Mathematics - II
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Lecture - 7
Residue Integration Method

Welcome to the lecture Series on Complex Analysis for undergraduate students. Today's lecture is on Residue Integration Method. Till now we had learnt, many integration methods are in many way. We had find out, how to solve the integration integrals for the complex functions. In all those methods, we had learnt that is the function has to be analytic in the region of integration. We had also done the Cauchy integral formula, where we had find out if the function is not analytic at complete domain.

But, it fails to be analytic at one particular point. We have taken that point out. And we had use the disk, where the function was analytic. And on that boundary, we had find out that function is analytic. And there, we had use one Cauchy formula. Today, we will learn, some more integration method, for functions which are not analytic at certain points. Then, what will happen?

We had learnt in last lecture, when the functions are not analytic at some points. Those points we have called singular points. We had classified the singular points as, poles and essential singularities. Now, we would have one more classification of isolated singular point that is, removable singular point.

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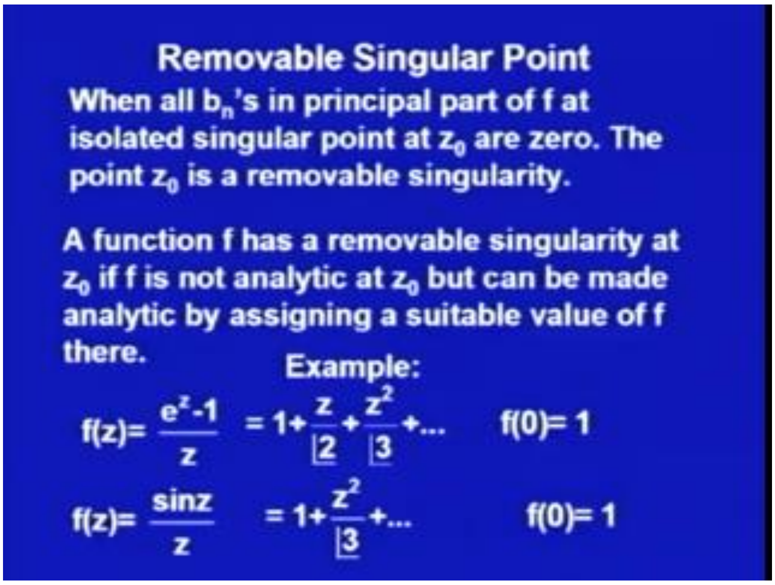

When all b n's in the principle part of f at isolated singular point, z naught or 0 the point z naught is called removable singularity. In other words, the definition of the removable singular point could be, why we are calling it removable? Because it can be removed. What it is saying is, a function f has a removable singularity at z naught. If f is not analytic at z naught, but can be made analytic by assigning a suitable value of f there.

What does it mean? It simply says is the, first definition says is that in the principle part all the b n's are zero. One, what it says it that, that singular point may be because, the function is not defined at that point. So, we would call it removable singularity because, if I could define a suitable value of the function at that point. We could make the function, analytic. See for example, what we are talking about here. Let us take this function, e to the power z minus 1 upon z.

As such it is looking is, that the numerator is our analytic function. A denominator of course, when z is equal to 0 I would not get it as analytic. But, see what is happening at z is equal to 0? Does this function is defined at z equal to 0. When z is 0, e to the power z would be 1. So, 1 minus 1 is 0 and the denominator is also 0. That is, the function is not defined at 0. Let us see, if I write the Maclaurin's expansion for e to the power z. That is, 1 plus z plus z square upon factorial 2 and so on.

So, 1 and minus 1 it cancel it out. And divided by z, we would get 1 plus z upon factorial 2 plus z square upon factorial 3 and so on. You see in this expansion, since at z equal is equal to 0. This function is not analytic. That is, z is equal to 0 is an isolated singularity. So, of course the Laurent series can be written. And if I am writing this Maclaurin series, I am finding out it is not having any principle part. That is none of the b n's are present over here.

Now, this is the case which we are talking about. That z is equal to 0 is an isolated singularity, which is removable. How it would be removed? You see, if I what we are saying is that the function is not defined at z is equal to 0. If I take limit of this function as z approaches to 0, what I would get? I would get this constant 1. And all other points would be, all other terms would become 0. So, now if I define the function, so the function is not defined at 0 that is, why it is singular.

So, if I define f 0 as 1. Then, you will find out that now function is analytic. Now, you can check that this function f z e to the power z minus 1 upon z. For all points z not equal to 0 and is equal to 1 for z is equal to 0. You would see is that is, this is an analytic function. One more similar kind of example is, sin z upon z. Again you find out that is, z is equal to 0 is an isolated singular point for this function sin z upon z.

But, we see what is f at 0? f at 0, sin at 0 is 0. So, it is again 0 by 0. That is in different form. Now, let us write again the Maclaurin's expansion for sin z. It is simply z plus z cube by factorial 3 plus z to the power 5 upon factorial 5 and so on. If I divide it by z, I would get 1 plus z square upon factorial 3 plus and so on. Now, again we find out as z approaches to 0. This limit of this f z would be 1. So, now what I would define? I would define f 0 as 1.

So, now this function f z sin z upon z for z not equal to 0. And 1 for z is equal to 0. This is now analytic function. So, now let us move to one more result, where we would relate the 0 and the poles for a function. If the function can be written as the rational of two functions poles and zeros.

