Mathematics - II
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Lecture - 5
Application of Cauchy Integral Formula

Welcome to the lecture series on complex analysis for undergraduate a students. Today's lecture is on application of Cauchy integral formula. In the last lecture, we had seen one application that evaluation of integrals. Today, we will go for the more applications in the analysis, as well as with there help how to evaluate certain integrals. So, let us go with the first application the derivative of analytic function.

With the help of Cauchy integral formula we will prove one important result that, analytic function has all the derivatives are all order derivatives and they are also analytic. So, here I am giving you the result.

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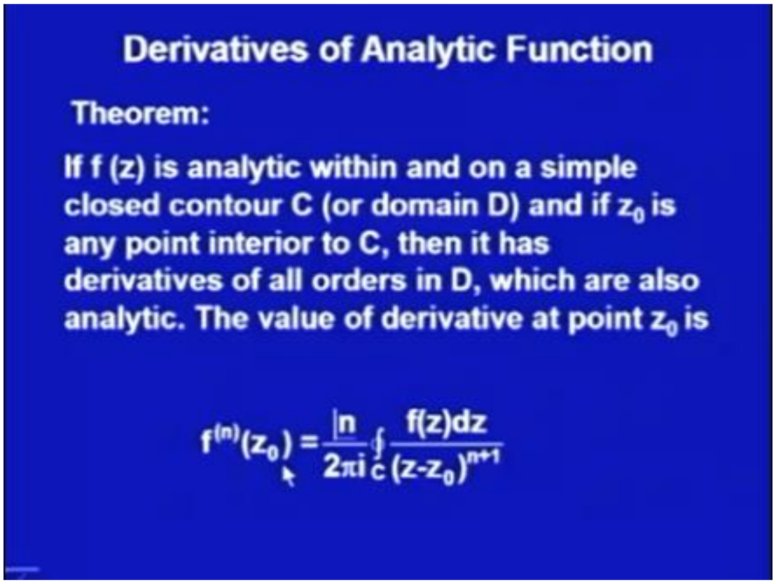

If f z is analytic within and on a simple closed contour C or in a domain D. And if, z naught is any point integral to C. Then, it has derivatives of all orders in D, which are also analytic. The value of the derivative at point z naught is given by the formula, the derivative f z naught as factorial n divided by 2 pi i integral on the closed contour C f z upon z minus z naught to the power n plus 1 with respect to z. So, when I am talking here the domain D, I would mean is that is the domain containing the points on C an interior to C.

If the function is analytic within that interior and on the contour, then it will have at all interior points, we will have that function would be analytic. And the formula for the derivative at that point could be given as the integral of this one. Let us see, that is how we can obtain it. So, first we will go for proving this result using the Cauchy integral formula. First we will go with the first derivative. And then second derivative and then we would just see if that is in the, we could find out this formula with induction. So let us, move to the proof.

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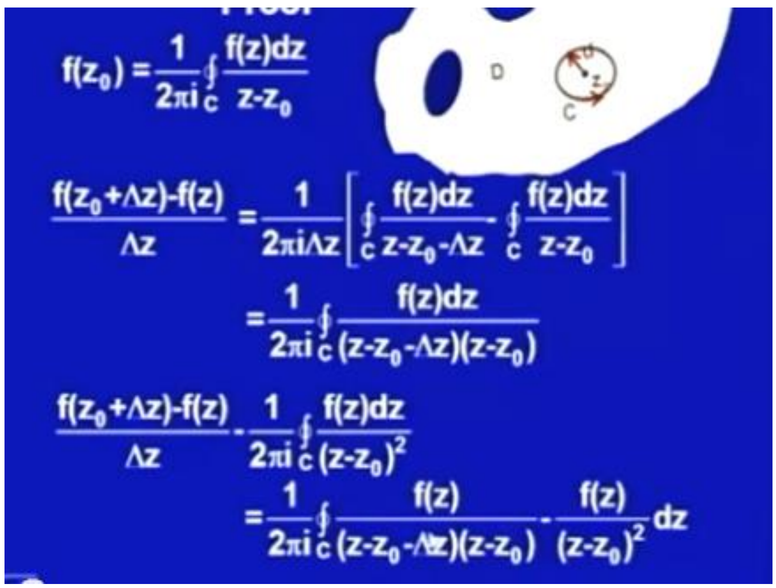

Let us see, this is a domain, so when I am talking about D. I am not talking about this complete one. I am talking about in that a statement with the theorem. The points on C and inside this one. So, let us see, if this is a contour and z naught is a point inside this contour C. So, what we are saying is we would just start with this Cauchy integral formula.

