Mathematics - II
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Lecture - 9
Evaluation of Real Integrals

Welcome to the lecture series, on complex analysis for under graduate students. Today's lecture is, Evaluation of Real Integrals. We are learning till now, the integration methods for complex integrals, contour integrals, complex functions. We had learnt one important result, residue theorem, and residue integration method, which was very helpful in evaluation of many contour integrals.

Today, we would see that, residue theorem, does not only help in evaluation of complex integral. But, it also helps to find out the integral, or evaluation of integrals of the real functions; and the real integrals. In this series, let us first see the definite integrals. First, we would see the definite integrals of those functions, which are involving sin and cosine function.

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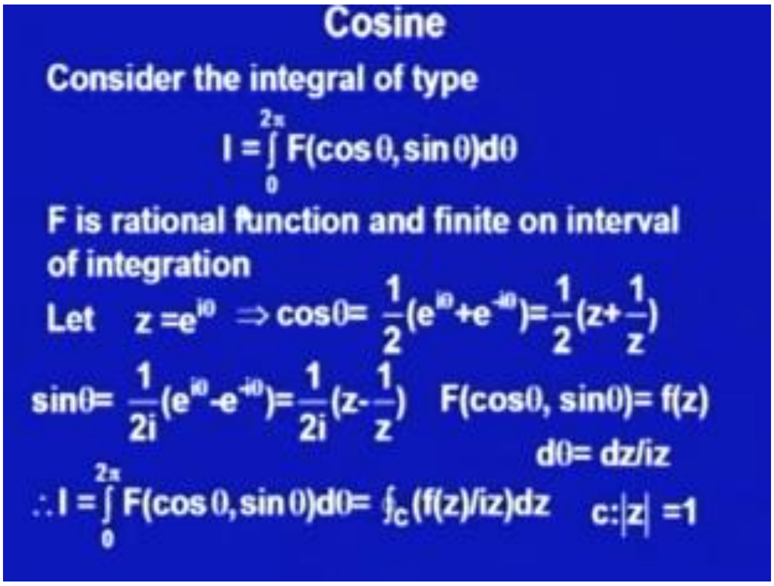

So, let us consider the integral of type, 0 to 2 pi a function of cos theta sin theta, integrated with the respect to theta. This F, cos theta sin theta, this is a real function. Moreover, we are assuming that F is a rational function. And it takes some finite values in whole interval of integration. That is it never goes, infinite at any point inside the interval of integration. Now, we would first actually change this real integral, in to contour integral. Or integral of the complex function.

And then, we would use this residue theorem to solve this integral. So, let us say we are making this transformation, z as e to the power i theta. We do know that, cosine theta is 1 by 2 e to the power i theta plus e to the power minus i theta. Or we could write it has 1 by 2 z plus 1 by z. Similarly, we do know that sin theta is 1 upon 2 i e to the power i theta minus e to the power minus i theta. That is we could write as 1 upon 2 i z minus 1 by z.

Now, what is this z is equal to e to the power i theta. For the range your having is, that is for theta is ranging from 0 to 2 pi. So, here if I take theta is ranging from 0 to 2 pi, z is equal to e i theta. This is nothing but, the parametric equation of unit circle. So, we are having is, that is now this function. If I substitute cosine theta and sin theta, as the functions of z and 1 of z, we would get actually a function of z only.

So, F cosine theta sin theta would be, say a small f of z some function of z of course, it involve z and 1 by z. Moreover, d z would be, actually d theta you could write as d z from here e to the power i theta d theta. So, d theta would be d z upon i z. What it say is that, we could write this integral 0 to 2 pi a function. That it is a capital F of cosine theta sin theta d theta, as a contour integration on the unit circle C of function small f z d z upon i z. That is d theta is d z upon i z.

So, rather I am writing it as integral of f z upon i z d z. On the contour C on where, what is this contour C? This contour C is nothing but, the unit circle mod z is equal to 1. So, you see, if I am having this definite integral, which is involving sin theta and cosine theta and ranging from 0 to 2 pi. Something like that, we could get is that, we can change it to the contour integral on the unit circle. Let us see one example to understand this method.

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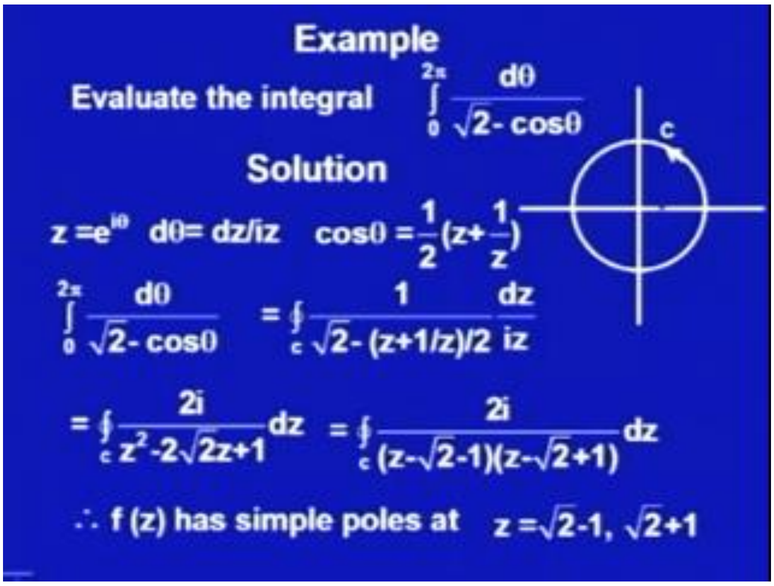

Evaluate the integral 0 to 2 pi d theta, upon is square root to minus cos theta, if you do remember your school mathematics. This integral is not really very nice integral, on which we would be happy to see that. We can do this integral. Let us see using this residue method, how we could find it out. And rather, actually we will do today examples, in which so many integrals you had use as the formulae.