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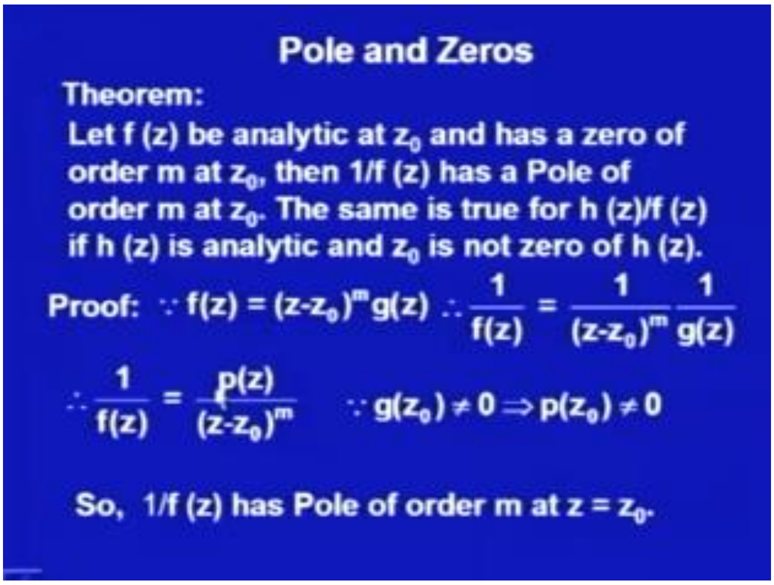

The result says is f z be analytic at z naught. And has a zero of order m at z naught. Then, 1 upon f z has a pole of order m at z naught. And the same is true for h z upon f of z, if h z is analytic and z naught is not zero of h z. So, what we are saying is, we are relating the zeros of analytic function with the poles of the analytic function. We are saying is if f z is analytic and has a 0 at some point. Then, 1 upon f z will have pole at that point.

And the order of that pole would be same as the order of that zero for f z. Now, this is simply for 1 upon f z. Now, if I do have that my function can be written as a rational, in the rational form. That is, it is ratio of two analytic functions h z upon f z, where both h z and f z are analytic. And f z has a 0 at z naught, but that z naught should not be the 0 of h. Then also, that z naught would be the pole of this new function h z upon f z. And the order would be again the same, for what is the order of 0.

Let us see, that is the how we can show that this will happen, just a proof of this. Since f z is having a 0 at z naught of order m. That says, we could write f z as z minus z naught to the power m into g z, where g z is an analytic function and g at z naught is not 0. So, 1 upon f z if I do write, I could write it as 1 upon z minus z naught to the power m into 1 upon g z. Since g at z naught is not 0. So, 1 upon g at z naught, that will also not be 0. So, write 1 upon g at z g z, as say p z.

So, what I would write? 1 upon f z I could write as p z upon z minus z naught to the power m where this p at z naught is not 0. What it says is, that I am having this new function 1 upon f z, which does has a principle part. The principle part contains the in the denominator z minus z naught to the power m. And its numerator is p z, which is not 0 at z naught. What it says is, that I could I am getting a principle part with m terms.

So, z naught should be the pole of this new function. And order of that pole must be m. So, since g z naught is not 0. So, p z naught will also not be 0. So, 1 upon f z has a pole of order m at z is equal to z naught by definition. So, now if I do have here rather than p z, if I do have h z. So, what I would have that p z would be now my h z upon g z. And at z is equal to z naught neither z naught g z naught g z is 0 nor h z is 0.

So, p z would also not be 0 at z naught. What it says is that, the function would remain defined. And so, the theorem would be true for this h z upon f z also. Now, from here because we are talking about all this isolated singularities, there kinds and the behavior of the function near them. Now, we want to move that integration method because, of this behavior of this function near these isolated singularities. For that, let us first again come to this Laurent series representation and this principle part.

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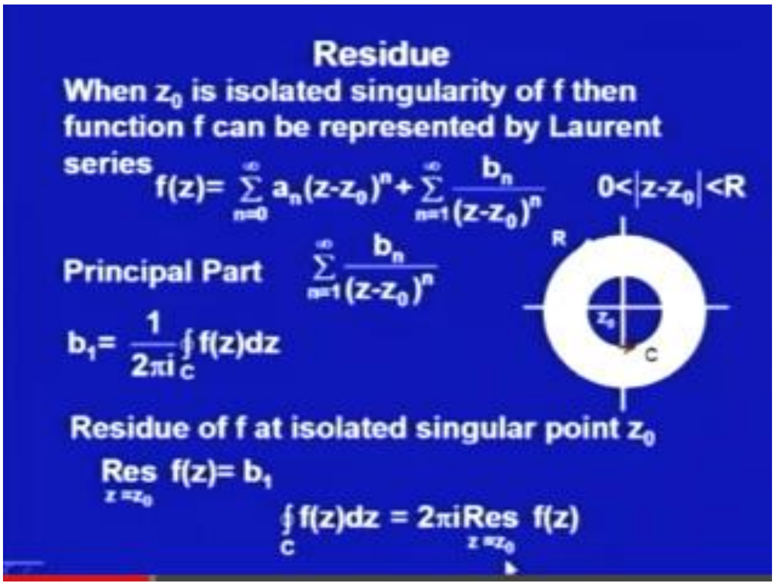

We do know that, when z naught is isolated singularity of f. Then, the function f can be represented by the Laurent series as summation n is running from 0 to infinity a n z minus z naught to the power n plus n is running from 1 to infinity, b n upon z minus z naught to the power n in the disk. 0 is less than z minus z naught is less than R. And this a n. So, this is the disk. And this principle part of this is, summation n is running from the n to infinity b n upon z minus z naught to the power n. And from this principle part, we have identified the isolated singularities.

We had also learnt in that, when we have defined this one, that b n by that is especially for n is equal to 1, b 1 would be 1 upon 2 pi i f z d z. Because b n's we had defined as 1 upon 2 pi i integral along this close contour f z upon z minus z naught to the power minus n plus 1. So, when n is equal to 1 I would get the denominator will not be there. I would get simply f z. We had called this b 1 as the residue of f at isolated singular point z naught. So, now we are defining residue.

Residue of any function f at isolated singular point z naught, is being given by 1 upon 2 pi i integral along this curve c of f z d z. So, we are having is that f z has a isolated singular point at z naught. Then, we are defining this one. Now, rewrite this one integral because, here what we are having is in the right hand side, we are involving an integral. Rewrite this one, what we could say? Integral f z t z on this close contour c is 2 pi i times b 1.

