Mathematics - II

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Lecture - 6
Zeroes, Singularities and Poles

Welcome to the lecture Series on Complex Analysis for undergraduate students. Today's lecture is on zeroes, singularities and poles. As such we do know, zero means for any function if it becomes zero at any point in a given domain. Then we call that point the zero of that function. Here we would talk about only analytic function. So, first zeros of analytic function.

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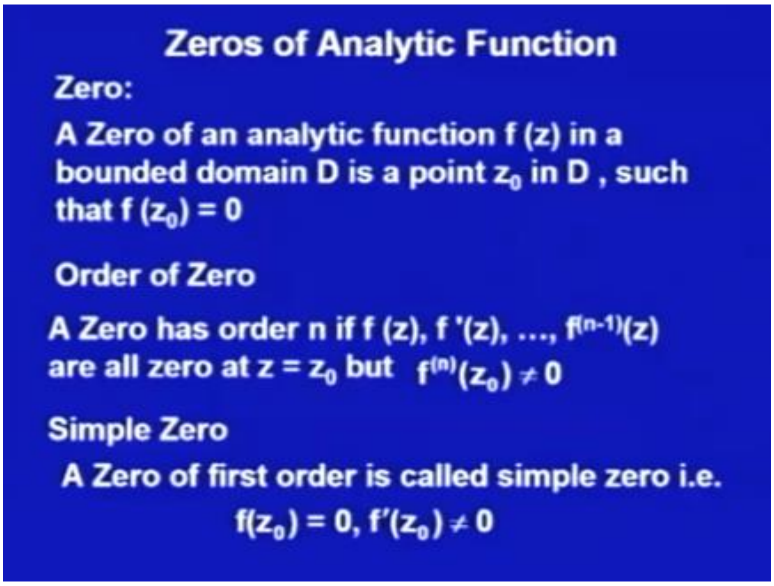

A zero of an analytic function f z in a bounded domain D, is a point z naught in D. Such that, f of z naught is 0. This is the simple definition of Zero. We say is zero of order n also. So, the order of zero, if a zero has order n. If the function, its first derivative second derivative and n minus 1th derivative at z naught are all zero. That is, a zero has order n if f z f dash z and so on; n minus 1 at derivative of f z at z equal to z naught are all 0.

But, the nth derivative that is f n z naught is not 0. Then, we call the order of zero has n. A zero of order 1that is called, Simple Zero. So, a zero of first order is called zero, simple zero. That is, where the z naught would be called simple zero. If f of z naught is 0. But, f dash at z naught is not 0. Let see some examples.

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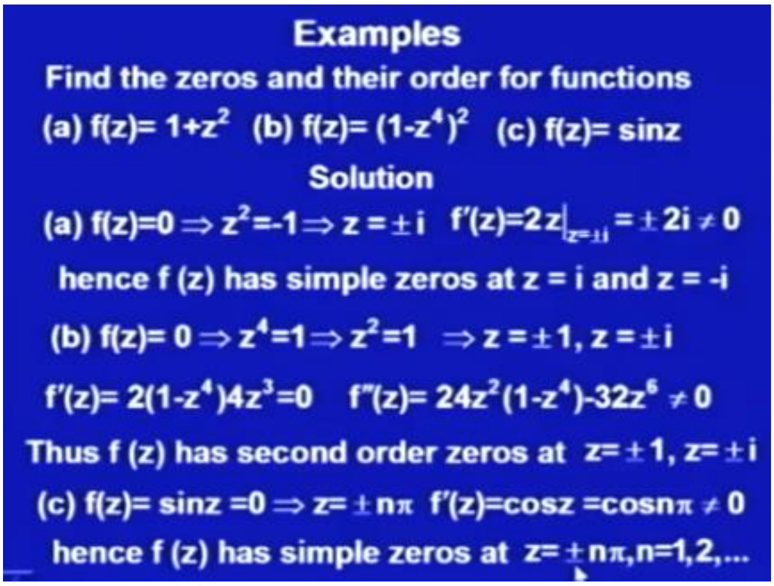

Find the zeros and their order for the functions. 1 plus z square 1 minus z to the power 4 whole square and sin z. Let us do one by one. Solution, f z is equal to 1 plus z square. When, it would become 0, so we would just equate it to the 0. And solve the equation. So, we would do get it that, 1 plus z square is 0. This says, this z square is equal to minus 1. This says this, that it would be 0 at z is equal to plus or minus i.

Now, if I take f dash z that would be 2 z. So, at z is equal to plus or minus i, that would be plus or minus 2 i, which is just not 0. So, f dash z at z is equal to plus or minus i would be plus or minus 2 i, which is naught 0. Hence, 0 is a simple zero. So, here plus and minus i are simple zeros of this function 1 plus z square. Now, let us come to the second function 1 minus z square z to the power 4 whole square.

If I equate it to 0, what I would get? I would get that z to the power 4 is equal to 1. This implies z square is equal to 1. Or this says as, why I am taking it plus 1 only. Why not minus 1? Because, just one side I am taking it. It could be z square would be actually plus or minus 1. When z square is plus 1, we would get z is equal to plus minus 1. And when z square is minus 1, we would get z as plus or minus i.

Now, let us see whether we have got that, 4 points at which this function is 0. The points are 1 minus 1 i and minus i. Let see that is, what is the order of the zeros at these points? So, find out what is f dash z? f dash z from here is 2 times 1 minus z to the power 4 into

4 z cube. So, if z is equal to plus 1 or minus 1 or plus i or minus i. I would get it 0 that is, all these 4 zeros f dash z is 0. Now, come to the second derivative.

Second derivative from here if you do see, it would be 24 z square into 1 minus z to the power 4 minus 32 times z to the power 6. Now, the first power portion of this second derivative, for z is equal to plus minus 1 or plus or minus i. This function would become, this power portion will become 0. But, this would not be 0 neither for 1 or minus 1 nor for i or minus i. So, second derivative is not zero at any of these points.

So, what will be the order of the zeros at these points? The order of the zero would be 2. Because, what we are having is f z is 0, f dash z is 0, but f double dash z is not 0. So, we do have for this function 1 minus z to the power 4 whole square. We do have 4 zeros at 1 minus 1 i and minus i. And all these zeros are of second order. Let us come to the third one, sin z. Sin z would be 0, this says as z should be plus or minus n pi.

