Mathematics - II
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Lecture - 8
Residue Theorem

Welcome to the lecture series on complex analysis for undergraduate students. Today's lecture is on Residue Theorem. We have learnt the method of integration for complex function. We had learnt contour integration and residue integration method. In the residue integration method, we are continuing today.

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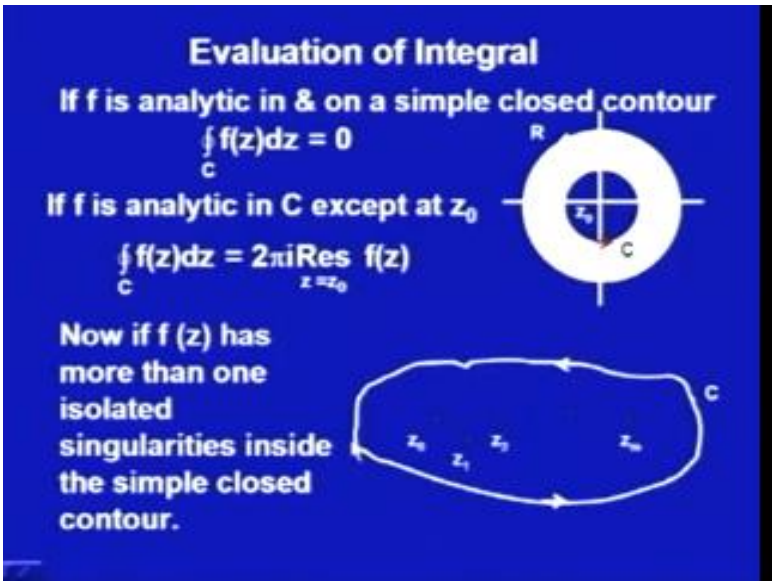

So, we had learnt that, if f is analytic in and on a simple closed contour C. Then, the integral of that analytic function f, along that close contour C is 0. If now is happening is that, we do not have the function analytic overall in the whole region. That is inside the C and on the C. But, we do have a singularity inside the C, that is into the interior of C. Then, what we had learnt, using the residue method.

We had learnt that, if f is analytic in C, except at z naught. Then, integral of the function f z, along that close contour C is 2 pi i times residue of f z at z naught. That is at the point of singularity. And that point singularity has to be isolated singularity. Now, if the thing is happening is that, rather than having only single isolated singularity. If my function is such that, that in the interior of the close contour C, I do have more than one isolated singularity.

Then what will happen? So, if f z has more than one isolated singularities, inside the simple close contour. That is say this kind of situation, we do have this a close contour, positively oriented. And we do have a number of singularities z naught, z 1, z 2 and so on, inside the C, then what will happen. Remember one thing, if you are saying is that function is having isolated singularities inside this one. There has to be finite, you have to think about it, why it will happen.

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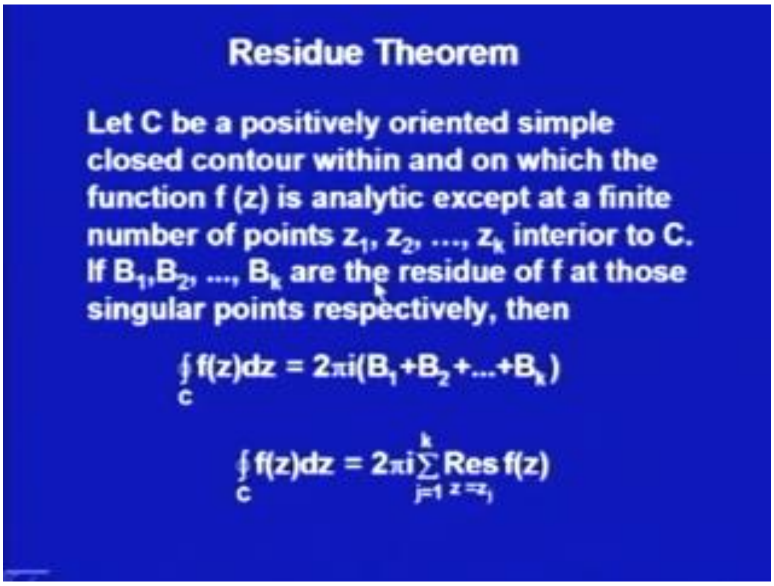

Here comes the answer in the form of residue theorem. Let C be a positively oriented simple close contour. Within and on which the function f z is analytic. Except at a finite number of points, z 1, z 2, z k, interior to C. And if capital B 1, capital B 2 and capital B k are the residues of f at those isolated singular points respectively. Then, we say that the integral of the function f z, along this close contour C, can be given as 2 pi i times B 1 plus B plus B k.

Or in other words, we are saying is that, if function f z is analytic on the contour C, close ((Refer Time: 03:36)) contour C. And it has a finite number of isolated singularities, inside that interior to the C. Then, the integral of that function along that contour C, can be given as 2 pi i times. Summation of residues of the function z, at those isolated singularities. Two things we have to just discuss here. First I said is that is, they are finite number of isolated singularities interior to the C.

What we said is that, function f is analytic, except at a finite number of singularities. And the singularities are isolated singularities. What we are trying to just say is that, isolated singularities, if they are interior to any close contour C; they has to be finite, because if they are not finite, then that singularity will not remain as isolated. So, isolated singularity, if the function has only isolated singularities, they must be finite. So, we are talking about the functions, which has finite isolated singularities. Let us prove this, we would prove it using the, our residue integration method or residue formula. Let us see, that is how we are proving it.

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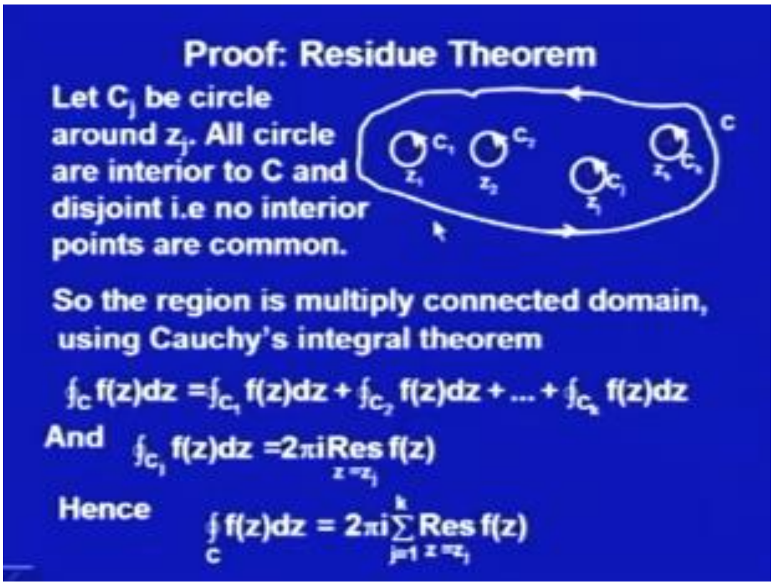

So, we are assuming that C j to be the circles, around each singular point z j. All the circles are interior to C and they are disjoint, that says is this was our close contour C. We were having k singular points inside the contour C. That is interior to C, z 1, z 2, say z j and z k. These are isolated singular points. Now, what I have done is, let us take a circle around this z j's, with the some radius. Let us say r j or rho j such that, since this is isolated singularity. So, we take this small circle such that, there is no other singular point interior to this circle C 1.