So, let us first find it out that. Using the method which just now we had learn. What we would do is, we would transform this theta 2 z. Using this transformation, z is equal to e to the power i theta. So, d theta would be d z upon i z. And here, it is involving only cosine theta. So, cosine theta we do know, we could write as half of z plus 1 by z. And then, this integral would be change to the unit circle.

So, integral 0 to 2 pi d theta upon square root 2 minus cosine theta. I could write as contour integral, on the close contour C, 1 upon root 2 is as such cosine theta. We are substituting as z plus 1 by z by 2. And d theta, we are substituting d z upon i z. And this contour C is your unit circle. Now, simplify this function. So that, we do have what this function is actually.

If we do simplify it, you do get it as upon 2 here is, and this i is the here. So, I would take this i up and I would write it as minus i. So, I would get 2 i and from here, we would get actually square root 2 z minus z is square minus 1. So, minus sign I am taking out, I would get z square minus root 2 z plus 1. So, what we have got, this integral is simplified form 2 i upon z square minus 2 root 2 z plus 1 d z on the contour unit circle. Now, this function is of the form is some constant upon a function.

So, certainly it would see is to be analytic at certain points, what would be those points. Of course, the points were this denominator is 0. So, we have to find out, where this denominator is 0. Let us see is, that is we could write it out as. Because, it is z square minus 2 root to z plus 1, we could just get it is fractions as of z minus root 2 minus 1 in to z minus root 2 plus 1. Now, you see is, this says is that 0's of this denominator would be at your root 2 plus 1; and at root 2 minus 1.

That say is this integrant is a function, which is having isolated singularities, at root 2 plus 1 and root 2 minus 1. My contour of integration is unit circle. So, certainly my root 2 minus 1 would be inside this was 1, but root 2 plus 1 would be outside. So, f z has simple poles at root 2 minus 1 and root 2 plus 1. So, root 2 minus 1 is certainly the point, which is inside the contour. So, now we are having is that, the given integral can be transform to a contour integral of a function.

And the function is such that, it has isolated singularity at a single point inside the given contour. And the contour is our units circle. So, now I can use the residue integration method. That say is, I have to find out the residue of this function, at this isolated singularity, root 2 minus 1, because this is outside our contour reason.

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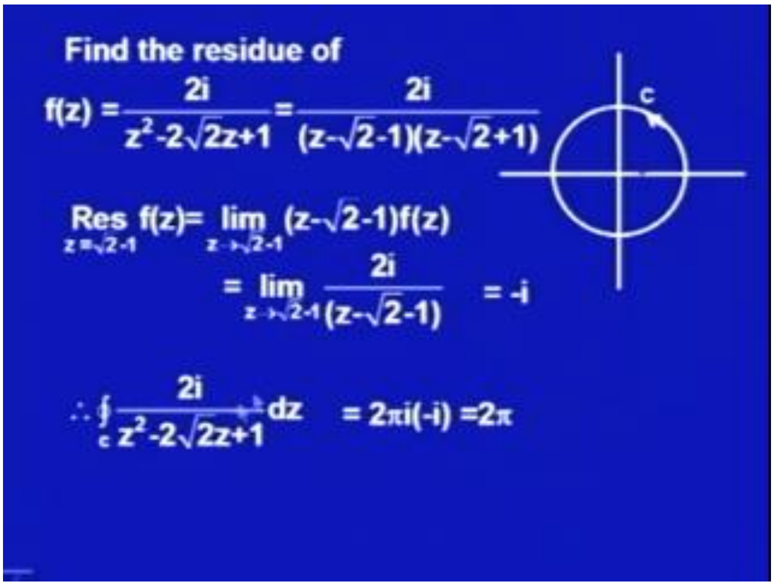

So, let us find out the residue of the function, 2 i upon z square minus 2 root z plus 1. That we are writing it as the fraction form, that is 2 i upon z minus root 2 minus 1 in to z minus root 2 plus 1. I have to evaluate it is residue. I have to calculate actually it is residue, at the isolated singularity root 2 minus 1. So, I would use our first formula for simple pole. For evaluation of residue, which says is that residue at z is equal to z naught, were z naught is a simple pole.

Is limit as a z approach is to z naught z minus z naught in to f z. So, by that formula, we would get residue at root 2 minus 1 of f z. As limit z is approaching to root 2 minus 1 of z minus root 2 plus 1 into f z. That would be actually from here, this portion would get cancel it out. Ad what we would get,2 i upon z minus root 2 minus 1 limit as z is approaching root 2 minus 1. So, this limit if we have to calculate, we have to simply evaluate at z is equal to root 2 minus 1.

So, if I write z is equal to root 2 minus 1 here. I would be getting is root 2 minus root 2, there would get cancel it out. And minus 1 minus 1, that is minus 2. So, I would be getting it as minus i. So, now using this residue method, we would get the integral of 2 i upon z square minus 2 root 2 z plus 1, on the unit circle. The contour is unit circle, is 2 pi i into the residue at the singularity. That is at root 2 minus 1 is minus i, we are getting is 2 pi. So, we have got this integral of the function 0 to 2 pi 1 upon this square root 2 minus cos theta d theta as 2 pi. So, you should find out, that is such a simple way, we could find out this integral using residue method.