Now, what will be the derivative of sin z? We do know, it is cosine z. And at plus or minus n pi, it would be cosine n pi and minus n pi. That is also cosine n pi. And we do know that, this is not 0. So, what we are having is that? All these points, plus minus n pi for n is equal to 1 2 3 and so on. This is and even for the zero. They are zeros of the function sin z. And all of them are simple zero. So, f z has simple zeros at z is equal to plus minus n pi for n is equal to 1 2 3 and so on.

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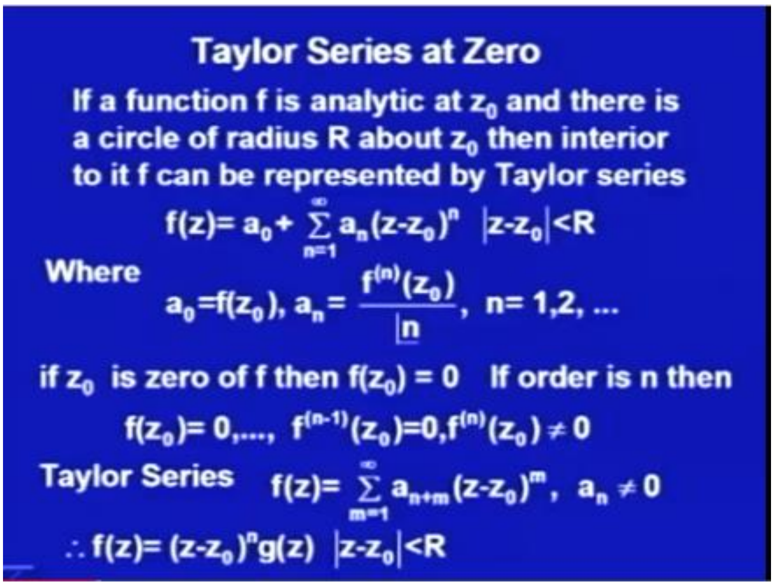

Now, let us move to the, what we have got Taylor series at zero. We do know that for any function, which is analytic or that is derivatives are existing at any point and in its neighborhood. We can write the Taylor series of that function, at that point. Now, if that point is a 0, then what will happen? If a function f is analytic at z naught and there is a circle of radius r about z naught. Then, interior to it f can be represented by the Taylor series.

That is what we are saying is, suppose f is analytic at z naught. That means, we can always find out a neighborhood in which the function would, the derivative would exists for all points at that is neighborhood. And this interior can be, and so we say is that is in this interior, if the function can be represented by the Taylor series. What was the Taylor series, you do know? f z is equal to a naught plus summation n is running from 1 to infinity a n, z minus z naught to the power n for the z minus z naught less than R.

We do know here that, this Taylor series expansion you have already done. We do know that a naught is nothing but, f of z naught. And a n are the nth derivative of f at z naught divided by factorial n for all n, 1 2 3 and so on. Now, if z naught is a zero of the function f. What it says? That, f of z naught would be 0. That says as a naught would be 0. Now, suppose z naught is a zero of order m. Then, what I would get? I would get that, f of z naught would be 0.

The first derivative, f dash z naught would be 0. The second derivative, f double dash at z naught would be 0. Until the m minus 1th derivative, of at z naught would be 0. What it says is, that if z naught is 0. Then, f of z naught is 0. If order is n, then what I would get? f of z naught 0 f of n that is, all these f of n minus 1 z naught is equal to 0. But, the nth derivative of f at z naught should not be 0. That is, f n z naught should not be 0.

What it says is, z naught f z naught is 0 means, a naught is 0. f dash z naught is 0 means, a 1 would be 0. And so on, I would get all a 1 a 2 till a n minus 1. All of them would be 0, because we are getting is here the n minus that is, n for a n, I am having is the nth derivative of f at z naught. All of them would be 0. But, a n would not be 0, because a n is f, the nth derivative of f at z naught divided by factorial n.

So, this will not be zero. What it says is, now my Taylor series because, I am having first n coefficients a naught a 1 a 2 a n minus 1, are all 0. My Taylor series would become now. Summation m is running from 1 to infinity, a n plus m z minus z naught to the power m for and a n is not 0, you see. When m is equal to 1, I would get it a 1 and so on. What it says is that, I could write it as z minus z naught to the power n, I can take out and then multiplied with g z.

So, what I could write. f z I could write as z minus z naught times to the power n times g z, in the region z minus z naught is less than R. What it says is, now if I take f at z naught here. I would get it from here. I would get 0 and here, it will be g of z naught. Since, I have written it in this manner, you see how I have started?

What I would get here? a n z minus z naught to the power n plus a n plus 1 z minus, z naught to the power n plus 1 and so on. So, z minus z naught to the power n I have taken out. Then, I would get a n plus a n plus 1 z minus z naught plus a n plus 2 z minus z naught square and so on. And a n is not 0. So, g of z naught would be a n and that is not 0.

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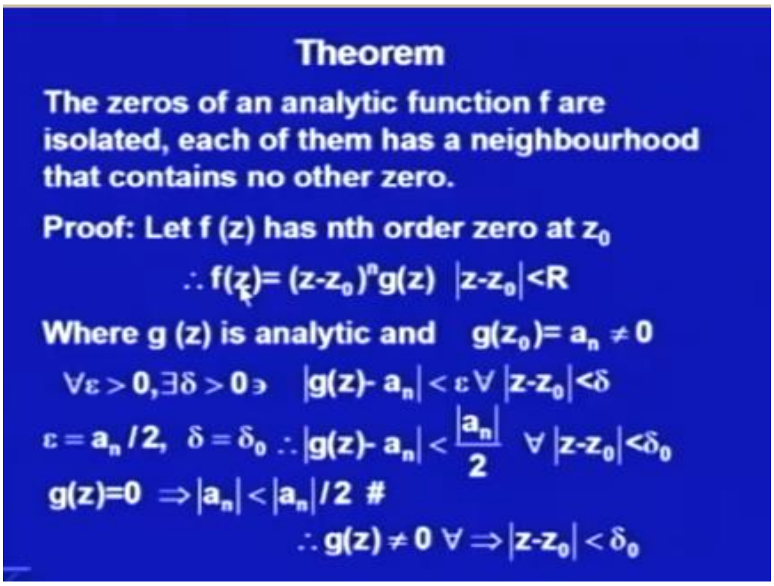

So, we are actually ready to state one result. Zeros of an analytic function, f are isolated. Each that is, each of them has a neighborhood that contains no other zero. So, we are saying is that, the zeros of analytic functions are isolated. Isolated means is, in each neighborhood I do find out there is no other zero other than the point itself. In every neighborhood of that z naught, they would be no other zero other than that point z naught.