That means, in general for each z j, we take a circle such a small, such that there is no other singular point interior to that C j. So, what we are taking and moreover, we want that the circles has to be interior to C. So, what we have got, that we have got now k more circles interior to C, which are disjoint. There is no point which is common. And since, they are not having any isolated singularity of this one. And moreover we do want them to be disjoint.

That is suppose, there is no singularity in between this. But, I do not want that, a circle is going up to here. And then, the circle is going to up to here. So, we want the disjoint also. Like this if we are having, then what we are having. Let us see this region, which is enclosed by this contour C, this region, this is interior to C. Now, this region I can see, it has a multiply connected domain, you are seeing it. What the definition of multiply connected domain is? That it is having p boundaries.

We are talking about a finite domain, or bounded domain. Then, we call it p connected, if it has p boundaries. So, here what we are having is, we are having is k plus 1 boundaries. So, this is a k plus 1 connected domain. Or in general, we simply say that this, we could treat as a multiply connected domain. Now, we want integral along the outer boundary of this multiply connected domain. Now, what we do have till now, we do have that for multiply connected domain.

There is one theorem, which is called the Cauchy theorem. For multiply connected domain, which says that the integral of a function f, which is analytic in the multiply connected domain. So, now you see, my f is such that, it has only this number of isolated singularities. So, if I am treating this, as a multiply connected domain. All the singular points, they would be outside the multiply connected domain. Because, the interior of this C and interior of this C j's, they would be out.

If I take in this negative direction. So, what we are having is that, this is a multiply connected domain, where my f is analytic. So, using that Cauchy theorem, we would say that integral of this function f, along the boundary would be 0. So, let us use it, region is multiply connected. So, use the Cauchy integral theorem, for multiply connected domain, which says is that, integral of analytic function along the boundary must be 0. Or what we say is that, along the boundary when we are talking about. Then, for multiply connected domain, inside these boundaries we have to take into the negative sense.

Or in other words, we do write that integral of f along this contour C. That is outer boundary, should be same as summation of integral of the f, on the inner boundary, for all this one. That is integral of f z along the contour C, is same as integral of f z, along the contour C 1 plus integral of f z along the contour C 2 and so on plus the integral of f z along the contour C k. Now, let us see these individual integrals. If I take the first integral let us say, this says is integral of f z along this contour C 1.

This contour C 1, if I do see in this positive orientation direction. We do have that, my function f is analytic throughout this region, on C 1 and interior to this C 1; except at that point z 1. Now, I can use the residue integration method, which says is that, if the function f is analytic in the interior to the C 1, except a single point z 1. Then, we could give the integral as 2 pi i times the residue of the function f, at that singularity point. That is z 1, this is true over here.

Similarly, since we have taken, this contour or the circles such that there is no other singular point inside it. And this f remains analytic at all the points, except at this point of singularity. So, for each C 1, C 2 and C j, Ck, this residual integral formula can be applied. So, what we are having is, for C j the integral of f z, d z would be 2 pi i residue of f z at z is equal to z j. Now, substitute this over here, for each one. What we are getting is that, integral of f z along this close contour C, should be 2 pi i times summation of residues of f z at the points z j.

And the summation is over all singular points, this is what is our residual due theorem. So, what now we have got actually, we are now not talking only a singular point. If it has more than one singular point, we could give the integral along the contour. If it contains a finite number of isolated singular points inside or interior to that contour. As the sum of residues into 2 pi i. Let us see some examples, how we are going to use this in our evaluation of integrals.

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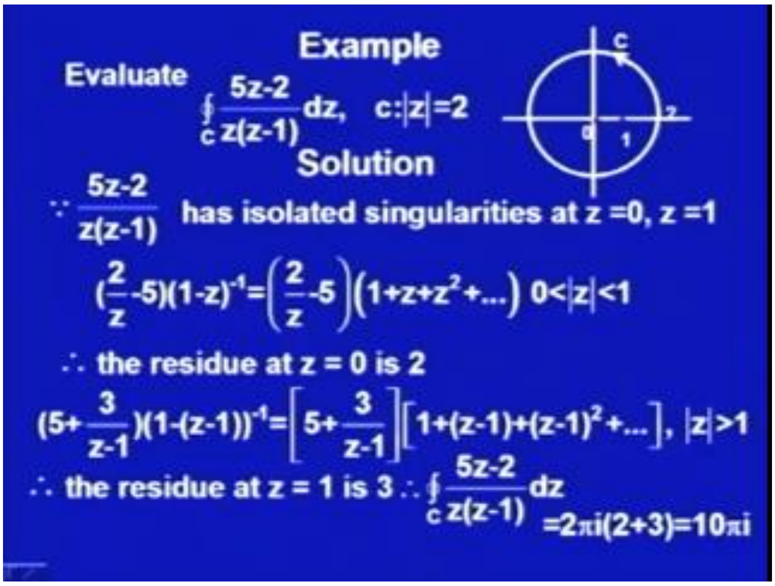

Let us do one example first here. Evaluate the integral of the function 5 z minus 2 upon z into z minus 1. And the contour C, where C is your circle mod z is equal to 2. That is the circle with radius 2 center 0, let us solve it. First see the contour, our contour is the circle positively oriented of the radius 2. Now, let us see this function f z, f z is my 5 z minus 2 upon z into z minus 1. We see this function, is of the form p z upon q z actually.

We do know that the function, if it is of the form of p z upon q z. And if q z has 0's, on which p z is not 0. Then, we do say all those 0's would be the poles of the function. And the order of the poles, would be same as the orders of the 0 of the denominator function. So, here we do have, from here we do see the 0's of this denominator are z is equal to 0 and z is equal to 1. At z is equal to 0, the numerator is not 0, at z is equal to 1 also the numerator is not 0.

