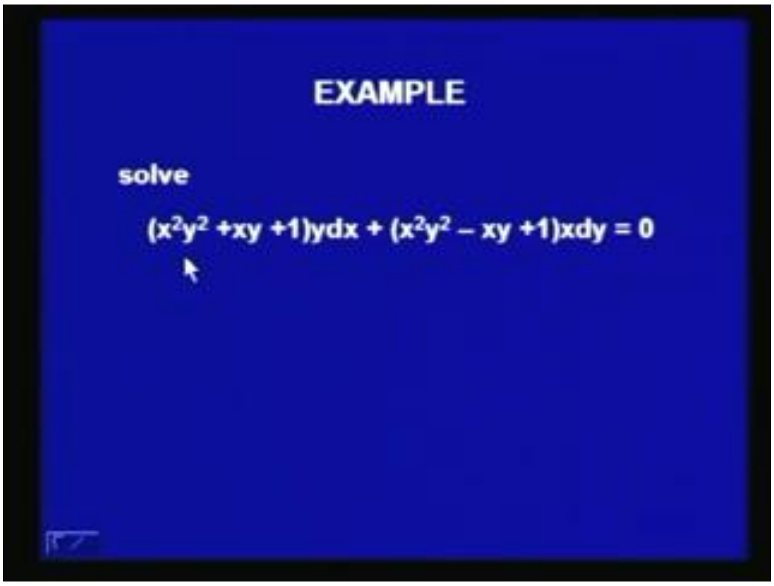


So, let us consider the case of the differential equation x square y square plus x y plus 1 in to y d x plus x square y square minus x y plus 1 in to x d y equal to 0. X square y square plus x y plus 1 is a function of x y, which we can denote as f 1 x y in to y d x plus x square y square minus x y plus 1 is another function of x y which we may regard as f 2 x y into x d y equal to 0. So, this is differential equation we can solve using the integrating factor 1 over M x minus N y.

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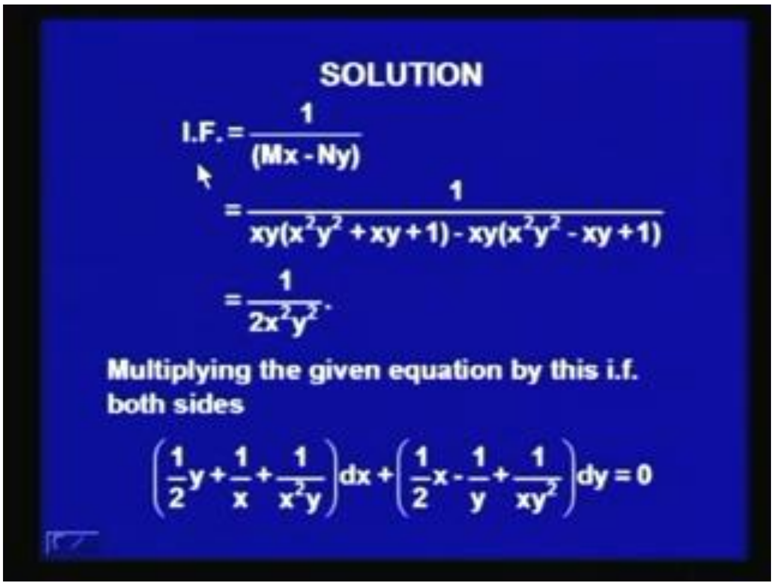


And the integrating factor when we find 1 over M x minus N y comes out to be 1 over 2 x square y square, so let us multiply the given equation by this integrating factor. And we shall have 1 over 2 y plus 1 over x plus 1 over x square y d x plus 1 over 2 x minus 1 over y plus 1 over x y square d y equal to 0. So, here M is equal to half by plus 1 by x plus 1 by x square y and N is half x minus 1 over y plus 1 over x y square, now this differential equation is an exact differential equation.

So, to find the general solution of this, we just integrate M with respect to x taking y as a constant that will give us half x y plus log x minus 1 over x y. And then we add to it the integral of those terms of M which do not contain x that is integral of minus 1 over y, which is minus log by N put it equal to a constant.