Let us prove it, just now what we have done. Suppose if z has nth order zero at z naught. That says as, I can write if z as z minus z naught to the power n g z, in the region z minus z naught is less than R. Just now we have shown that is, if f z is analytic and it so. So, we can write a Taylor series expansion at the point z naught. And if it has nth order zero then the Taylor series expansion could be written over here.

Where g z is also an analytic function because, what would be the g z? g z would be a n plus a n plus 1 z minus z naught to the power 1 plus a n plus 2 z minus z naught to the power 2. And so, that is again power series. So, that is also an analytic function. Moreover, g at z naught is not 0. Now, so g of z is analytic and g of z naught is a n which is not 0. Now, we are having a since g z is analytic what it says is, at z naught g is also continuous. What does it says?

The definition of continuity says, for every epsilon greater than 0 there does exists a delta greater than 0. Such that g z minus a n because, g of z naught is a n. g z minus g z naught are g z minus a n is less than epsilon, for all z minus z naught less than delta. So, I have taken a delta neighborhood you could say. Because, g z is analytic, they are analytic everywhere. So, at z naught also it is analytic. Analytic means that, it is continue at z naught. So, by the definition of continuity if I take any neighborhood delta neighborhood of z naught, he difference of g z with d z naught that is, a n would always be less than epsilon.

Now, choose this epsilon as a n by 2. And suppose for that epsilon, the delta is delta naught. What we are saying is the definition says is, for every epsilon there does exist a delta. So, I am saying is of chosen epsilon as a n by 2. So, the delta would be delta naught. Here is actually, I should have chosen epsilon is mod of a n. If a n is negative then because epsilon we want to positive quantity. So, ideally it should have be mod of a n. Here I am taking precaution that is, of a n is possible it is a n is positive.

If this is happening, then what will happen? We say from here that, g z minus a n this is less than mod of a n upon 2. For all z minus z naught in the less than delta naught because, corresponding to this epsilon this delta is this delta naught. Now, what I want to say that, in this small neighborhood delta naught neighborhood of z naught, g z will not be 0. Why? I am saying is that, they would is no z in this small neighborhood z naught such that, g z is 0.

If g z is zero, then what will happen? From here, what I would get? If g z is 0, what I would get is here, this g z minus a n it is a absolute value would be actually absolute value of a n. And that, I would says is a smaller than absolute value of a n by 2. But, this is the contradiction. a n I am taking a positive value, a mod of a n is a positive value. I am saying is a positive number, which is less than its half of half the number, that is not possible. So, this is the contradiction. Hence, g z cannot be 0 in this neighborhood.

What it says is, I have find it out. If z naught is a 0 of f, then for of order n. Then, I can write if z as z minus z naught to the power n into g z. And g z is not 0 in a small delta naught neighborhood. This is this particular example, I have taken. Actually, what we are saying is that this epsilon is arbitrary. So, we could use any particular neighborhood or we can use any other neighborhood. We are getting is that g z is not 0.

Since g z is not 0, that says is that my f z will not be 0 for any other z in this delta naught neighborhood. Because, this would be this is this term would be 0 only at z is equal to z naught. Any other point is zero of f in the z naught neighborhood delta naught ever neighborhood of z naught. That says as, I should have for some other z also, it is zero.

For some other z, it would be zero only if g z is 0 for some other z. But, what we had find it out here that, in this is a small delta naught neighborhood of z naught. I am finding that, g z cannot be zero. Hence my f cannot take zero value, in any a small neighborhood of delta naught z naught that says is my z is isolated, z naught is isolated zero. Now, come to the other term called singularities.

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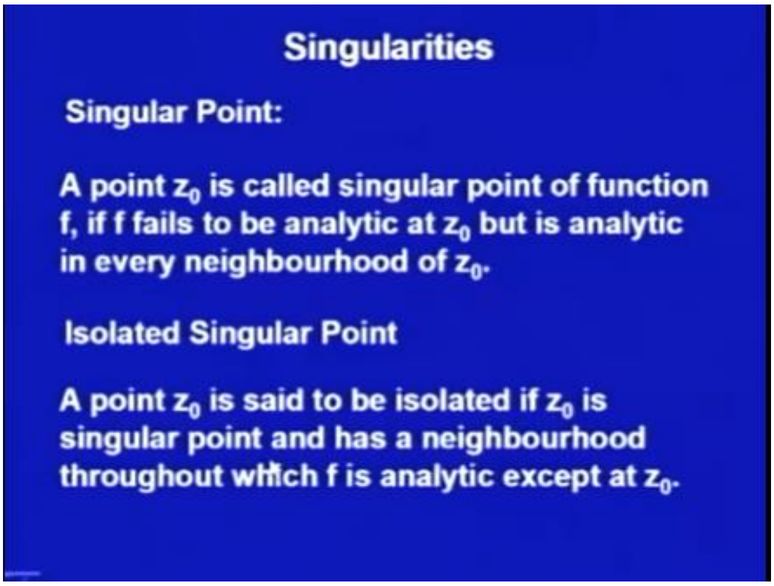

Singularities are also known as singular point. Say a point z naught is called singular point of function f, if f fails to be analytic at z naught. But, it is analytic in every neighborhood of z naught. So, what we are saying is that, if we do have an analytic function fails to be analytic. But, it is analytic in every neighborhood of z naught. Then, we are calling it as a singular point. We call it isolated singular point, a point z naught is said to be isolated if z naught is singular point.

And has a neighborhood throughout, which f is analytic except at z naught. So, we have defined a singular point where the function is analytic. It fails to analytic that is called singular point. It is called isolated singular point, if it has a neighborhood throughout which f is analytic except at z naught. Then, in that neighborhood we call it as isolated singular point. That is, it is only singular point the over there. Now, let us see what we are having? We would relate with the Laurent series. See example.