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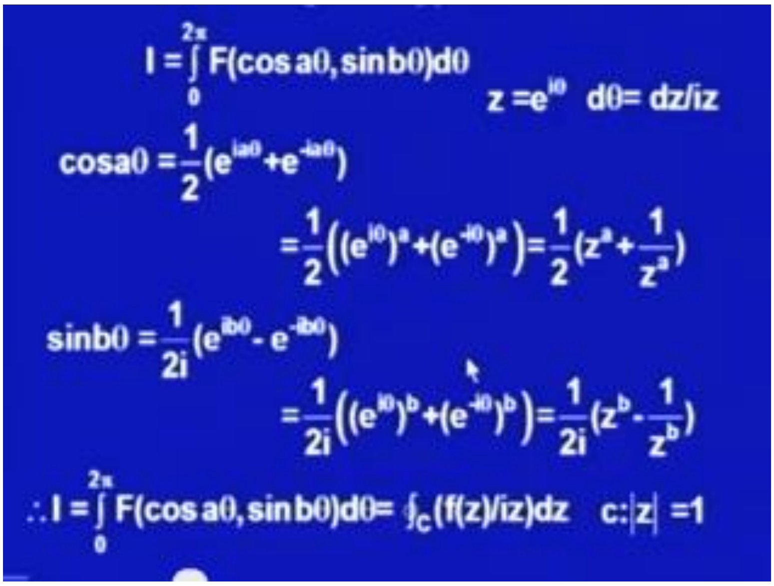

Now, let us consider the integrals of this type. 0 to 2 pi F a function of cos a theta and sin b theta. First one I have take is that the function of cos theta and sine theta. Now, we are having is cos a theta and sin b theta. Can we change this kind of integrals, also in contour integrals, so that we could use the residue theorem. Let us see again, what we are assuming is that my F is rational function. And it does not becomes infinite, in the interval of integration.

Use the same transformation z is equal to e to the power i theta. Then, we do know that d theta as such this one. So, we do know that cos a theta, we could write as 1 by 2 e to the power i a theta plus e to the power minus i a theta. Now, replay little be told, because this is exponential function. And with many good properties of exponential function. We could write it as, e to the power i theta to the power a plus e to the power minus i theta to the power a.

E to the power i theta from here is z. So, it would be actually z to the power a. And e to the power minus i theta would be actually 1 upon z. So, it would be actually 1 upon z to the power a. So, what we have got, cos a theta we could write as, 1 upon 2 z to the power a plus 1 upon z to the power a. Similarly, this sin b theta, we could write as 1 upon 2 i e to the power i b theta minus e to the power minus i b theta; which I could write as, this 1 upon 2 i such e the power i b theta.

We could write e to the power i theta to the power b, and minus e to the power minus i theta to the b. So, what we would be getting is, z to the power b minus 1 upon z to the power b into 1 upon 2 i. That says is that, again we can transform this kind of integrals, with taking this transformation z is equal to e to the power i theta. And theta is ranging from 0 to 2 pi, says is that on the contour integration on the unit circle. So, integral on the unit circle of, now this function is... Suppose this function is a function of z is a small f z point i z d z. Let us see an example, again for this kind of things.

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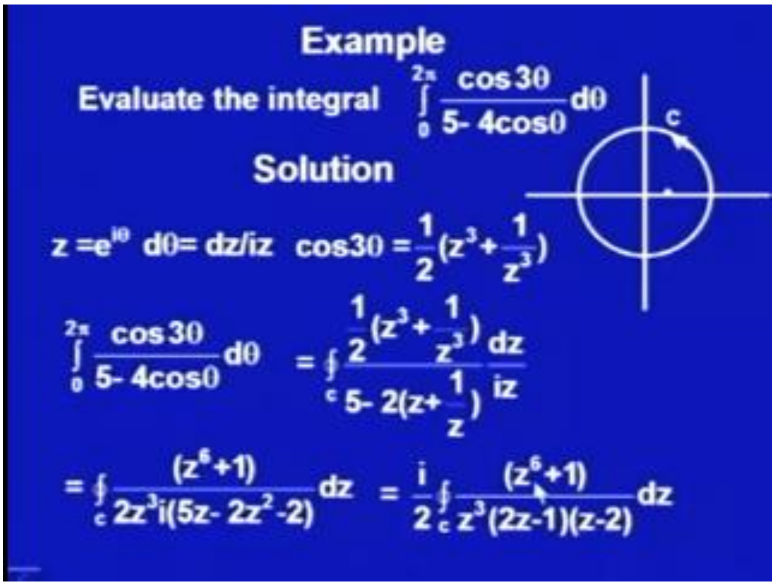

Evaluate the integral 0 to 2 pi cos 3 theta upon 5 minus 4 cos theta d theta. I thing this you might be remembering that, this integral. We have solved somewhere, in your solution. we would transform this integral into contour integration. For that, we will transform this theta to, as using the transformation z is equal to e to the power i theta. Then, d z d theta is nothing but, d z upon i z. Now, cos 3 theta is 1 upon 2, cos 3 theta we do we have formula that, cos a theta is z to the power a plus 1 upon z to the power a into 1 by 2.

So, here your a is 3, so it would be half z cube plus 1 upon z cube, and cos theta is of course, half z plus 1 by z. So, now substitute all these value, so in this integral. So, the integral 0 to pi cos 3 theta upon 5 minus 4 cos theta d theta, can be written as the contour integral. Along the unit circle C, as half z cube plus 1 upon z cube upon 5 minus 4 times.

So, 4 is here 1 by 2, so minus 2 times z plus 1 by z into d z upon i z d theta, we are replacing by d z upon i z.

This would be on our unit circle. Now, let us simplify this function see the numerator. Numerator, here is we would get is z to the power 6 plus 1 upon 2 z cube. And in the denominator, we would get from here 5 z minus 2 z square minus 2 upon z. And this 1 upon i z is also there. So, this z would get it cancel out with this z. And this is z cube would get in the denominator, moreover this 2 also will get commons in the denominator.

So, what we would get actually in the numerator, z to the power 6 plus 1. 2 z cube, that is coming in the denominator. I is also going in the denominator, 5 z minus 2 z square minus 2. Now, this I will take in to the numerator, that says is 1 upon i is actually minus i. And this minus sin I would take inside this one. So, I would get this denominator as 2 z square minus 5 z plus 2. So, I could rewrite this integral as i upon 2. Contour integration on the unit circle C, z to the power 6 plus 1 upon z cube.