What is b 1? b 1 we called as the residue of f at this point. This another notation for this is, we are writing. That residue at z is equal to z naught of f z. So, 2 pi i b 1 or 2 pi i residue of f z at z is equal to z naught. Now, what we have got? we have actually got 1 integration formula. Till now the integration formula, we had got is that Cauchy theorem said. If we are having f as analytic in the whole domain, then integral along any close contour would be zero for analytic function.

If the function is analytic, inside that close contour and on that close contour. We had Cauchy principle, we had Cauchy integral formula which said is that is if we do have z naught as an interior point of your c. We had got that is, f of z naught we could define as integral along that close contour f z d z with 1 upon 2 pi i. Now, there also f was analytic completely. Now, we are having is this reason, this close contour c. The function is analytical on this contour.

And in this interior except at point z naught because, z naught is isolated singular point. This c I could take any circles such that, it is a small neighborhood where it makes is that z naught is isolated singularity. So, z naught is isolated singularity of the function f. So, now, f is not analytic. So, when f is it is fails to be analytic or f has 1 singular point inside in the interior of this close contour c. We could evaluate, this integral as 2 pi i times residue of f at this one.

If it is a singular point, we do know we could write it as Laurent series or we could find out its residue. That is the principle part the first, the coefficient of 1 upon z minus z naught. That is we have called residue and that we are getting from here. Now, since we do know that, Laurent series we are not always obtaining by those integral formulas. We are obtaining the Laurent series in many manners. That says as, this residue can be obtained without really integrating it. Thus it says is, that this integral can be obtained using the Laurent series and the residue at that isolated singular point. Let us see is, that is what I am saying is how we are doing it with the help some examples.

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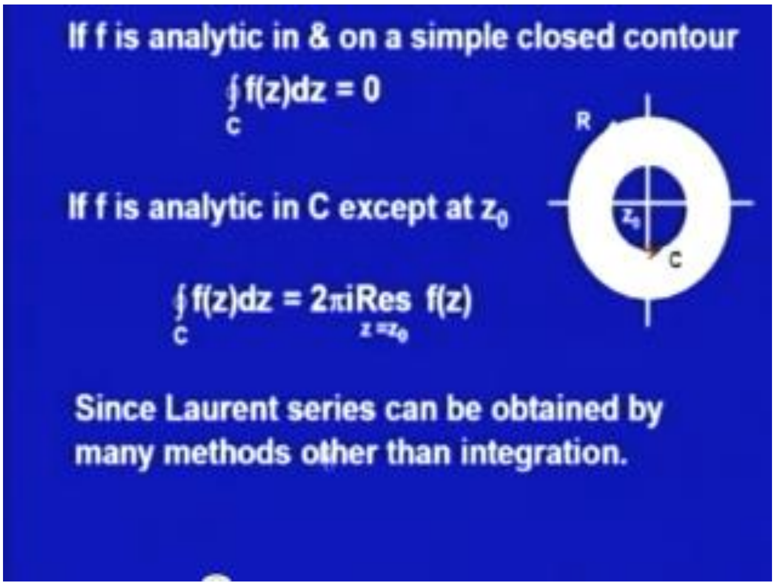

If f is analytic in and on a simple close contour, then function then integral along that contour f z d z would be 0. This we do know this is, what is our Cauchy integral theorem. Now, if I do have f is analytic in C except at z naught. Then, this just now the result which we had obtained, that said is integral along this contour of f z d z is 2 pi i times residue of f at z is equal to z naught.

So, now we have got new formula. And as I said, this Laurent series can be obtained by many methods. Other than the integration this could help us in finding out the integration or the evaluation of this formula, would help us in evaluation of integral. Let us see one example.

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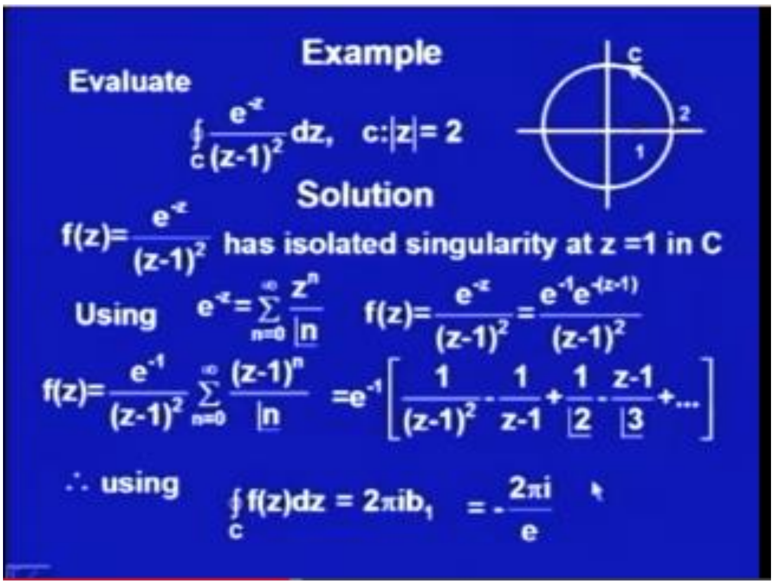

Evaluate this integral, e to the power minus z upon z minus 1 whole square d z on the circle, mod z is equal to 2. That is, it is a circle with the radius 2 and center origin. Let us see, the circle with radius 2 and center origin. Let us see this function, e to the power minus z upon z minus 1 square. The function is ratio of two functions, the numerator e to the power minus z. This is entire function. The denominator is z minus 1 whole square. This would have 0 at 1 and the order of the 0 would be 2.