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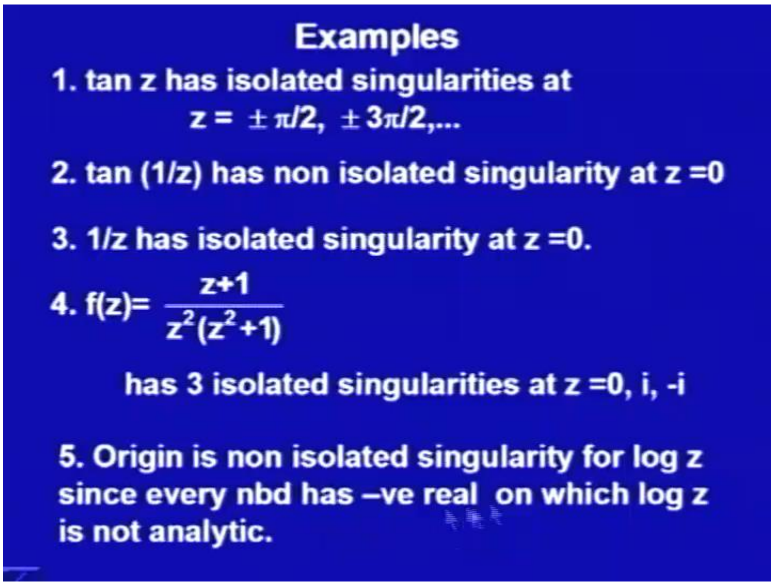

Tan z has a singularities isolated singularities at plus minus n pi plus minus 3 pi by 2 and so on. You see, tan z is naught defined at pi by 2 or minus pi by 2 or minus 3 pi by 2 or 3 pi by 2. But, is define everywhere else and is analytic also everywhere else. So, we do have that is if I take pi by 2, so if it example. Then, tan z is not analytic at pi by 2. But, pi by 2 is the only point at which it is not analytic. Any point in the neighborhood of pi by 2, the tan z is analytic.

So, it has isolated singularity at pi by 2. Similarly, we can check for each of these points. Let us take one more example. Tan 1 by z. Now, this has non isolated singularity at z is equal to 0. That means at z is equal to 0, what we do have? x non singular, but in that neighborhood also we do find out the points because, we to find out. That is, since tan z has singularity at. You could says 2 n plus 1 pi by 2. So, I could always find out many points in small neighborhood of z.

Such that, tan 1 by z is not analytic at those point at tan 1 by z has singularities, at those point in a small neighborhood of zero. So, tan 1 by z has non isolated singularity at z is equal to 0. Now, 1 by z has isolated singularity at z equal to 0. 1 by z is of function, which has singularity at z is equal to 0. If I take any neighborhood of z is a 0. For this function, we will find it out that the function 1 by z would be analytic, in all that neighborhood of 1. So, 1 by z has isolated singularity at z is equal to 0.

Now, let us take these are one simple example. Now, let us take another example. f x is z plus 1 upon z square into z square plus 1. We see, this is a rational function you could say, where the numerator is a polynomial as well as the denominator is also polynomials. You see simply from here, that this function will not be defined for z is equal to 0. As well as, it would be defined for z square is equal to minus 1 or z is equal plus or minus i.

So, we do have 3 isolated singularities 0 i and minus i. Whether they are singularities are all right, but whether they are isolated or not. Let us take z is equal to 0. If I take in a small neighborhood of z is equal to 0. You do find it out, that this function would remain analytic at all those points in the neighborhood of zero. Similarly, if you do take z is equal to take plus or minus i. And any neighborhood of i or minus i, this function will remain analytic in that one.

That you can check by the first definition also. Rather than the some simple methods of checking about the singularities and isolated singularities, we will discuss a little later on. Now, if I take the consider the function log z. Then, origin is non isolated singularity for log z. Why? We do know that, the log function is defined for all positive real x. Now, if I take origin and take a neighborhood in the, on the real line itself only. Then, for all negative points log is not defined.

So, log z is not analytic over there. What it says is, since every neighborhood of origin will contain some negative real, for which log z is not analytic. So, origin is non isolated singularity. That is, every neighborhood of zero will contain some points at which the log z is not analytic. That is, it is not isolated singularity. It is non isolated singularity. This isolated singularity has some importance, we would find out little later.

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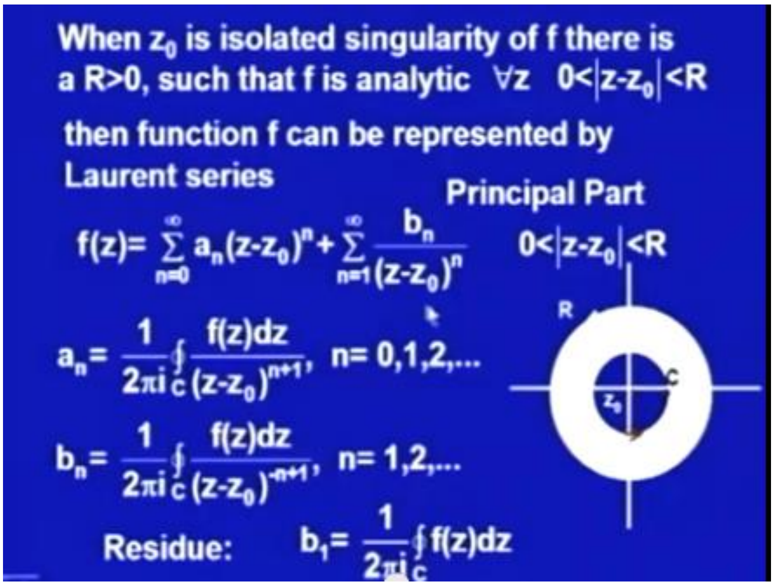

If it is an isolated singularity, then as I said we can related with the Laurent series. We do know that, when z naught is isolated singularity of f. There is always an R, which is positive such that f is analytic in that reason. Isolated similarity, a similarity simply means is that in a neighborhood, I could find out a neighborhood of z naught in which f is analytic throughout that neighborhood, except at the point z naught.

That is, what we are saying if it is an isolated singularity. I could always find out number R such that, f is analytic for every z in that neighborhood. So, we are saying yes. Because, z minus z naught the absolute value this has always to be positive. So, I am writing it 0 is less than mod of z minus z naught is less than R. Let us see, what this is? This is you see, you can recognize very easily. It is a disc, which is not taking or this is you could say a circle centered at z naught with radius R, 0 is out of this one.

So, we could have as it is disc. So, we do know that, in a disc if the function is analytic, then the function can be represented by the Laurent series. This you had already done the Laurent series. What is the Laurent series? Laurent series is that a function f z can be written as, summation of n is running from 0 to infinity a n z minus z naught to the power n plus summation n is running from 1 to infinity, b n upon z minus z naught to the power n for all z lying between, 0 is less than z minus z naught is less than R. That is in the disc.

What we are saying is, if f z is analytic we had find it out. That it can be presented by a Taylor series or that power series. If it is has an isolated singularity at z naught then in the disc we could write it as a Laurent series. Laurent series has two kinds of terms, one is the power series terms. Another is the terms, where we are having is that b upon z minus z naught to the power n. So, this we are having is the disc this is R. And we are having is z naught is the point, where this is fail to be singular.