Now, 2 z is square minus 5 z plus 2. We can factorize it, we could get 2 z minus 1 into z minus 2. So, this given real integral, we have changed to the contour integral. And the contour integral of a function, which is a of the form p z upon q z. So, certainly this function will have some, because this denominator is having some 0's. So, this will have isolated singularities, let see what are the 0's. The denominator does has 0 at z is equal to 0, z is equal to half and z is equal to 2. Now, let see what kind of this denominator, this 0's. And so accordingly what kind of isolates singularities. We are having for this function. And how many of those isolated singularities are lying inside our contour reason.

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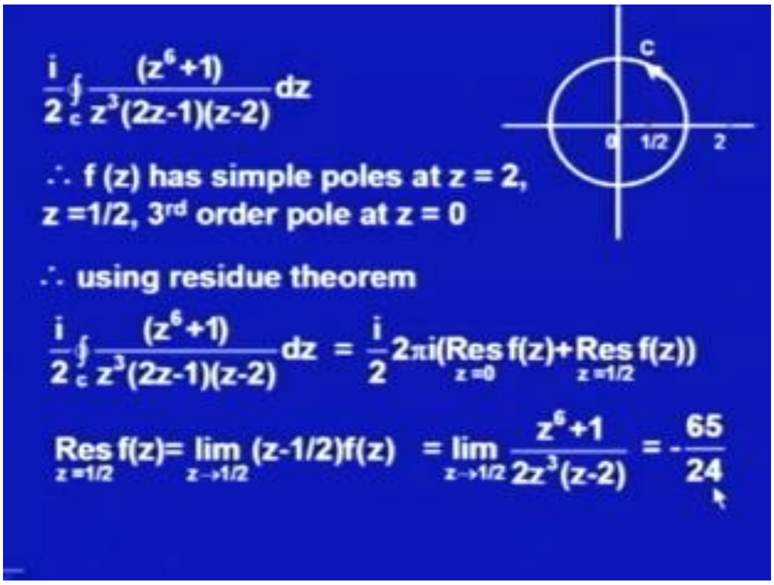

So, i upon 2 z to the power 6 upon z cube 2 z minus 1 z minus 2 d z. This we have to evaluate on the unit circle. Now, this has simple pole at z is equal to 2, on the z is equal to 1 by 2. But, you see that denominator has this another 0, 0 which is of the order 3. So, this f z will have a third order pole, at z is equal to 0. So, now what we are having is, we are having three isolated singularities, 0, half and 2.

Out of these three isolated singularities, if we see in our region of this contour. Contour is unit circle. Unit circles means, this 0 would be inside this unit circle. Half would be of course, certainly inside the circle. But, this 2 would be outside this unit circle. So, we have got that this contour is having two isolated singularities, inside the contour. So, I am evaluating a contour integration on the close contour; where the function f z is analytic, except at some finite number of isolated singularities.

That says is, we can use the... And those isolated singularities are actually in this particular case, are the poles y less. Actually at z is equal to half, we do have simple pole. But, at z is equal to 0, we are having a third order pole. So, we can use the residue theorem. And for that, we require the poles at 0 and that half. So, let us see that, using the residue theorem. We could say this integral i by 2 contour integration on C of z to the power 6 plus 1 upon z cube in to 2 z minus 1 in to z minus 2 d z, should be i by 2 times.

This integral is using the residue theorem, 2 pi i times residue at z is equal to 0 and at z is equal to half. So, we have to calculate the two residue, that z is equal to 0 and z is equal to half. Let us first see, this z is equal to half. Because this is a simple pole, because it is a simple pole. And my function is z to the power 6 plus 1 upon z cube in to 2 z minus 1 in to z minus 2. I would simply because in this fraction form, I would simply use the first method of calculation of simple residue at simple pole.

As limit z is approach in to half z minus half into f z. If I am multiplying the z minus half, z minus half we could write as 2 z minus 1 upon 2. What I would get is, z to the power 6 plus 1 upon 2 z cube in to z plus 2. So, limit as z is approach in to half of z plus 6 plus 1, z to the power 6 plus 1 upon 2 z cube into z minus 2. So, as z is approach in to half, this limit would be just evaluating, it at z is equal to half. Z is equal to half says is, z to the power 6 would be 1 upon 2 to the power 6 plus 1.

And here, what we would get is 2 in times 1 upon 2 to the power 3. And here is half minus 2, that says is minus 3 by 2. So, minus 3 by 2 and this 2, I would get it cancel it out, I would get here minus sign. And 1 by 2 cube, here we are getting is 1 by 2 6. So, we write it 1 plus 2 to the power 6, upon 2 to the power 6 into 1 by... So, we would get 2 to the power 3 here into minus 3. So, finally what we would get it is, actually 2 to the power 6 plus 1.

2 to the power 6 is nothing but, 64 it is very easy. We always do this kind of calculation. So, 2 to the power 6 is 64 plus 1 is 65, upon your 2 to the power 3. That is your 8 into minus 3 that is minus 24. So, what we are getting is, minus 65 upon 24. So, residue at the simple pole, we had calculated that z is equal to half is minus 65 by 24. Now, what is remaining is residue at z is equal to 0. 0 is not a simple pole, it is a third order pole.