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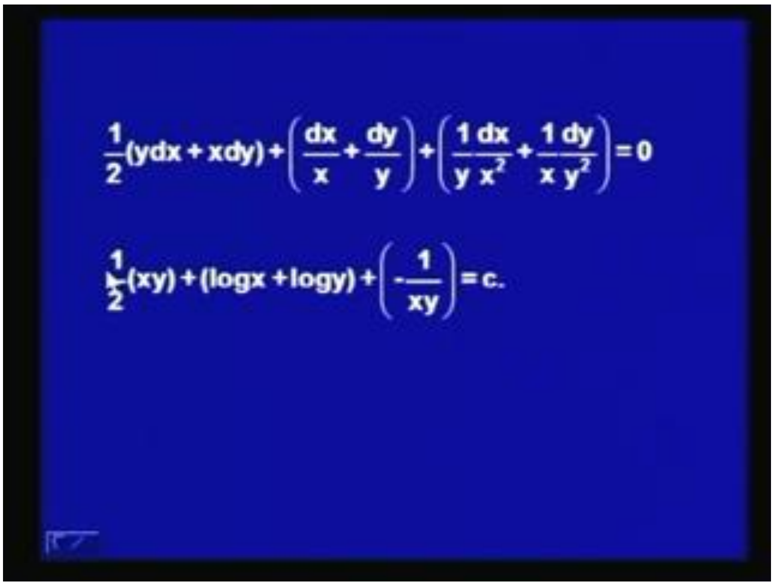


So, we shall have the integral as half x y plus log x plus log y minus 1 over x y equal to c, which is the general solution of the given differential equation. Here we can also find it by regrouping the terms and then writing them as exact differentials of some functions of x and y, so that way also we can do. So, the general solution is half x y plus log x plus log y plus minus 1 up on x y equal to c.

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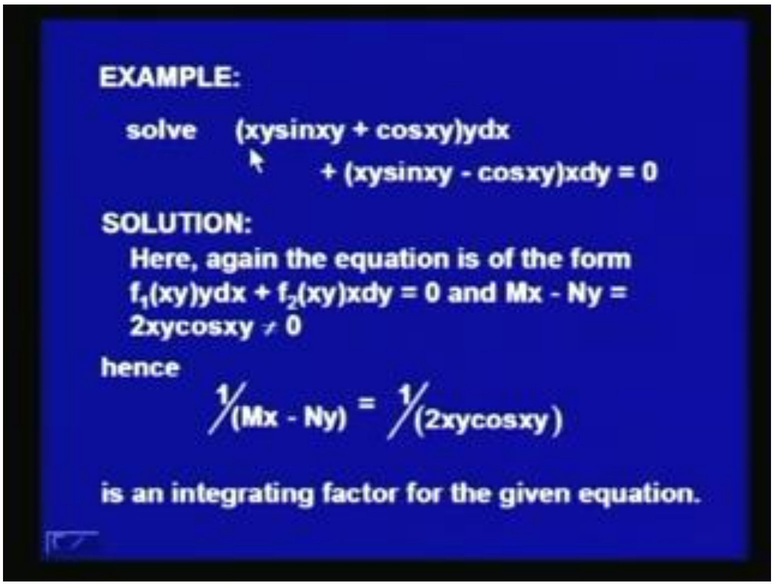


And then we take another example of a differential equation of this type x y sin x y plus cos x y in to y d x plus x y sin x y minus cos x y in to x d y equal to 0. So, here again we have the differential equation as a function of x y in to y d x plus a function of x y in to x d y equal to 0. And we note that 1 over m x minus n y is equal to 1 over 2 x y cos x y, so it is an integrating factor for the given equation.

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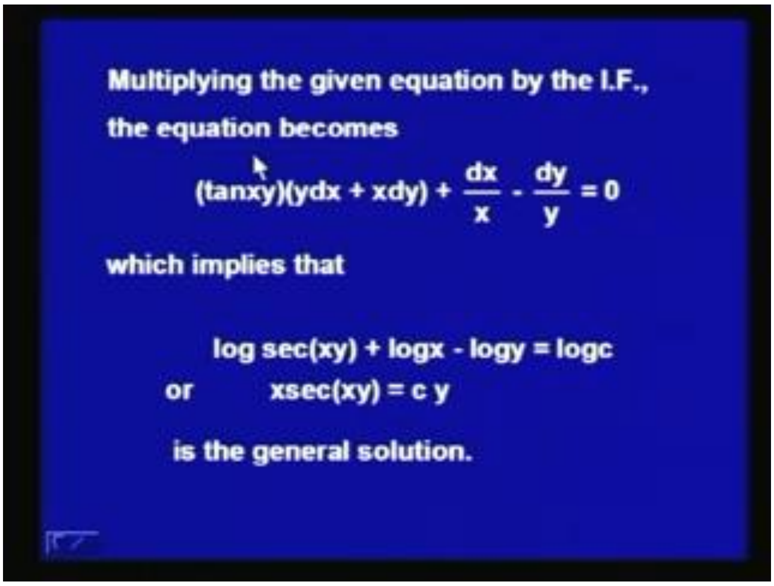