That is, there is a singular fails to be analytic. So, this is the singular point. So, if I do have any disc over here, in this disc we can define this has this one. Now, where this a n you had done it can be obtained by these integrals, as 1 upon 2 pi i integral along this curve c. The curve c that is we are taking is, that this disc. So, whenever I am taking this 0 is less than z minus z naught.

So, wherever this point. This is not necessarily that, this c this could be any c. But, it has to be exclude your z naught. This can be given as 1 upon 2 pi i. The integral on the close contour c of f z upon z minus z naught to the power n plus 1, for n is equal to 0 1 2 and so on. And b n is been given as 1 upon 2 pi i integral along this close contour c, f z upon z minus z naught to the power minus n plus 1.

You see, for a n we are having is in the denominator z minus z naught to the power n plus 1. And in the b n we do have is, z minus z naught to the power minus n plus 1, for n is equal to 1 2 and so on. Now, this when n is equal to 1 this b 1 that would be actually 1 upon 2 pi integral f z d z. This has a special significance, this b 1 is called residue also. This portion, which is involving the z minus z naught to the power in the denominator, this is called the principle part.

Now, what we have come up? We had come up with isolated singularities. We had shown that, the Laurent series when we do have isolated singularity. We can find it out, that we can write the function as in the form of R. The function can be represented in the term in the form of Laurent series. In the Laurent series, we do have two parts. One is part you could say, power series part.

Another part is, that is in the denominator we do have the powers of z minus z naught. This second part is called the principle part of f z actually. This principle part is important why? This principle part is going to classify my isolated singularities because, my function f is having isolated singularity at z naught. The kind of singularity is that, we would identify using this principle part. Let us see how we are going to use this principle part.

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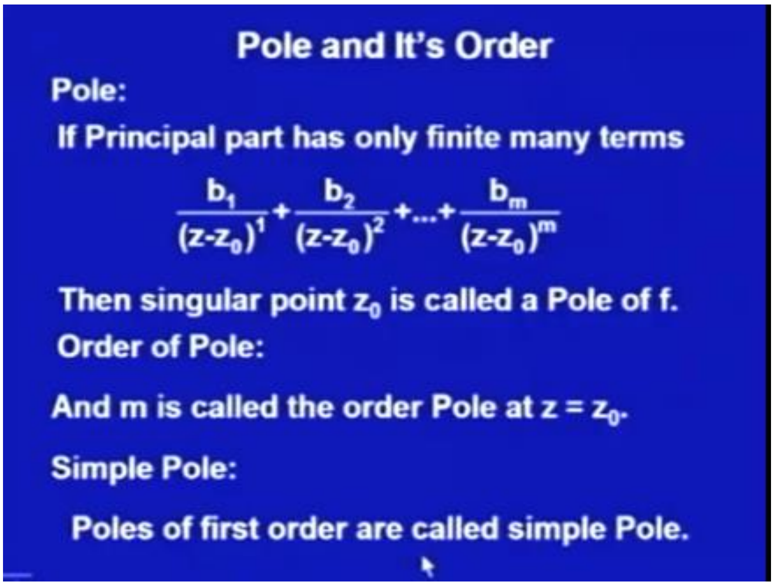

We do define a pole. If principle part has only finite many terms, that is if I do have the principle part such that b 1 upon z minus z naught plus b 2 upon z minus z naught square and so on. b m upon z minus z naught to the power m. And then b m plus 1 b m plus 2 and all those are zero. Then, we say that the isolated singular b z naught is a pole and order of that pole is m. So, what we say is that, then the singular point z naught is called the pole of f.

Now, say you have got. Now, the singularity I have isolated singularity. I have given one more term pole. What we are calling pole? If the principle part has finite many terms only. And if this b m is not zero, if b m is not zero then we would call this pole of order m. So, m is called the order of pole at z is equal to z naught. So now, if my m is equal to 1 that is, if I do have only simple single element, single term in the principle part b 1 upon z minus z naught.

Then, it would be called simple pole. So, if pole of first order are called simple pole. Now, if so we have come up that to finite. Then, it is a pole of order m. And of course, first order pole we call the simple pole. But, if the principle part does has infinite many terms, but before answering that question, let us do some examples for the poles.

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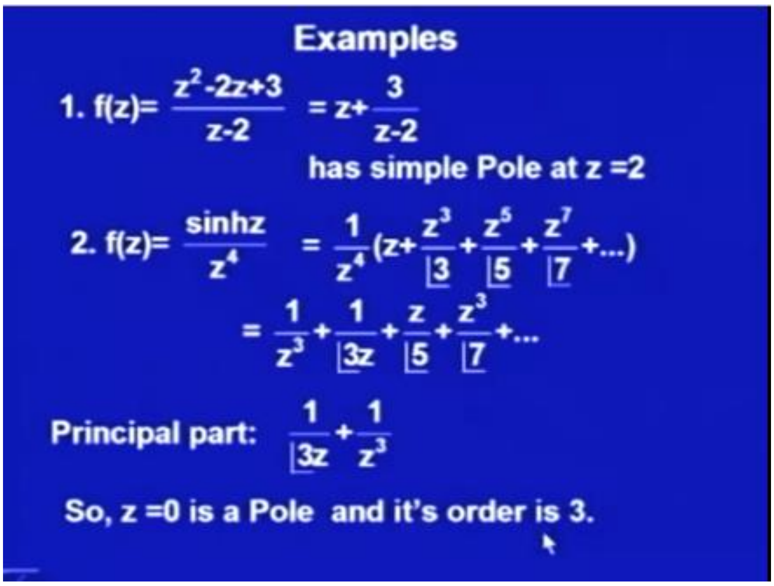

First f z is, z square minus 2 z plus 3 upon z minus 2. We have to find out the poles of this function and order of that pole. Here we see is, that is this function I could rewrite as z square minus 2 z upon z minus 2 that is z plus 3 upon z minus 2. What we can write? f z as z plus 3 upon z minus 2. So, the principle part is 3 upon z minus 2. This has a simple pole at z is equal to 2 because, the principle part contains only single term.

So, it is a pole at 2 and that is a simple pole because of the first order. Let us do some more examples, sin hyperbolic z upon z to the power 4. Let us write the Maclaurin series for sin hyperbolic z. It is of z plus z cube upon factorial 3 plus z to the power 5 upon factorial 5 and so on. And z to the power 4 is in the denominator, so 1 upon z to the power 4. Let us multiply it, what I would get? I would get 1 upon z cube plus 1 upon factorial 3 times z plus 5 plus z upon factorial 5 and plus z cube upon factorial 7.