So, we do have that, this integral the function 5 z minus 2 upon z into z minus 1. This has isolated singularities at z is equal to 0, and z is equal to 1. Moreover these singularities are poles. And they are simple pole actually. Now, let us see what this singularities, where they are lining with respect to this given contour 0, is the center. And 1 is certainly inside, the circle z is equal to 2. So, we are having that, the both the singularities are lying interior to the given contour.

Now, that says is, if I have to find out this integral of this function, along this contour. Then, we do have that, function is analytic everywhere, except at the two points, 0 and 1.

And this points 0 and 1, they are isolated singularities. And moreover these isolated singularities are actually poles on simple poles. Now, it says is that, now we can apply our residue theorem; which says is that, integral along this should be same as the summation of the residues at 0. And at 1 and multiply with the 2 pi i.

So, for that what we require, we require the residue of this function at 0 and at 1. Let us first find out the residue. The first method I am applying here is, finding the residue using this Laurent series. This function, we are writing as 2 by z minus 5. This is z I am taking from here, so 2 by z minus 5 into 1 minus z to the power minus 1. You see, if I am using this Laurent series method. I have to find out the residue at z equal to 0; and z is equal to 1

That says is, once I have to write the Laurent series expansion, in the region 0 is less than z is less than 1. And another is, that is where I would have 1 is outside this one, that is 1 to 2 are greater than 1. So, first I am writing it between 0 and 1. For 0 and 1, here we are having is z minus 1. So, if I write it 1 minus z to the power minus 1. We do know that, Maclaurin's expansion, for this is valid when mod z is less than 1. That is 0 is less than, mod z is less than 1.

That we are talking about the residue at, or Laurent series with respect to singularity 0. So, we do have 2 by z minus 5 into the Maclaurin's expansion of 1 minus z to the power minus 1, as 1 plus z plus z square and so on. This is valid for mod z lying between 0 and 1. From here, if I do see, this is the part which would going to be the principle part 2 upon z. So, here I would get 2 by z, then I would get 2. And then, would get is all the terms would be on the powers of z.

So, the coefficient of 1 by z is only 2. So, from here what we are getting is, that B 1 is 2. So, the residue at z is equal to 0 is 2. Now, we have to find out, the residue at z is equal to 1. For this what I want, I want that my expansion should be valid from, for mod z greater than 1. Or moreover, we would like mod z is, one is less than this one. So, what we do take, we will try to write the expansion such that, this is valid. So, for that let us see, how we are going to write.

We would use this z minus 1, divide it by the z minus 1, I would get it 5 z minus 5. So, here we would get plus 3. So, 5 plus 3 upon z minus 1. And then here it is z, z I am writing as 1 minus bracket z minus 1. So, now 1 minus z minus 1 is to the power minus

1. The Maclaurin's expansion, for this function would be valid, when mod z minus 1 is lying between 0 and 1 or in other words, when mod z is lying between 1 and 2.

So, let us just move it from here, writing the Maclaurin's. So, the first part is as such 5 plus 3 upon z minus 1. Maclaurin's expansion for 1 minus z minus 1, whole to the power minus 1, that is z minus 1 is now my, let us say y. We would get it, 1 plus z minus 1 plus z minus 1 whole square and so on. And this is valid for mod z is greater than 1 and less than 2 actually. So, we are more interested that, if for the singularity I have to have outside this one.

So, mod z is greater than 1. What we want, we want the residue at z is equal to 1. That is we want, the coefficient of 1 upon z minus 1. If we see the coefficient of 1 upon z minus 1, from here we would get it 3. At any other place I would not get 1 upon z minus 1. We would get actually all the powers of z minus 1, to the positive powers. So, we have got the coefficient of z minus 1, 1 upon z minus 1, that B 1 is 3 here. So, Residue at z is equal to 1 is 3.

Now, if I use this residue theorem. That says is that integral of this function along this contour C, which has two isolated singularities. I should have it as, 2 pi i into the residue at of f z at z is equal to 0. And at z is equal to 1, that is we are getting is 10 pi i. Now, here I had gives the first way of finding out, the residue at different isolated similarities using the Laurent series method. Since, we had learn many other methods of finding out residue; does it simplifies our calculations. Let us see with the help of same example, because this example we have done. So, let us move do this same example, with alternative methods.

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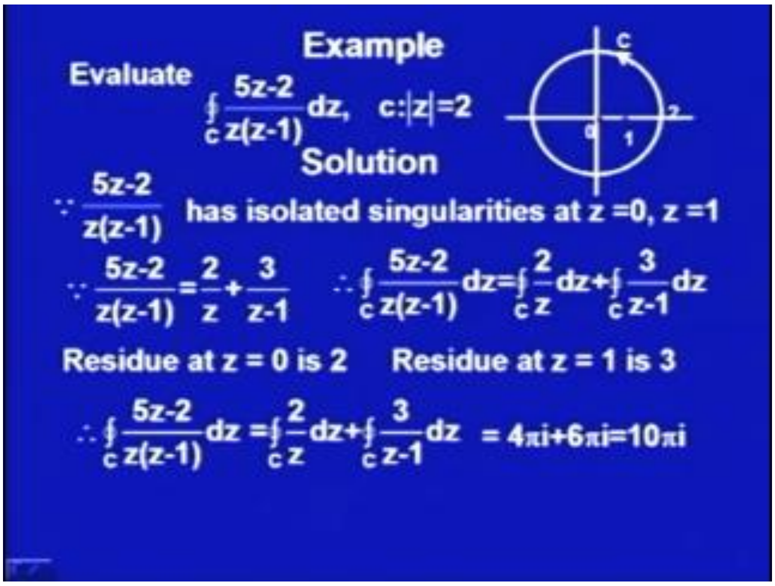

So, we do have the same example, that is evaluate this integral, along the circle of radius 2 center 0. We do know that, the function is same, it has isolated singularities between 0 and 1. So, using residue theorem, I have to find out the residue of the function 5 z minus 2 upon z into z minus 1. At z is equal to 0 and z is equal to 1. Since, we had already identified, that this 0 and 1, they are simple poles. And this function is of the form of p z upon q z, we can use the method of finding out residue at simple pole.

Or rather first, here first this question is coming, where rather than, using it as one integral, we are using it as two integral. So, I can use it using the partial fraction. 5 z minus 2 upon z into z minus 1. We could write as 2 upon z plus 3 upon z minus 1, you could do the partial fraction. So, what I would get this integral, integral along this contour C of 5 z minus 2 upon z into z minus 1 d z. We could do write us the sum of, because this is sum of two functions.

So, we do know by the integral property is that, it could be braked into two integrals. So, we do get as the integral of the 2 by z, this new function, along this contour C, and the integral of the function 3 upon z minus 1 along this. Now, we have to evaluate two integrals. Let us see the first integral, it has the function 2 upon z. This has isolated singularity at z is equal to 0.