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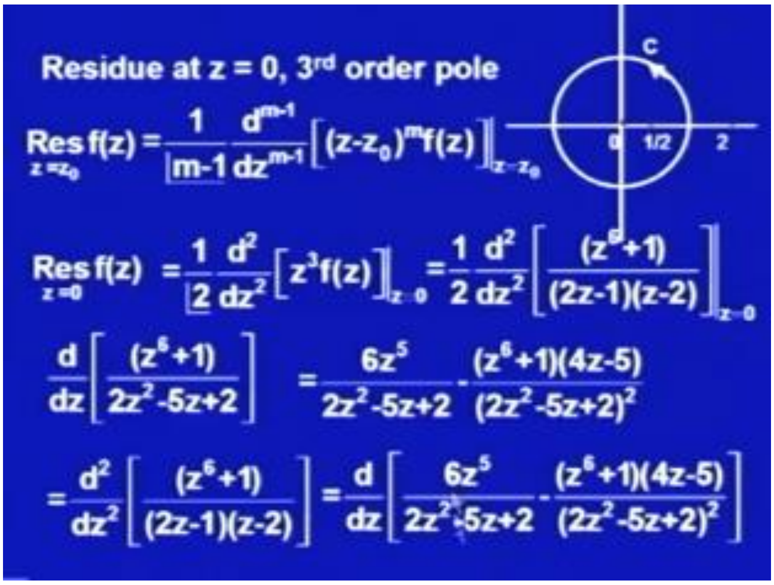

So, we have to find out the method of calculation residue, of the third order pole. We would use the formula, that is for residue. To calculate a residue at a m th order pole at z naught. It is 1 upon factorial m minus 1, the m minus 1 of derivative of z minus z naught to the power m into f z evaluated at z is equal to z naught. So, here my m is your 3 and z naught is 0. So, what we would get from here, that residue at z is equal to 0 for our given function f z, would be 1 upon factorial 2.

The second derivative of z minus z naught, that is z cube into f z evaluated at z is equal to 0. F z is a z to the power 6 plus 1 upon z cube, in to 2 z minus 1 into z minus 2. So, z cube would get cancel it out. We would get here, 1 upon 2 second derivative z to the power 6 plus 1 upon 2 z minus 1 into z minus 2, evaluated at z is equal to 0. Now, let says is, we have to find out the second derivative of this function, we will go 1 by 1. Let us first find out the first derivative and then, the second derivative.

So, the derivative of z to the power 6 plus 1 upon 2 z square minus 5 z plus 2. We will simply use, the method for the multiplication of two functions. That would be 5, 6 z to the power 5 upon 2 z square minus 5 z plus 2 as such minus the integral of this function as such. And derivative this, because there is 1 upon, so that is why it is minus upon 2 z square minus 5 z plus 2 whole square.

And the derivative of this is, you see 4 z minus 5. That would come in the numerator. So, we have got 6 z to the power 5 upon 2 z square minus 5 z plus 2 minus z to the power 6 plus 1 into 4 z minus 5 upon 2 z square minus 5 z plus 2 whole square, this is the first derivative. The second derivative would be the derivative of this first derivative. So, now let us calculate the derivative of this, this is again the derivative of two fractions. So, again we will go with the multiplication of functions, rather than the formula for the fraction.

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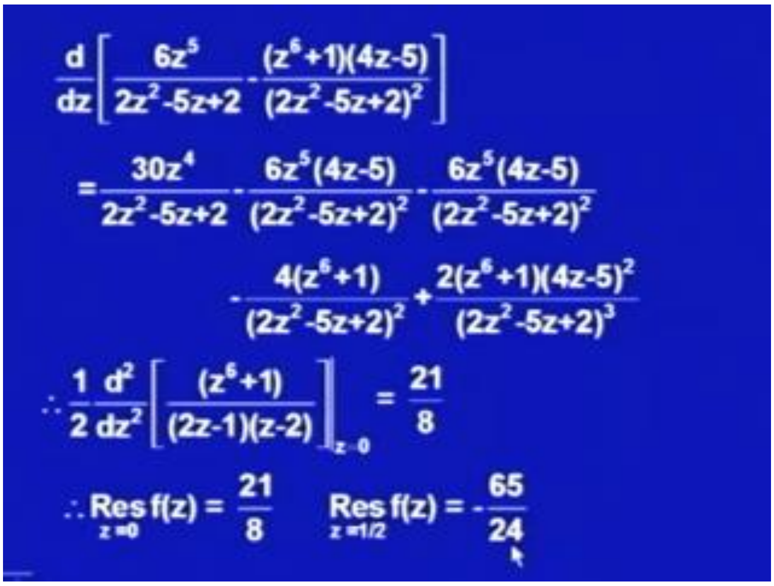

Calculate this derivative, so first for this first one. Derivative of this would be, here the first function. That is 6 into 5, that is 30 times z to the power 4 upon this function is as such. Then, this function as such that, is 6 z to the power 5 is as such. And derivative of 1 upon 2 z square minus 5 z plus 2. Just the last integral, we had calculate this derivative. That is minus 1 upon 2 z square minus 5 z plus whole square into 4 z minus 5. Now, come to the derivative of this function.

This function, we are having is at ((Refer Time: 25:52)) the multiplication of the two function, in the numerator. And the one function in the denominator. So, first the numerator one and denominator, we will treat as multiplication 1 upon this one. So, first function this one is z to the power 6 plus 1, it is derivative is 6 z to the power 5. And all other functions are as such.

Then, will go for the derivative of the second function. And all other functions are such the derivative of the 6, second function is 4. So, we would get 4 times z to the power 6 plus 1 upon 2 z square minus 5 z plus 2 whole square. Now, will come to the derivative of 1 upon 2 z square minus 5 z plus 2 whole square. This we will treat as 1 upon. So, it is derivative has to be minus 2 upon, this whole function cube and the derivative of this one.

And all the functions remain as such. So, we would get is minus and minus, will get plus. And 2 upon 2 z square minus 5 z plus 2 whole cube. And the derivative of this inner one is 4 z minus 5. That we have to multiply to the numerator, z to the power 6 plus 1 in to 4 z minus 5, multiplied with that is it is square. Now, what we have to actually evaluate. We have to evaluate the second derivative, at z is equal to 0 multiplied with half.