That is, now these terms are your terms of summation a naught or as summation a naught z minus z naught to the power n. And here is the principle part. So, the principle part contains two terms here. But, you see it is not only the two terms. You do remember, that is principle part we say it is, b 1 upon z minus z naught plus b 2 upon z minus z naught square plus b 3 upon z minus z naught cube and so on.

Having is here is, z naught you could see is that is we are having zero. And what we are having till z cube. That is the term b 3. So, we are having actually three terms in our principle part, where b 2 is actually zero. So, we would have a pole at zero for this function and order of that pole would be 3. So, principle part is this one. So, z is equal to zero is pole and it is order is 3. Now, come to the question that is, if my principle part contains infinite many terms, then we called that isolated similarity as essential similarity.

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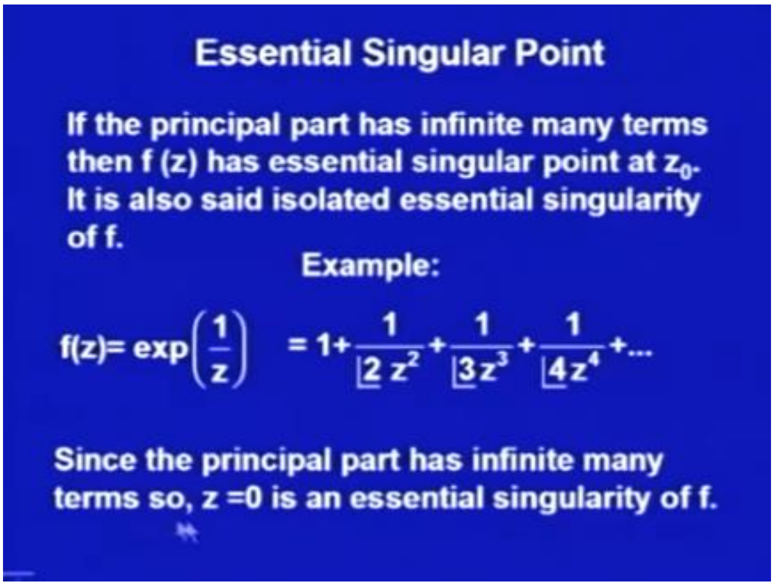

So, we define the term essential singular point. If the principle part has infinite many terms then f z has essential singular point at z naught. It is also said, isolated essential similarity of f. Say for example, if I take the function exponential of 1 upon z that is, e to the power 1 by z. If I write it Maclaurin expansion, we do know it is 1 plus 1 upon factorial 2 z square plus 1 upon factorial 3 z cube plus and so on. Of course, we will have the term 1 upon 1 plus z also.

Now, you see is we will have infinite many terms in this one. That says as, at z is equal to 0 it is essential singularity. So, it means principle part has infinite many terms. So, z is equal to 0 is an essential singularity of f. Now, we had use this Laurent series or the principle part to define or to classified the, isolated singularities. We had I made it actually three kind of classification. One is essential singularity, another is poles.

And in the poles, again we have made two things that is sample pole and pole of a mth order. It is not only that, we are just classifying this for the sake of classification. Actually at all these three points that is, if the essential if the isolated singular point is essential or simple pole or m th order pole. The behavior of the function near that singular point, changes very dramatically. So, let us see the function of the behavior of the function, near poles and near this isolated, this is essential singularities. So, the behavior of the function at pole.

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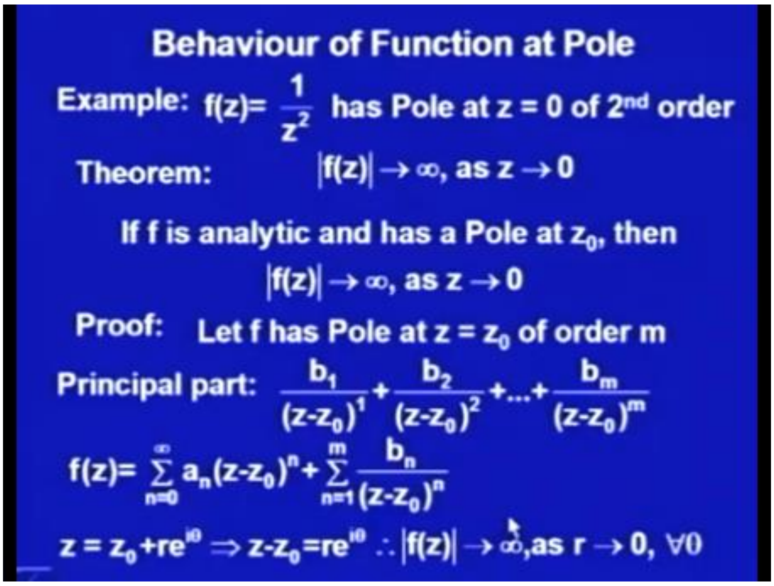

Let us first see it, by the example. First, suppose I am taking the function 1 upon z square. If I take this one, we do know that 1 upon z square. So, this is just you could say is principle part, which is having the second term. Till second term b 1 is 0, but of b 2 is 1 and we do have the second term. So, the it would have pole at 0 and it is of the second order. Now, how it is behaving near this zero? The function if z is approaching to 0 from any side.

Now, you see that is, we are not on the real line. We are on the complex plane. In the complex plane that is says is that, we are talking about the function of two variables equivalent to the talking about the function two variable, always in the complex plane. All these things you had of course, learn in the limits and although derivatives. So, we do know that, f z will approached infinity as z approaches to 0 in any manner.

What we have got, if it is a pole? Then, at pole it is approaching to the function will approach to infinity from any manner, z is approaching to 0. From here, we do get one result. Now, I am just stating this is at, if f is analytic and has a pole at z naught. Then, f z will approach to infinity as z approaches to 0, in any manner. That is from any direction, if z approaching to 0, f z will always approach to infinite.

You see the proof of this, how we are doing? Suppose f has a pole at z is equal to z naught of order m. What it says is, that the principle part we would have m terms b 1 upon z minus z naught b 2 upon z minus z square plus b m upon z minus z naught to the power m. Where certainly b m should not be zero, with any of these terms could be zero. But, b m should not be zero.

Now, you see this portion will approach to zero, from any direction as z approaches to z naught. See f has the pole of this once, so what we are saying is that is a poles f can be written as this manner. We do not know, whatever be this a n. So, this is what we are saying? Because, it is z naught is a pole. That means, z naught is an isolated similarity. That says is in a disc, zero is less than z minus z naught is less than R.