Now, the contour is, this z is equal to, mod z is equal to 2. For this function, there is only one isolated singularity. I can just simply use my residue method, which says is the integral of this function, should be 2 pi i times residue at z is equal to 0 of the function 2 by z. 2 by z itself is written as in the form of principle part, for only one term 2 by z, so the residue is 2. So, this integral, this would be 2 pi i into 2.

Similarly, if I see for the second one. This is function is 3 upon z minus 1, this has isolated singularity at z is equal to 1. Now, if I see this second integral. This second integral is, C is my this circle with radius 2. Z is equal to 1, that is the only single singularity isolated one. And this is inside this one. Again I could use the residue. And the function is also simple, it is having only principle part say 3 upon z minus 1 with single term. So, B is here 3.

Now, with the residue is very easy to find it out. So, residue at z is equal to 0, for this function 2 by z is 2. And for the function 3 upon z minus 1, the residue at z is equal to 1 is 3. Hence, this integral would be 2 pi i into 2, that is 4 pi i. This integral would be 2 pi i into 3, that is 6 pi i again 10 pi i. So, now you have to seen, that is we are not going it, doing it, using the more than one singularity. We can use here, because the function is simple partial fractions are such that, we could get the functions in easy form. Or easy form we could say is, that is for the residue, we do not have to do any calculations. And we have to apply only the residue method, rather than the residue theorem. Even if finding out residue by another methods.

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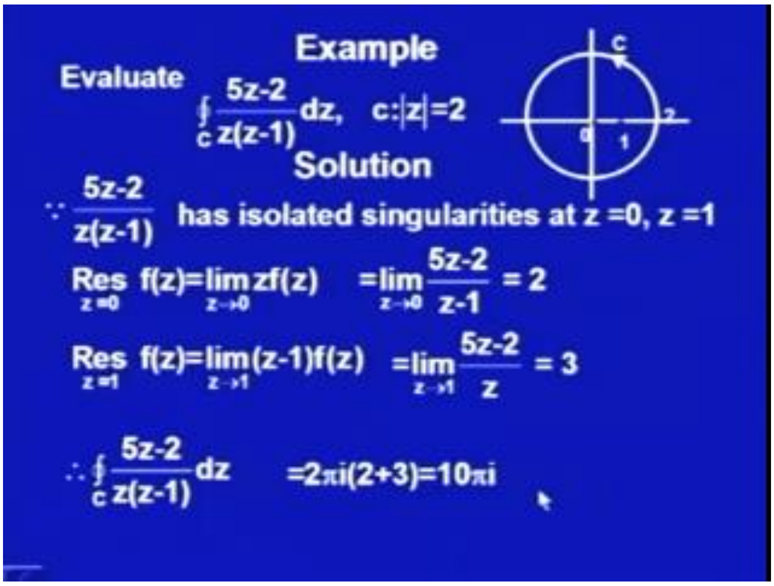

So, if I just go with the same example, we do know that is poles. So, now you will come up our, that is again the same example. I am doing is now, finding the residue, using that alternative methods, rather than the Laurent series. We do have that z is equal to 0 and z is equal to 1. They are simple poles for this function. And to calculate the residue at the simple poles for the function. This we do have the methods, which says is that, residue at z is equal to z naught.

If z naught is a simple pole, should be limit of z approaching to z naught z minus z naught f z. So, first we are calculating it at 0. So, residue at z is equal to 0 of f z, should be limit as z approaches to 0, z minus 0 that is z into f z. If I multiply z with f z, I would get it 5 z minus 2 upon z minus 1. We have to find out limit, as z is approaching to 0 5 z minus 2 upon z minus 1. That says is evaluate this function at z is equal to 0. When z is equal to 0, I would get the numerator as minus 2 and denominator as minus 1.

So, the value is 2, so residue at z is equal to 0 is 2. Now, I have not written the Laurent series. We are not worrying about, at which Laurent series I have to use. Whether, it is valid in the region, where I have to find out that. In that region, where this isolated singularity is there or not, we just write this one. So, we can calculate it. Similarly, for residue at z is equal to 1. By the same definition, it should be that limit as z approaches to 1 of z minus 1 into f z.

If I multiply this f z by z minus 1, I would get 5 z minus 2 upon z. So, we have to evaluate the limit as z approaching to 1 of 5 z minus to upon z. Now, again it says is that, the simply evaluate at z is equal to 1. Z is equal to 1, the numerator is 3 denominator is 1, so we are getting it 3. So, this is much simpler to find out, the residue by this alternative methods. Because, the poles are simple and the function is also simpler one. So, we could get directly over here.

Thus using the residue theorem, my integral would be 2 pi i into 2 plus 3 is equal to 10 pi i. So, we have find it out that, we have to find out, whether the function is having more than one singularity. So, shall we apply residue theorem. Or if we could see that, function is easy, we could do the partial fraction. As in this particular example, we could do. We do not require to use a residue theorem, we can directly do using the residue integration method. And finding out residue at z is equal to up, at any simple poles. It is much easy to us these alternative methods, rather than these Laurent series methods.

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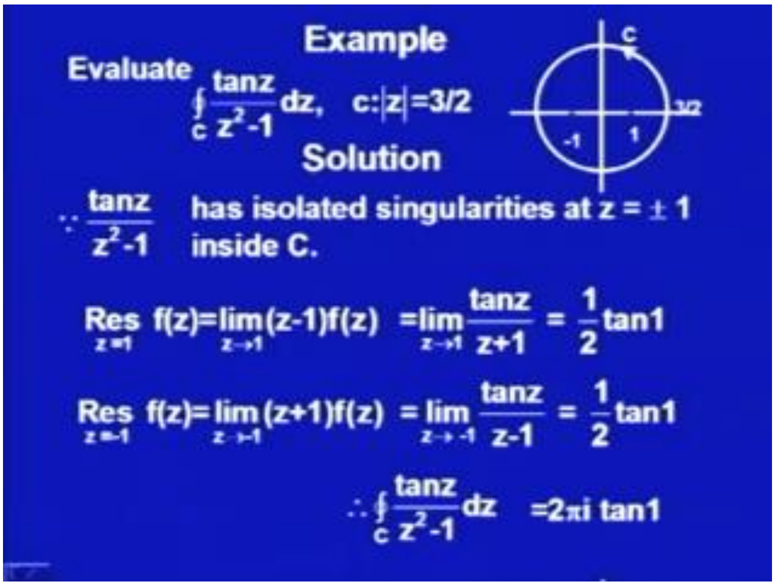

Now, let us do one more example. Evaluation of the integral of tan z upon z square minus 1 on the contour C; where the contour C is the circle centered 0 with radius as 3 by 2. See how we are going to solve it. Let us first see is, that is what is our contour? Our contour is this circle with the radius 3 by 2 centered at 0. Let us see now this function, tan z upon z square minus 1 this integrant. We see that, this function is again of the form p z upon q z. So, it will have singularities, whenever this I do have here 0's.