Second derivative, we had calculated, now we have to evaluate it at z is equal to 0. If I have to evaluated z is equal to 0, you see that is as such we have got very lengthy expression. But, with z equal to 0 the first expression is 0, with z is equal to 0 the second expression is 0. As well as the third expression is also 0. So, the first three expressions would vanished at z is equal to 0. When in the fourth expression, we have keeping z is equal to 0.

What I would get in the numerator as 4, in the denominator this is 0, this is 0, here is 2, 2 square is 4. So, what we would get minus 4 upon 4. And the last expression this is 2, here we would get 1, here we would get 5 square, so 2 in to 5 square. And denominator is your 2 to the power 3. So, 1, 2 is getting cancel out, we are getting as a 4. And the numerator is 5 square, that is 25. So, here we have got as minus 4 upon 4. And here, we have got is plus 25 upon 4.

That is net solution would be actually 21 by 4 multiplied with half, so we get 21 by 8. So, we have got the residue at z is equal to 0 is 21 by 8. While as that, the residue at z is equal to half plus minus 65 by 24. Now, we had calculate our both residues.

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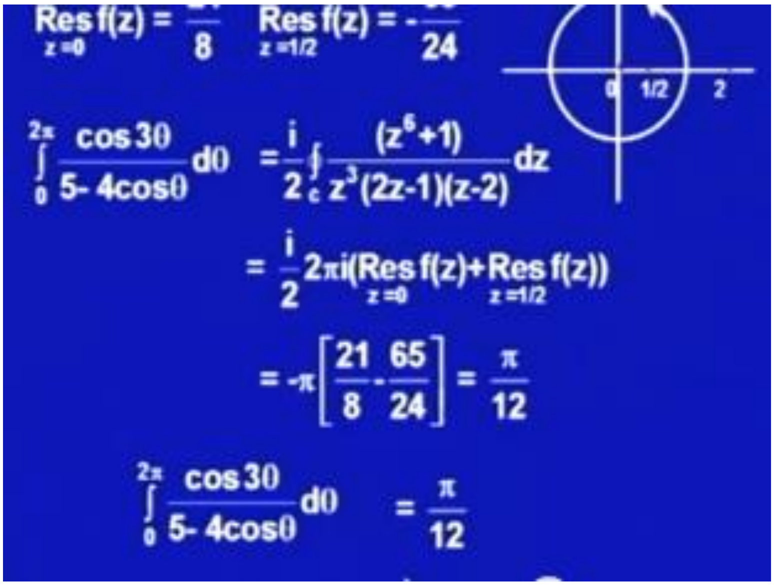

So, now we are ready to answer our questions. So, we are having is that the residue, at 0 is 21 by 8. Residue at half is minus 65 by 24. These are the two isolated singularities, inside this unit circle. So, the integral of 0 to 2 pi cos 3 theta upon 5 minus 4 cos theta d theta; which we have change into the contour integration as i upon 2. Integral along the unit circle of z to the power 6 plus 1 upon z cube into 2 z minus 1 into z minus 2 d z.

This is by residues theorem, has to be i by 2 times 2 pi i residue of f z, at z is equal to 0. And of f z at z is equal to half. We had just substitute this 21 by 8 minus 65 by 24. If we make it one this would be 63 by 24, that is we are getting minus 2 by 24. This i and i would give minus 1, 2 is getting cancel it out. So, I am getting is minus pi outside, here i am getting is minus 2 upon 24, that it is pi by 12. So, finally we have got the answer. That integral of 0 to 2 pi cos 3 theta upon 5 minus 4 cos theta d theta is pi by 12. Now, let us see one more formula. Let us see one more example.

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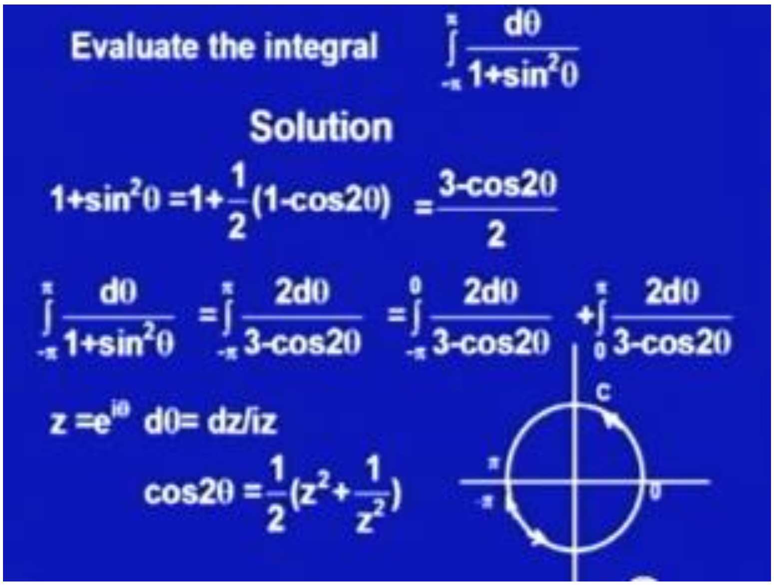

Evaluate the integral minus pi to plus pi d theta upon 1 plus sin square theta. Now, you see in this integral, our interval of integration is from minus pi to plus pi. Integrand is also having a square of sin function. So, first we would, make it in to the form of either cosine theta sin theta form of. Or cosine a theta and sin a theta form, only then we would apply our results. So, first take this function, the denominator 1 plus sin square theta.

This we do know, we could write sin square theta as 1 minus cos 2 theta upon 2. So, 1 plus 1 upon 2 1 minus cos 2 theta, this says is it is 3 minus cos 2 theta upon 2. So, what we have got now, our integral minus pi to plus pi d theta upon 1 plus sin square theta. We could write this as integral, from minus pi to plus pi 2 upon 3 minus cos 2 theta d theta. Now, this has come to the form of, f cos a theta sine b theta.