We could write f as the Laurent series. Summation n is running from 0 to infinity a n z minus z naught to the power n plus n is running from 1 to m, b n upon z minus z naught to the power n. Because, we are assuming that is z naught is a pole of order m. This is for this one now. Let z is z naught that is, on that disc we are talking about. And we do know that, b n's and a n's we are finding out as the contour integral, along the simple close contour and that, we are taking as that circle.

So, let us take that circle centered at z naught as r e to the power i theta. This says as, z minus z naught would be r e to the power i theta. Now, as z approaches to z naught, what I would get is, that mod of z minus z naught would be actually r. So, as z approaches to z naught mod of z naught minus z naught, should approach to 0. That says as, r should approach to 0. Now, here what we are getting is all the terms, if I write it out.

I would get r e to the power i theta r square e to the power 2 i theta r cube e to the power 2 i i m theta. Now, whatever be this theta as r approaches to 0 for every theta, these terms would approach to infinity. That is this principle part, will approach to infinity. As r is approaching to 0, this principle part would be approaching to infinity for any theta. Theta means is that, if you to remember that we are having this kind of circle.

So, what we are having is that is either r is approaching to 0 in this direction, this direction or this direction that is from in any manner, if r is approaching. And this z is approaching to z naught. This principle part would approach to infinite or hence, f z will approach to infinity. So, this is what we are saying is if it has a pole of any order, then f z will approach to infinite now. So, this is happening for every pole at now.

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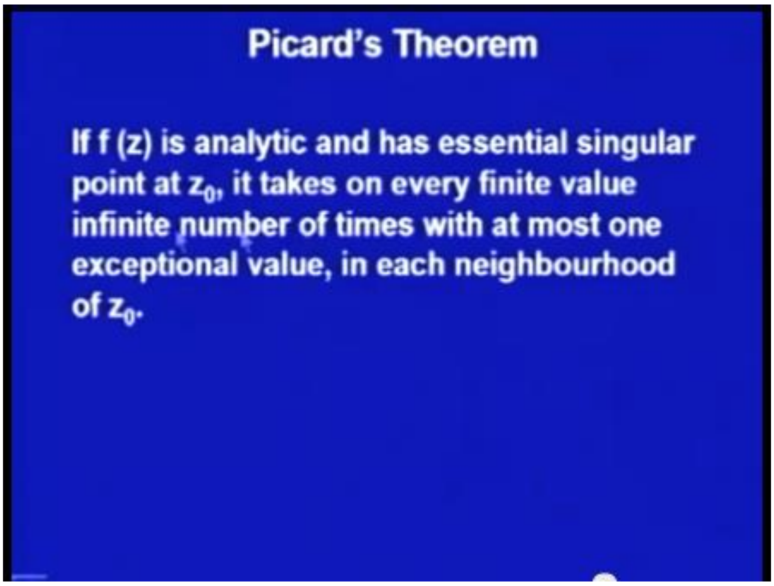

What is the behavior of the pole at essential similarity. This is being actually done by, Picard's theorem is been explained. So, let us first write this. I will just give you the Picard's theorem. Of course, we are not going to prove it. But, I would show you that is how this is happening. If f z is analytic and has essential singular point at z naught. It takes on every finite value, infinite number of times with at most, one exceptional value in each neighborhood of z naught.

Let us first understand, what this is statement actually saying. This statement is actually telling us, about the behavior of the function f near essential singularity z naught. What it is saying is that, if I take any neighborhood of z naught. I would find out, I can find out that every finite value that the function can take, could be taken infinite number of times. That is suppose f of z is the say, let say is 1.

Then, they would be infinite many z's for which f of z would be 1 in a neighborhood of z naught. And this would happen for every finite value, except one value. That except one value, that value they would be one value for which it, this function f cannot take that value. So, that is called the exceptional value. And this is happening in each neighborhood of z naught. Let us see, what it is saying.

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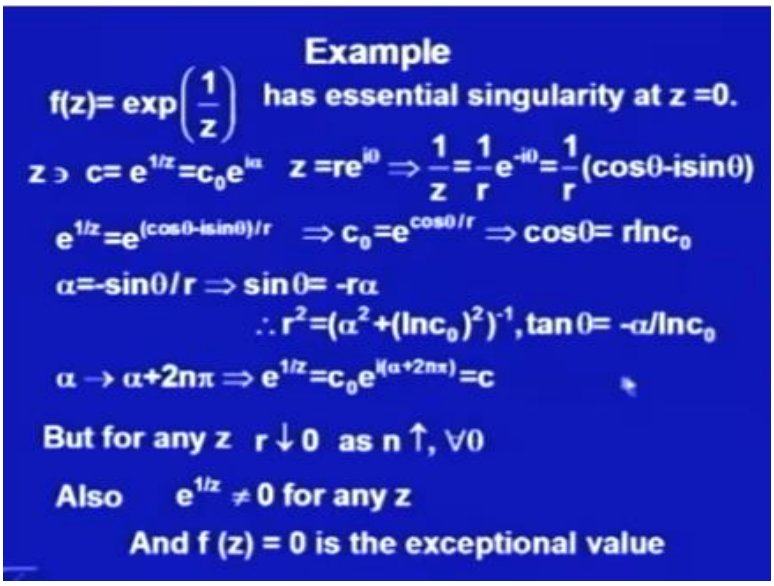

It would explain this behavior, through one example. The example of essential singular function essential singularity for a function e to the power 1 by z, we had already done. So, let us come back to the same example. We do have exponential of 1 by z. This function, we do know that z is equal to 0 is essential singularity for this function. So now, because it is essential singularity and now we have going to use that disc, that 0 is less than z is less than R.

So, I would chose a z such that, what we do say is that is for every finite value, it will take infinite through. Now, I am taking a particular point c a particular value c for f z. So, I am choosing a z such that f z that is e to the power 1 by z, is a constant c. Since it is a constant, it has to be a complex number we are talking in general. If it is not complex in it, it would be real. So, of this complex number let us write it out as c naught e to the power i alpha t, where c naught is some constant. Alpha is some constant.

Now, if I represent my z by the polar coordinates rather than x and y. If rather than writing z as x plus i y, if I write in the polar coordinate then z can be written as r times e to the power i theta. This says as 1 upon z would be 1 upon r e to the power minus i theta. We do know that by cos value that is Eulers formula that e to the power i theta or e to the power minus i theta. I could write as cos theta minus i sin theta.