If I take the integral along this contour of the function f z upon z minus z naught by Cauchy integral formula, it is 2 pi i f z naught or f z naught is 1 upon 2 pi i integral on this close contour C of, f z upon z minus z naught. What is this contour C, this contour C now I have taken as one, you could say looking like a circle with radius D. But, it is not actually it could be any contour C. Now, this Cauchy integral formula I would apply to any other point f z naught plus delta z.

So, what we do is I take 1one point z naught plus delta z in a small neighborhood of this z naught. That is small neighborhood of z naught, if I do take the point z naught plus delta z. So, that is that is small neighborhood, I have taken in a manner such that that is also inside this contour C. So, again using the Cauchy principle formula for that point. I could write that f z naught plus delta z, as f z upon z minus z naught minus delta z d z integral along the contour C 1 upon 2 pi i.

So, now the difference of f z naught plus delta z minus f z upon delta z, I would write that is 1 upon 2 pi i integral f z upon z minus z naught minus delta z minus of f z upon z minus z naught. So, this is what we are writing. So, for this f z naught plus delta z this is integral f z d z upon z minus z naught minus delta z. And for f z naught it should be f z naught it is f z d z upon z minus z naught, where I have taken z naught plus delta z in a small neighborhood of z naught. Such that, this z naught plus delta z is also inside the contour c r is also interior to C.

Now, from here if I just write, because there is other same contour so this integrant we can joint and we could get it of simplifying it. I could write it out that is the f z upon this one we would get is it is only delta z upon z minus z naught delta z into z minus z naught into f z. So, delta z and delta z that is getting cancel it out. And I am getting it is equal to 1 upon 2 pi i integral along the contour C of f z upon z minus z naught minus delta z into z minus z naught with respect to z.

What we are goal is we will first show that, the limit of this is given by the formula, that is 1 upon 2 pi i integral along the C f z upon z minus z naught square d z. So, that we could show that this limit of this function is that. So, we could show that by the definition of derivative that the derivative of this does exist. Hence, it would be analytic at z naught. So, we will go in that manner, so let us just write it out, what we want we will write f z naught minus delta z minus f z naught upon delta z minus 1 upon 2 pi i integral f z d z z minus z naught square.

So, this function we had all ready find out, that this is equal to this integral minus, I am subtracting this integral this integral is again on the same contour. 1 upon 2 pi i is common so again, what we do have is integrant is here f z upon z minus z naught minus delta z into z minus z naught. And here, the integrant is f z upon z minus z naught whole square. So if I take the difference, I would again get the difference as delta z upon z minus z naught whole square z minus z naught minus delta z. So, what we would get is 1 upon 2 pi integral along the contour c f z z minus z naught minus delta z minus z naught minus f z z minus z naught whole square d z. Now, let us do.

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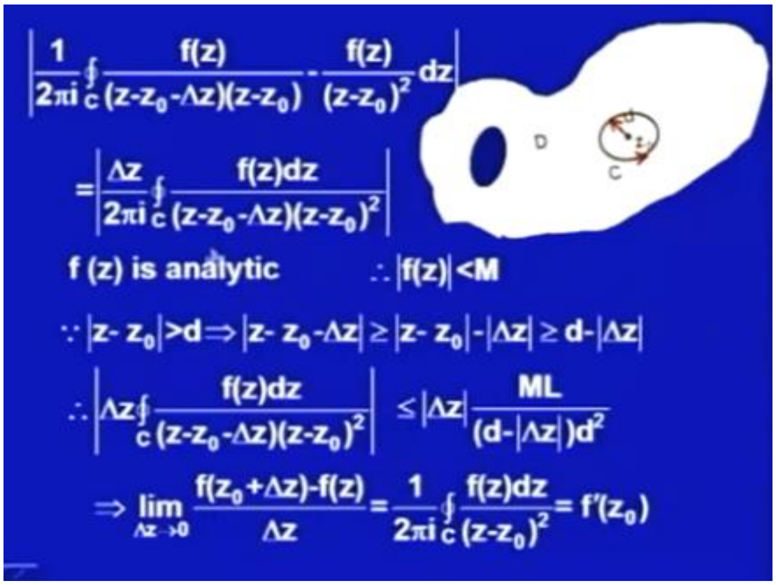

Find out, this absolute value of this difference is delta z upon 2 pi i f z upon z minus z naught minus delta z into z minus z naught whole square. Now, what we would like to show is that this integral should go to C also only then, we would be able to say that the limit is existing. So, we are going to prove that as delta z is small enough, that is from z naught to delta z, z naught plus delta z, if z delta z is small enough this integral goes to 0.