Let us multiply the given equation by this integrating factor, the equation becomes tan x y in to y d x plus x d y plus d x over x minus d y over y equal to 0 and which will give us, now y d x plus x d y is the exact differential of x y. So, this is tan x left first term can be written as tan x y in to differential of x y and therefore, the when we integrate it will get log of sec x y, integral of d x over x will give us log x, integral of d y over y will give us log y. So, we will get the general solution as log sec x y plus log x minus log y equal to log c, which after simplification we can write as x sec x y equal to c y, this gives us the general solution of the differential equation.

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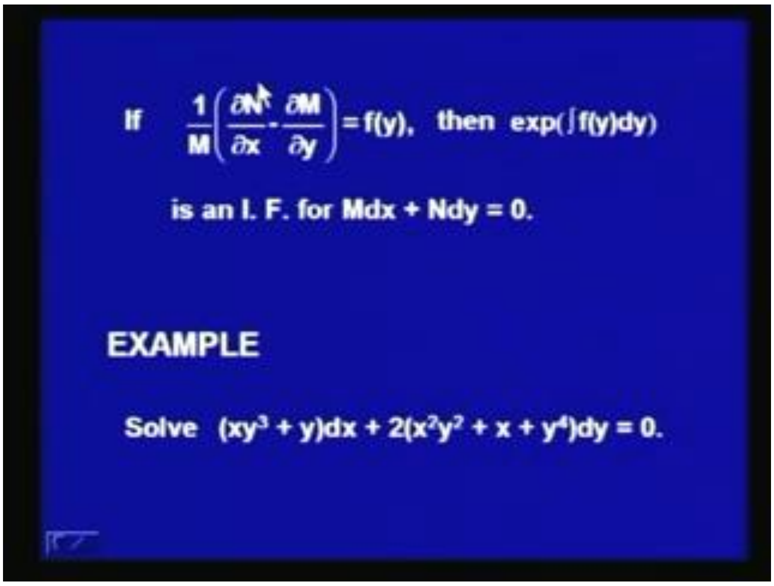


Now, let us consider the case of those differential equations M d x plus N d y equal to 0, there when we find 1 over M in to delta N over delta x minus delta M over delta y and it turns out to be a function of y only. Then exponential of integral of f y d y is an integrating factor for M d x plus N d y equal to 0, for example let us consider the differential equation x y cube plus y d x plus 2 times x square y square plus x plus y 4 d y equal to 0. So, here M is x y cube plus y and N is 2 times x square y square plus x plus y 4 equal to 0.

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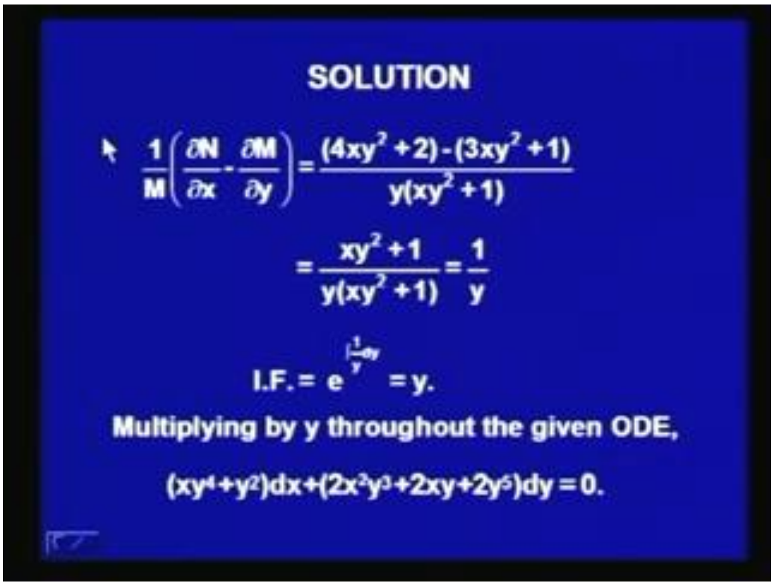

And when we find 1 over M in to delta N by delta x minus delta M over delta y, we shall have 4 x y square plus 2 minus 3 x y square plus 1 over y times x y square plus 1, which after simplification gives us 1 over y. And therefore, the integrating factor is e to the power integral of 1 over y d y which is equal to e to the power log y end it power log y is y. So, we multiply the equation by y throughout and have x y 4 plus y square d x plus 2 x square y cube plus 2 x y plus 2 y 5 d y equal to 0, now this is an exact differential equation.