Or we simply directly see is that, this function fails to be analytic at z is equal to plus and minus 1. So, it has isolated singularities at plus and minus 1. And both this singularity, if I see over here. Because, this radius is 3 by 2, so this is plus 3 by 2, this would be minus 3 by 2. So, 1 and minus 1, both would be inside the contour C. That is they are interior to contour C. So, now we are having two singularities inside the contour C. Now, that says is again, we are going to use our residue method, and residue theorem. For that, we do require the poles at z is equal to plus 1 and z is equal to minus 1. Let us find out the poles. Of course, we are not going to write the tables, this Laurent series. We would just use simple formula, because both the poles are simple poles. So, we will just simply use the formula for simple pole at a point. So, residue at z is equal to 1 of f z. By definition it goes that limit as z approaches to 1 of z minus 1 into f z.

If I am multiplying this denominator, we could write as z minus 1 into z plus 1. So, what we would get is, our function z minus 1 into f z as tan z upon z plus 1. We have to find out the limit, as z is approaching 1. We simply say is that, evaluate this at z is equal to 1, at the denominator would be 1 plus 1 as 2. And the numerator would be tan 1. So, we have got it 1 by 2 tan 1. Now, we have to find out the residue at z is equal to minus 1.

By the same definition, it says is should be the limit z approaches to minus 1, z plus 1 of into f of z. When we are just multiplying this f of z by z plus 1, I would get tan z upon z minus 1. I have to find out the limit of this function, as z is approaching to minus 1. This is simple as evaluating at z is equal to minus 1. So, the denominator, we would get minus 2. Numerator what we would get, tan of minus 1, we do know tan of minus theta or tan of minus theta is minus of tan theta. So, we again get 1 by 2 times tan.

Now, using the residue theorem, we say that the integral of tan z upon z square minus 1. Along this contour C, where the contour C is this circle, mod z is equal to 3 by 2. This would be 2 pi i into 1 by 2 tan 1 plus 1 by 2 tan 1, that is tan. So, we have got, if do remember you might have come across this kind of a function, which is much harder. If you apply some other way of integration, at say simple with we have got over here. Let us try some more interesting kind of examples.

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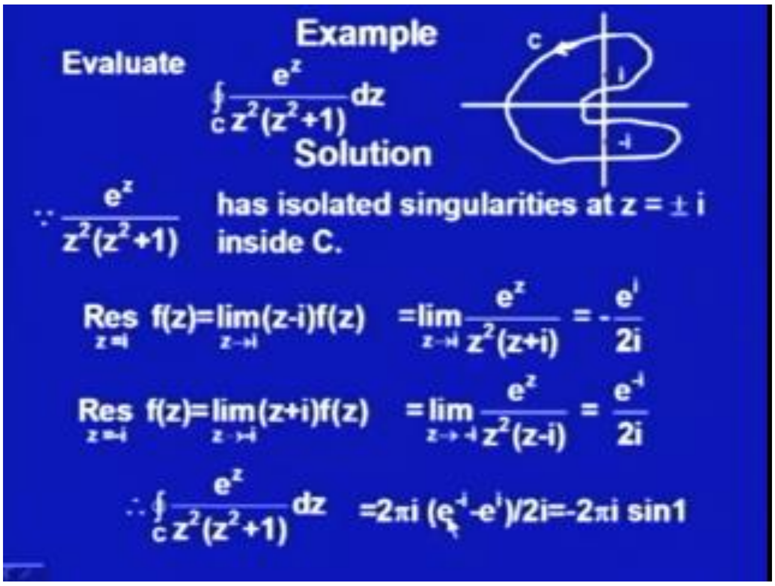

Evaluate the integral along the curve C, of the function e to the power z upon z square into z square plus 1, with respect to z; where C is a close contour given by this figure. Now, we see here, we have to evaluate along this given path this C. If we see, this is a closed contour. From many way we could start, it is a close contour. So, let us use and we are finding out, that the function which is given here. This is e to the power z upon z square into z square plus 1.

So, simply it would have singularities at certain points. So, see that is e to the power z, this is an entire function. But, in the denominator I do have z square and z square plus 1. So, at it would have singularities at z is equal to 0. And z is equal to plus minus i. But, z is equal to 0, if I to see 0 is this one. Because, this is the contour is given to us. And plus and minus i, this is plus i and this is minus, that is on the imaginary axis at plus 1 and minus 1.

So, we are having is that, singularities at 0. And plus and minus i, in which in the given contour, we are having only two singularities. The singularity 0 is outside this contour. This is not inside this contour. So, see that is here we would again, apply the residue theorem. But, it does not says, that we do not have to see that contour. We have to always see on which contour, we are evaluating this integral. And accordingly we have to decide that is...

Accordingly we have to find out, that is where this residue theorem, if we are applying. And which points we have to find out the residue. Here, we are getting is that 0, this is a singularity point, but this is outside this one. So, we would not use it. We are having only two singularities inside, again this is more than one. So, we have to apply the residue theorem. For that, we have to calculate the residue of this given function. So, first find out the residue at z is equal to i.

Again, we are going to use the first method. That is residue at any point z naught of f z is limit. As z approaches to z naught z minus z naught into f z, here z naught is your i. So, this would be limit as z is approaching to i, z minus i into f z. Now, let us see this function f z. We could rewrite it as e to the power z into upon z square into z minus i and z plus i.. So, z minus i, when we are multiplying, what things would remain is e to the power z upon z square into z plus i.

We have to find out it is limit as z is approaching to i. This says is evaluation of this function, at z is equal to i. When z is equal to i, the numerator would be e to the power i. And the denominator z square, that is i square, it should be minus 1. And z is equal to i plus i, that is 2 i. So, what we get, minus e to the power i upon 2 i, this is the residue at z is equal to i for this function. Then, we have to find out the residue at z is equal to minus i also; so let us find out residue at z is equal to minus i.

Again by the same definition, it should be limit as z is approaching to minus i z plus i into f z. Now, if I multiply this z plus i with this function, what I would get, e to the power z upon z square into z minus i. So, what we have to evaluate the limit, as z is approaching to minus i of e to the power z upon z square into z minus i. That says is simply evaluate this function, at z is equal to minus i. Z is equal to minus i says, numerator as e to the power minus i.