So, now let us come e to the power 1 by z, e to the power 1 by z. I could write us e to the power cos theta minus i sin theta upon r. Now, break it into this part because, what we want e to the power z is constant c. So, I would like to equate and this constant c. I am writing as c naught e to the power i alpha. It is so I would try to find out what is this real part and what is this c naught and what is this e to the power i alpha d.

So, from here if I do rewrite it, I would get c naught as e to the power cos theta by r. And into e to the power minus i sin theta by r. So, what I would be getting is that is first c naught I would get as e to the power cos theta by r. What it says is from here, cos theta by r would be logarithmic of c naught or cos theta would be r times logarithmic of c naught. So, we are writing r l n c naught.

And the comparing it with this e to the power i alpha t, I would get it here into e to power minus i sin theta by r. That say as, my alpha would be minus sin theta by r. What it says is that, sin theta would be minus r alpha. Now, we do know sin square theta plus cos square theta is equal to 1. So, from here if I add it up, what I would get? r square log c naught square plus r square alpha square is equal to 1 r.

What we are getting is r square from here? alpha square plus log c naught whole square whole to the power minus 1 r 1 upon. That is, what we are getting is r square times alpha square plus log c naught whole square is equal to 1. So, r square would be 1 upon alpha square plus log c naught whole square. So, I am writing it in this manner. And what will be the tan theta? That is sine theta upon cosine theta. r and r would cancel it out. I would get minus alpha upon log c naught.

Now, we have to find out the behavior of, what we are saying is the Picard's theorem says is that. This value c would be taken by infinite many z's. I have taken this 1 z. And I am representing that z by the polar coordinates. So, what I have to show is, that for many values of r and theta, I would get this c as here. Now you see, here if I chose my alpha change my alpha to alpha plus 2 n pi. Suppose I change my alpha 2 alpha plus 2 n pi, what will happen?

This c value, c value would be e to the power 1 by z. That is c naught e to the power i alpha plus 2 n pi t. So, alpha i alpha plus 2 n pi, i alpha is as such, i 2 and n pi that we do know is 1 because, cos 2 n pi plus i sin 2 n pi and cos 2 n pi is always plus 1. So, this is same as c naught e to the power i alpha that is c. So, if I am changing alpha to alpha plus 2 n pi, suddenly this value c is not changing. Then, what is happening?

If I take this r, what will happen? If I am making alpha 2 alpha plus 2 n pi. That means, my r is changing r would get, this denominator I would get. That alpha is increasing, if denominator is increasing. Then, suddenly my r would go to zero. We want that is, how this function will behave as this is going to zero. We are saying is that is, in that neighborhood actually I would find out, infinite many points for which this is happening now.

For each and I would get a different term. What will happen to theta? Theta is minus alpha plus 2 n pi. That would not change you can check it that is it is not going to change. So, what I would get for any z as r approaches to 0 or as n is the n would increase for all theta. What it says is that, e to the, so what we are getting is. Let us first, I explain this term. For any z as r approaches, r decreases towards to 0, r will decrease towards 0 and n 1 increase. And this is will happen for all theta.

Whatever be value of theta, I do take. If I take alpha plus 2 pi, I would get one value r here. And for that actually you to see, that is for every theta because, this is now fixed up. We do get is that is for every theta I could find out a different r. That says is, what we are finding it out. I could find out in polar once, what we are saying is theta is one direction different r. And for one fixed r, we do have different theta.

So, what we are saying is for one fixed r, I do have different theta one over there. And for all those theta, the value of the function f z that is see the, when I am changing it to 2 n pi some. Let us, I am first taking the example of 2 pi. I do know that, whenever I am talking about tan theta plus alpha plus 2 pi, I would be changing towards only that making whole circle, but r values getting changed.

So, for different values of n I would be getting different task. For all theta as if, they are infinite many points in that small neighborhood. Such that, the function is taking value the same value c. And this c naught and this c I have taken as arbitrary. This alpha is also arbitrary. That says is that, term for any arbitrary finite value. The only condition is that is, I have to take the finite that c naught has to be finite.

So, for any finite value we are getting is that they would be many z for which f of z would be that c. Moreover, we do know e to the power 1 by z. This cannot be 0 for any z. So, in the Picard’s theorem that, what they are talking about the exceptional point. That 0 is the exceptional point here or the exceptional value, rather it is not the point it is the exceptional value. So, we do say is that is because, f z that is 0 cannot be taken for any z.

So, we do have if the function has essential singularity, at any point z naught. Then, every neighborhood of z naught we could find out that, it will take all finite values infinite many number of times. But, one value which it cannot take. And that value is called the exceptional value. So, we have find it out that the behavior of function at the isolated singularities is different, according to the classification of singularity.

If it is a pole then the function will approach to zero near the pole. If the function is, if the point is not pole, then actually we are finding it out that, it is not reaching to that point for in any manner. Rather it is having all the values in that neighborhood. It is having, but it is never reaching to that point. So, this is one example that is what the Picard's theorem had said. So, we have got that if point has the Picard's theorem, which is telling the behavior of the function at essential singularity.

We had one more result, which was telling about the function of the behavior of the function at poles. So, you do find it out. Today we had learn for an analytic function, what is the zero of the function. And we had find it out, that the zero of an analytic function is always isolated. That is, it will have in a neighborhood the single zero at that point, no other zero. Then, we had find it out that if suppose function is naught analytic at any point, that points we called the singular point.

We have defined isolated singular point. That says as, if in a small neighborhood I do find out the function is analytic throughout that neighborhood, except at that pint z naught. We called it isolated singular point. Then, this isolated singular points we have defined that, we could use the Laurent series to define the function in a disc, which is including that excluding that isolated singular point.

And for that, from there we had find it out that the principle part of that series at that the function, that can be use to classify those essential, this isolated singularities. That classification was necessary, because the behavior of the function is changing dramatically or those points as they are classified. That is at the poles, we do find it out. That the function will approach to infinite as z is approach into that isolated singular point, when the point was a pole in any manner.

Then, we had seen that if it is all essential singularity. It is not necessary that the function will approach to or you may have found it out, that in a small neighborhood. We can find out that, every finite value is being taken by the function. And the function you could says, you can find out infinite many points z in that small neighborhood, for which that particular value would be taken.

And one more thing, which was crucial for that. That there is one exceptional value, that there may be one value in that a small neighborhood, that is in all finite values I could find out one value, for which this function will never take that value. That value has been named as exceptional value of the function. So, we had learn the poles or the singularities and which the behavior of the function, at those singularity points. This is one more characteristics of the singularities. But, today we will not go ahead for that. We will do those things in the next lectures. So, today is that is all for all.

Thank you.