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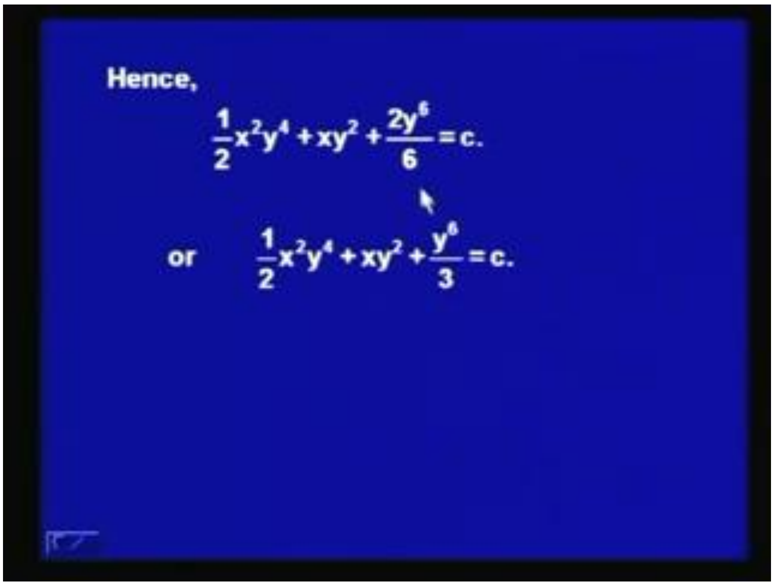


And so the general solution will be integral of M with respect to x, taking y as a constant that is x square by 2 into by 4 plus x y square plus integral of those terms of N, which do not contain x, that is integral of 2 y to the power 5 with respect to y which will give us 2 times Y to the power 6 over 6 equal to c. And hence, we have half x square y 4 plus x y square plus y to the power 6 over 3 equal to c which is the general solution of the given differential equation.

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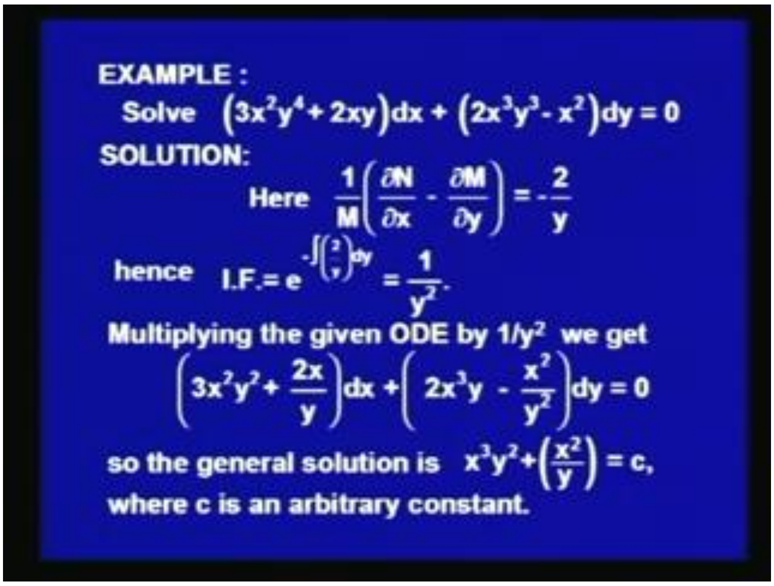


Let us take one more example of a differential equation of this type, let us solve 3 x square y to the power 4 plus 2 x y in to d x plus 2 x cube y cube minus x square d y equal to 0. Here, we find 1 over M delta N over delta x minus delta M over delta y equal to minus 2 over y and therefore, integrating factor is e to the power minus integral of 2 over y d y, which is equal to e to the power minus 2 log y.