In the denominator z square, minus i square is again minus 1. Because, it is minus i into minus i, it is i square, so it is minus 1. Z is equal to minus i minus i, that is minus 2 i. So, I would get as 2 i only, because minus 1 into minus 2 i. So, my limit would be e to the power minus i upon 2 i. Now, I will use the residue theorem, for this evaluation of this integral you see. So, it integral of the function e to the power z upon z square into z square plus 1.

Along this given contour C, would be 2 pi i times residue at z is equal to i plus residue at z is equal to minus i. Residue at z is equal i is minus e to the power i upon 2 i. And residue at z is equal to minus i is e to the power minus i upon 2 i. So, upon 2 i, that has been taken common. We are getting is e to the power minus i minus e to the power i. Or rather you could write it out, minus of e to the power i minus e to the power of minus i upon 2 i. That we do know is nothing but, the sin 1.

So, we are getting it minus 2 pi i sin 1. So, the integral along this contour of the function is coming as minus 2 pi i sin 1. Now, you see is that is if, even if we try to write out first this definition of the contour. Or define this contour along this curve, it is really very difficult one. But, using the residue theorem, for integration process you could do. And the function is also, very nice kind of function you could see. We are able to do this difficult integrals very easily.

The only thing which we are remembering is, whenever we are applying the residue method. We have to have that isolated singularity inside, or interior to the given contour C. Whether, we are using simple residue integration method. Or we are applying this residue theorem, where we are having multiple isolated singularities. But, all of them has to be interior to this one. Now, we do have here in this particular example.

Three singularities, 0, plus i and minus i. But, the given contour on which we are integrating, 0 was not interior to that contour. So, for that we are not going to add that residue at 0. We are not going to add in the evaluation of integral. Let us see, if it is not only that isolated singularities. And this isolated singularities, we have got all the examples where they are simple poles. If it is not only the simple poles, or the poles of any finite order, if it is a essential singularity.

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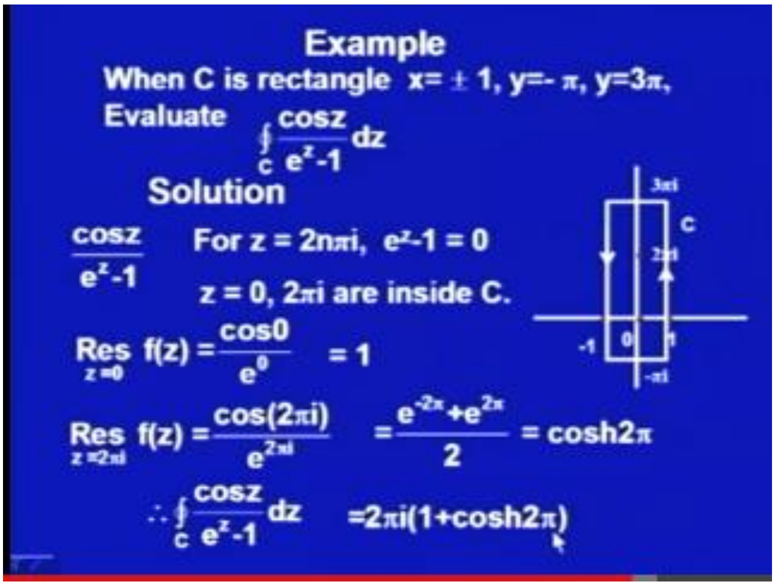

Then what will happen, let us see when C is rectangle x is equal to plus minus 1, y is equal to minus pi and y is equal to 3 pi. Then, evaluate this integral cos z upon e to the power z minus 1 d z. So, now you see is, that is we are not having any circular or any given curve. We are having is that our contour is a rectangle. Again when we are talking about the close contour as a convention, without writing here, we would be talking about positive direction. That is counter clockwise, try to solve it.

First see this function, cos z upon e to the power z minus 1, cos z is an integral function. In the denominator I do have e to the power z minus 1. So, this function would be not analytic, or would have singularities at all those points, where e to the power z is 1. If e to the power z, where we could have e to the power z is equal 1. If you do remember, e to the power 2 n pi i, that is always 1 for all n. So, what we do have here is, for z is equal to 2 n pi i.

I would get e to the power 2 n pi i, that is always 1, so 1 minus 1 that could be giving me 0. That says is e to the power z minus 1, this is always 0 for all z is equal to 2 n pi i, for all n 0, 1, 2 plus minus 1 2 and so on. That says is we are having many isolated singularities for this function. And also see, all these are isolated singularities. We are not having any other point, in the neighborhood, where this function is singular.

Let us see, that is how many singularities, because with the contour on which we have to integrate. Now, let us first examine this contour. We are having this contour, which is a rectangle, given as x is equal to plus or minus 1. That is this line x is equal to plus 1, this line is x is equal to minus 1. Then, the line is y is equal to minus pi and y is equal to 3 pi. So, either you say x y plane, or you say z plane. In z plane, we would write this y axis as a imaginary axis, so we add i over there.

So, we are having this is minus pi i, this is plus 3 pi i. Now, here if I see which points, or which singularities would be coming interior to this given curve. So, the singularities we would have, when n is equal to 0, z is equal to 0. Z is equal to 0 is origin, certainly it is interior to this given contour. Then, for n is equal to 1, the z would be 2 pi i. That is on the imaginary axis at 2 pi.

Since, by I am having this from y is equal to minus pi to y is equal to 3 pi y is equal to 2 pi. This is on the imaginary axis. And this is certainly inside the or interior to the C. Then, for n is equal to minus 1. Because, we are having this n is equal to plus minus 1. N is equal to minus 1, we would have minus 2 pi i. You see that lower line of the this rectangle is, y is equal to minus pi i. So, minus 2 pi i would be outside this.

Now, if I increase in, go with n is equal to plus minus 2. We would get as plus minus 4 pi i, certainly both these things would be outside. So, what we are having is, only two isolated singularities. One is at 0, another is at 2 pi i, they are interior to this given contour. So, again you are having an example, where the function has many isolated singularities. But, a given contour does have some of them interior to it. So, we are interested only at these two isolated singularities.

So, they are two, so we can use the residue theorem. And we will find out the residue at 0 and at 2 pi i. Let us see this singularities are actually again the simple poles. We would use, so we are having this residue at z is equal to 0 of f z. We would use the method p z upon q z, where residue at z is equal to z naught is p z naught upon q dash z naught.

What will be q dash z over here, that would be e to the power z. So, what we would get at z is equal to 0, p at z naught, that is cos 0, q dash is e to the power z.