For doing it, we would again use this ML inequality, you do find it out that is how we are using this Cauchy integral formula and this ML inequality. f z is analytic, so f z would be less than would be bounded. Because, f z is analytic inside this so I could find out to one point, that is such that f z naught is less M. Because, it is continuous so in this neighborhood I could find it out that is less than and that is very simple result, we have done many times till now.

Now, since z minus z naught, any point on this C is greater than d, now d I am taking some distance or you could say is d is the smallest distance from z naught to this contour

C. Then for any z on this contour C, z minus z naught would be greater than d. This says is z minus z naught minus delta z that is z naught plus delta z suppose this is the point here, would be greater than just using the absolute value inequalities. Would be greater than or equal to modulus of z minus z naught minus modulus of delta z.

z minus z naught modulus of this is greater than d. Hence, this would be greater than or equal to d minus mod of delta z. Now, what it says is my f z is less than M, z minus z naught is greater than d, z minus z naught minus delta z is greater than or equal to d minus delta z. If substitute all these things, what I would get this integrant f z upon z minus z naught minus delta z into z minus z naught square, would be actually bounded by M upon d square into d minus mod delta z.

So, now what we do get is, this would be less than or equal to delta z upon that is M is this one M upon d square into d minus delta z, this delta z is as such here. And L, what is L, L will be the length of this contour. Now, whatever be this L this is fixed one. Because, this contour C is fixed one, so this L is fixed one. Now, if delta z is small enough. What I would get is that ((Refer Time: 10:48)) this would go 0 and this is again a fixed quantity, M is fixed quantity, L is a fix quantity.

I could make this delta z, such a small. Such that, it approaches to 0 or what we are saying is the left hand side. This is the absolute value of this integral this has to be positive or you could say it is nonnegative. This is less than or equal to a value, which can be made arbitrarily small, such a small that it can move to 0. So, this must be equal to 0.

If this is equal to 0, then what we have got by this definition of now, we had proved that f z naught minus delta minus f z naught upon delta z, which is minus 1 upon 2 pi i integral c f z upon z minus z square d z. That is, what this absolute value difference was from the last slide if you do remember. This is by the definition of or integration this differentiation this is nothing but, the f dash or the derivative of at z naught. So this is equal to this one.

So, what we had proved is that, Cauchy integral from the Cauchy integral formula. That if, f is analytic in a domain D. Then, f dash or that is f is dash z is there. And f dash z naught is again, we are getting in the form this integral. You see, f of z naught we had started, f of z naught at is the form of integral using the Cauchy principle. The Cauchy integral formula that 1 upon 2 pi i integral over the close contour C f z upon z minus z naught. That was where f z was analytic.

Since, the f z was analytic, we had shown that now. This function we are getting is that is f dash z naught is this one. Now, f dash z naught is again coming in the same form the only thing is that is here we had made at z minus z naught whole square rather than z minus z naught only. Function f z is analytic, again in the whole domain D in the contour C. And inside this C, only point of discontinuity or this where this analyticity will break for this function f z upon z minus z naught square, would be only z naught.

That is again the same kinds of conditions are being satisfied. So, in the similar lines, if i move, that is if i again take f dash z naught minus f dash z naught plus delta z, using this formula. And then, use it factorial 3 upon 2 pi i and like that one. We would get it that this is again going to prove that is you would get that this limit is going to 0. Or you are will be getting is that the formula would be satisfied. The only thing is that is you have to use that distance, the minimum distance of z naught to the contour C.

That we had already used in the similar manner, you can move with certain modifications, small modification that is how to get it these points. You will get it square and cubes and all those things, so here you will get cube and these points as such, so we would get that is ah f dash z naught in the similar manner is also analytic. And the formula for f dash z naught we would get it like this one. What we are getting is actually? z naught the point I have take it any arbitrary point in the interior of C.

What it says is, if f z is analytic in this whole reason. Then, for any z which is interior to this C. Because, z naught was not arbitrary, I have not chosen any particular z naught. So for, every interior point in this region f z would be the formula would be given by the same kind of thing for any z. And since, z naught is arbitrary what we do say is f dash z will also be analytic in whole region. In a similar manner, we will go again for that if double dash z.

So, what we are getting is because z naught is arbitrary. We are getting that if f z is analytic in f within and on a simple close contour C. Then, it is all ordered derivatives, that is first order derivative, second order derivative and in a third order derivative. They would be existing. And they would also be analytic. Because, all of them we would be getting is in the form of this integral formula. So, we had prove this result that this would be analytic and this one. So, now let us see, is that is how this formula or this theorem is going to helpful in evaluation of integrals.