Or we can say e to the power log y to the power minus 2, which is equal to 1 over y square, so let us multiply the given differential equation by 1 over y square we shall have 3 x square y square plus 2 x over y d x plus 2 x cube y minus x square over y square d y equal to 0. We will integrate M with respect to x that will give us x cube y square and then we will have x square over y, now we will take all the those terms of N which do not contain x and we can see here that in N both the terms contain x. So, the contribution from N will be 0 and therefore, we have the general solution of the differential equation as x cube y square plus x square over y equal to c, where c is an arbitrary constant.

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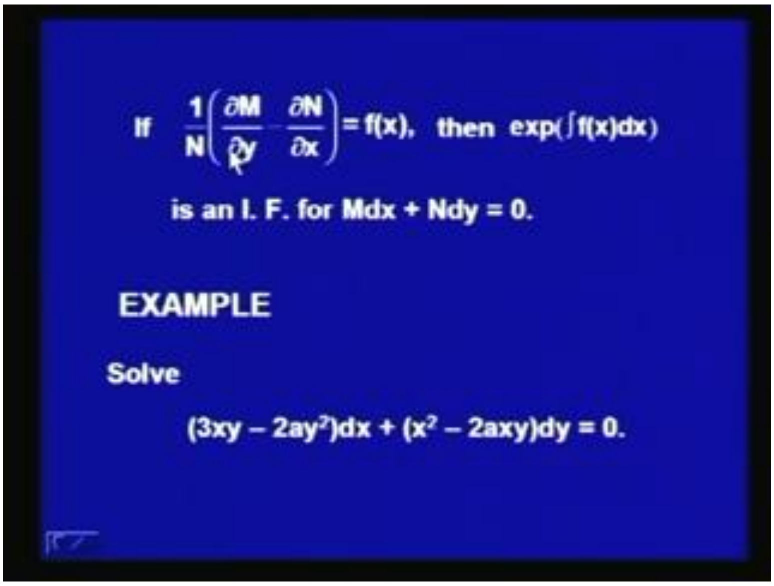


Now, if it ordinary differential equation is of the type where 1 over N in to delta M over delta y minus delta N over delta x comes out to be a function of x only, then the exponential of integral f x d x is an integrating factor, for the differential equation M d x plus N d y equal to 0. So, let us take an example of a differential equation of this type, let us consider 3 x y minus 2 a y square d x plus x square minus 2 a x y d y equal to 0.

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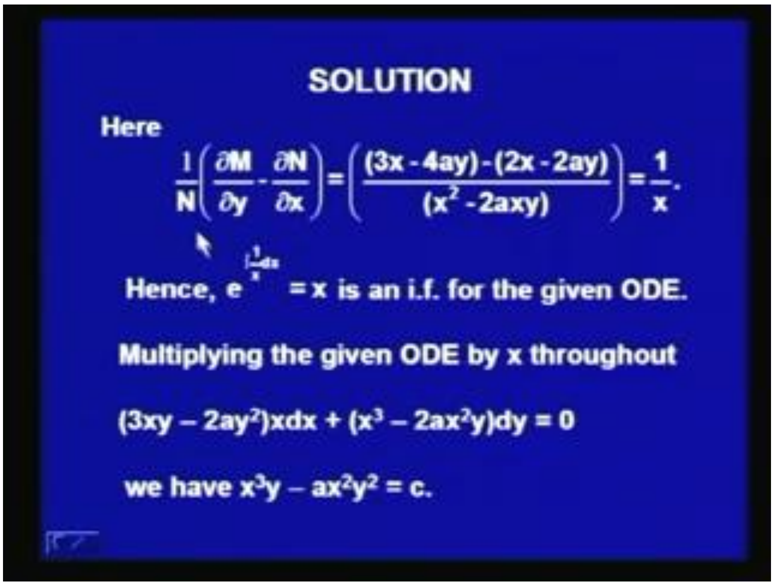


So, here we will find 1 over N into delta M over delta y minus delta N over delta x, this will be equal to 3 x minus 4 a y minus 2 x minus 2 a y over x square minus 2 a x y which is equal to 1 over x and hence it is a function of x only. So, e to the power integral 1 over x d x, which is e to the power log x and e to the power log x is equal to x is an integrating factor for the given ordinary differential equation.