So, q dash at z is equal to 0, would be e to the power 0. E to the power 0 is 1, cos 0 is 1, so I do get is as 1. So, residue at z is equal to 0, for this function is 1. You can check that this, these are also simple poles. Now, we want the residue at 2 pi i. Again I would use the same definition or same formula, for calculation of residue. That is p z naught upon q dash z naught. So, residue at z is equal to 2 pi i, would be cos at 2 pi i divided by e to the power 2 pi i. Certainly e to the power 2 pi i, we do know is cos 2 pi plus i sin 2 pi, that is 1.

And cos 2 pi i, we could write it as, if you do remember that, cos theta we could write as e to the i theta plus e to the power minus i theta by 2. So, using that we could write it as, e to the power i theta. Theta is here is now 2 pi i. So, i times 2 pi i, what we would get minus 2 pi. So, e to the power minus 2 pi into plus e to the power minus i theta. So, minus i times 2 pi i, I would get e to the power plus 2 pi i divided by 2 e to the power 2 pi i is 1.

What is this, this is nothing but, cos hyperbolic of 2 pi. So, we have got the integral of cos z upon e to the power z minus 1, on this rectangle C; which is bounded by a plus and minus 1 on the x side. And minus pi with 3 pi on the y side is nothing but, 2 pi i. 1 plus that residue at z is equal to 0, that is 1. Residue at z is equal to 2 pi i is cos hyperbolic 2 pi. So, we have got this integral, then... Now, let us do, as I promise that is, we would do one example where we are not having only poles, if it is essential singularity.

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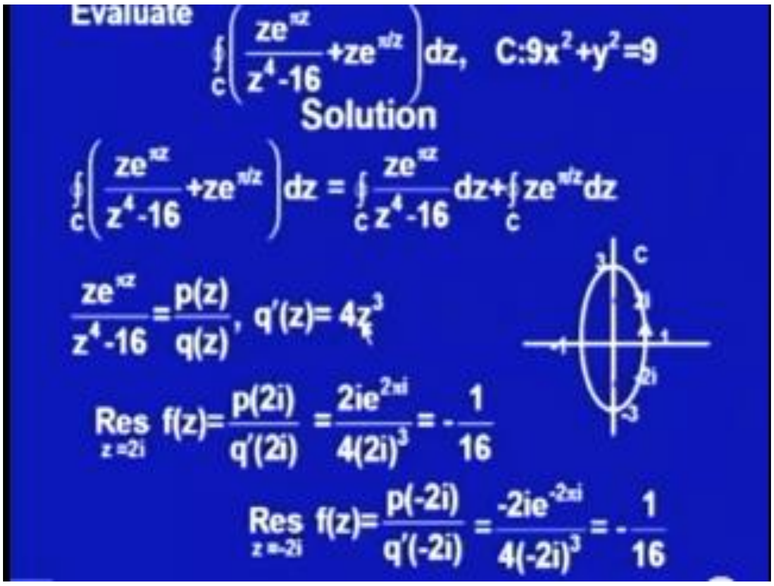

Evaluate the integral z e to the power pi z, upon z to the power 4 minus 16. Plus z times e to the power pi by z, with respect to z on the contour C, where C is an ellipse. If I write this equation in the x y coordinates, it is 9 x square plus y square is equal to 9. Of course, we do know that in the complex number, either we call z plane or we call x y plane. When we call it complex plane, that is our z plane. We take that y axis as imaginary axis. And we use that z is equal to x plus i y; that is here whatever the coordinate we are getting is, that we include i over there.

Let us see, how we are going to solve this. See this function first, this function is sum of two functions. So, first we will write this integral as the sum of two integrals. Z e to the power pi z upon z to power 4 minus 16 plus z times e to the power pi by z. This integral with respect to d z on this contour C, the contour C is ellipse. And of course, whenever we are talking about the curves, we are talking as a convention positive orientation. I am writing this as two integrals.

One in is integral of the function, z times e to the power pi z upon z to the power 4 minus 16. And another as integral of the function, z times e to the power pi by z on the contour C. Let us see one by one these functions. The contour is now my ellipse, 9 x square plus y square is equal to 9. So, if you see is, that is the x would vary from minus 1 to plus 1. And y would vary from minus 3 i to plus 3 i could see.

Now, let us take the first one. Z times e to the power pi z upon z to the power 4 minus 16, this is of the form of p z upon q z. This would be having singularities at your z square is equal to plus minus 4. Or when z square is equal to 4, z would be plus minus 2. And when z square is equal to minus 4, you would get z is equal to plus minus 2 i. So, we have got that, this function has four isolated singularities, one is plus minus 2, another is plus minus 2 i.

Plus minus 2 means is, on the real axis minus 2 and plus 2. Certainly those points are outside this given contour. Then, we do have another isolated singularities plus minus 2 i, that means minus 2 and plus 2 on the y axis. So, minus 2 and plus 2, since on the y axis, this contour is ranging from minus 3 to plus 3. So, minus 2 i and plus 2 i, they would be inside the given contour. So, now what we are having is, for this first integral.

We have to evaluate this integral for a function, which has two isolated singularities inside the contour. So, we would have to find out use the residue theorem here, for a finite number of isolated singularities. Now, moreover we see is that, all these points are actually simple poles. So, a function is of the form p z upon q z. So, we would use the formula for calculation of residue, at any isolated singularities z naught. As p z naught upon q dash z naught.

So, q z here is z to the power 4 minus 16, so q dash z would be 4 z cube. So, now residue at z is equal to 2 i, would be p at 2 i divided by q dash at 2 i. So, p is your z times e to the power pi z. So, what we would get 2 i e to the power 2 pi i upon 4 z cube, z cube means is your 2 i cube. Since you are having here also z. So, you could write it as actually p z naught upon q dash z naught. You may write it e to the power pi z upon 4 z square also.

Or whatever manner, you try to solve it. We do get as 2 i e to the power 2 pi i upon 4 times 2 i cube. So, once 2 i will cancel it out, here what we would get 2 i square. That is 4 i square, that is minus 4. 4 into minus 4 would be minus 16, e to the power 2 pi i we do know is 1. So, what we would get minus 1 by 16. Similarly, the residue at z is equal to minus 2 i with the same formula, it should be p at minus 2 i upon q dash at minus 2 i.

Now, what is p at minus 2 i, it would be minus 2 i into e to the power minus 2 pi i, divided by q dash at minus 2 i. That is 4 times minus 2 i q 1 2 i and 1 2 i all go here. I would get in the denominator minus 2 i whole square, that would be same as 2 i whole square. So, it would be again minus 4. So, the denominator again we would get minus

16. In the numerator what we are getting is, e to the power minus 2 pi i. We do know e to the power minus 2 pi i also 1. So, what we are getting, residue at z is equal to minus 2 i is also minus 1 by 16.