So, what we do have is. This has a pole of order 2 at 1. Or it is isolated singularity at 1. 1 is inside this region of integration or this contour. Now, I can apply my residue result. Using this e to the power minus z as this Maclaurin's expansion, I could write the Laurent series expansion of f z. As we see here, e to the power minus z upon z minus 1 whole square. Now, you see is that is how I am writing the Laurent series.

I am not writing this Laurent series, finding out the integration methods e to the power minus z i could write as e to the power minus 1 into e to the power minus z minus 1 upon z minus 1 square. Now, e to the power minus 1, I will take common. And I would write the Maclaurin's expansion of e to the power minus z minus 1 using this one. So, what it would be? Summation n is running from 0 to infinity z minus 1 to the power n upon factorial n.

So, now if I take this one, what I would get? I would get e to the power minus 1.1 upon z minus 1 square minus 1 upon z minus 1 plus 1 upon factorial 2 minus z minus 1 upon factorial 3 plus z minus 1 whole square upon factorial 4 and so on. So, after that these terms from, here the terms would be the summation a naught, summation a n z minus z naught to the power n and a naught is 1 upon 2. This is the principle part, 1 upon z minus 1 1 upon z minus 1 whole square and so.

We of course, as I said is from here. This has a pole order 2 at z is equal to 1. Now, what is the b 1 that residue? Residue is the coefficient of z minus z naught 1 upon z minus z naught. The coefficient of z minus 1 is here minus 1 and of course, this e to the power minus 1. So, minus e to the power minus 1, so what we have got? This residue b 1 is minus e to the power minus 1. If I use this formula, which says integral along this contour c f z d z is 2 pi i b 1.

So, 2 pi i into minus e to the power minus 1 or minus 1 upon e. I have got the integral minus 2 pi i upon e. You see that is, because for this function we could write its expansion Laurent series expansion, or we could find out the principle parts or we could find out the residue. So, we could use this integral. We could find out the integral, where the function is not analytic at one point inside this 1. Let see one more example.

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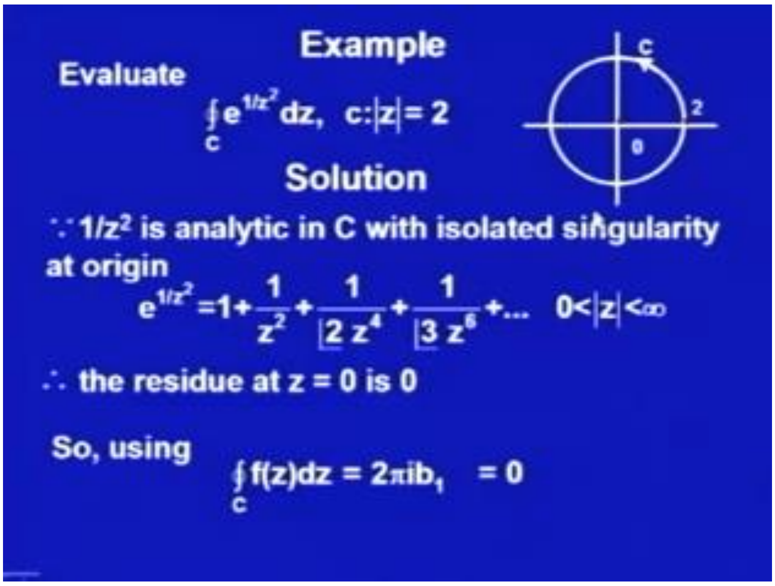

Evaluate the integral along the contour c e to the power minus 1 upon z square d z again the contour is that circle with the radius 2 and center 0. Let us see this function, e to the power 1 upon z square. Now, 1 upon z square is analytic in C with isolated singularity at origin. So, e to the power 1 upon z square will also be analytic, except at origin. And that origin would be actually isolated singularity. So, again I would use my result of this residue 1. And for that, we would use the integral.

So, for that we have to find out, first we have to write out the Laurent series expansion for e to the power 1 by z square. So, that we could find out the residue at z is equal to 0. If I write e to the power 1 by z square, it would be actually 1 plus 1 upon z square plus 1 upon factorial 2 z to the power 4 and so on. And this expansion is valid, for all z between 0 and infinity. From here, if you see there what is the residue?

Residue should be 1, the coefficient of 1 upon z that is 0 here. Because, I do not have any term of 1 upon z that says is, this integral would be 0. So, we have got that the function is used to be analytic at origin, which is inside this contour simple close contour. But, it still the for this function e to the power 1 by z square, the integral along this close contour is 0. It is just because, the residue at that point is 0.

Here, we cannot use this Cauchy integral theorem which says is along any close contour. It would be 0, because then the function f has to be analytic throughout the region. But, this function is C is having a singular point at origin. One more thing is that is, whenever we are trying to find out this integral, we have to be very cautious. That is find it out, whether that singular point is inside the contour region or not, and whether the Laurent series which I am writing, because Laurent series is always on a disk. That Laurent series is whether it is valid in that region, in which that isolated singularity is lying or not that is very important one. Let us see with one more example that, this clearly.

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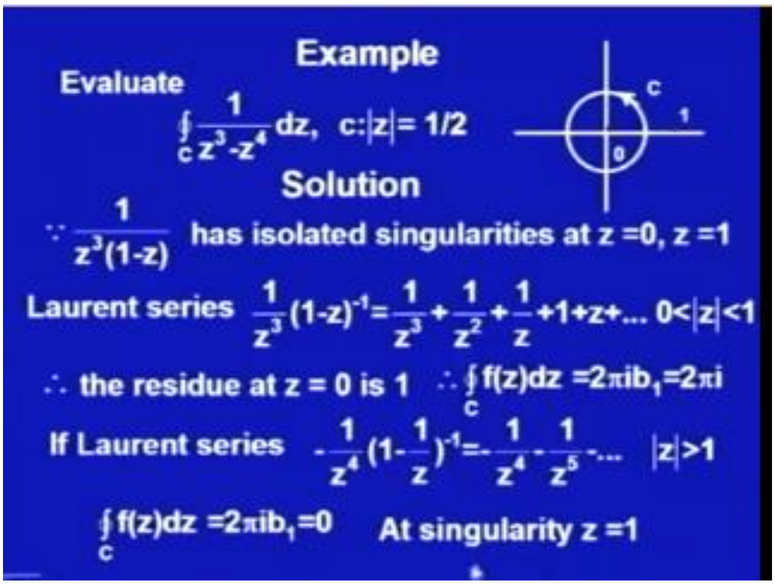