Hence multiplying the given ordinary differential equation by x throughout, we shall have 3 x y minus 2 a y square in to x d x plus x cube minus 2 a x square y in to d y equal to 0, now here M is equal to 3 x square y minus 2 x y square. So, when we integrate it with respect to x, keeping y as a constant we shall have x cube by minus a x square y square and in N both the terms contain x, so the contribution from N will be 0. And therefore, we shall have the solution of the given differential equation as x cube y minus a x square y square equal to c.

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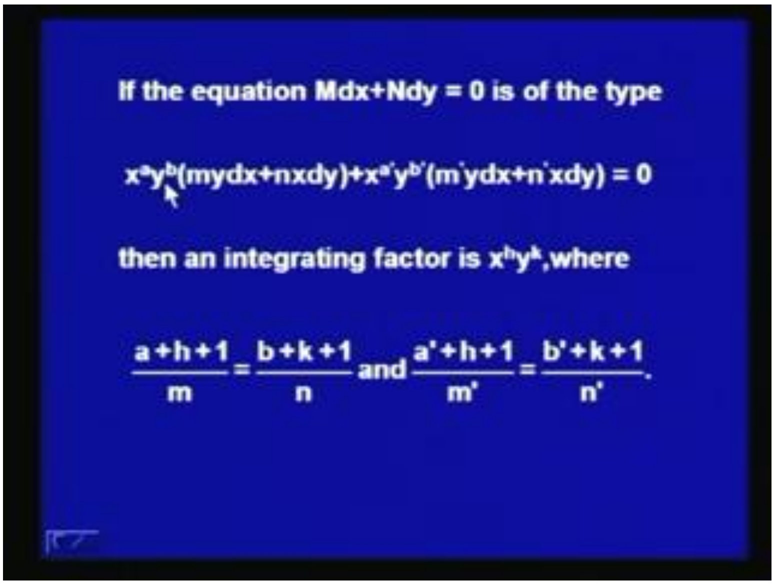


Now we consider the equation M d x plus N d y equal to 0, which is of the type, x to the power a in to y to the power b in to m y d x plus n x d y plus x to the power a dash in to y to the power b dash into m dash y d x plus n dash x d y equal to 0. An integrating factor for such a differential equation is x to the power h in to y to the power k, where a plus h plus 1 is equal to b plus k plus 1 over n and a dash plus h plus 1 over m dash is equal to b dash plus k plus 1 over n dash.

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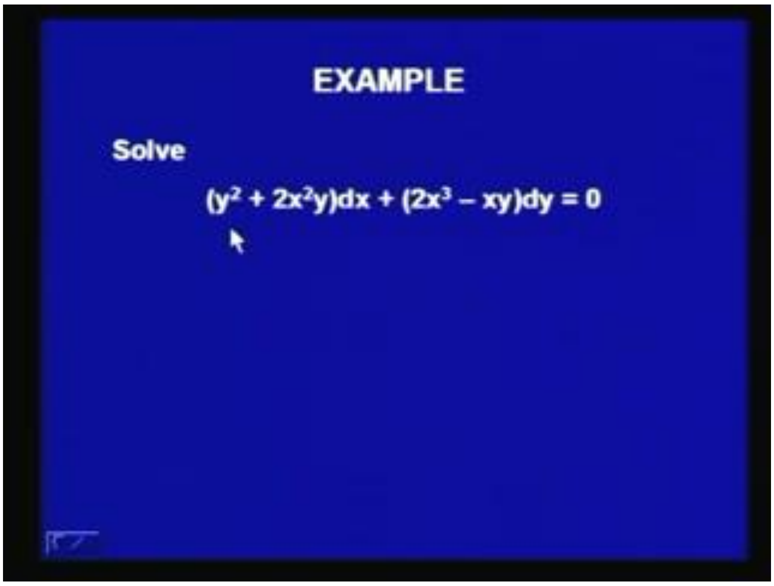


Let us take up an example of a differential equation of this type, let us consider y square plus 2 x square y d x plus 2 x cube minus x y d y equal to 0.