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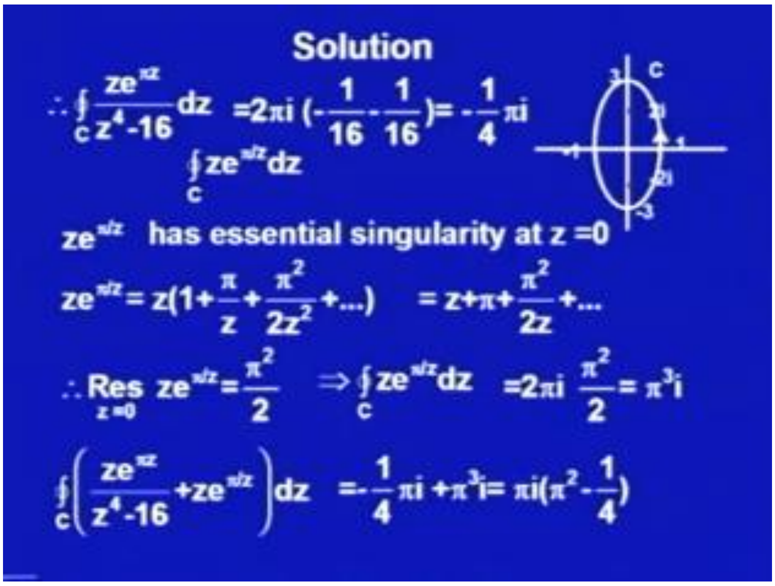

So, what we get, the integral of z e to the power pi z upon z to the power 4 minus 16. As using the residue theorem, 2 pi i into the sum of residues at on isolated singularities. We have two isolated singular case. So, at z is equal to 2 i, it is minus 1 by 16. At z is equal to minus 2 i, it is minus 1 by 16. So, added up what we are getting is minus 1 by 8 into 2 pi i. That is minus 1 by 4 pi i, this is about the first integral.

Now, move to the second integral. Second integral was on the same contour of the function z, into e to the power pi by z. Now, pi by z has singularity at z is equal to 0. So, e to the power pi by z. Actually would have an expansion, in which all the principle part will have infinite many terms. So, we do have that z is equal to 0 has, is an essential singularity for the function, z into e to the power pi by z.

For essential singularity, we are having that formula for the poles only. For the essential singularity, we have to move to the Laurent series. So, let us for this kind of function, it is we do know is. That is writing the Maclaurin’s expansion of e to the power x, is much easy to get the Laurent series expansion. So, we get z into e to the power pi by z, using the Maclaurin’s series. Z into 1 plus pi upon z plus pi square upon z square into factorial 2 plus pi cube upon z cube into factorial 3 and so on.

And this expansion is valid for all z greater than 0. So, what we get it is z plus. And the positive 1, so z plus pi plus pi square upon 2 z and so on. Now, here at z is equal to 0, z is equal to z. We do have the residue, the coefficient of 1 upon z is pi square by 2. Of course, the next term would be your pi cube upon factorial 3 z square and so on. So, we do have our B 1 as pi square by 2. So, the residue of this function z e to the power pi by z, at z is equal to 0 is pi square by 2.

So, what we have got, the integral of this function e to the power. Because, it has only single singularity point, a single isolated singularity. Using the residue integration method, it should be 2 pi i times residue, at that isolated singularity. So, integral along this contour C of z e to the power pi by z d z would be your 2 pi i pi square by into pi square by 2. That is you are getting is pi cube i. Now, what we have got the first integral, we have got as minus 1 by 4 pi i. The second integral, we are getting is pi cube over i.

So, now the integral of our given function z e to the power pi z upon z to the power 4 minus 16 plus z times e to the power pi by z d z. This is actually the sum of both the integrals. So, first integral value is minus 1 by 4 pi i. The second integral value is pi cube over i. So, we are getting is and a manner, you can simplify it, you to write it out i pi times pi square minus 1 by 4. Or whatever way you find out simplified. You can use the values of pi as 3.1417. And then, you can evaluate this, if you do have the facility for using calculators; you can evaluate this value of this integral.

So, now we had learned that residue theorem. Or this residue integration method, this is not only applicable to one kind of integrals. Actually to a whole lot of functions, we could apply this one, this method. The condition only we are having is that, inside the contour say, it must have a finite number of isolated singularities. So, one integration method, you have learnt for the analytic functions. We had learnt about, it as Cauchy integral formula.

Then, if when function f is analytic, except at certain some finite number of isolated singularities. We had learnt this residue method, which says is that, you can... If it is single isolated singularity, we can write this 2 pi i times, the residue of the function at that isolated singularity. If it has more than one isolated singularities, we had got one result, which we called residue theorem, which has given us very strong result.

That is we can use, all those finite isolated singularities points. We can calculate the residue and the integral should be 2 pi i times sum of all those residues. So, we had learnt today, very nice method for calculation of integrals or evaluation of integrals. In these examples, we have tried to find it out that, or try to show that. We can apply this residue method in many different manners.

It is just up to me, that how we do find out the function, whether we could use the direct residue one. That is isolated singularities, if the functions are simple. Or if we are having simple poles, we can calculate the residues, either using the Laurent series. Or it is much easy to use the alternative methods. That is using limit as z is approaching to z naught of z minus z naught into f z. Or if it is of the form of p z upon q z, then using the formula that residue at z is equal to z naught is p z naught upon q dash z naught.

Or if it is of order m, we can use the formula for residue at z is equal to z naught. As 1 upon factorial m minus 1 is derivative of z minus z naught into f z. So, like that, we could use that, moreover we had learnt. If it is, the function if I could break that function not only into the two functions, as in the first example we had seen that there was the one function which has two isolated singularities.

But, using the partial fraction. We can break it into two functions; where each function on that given contour, was having only singular isolated singularity. And the function was also turning out to be very simple. So that, the residues really we do not have to calculate. Or in this last example, what we have seen, the function was really very large one or tedious one or to see at the first point. But we had braked it into two functions.

Of course, here it was easy that in this example, the function has already been braked into two function. So, what we have done is, rather than evaluating the integral as one whole integral. We have divided into using the properties of integrals, that we can divided into two integrals. And for both the integrals, we had find out that the function was having different singularities, inside the contour. And the singularities are all also of different kind.

In one we are having the singularities as simple poles. While the in the other function the singularity was isolate, this essential singularity. For poles we do know that, we do have simple formulas. But, for essential singularity, we do not have any simple formula to calculate the residue. For that, we directly go or normally, we do go only with the expansion part. And find out the coefficient of 1 upon z minus z naught, that is B 1. So, today we had learn the residue theorem. It is application in evaluation of contour integrals. You could say that, where the function may have some isolated singularities. So, that is all for today, we learn some more applications of this residue theorem.

Thank you.