Suppose, I want to find out the integral of this function 1 upon z cube minus z to the power 4, on a contour C where this contour is actually circle with radius half center at 0. So, this is my circle with radius half and center 0. Now, let us see this function 1 upon z cube into 1 minus z. We find it out, that at z is equal to 1 and z is equal to 0. Both are singular points, z is equal to 1 and z is equal to 0. Both are singular points and both are actually isolated singularities.

Now, let us see here. z is equal to 0 means, this origin z is equal to 1 this, the point on the real axis. My contour is, z is equal to mod z is equal to half that is a circle with radius half. So, certainly the point non singular point 1, the singular point 1 is outside my region of interest. Inside the contour is, only 1 singular point 0. So, I should use a Laurent series which is valid in the region containing this one. So, let us see if I use this Laurent series.

This Laurent series is 1 upon z cube into 1 minus z to the power minus 1. We do know again by the Maclaurin's expansion, that 1 minus z to the power minus 1. This is 1 plus z square plus z cube and so on. So, 1 upon z cube I would get 1 upon z cube plus 1 z square plus 1 upon z plus 1 plus z and so on. And this expansion is valid for all the z lying between 0 and 1. That is absolute value of z or the mod of z is lying between 0 and 1.

That is, the disk which we are having is containing this, my region of a integral. This contour and the singularity is inside that contour. So, the region which we are taking is that is inside this one. If I use this one from here, if I see what will be my residue? Residue is the coefficient of 1 upon z residue at 0. We want the residue at 0. So, residue at 0 is 1. That is my integral would be 2 pi i. So, the integral of this would be simply 2 pi i.

Now, suppose I had made this function in some different manner, let us see. If the Laurent series I had use this as, 1 upon this z to the power 4 I have taken common from here. Then, what I would get from here minus z to the power 4. If I do take common I would get 1 minus 1 upon z. So, now if I write the function in this manner, minus 1 upon z to the power 4 into 1 minus 1 upon z, to the power minus 1.

We do know that, expansion of this function is also valid. But, for that 1 upon z has to lie between 0 and 1. That says as, my mod z has to be greater than 1. So, if I am writing this expansion, this expansion is 1 plus 1 upon z plus 1 upon z square and so on. So, if I am multiplying it minus 1 upon z to the power 4, I would be getting is minus 1 upon z to the power 4 minus z 1 upon z to the power 5 and so on.

What it says is, I am not having any term of 1 upon z. That is the coefficient of 1 upon z is 0. This expansion, this series is valid for the disk mod z greater than 1. That says is, that is I would be going outside this one that is outside this one. While I am actually integrating on z is equal to half. And this series is valid for z greater than 1. That is, it is not valid in this region. So, if I am using this series. This would be wrong. So, we have to be very careful.

That is, what is the region of integration that is which is the contour? And the Laurent series which I mean using is, that should include that contour. So, here it would be 0 which is really wrong. This is actually at a singularity z is equal to 1. And of course, it is not this c. It will not happen. So, this way we have to be a little bit careful, which Laurent series we are using. Now, since I said is that the Laurent series we could find out in many methods.

And in all these examples, I had use simple functions for which this Maclaurin’s expansion is valid. And we could get the Laurent series where simply. But, are there some methods which says is I could find out the residue without. Really you writing this expansion, this is a lengthy job that can happen. Let see.

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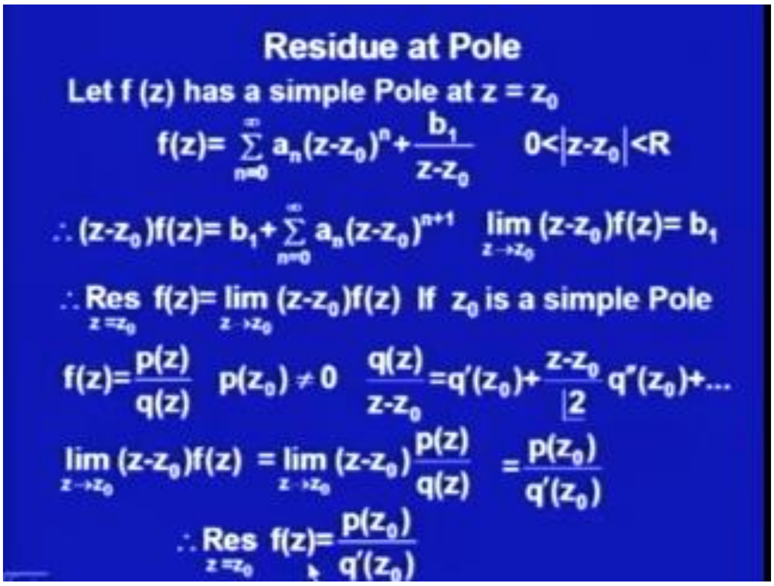

So, we are just finding out the residues at pole. So, if f z has this z naught is a singular isolated singular point of f z, then we identify it as a pole. If I do have only one term in the principle part, b 1 upon z minus z naught. So, if it is a simple pole at z is equal to z naught, my residue would be simply b 1. That the coefficient of 1 upon z minus z naught. Let us see that is, how I am writing it.

What it says is that, f z would be n is equal to 0 to infinity a n z minus z naught, to the power n plus b 1 upon z minus z naught for the region, 0 is less than mod of z minus z naught is less than R. Now, b 1 is the residue. Let us multiply this whole thing by z minus z naught. What it says is, z minus z naught into f z would be b 1 plus summation n is running from 0 to infinity a n z minus z naught to the power n plus 1.