So, we could use our transformation, or we could... But, here the interval of integration, is actually from minus pi to plus pi. In our integrals, we have take the integration from 0 2 pi. Now, let us see is that is what this minus pi to plus pi is. I can rewrite this integral as the sum of 2 integrals. For the first integral, it should be minus pi to 0 of the function. And in the second one, this integral should be from 0 to pi.

Now, if I rather than, having it as from 0 to 2 pi. If I am breaking it in to two parts; one is from 0 to pi, another is minus pi to 0. So, if I apply z is equal to i theta, my theta would range say in the second interval as 0 to pi. Only that says is we would have only half of the unit circle. What would be the range minus pi to 0, let see it here. So, first we are going to make the same our substitution, that z as e to the power i theta.

So, my d z, d theta would be d z upon i z. And cos 2 theta from here would be 1 upon 2 z square plus 1 upon z square. Now, we see these limits, which we are having minus pi to plus pi. This I am breaking as minus pi to 0 and 0 to pi. So, let us first see this 0 to pi, 0 to pi... So, now this we are saying is that, we are talking about these say polar coordinates. So, I would have R is 1. And my theta is ranging from 0 to pi. That means, on this unit circle we are going it is counterclockwise.

And the first integral, actually we are ranging from minus pi to 0. So, if we see in the polar coordinates, this theta if we are measuring. This way of the measurement is call the positive. And this way of the measurement is called negative. So, minus pi would be actually, we starting from 0 and running over here. So, this point would be minus pi. So, when we are saying is minus pi to 0. It says is we have to move in the circle, unit circle from minus pi to this 1.

That says is the this integral 0 to pi, we have covered. This half circle, upper half circle. And from minus pi to 0, we are covering lower half circle. What is the direction, 0 to pi the direction is counterclockwise, minus pi to 0 again the direction is counterclockwise. That says is actually this integral, if I join it minus pi to plus pi. We are actually moving from here to here, that is the full circle. That says is, it is same thing as having, we can again change it to the unit circle.

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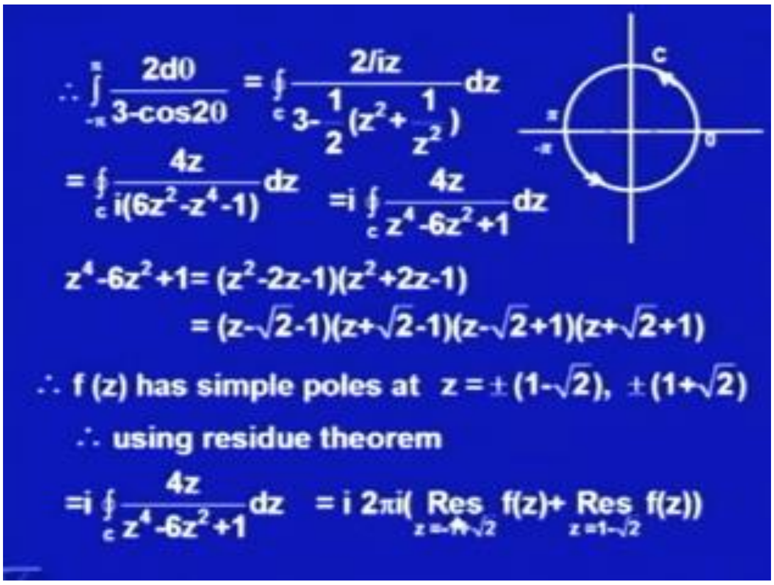

So, let us see, we do have integral minus pi to plus pi 2 d theta 3 minus cos 2 theta. Changed as the integral, contour integral on the unit circle. Here, we are substituting this d theta as d z upon i z. So, that i z have written in the numerator 2 upon i z, and 3 minus cos 2 theta, cos 2 theta is 1 by 2 z square minus plus 1 upon z square. Simplify it, this would be z to the power 4 plus 1. And here, we would get 3 z square. And 2 is also, so 6 z square.

And this two we would get up side, that is it would be 4. Then, z square will be go up. So, this we would get only z, because z square upon z is z. So, what we are getting in the numerator, 4 z. In the denominator, i is coming in the denominator, 6 z square minus z to the power 4 minus 1. Again 1 upon i, I would write as minus i. And that minus sin i would change here, that is z to the power 4 minus 6 z square plus 1.

So, we do write that our integral change as to, i times the contour integration on the unit circle of the function. 4 z upon z to the power minus 6 z square plus 1. Again my function is coming as a rational function. That is which is the ratio of 2 functions p z and q z. P z is 4 z, q z wherever this would have 0's at those points, this function would have singularity. We will find out, that is what kind of those singularities are. And we have to find out the integral.

And of course, it is looking that, this would have 0's are always isolated 0. So, the singularity will also be always are isolated singularities. Let us see z to the power 4 minus 6 z square plus 1. We would write it as z to the power 4 minus 2 z square plus 1 minus 4 z square. So, the z to the power 4 minus 2 z square plus 1, would be z square minus 1 whole square minus 4 z square. That is minus 2 z whole square, we would use x square minus a square as a x minus a in to x plus a.

So, we would get z square minus 2 z minus 1 into z square plus 2 z minus 1. Now, let us see one by one this, z square minus 2 z minus 1. So, if I take it here plus 1, I would get minus 2. Z is square minus 2 z plus 1, that is z minus 1 whole square minus a square 2 of whole square. Again we would same in equality. So, first one I would get, z minus 1 minus root to z minus 1 plus root 2.