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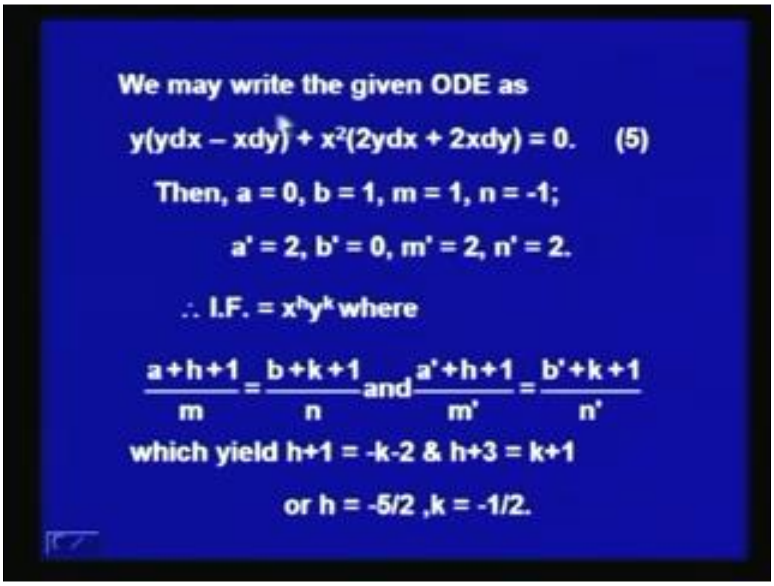


This differential equation may be written as y in to y d x minus x d y plus x square in to 2 y d x plus 2 x d y equal to 0 and when we compare it with this standard form of a differential equation of this type. We find that here a is equal to 0, b is equal to 1, m is equal to 1, n is equal to minus 1, a dash is equal to 2, b dash is 0, m dash is equal to 2 and n dash equal to 2.

And therefore, integrating factor is x to the power h in to y to the power k, where h and k satisfy the two equations a plus h plus 1 over m equal to b plus k plus 1 over n, which gives us h plus 1 equal to minus k minus 2. And a dash plus h plus 1 over m dash equal to b dash plus k plus 1 over n dash gives us the other equation involving h and k as h plus 3 equal to k plus 1, when we solve these two equations we have the value of h and k as minus 5 by 2 and minus half.

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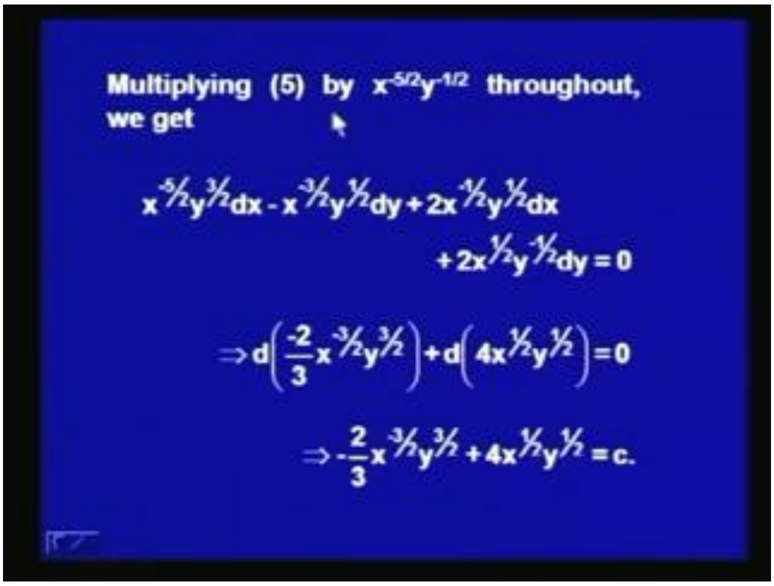


So, let us multiply the equation given equation by x to the power minus 5 by 2 n into y to the power minus half throughout. We shall have x to the power minus 5 by 2 y to the power 3 by 2 d x minus x to the power minus 3 by 2 into y to the power half d y plus 2 x to the power minus half in to y to the power half d x plus 2 x to the power half y to the power minus half d y equal to 0, which is an exact differential of minus 2 over 3 x to the power minus 3 by 2 into y to the power 3 by 2 plus an exact differential of 4 x to the power half y to the power half equal to 0.

On integration both sides, we have the general solution of the given differential equation as minus 2 by 3 x to the power minus 3 by 2 in to y to the power 3 by 2 plus 4 x to the power half y to the power half equal to c. In our lecture today, we have covered the cases of all those differential equations of first order and first degree, where the variables can be separated. The homogeneous equations and the exact equations and also those differential equations which can be reduced to any one of those three forms.

In our next lecture, we shall consider the case of linear ordinary differential equation of first order and first degree. And also those differential equations of first order first degree which can be reduced to the linear form, we shall also consider the orthogonal trajectories.

Thank you.