Now, if I take the limit as z is approaching to z naught. Then, we do know as that is a if z is approaching to z naught z minus z naught, will approach to 0. That is all these terms will go to 0. And what I would, it will be get left is only b 1 that is the residue. So, what we have got? One simple method is to find out the residue. If the function has a simple pole at z naught, that says is residue at z is equal to z naught.

When z naught is a simple pole of f, is limit as z approaches to z naught of z minus z naught into f z. It gives very simple method to calculate the pole. Calculate the residue at a simple pole. Now, if it is a simple pole we can have one more result. Suppose, my f is of the form of rational form, that is f is a ratio of two functions p and q. Then, what will happen? If f z is p z upon q z and p of z naught is not 0.

That is z naught not a 0 of p but, z naught is a 0 of q. So, that I do and that is a simple 0. So, that I do have a simple pole at z naught for f z. If this is happening then what we could say is because q z is having a 0 or z naught. That says is, I could write q z as q z naught plus z minus z naught times q dash z, that is Taylor's expansion. Sense q is z naught is 0 of q. So, q z naught would be 0.

That says now Taylor's expansion I am writing in different manner. Or rather you could say as I am dividing by z minus z naught. I would get q z upon z minus z naught as q dash z naught plus z minus z naught upon factorial 2 times q double dash z naught and so on. This is the Taylor's expansion of q z at z naught. And since z naught is 0. So, q of z naught would be 0. Now, you saw this is same result.

That residue at z is equal to z naught of f z is limit z as approaches to z naught of z minus z naught into f z. f z is here p z upon q z. So, if I am writing z minus z naught of p z upon q z, it is simply I could write p z upon q z upon z minus z naught. And q z upon z minus z naught is q dash z naught and so on. Now, if I take the limit as z approaches to z naught. So, now what I am writing is I am writing p z. p z also you could write as a Taylor's expansion because, the z naught is not 0.

So, p z naught would not be 0. So, I would get p z naught plus z minus z naught times p dash z naught and so on. The terms or which are not involving z minus z naught. They are p z naught in the numerator. And q dash z naught in the denominator. All other terms would involve z minus z naught. So, as approaches to z naught all these terms will vanish. What it says is, I would get p z naught upon q dash z naught.

This is still more simple, we do not have to really find out the limit. What we do if my function is of the form p z upon q z? Such that, z naught is 0 of q, but not a 0 of p. That is z naught is the simple pole of f. Then, residue at z is equal to z naught of f can be given as p z naught upon q dash z naught. That is the derivative of q at z naught. These are the simple formulas. Let us use these simple formulas for the evaluation of our integrals. So, let see one example.

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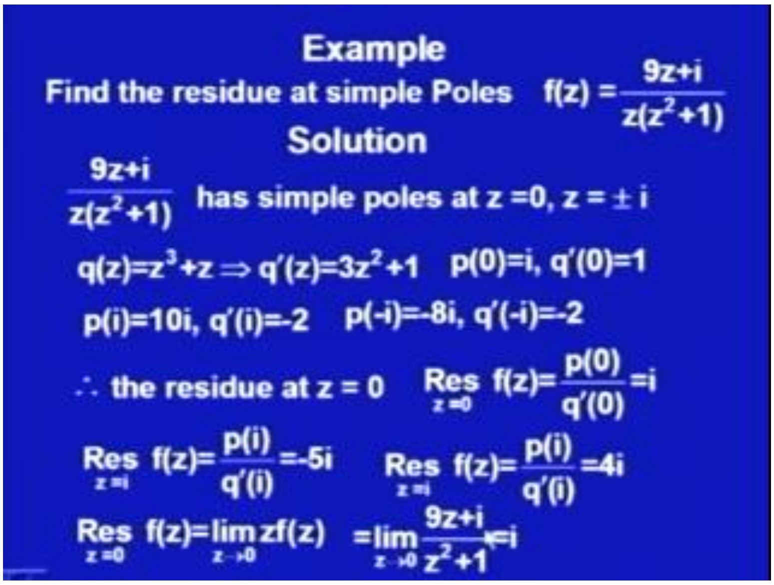

Find the residue at simple poles, for the function f z at 9 z plus i upon z into z square plus 1. So, we are just going to find out the residue without really finding out the Laurent series or without really integrating the function. See, the function is 9 z plus i upon z into z square plus 1. The function is of the form p z upon q z. If I see this q z, this has a zero at zero, z equal to 0 and at z is equal to plus and minus i. All these zeros are simple zeros.

Since f dash z, would be actually this is z cube plus z. So, f dash z would be 3 z square plus 1. So, for z is equal to 0 or plus minus i, that would be 0. So, we are getting is that the denominator z into z square plus 1 is having simple zeros at zero and plus and minus i. What it says is, that my this function 9 z plus i upon z into z square plus 1 will has simple poles at z is equal to 0 and plus minus i.

So, you see that is I have used this, a relationship between the poles and zeros to find the poles. And we had find out that this as simple pole at 0 and plus and minus i. So, let us find out the residue at all these isolated in singular points or that is at all this simple poles. We would use the formulas, just we had find it out. So, if I find out the residue at z is equal to 0, what we see is that my denominator is z cube plus.

So, rather than having a if I do have multiplication of z minus z naught f z that also is, but there I have to find out the limit. If I do use the second method, where it is in the rational form I could use this, the derivative of the denominator at that pole point only and we would get it. So, I am using that method that is more simpler. So, q z is z cube plus z. This says is q dash z would be 3 z square plus 1.