Similarly, here we would get z plus 1 minus root to z plus 1 plus root 2. So, I am getting all the factors, as z minus root 2 minus 1 into z plus root 2 minus 1. Here z minus root 2 plus 1 z plus root 2 plus 1. So, what we have got this denominator is having 0's, at 1 plus root 2, at 1 minus root 2. Here again 1, minus of 1 plus root 2 and minus of minus 1 minus root 2. That is we are having, that f z has simple poles and all the 0's are simple 0's. So, we would have the simple poles are plus minus 1 minus root 2. And plus minus 1 plus root 2.

So, my function is having 4 isolated singularities. At plus minus 1 minus root 2 and plus minus 1 plus root 2, our contour is unit circle. So, certainly plus minus 1 plus root 2, 1 plus root plus side would be outside this one, minus would be outside this one. So, these two isolated singularities, would be outside our pole, our reason from the interior of the contour. We would have only two singularities, which are inside the interior.

One is your 1 minus root 2 and another is minus of 1 minus root 2, that is root 2 minus 1. So, we are having now, we have to evaluate a contour integral of a function, which is analytic on inside to the unit circle. Except at two points, two isolated singularities. So, I can use the residue theorem. And find out the integral using the residues, at both the isolated singularities. Let see, using residue theorem, I would get this integral, i times in contour integral of 4 z upon z to the power 4 minus 6 z square plus 1 d z. As i in to 2 pi i times, this integral is 2 pi i times, residue at z to the r z is equal to minus 1 plus root 2. And residue at z is equal to 1 minus root 2 of the function f z, were f z is this function.

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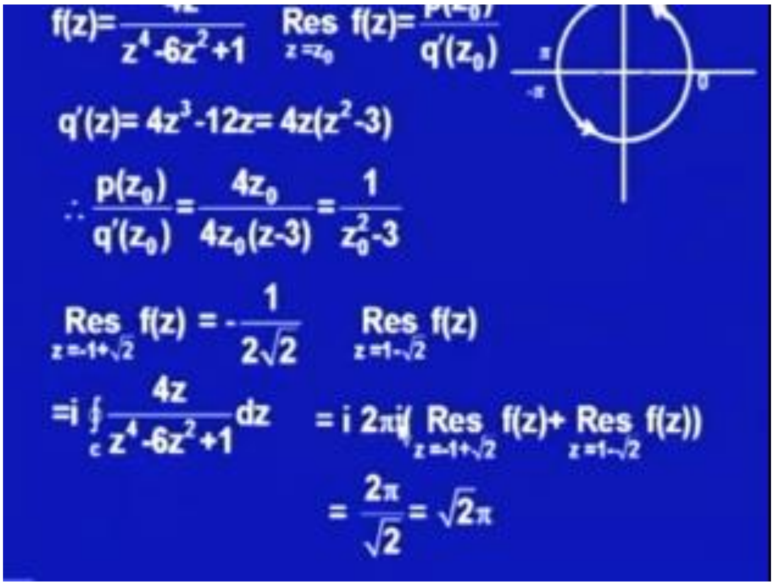

So, now first will calculate our residues. My function is 4 z upon z to the power 4 minus 6 z square plus 1, this of the form p z upon q z. We would use and our isolated singularities, both are simple poles, that says is, we would use the formula for p z naught upon q dash z naught. So, let us find out. So, will use this formula, residue at z is equal to z naught of a f z is p z naught upon q dash z naught, when z naught is a simple pole.

Q z, q dash z would be 4 z cube minus 12 z. We could rewrite it as, 4 z times z square minus 3. So, what will be p z naught upon q dash z naught, this would be p z is 4 z. So, 4 z naught upon 4 z naught into z naught minus 3, z naught is square minus 3, this would should be z naught square minus 3. So, what we would be getting 1 upon z naught square minus 3. This is what is p z naught upon q dash z naught.

So, the residue at z is equal to z naught, would be actually 1 upon z naught square minus 3. Now, use this simple formula, for calculation of a residue at both the points. That is minus 1 plus root 2, it would be 1 upon minus 1 plus root 2 whole square. What we would get, 1 plus 2 minus 2 root 2. 1 plus 2 is 3, 3 minus 3 is canceled out, I would get 2 root 2 with the minus sign. So, we would be getting minus 1 upon 2 root 2.

Similarly, the residue at z is equal to 1 minus root 2. Square of 1 minus root 2, again 1 plus 2 minus 2 root 2. 1 plus 2 and 3 we canceled out, we would again get minus 1 upon 2 root 2. So, now we are getting... So, use this residue theorem, the integration would be i times 2 pi i residue at minus 1 plus root 2 plus residue at 1 minus root 2; which is i into i. That is minus 1, 2 pi.

Here what we are getting is, minus 1 upon 2 root 2, again minus 1 upon 2 root 2. We would be getting it as, minus 1 upon root 2, minus 1 is already outside. I would get 2 pi upon root 2, that is square root 2 pi. So, we have got one more formula. That integral of minus pi to plus pi 1 upon 1 plus sin square theta is square root 2 pi. This you would remember, that you have used it as formula. And prove of this was tedious.

Of course, here also it is not very simple, but the methodology is much simpler. So, we had seen today that, the residue theorem or residue integration method is not helpful, only in evaluation of complex integrals. Rather, we can use it in evaluation of real integrals as well. Today, we had seen the evaluation of definite integral. And those definite integrals, we had seen which are involving the sin and cosine function.

Now, from here you could see, we could actually solve a lot kind of real integrals; where the function of that variable real variable. We are changing it 2 into sin theta and cosine theta. And we are getting the integral of the form 0 to 2 pi. And function of sine theta cosine theta. Whether, it is simply sin theta cosine theta. Or in the power of sin square and cosine square, are we changing them it to the sine m theta cosine m theta.

All those integrals, we could solve it using this residue integration method. And it gives a simple way of solving or you could say is, one more way of solving the real integrals. We would learn some more applications of residue integration method; and residue theorem, in coming lectures. Today, we have learn only for the definite integrals. So, that is all for today's lecture.

Thank you.