So, p 0 is i and q dash 0 is 1 here. p 0 is would be i and q dash. So, we are getting is that at z equal to 0, p z is not 0 all these conditions are satisfied. So, my residue similarly at z is equal to i, what I would get pi that is 9 i plus i that is ten i. And q dash at i I would get as 3 i square minus 3 plus 1 that is minus 2. Similarly, if I just go for minus i I would get p at minus i as minus 8 i and q dash at minus i is minus 2.

So, now what I have find it out, that is this function is of the form p z upon q z. It has simple poles at 0 and plus and minus i. Moreover, at all these pole points that is 0 and plus i and minus I, the numerator is not 0 at those points. So, we just go with the second one. So, the residue at z is equal to 0 first. p 0 upon q dash 0, now p is i q dash 0 is 1. So, this is i. Residue at z is equal to i. That should be pi divided by q dash at i. p at i is ten i, q dash i is minus 2. So, I would get minus 5 i.

Similarly, residue at z is equal to minus i. p at minus i is minus 8 i, q at minus q dash at minus i is minus 2, I would get it 4 i. So, we have calculated the residue without really expanding this function or finding out any kind of expansions and worrying about. What will happen that is, we had just find out at all singular points what is the residue. Let us use that another method, that limit method. As I said is that is also simple. But, here this was more simpler, because the function is of this form.

So, f z residue at z equal to 0 of f z would be limit as z approaches to 0 z into f z. If I write z into f z, I would get 9 z plus i upon z square plus 1. Limit as z approaches to 0. There is a much simpler 1 as z is 0, I would get here i and here I would get as 1. So, it is i. Similarly, for others you can find it out.

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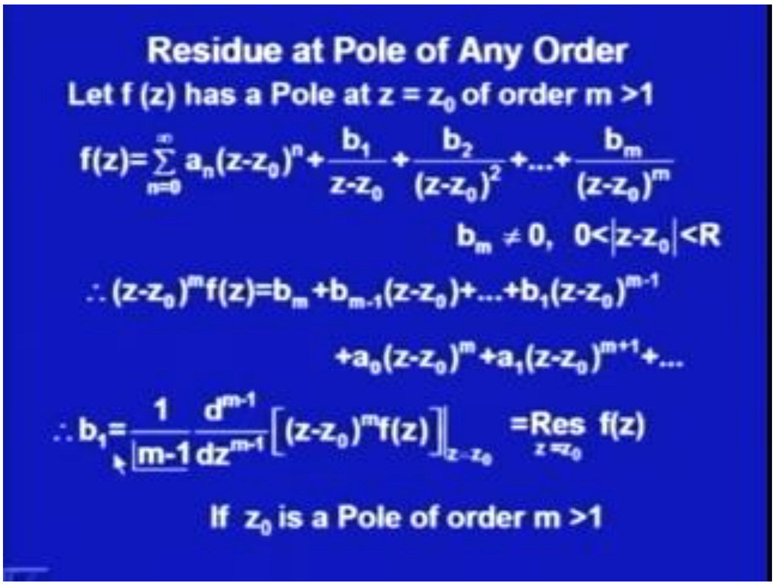

If I do have a pole of higher order higher than 1, it is not a simple pole. Is there a simple method, in which I could find out the residue? Of course, now we are going to discuss the method for finding out the residue at pole of any order. So, let f z has a pole at z is equal to z naught of order m. Certainly, I am taking m is strictly greater than 1. It is not for the simple pole. If this is happening then we do know that f z can be represented by the Laurent series.

Summation n is running from 0 to infinity, a n z minus z naught to the power n plus b 1 upon z minus z naught plus b 2 upon z minus z naught square and so on, plus b m upon z minus z naught to the power n, because it is of pole of order m. So, my principle part will take m terms only. Now, let us multiply this whole term by z minus z naught to the power n, on both the sides.

So, what we do get z minus z naught to the power m f z is equal to, now I am writing from this side. b m plus b m minus 1 times z minus z naught and so on. b 1 to the power times z minus z naught to the power m minus 1 plus. Here again, I would have multiplied by z minus z naught to the power m. So, a naught multiplied with z naught z minus z naught to the power m plus a 1 times z minus z naught to the power m plus 1 and so on.

Now, since z naught is a pole of order m it says is b m should not be 0. And this expansion is valid for all z minus z naught less than R mod of z minus z naught is less than R. That says is, when I take z is approaching to z naught. All these terms will approach to 0. The only term which would be left is this b m. And that is not 0. But, what we want actually? We want the residue. Residue is b 1.

In this one if I do see, the residue is coefficient of z minus z naught to the power m minus 1. So, here I cannot go with the simple method that is limit taking, limit as z approaches to z naught because, I want b 1. Now, let us again see this expansion. What is this? I am having it as you see, let us take it as a sum d. This is d naught, this is d 1. So, that d m minus 1, this is d m. So, what I am having? This is Taylor series expansion of the function z minus z naught to the power m into f z.

I am having constant terms, then the terms involving the powers of z minus z naught. That says is, my residue for this function f at z equal to z naught, where z naught is a pole of order m. It is actually the coefficient of z minus z naught, to the power m minus 1 in the Taylor series expansion of z minus z naught, to the power m into f z that new function g z. So, what it would be?

It should be, if you do know we do know that Taylor series expansion of any function g z. The coefficient of z minus z naught to the power m is nothing but, the mth derivative of the function g z, so and divided by factorial m. So, here we want m minus 1th coefficient of z minus z naught to the power m minus 1. So, it should be 1 upon m minus 1 factorial. The derivative m minus 1th derivative of the function z minus z naught to the power m into f z, evaluated at z is equal to z naught.

So, now what we have got? We have got a formula for the pole of order m. How to find out the residue? So, this is the residue of z equal to z naught of f z at z is equal to z naught. Let us use this formula for finding out, the residue for some functions. You see this formula would be valid, if m is greater than 1 because, if m is equal to 1 I would get here 0.

And here, I would get is a d 0 upon d z 0 and z minus z naught times f z at z is equal to z naught. I would always get it 0. So, this is not that will not be valid for our simple pole. This formula is valid for the pole of order higher than 1. So, it is let see the example.

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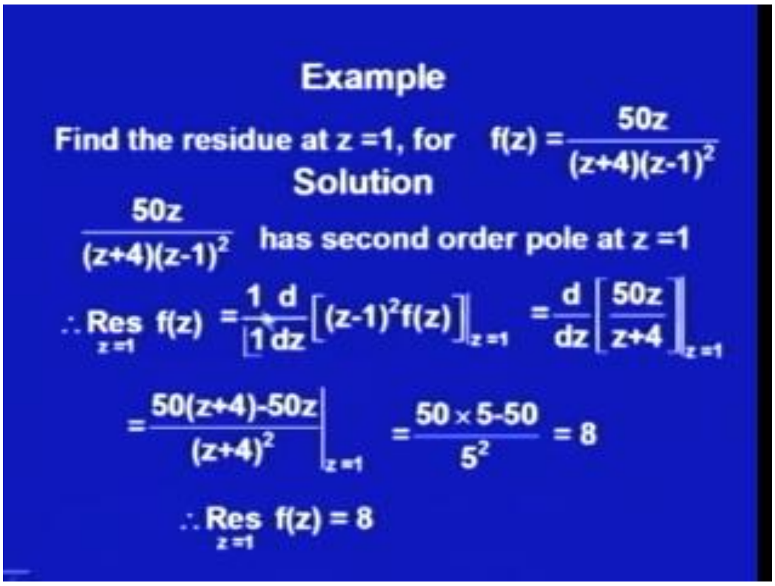

Find the residue at z is equal to 1 for the function, 50 z upon z plus 4 into z minus 1 square. Let us see this function, this function 50 z upon z plus 4 into z minus 1 whole square. We see the function is of the form p z upon q z. And q z has zeros at 4 and at 1. Moreover you could see is that, this 0 at 1 is of order 2. Or in other words, we do say is that this function has 2 has poles at minus 4 and 1. And the pole at 1 is of order 2.

So, this is second order pole at z is equal to 1. And we have to find out the residue at z is equal to 1. So, this here we are having this case, where the pole is of order greater than 1. So, I would use this second formula just now we had obtained. Which says is that, residue at z is equal to 1 of f z should be 1 upon factorial m minus 1 m is 2 norm. So, 1 upon factorial 1 d m minus 1 that is d upon d z of z minus 1 square f z, evaluated at z is equal to 1.

If I multiply this function f z with z minus 1 whole square, I would get 50 z upon z plus 4. So, I have to find out the derivative of 50 z upon z plus 4. You see is that is whenever the functions can will be of the form, this z minus z naught to the power m into f z. That would give actually as a simple function for the differentiation. Now, if I differentiate it once with respect to z, I would get it as 50 times z plus 4 minus 50 z upon z plus 4 whole square, evaluated at z is equal to 1.

When I put z is equal to 1, here I would get 50 into 5 minus 50. That simply says or you could says 50 z and 50 z is canceled out. You are getting is 50 into 4 divided by 5 whole square. So, that is 25 into 4 that is divided by 25 you are getting is 8. So, residue at z is equal to 1 is 8. Now, we had learnt here, that we can find out the residues using the simple formulations. And we are using these residues for evaluation of integrals. What the evaluation of integrals, we had done till now.

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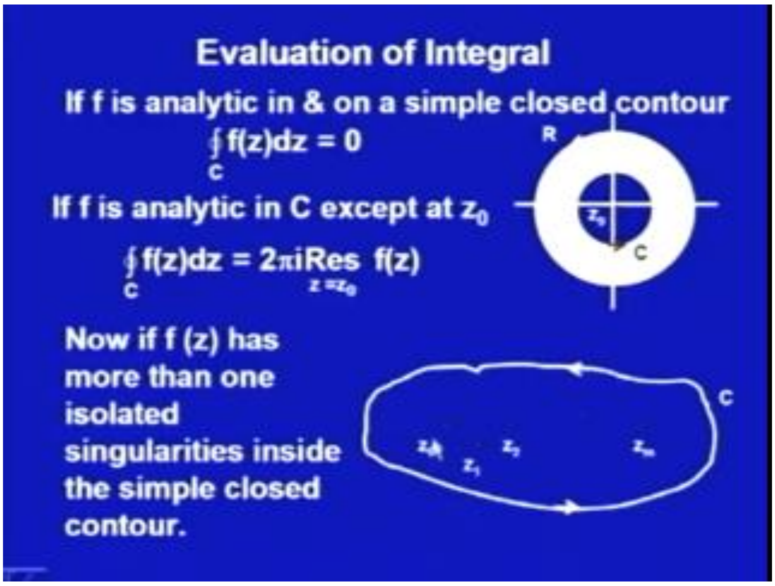

If f is analytic in and on a simple close contour, we do know that integral f z d z should be 0. Moreover, we had find it out that if f is analytic in C except at z naught, then the residue method has given that integral, along this close contour c of f z d z is 2 pi i times residue z is equal to z naught or residue of f z where I was having a single nonsingular or a single singular point or isolated singularity, inside the contour C.

Now, suppose this contour I do have more than one isolated singularities in a simple close contour. That is if this kind of thing that is, I do have this I have to find out the integration along this contour. And inside that, the function is having isolated singularities at the point z naught z 1 z 2 z n and so on. Then, what will happen? Can we still find out the result residue or something? We can do. But, that we will do in the next class. Here we would be.

So, today we had learnt one residue integration method, which said is that if the function is having isolated singularities, single isolated singularity at a point inside interior to the contour, simple close contour C. We can find out the integral along that contour close, contour C as 2 pi i times residue of the function f at that isolated singular point. This can be extended, if I do have more than one singular points. That we will do next time. So, today we had learnt one integration method where function is having isolated singularity inside the simple close contour, that is all for this lecture.

Thank you.