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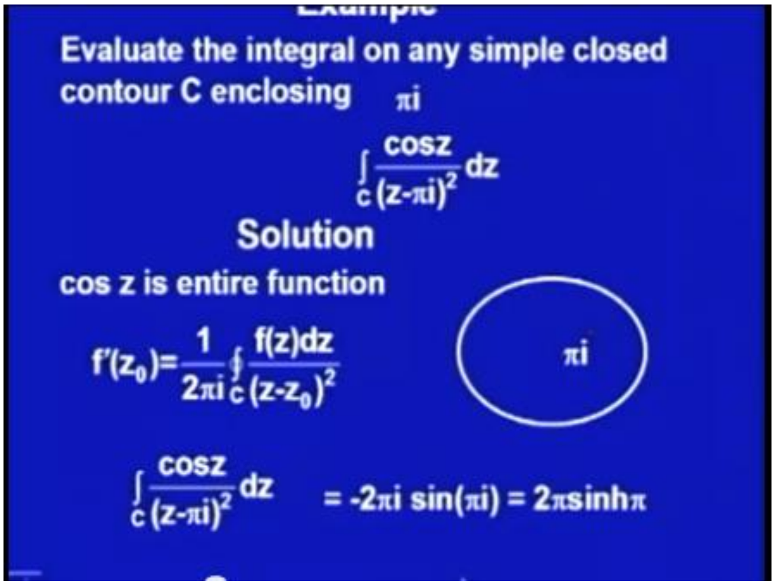

Let us say, the evaluate the integral on any simple close contour C, enclosing pi i of this integral cos z minus pi i whole square d z on a close contour C, which is any close contour C enclosing pi i. So, let us see this suppose this is a close contour, where this pi i is the interior point. Now, I would use this analyticity of the f z. And what we have got that formula that f ((Refer Time: 16:40)) nth derivative of z or rather you could say just I would use f dash z naught is equal 1 upon 2 pi i integral along the close contour C of f z upon z minus z naught whole square d z.

Here, you see what the condition I do require is that my f z has to be analytic in the whole region that is inside the C and on the C now. I am taking any simple close contour now lets see, what is this function cos z. cos z is actually, an entire function. And that is why, if I do take any simple close contour it does not matter, because it is entire function. So, till it is or until it is containing this pi i as interior my this result would hold true. So, f dash z naught 1 upon 2 pi integral along this contour C, f z upon d z z minus z naught whole square.

So now, what is f dash z naught, f is f z is cos z. So, f dash z would be minus sign z. So, at z naught z naught is the point pi i. So, I would get it minus the 2 pi i minus sign and that pi i. So, this is 2 pi sin hyperbolic pi, you find it out that is we could evaluate this integral using this formula for the derivatives. Let us see, some more examples.

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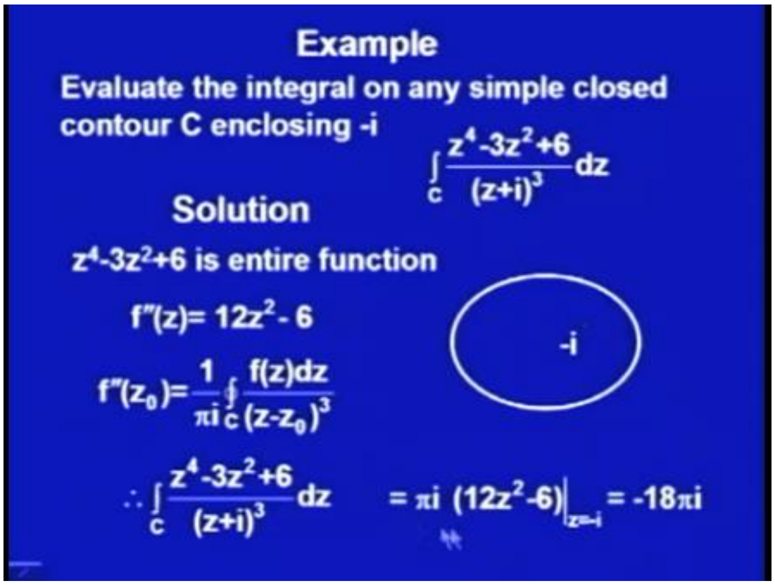

Evaluate the integral on any simple contour close contour C, enclosing minus I, for the function z square minus 3 z square plus 6 upon z plus i cube d z. Now you see, this integral of this function, the function which is on the numerator. z to the power 4 minus 3 z square plus 6, this is the polynomial we do know that z to the power 4 z square all these are entire functions, so this is an entire function. So, that is why we are able to do it, in general that is any close contour enclosing i.

So, let us see, this is the close contour and a, which is having minus i as to the interior of this one. Then, here what I do have is z plus i cube. That says is how to use the second derivative formula. So, this is an entire function, f double dash z of this. If I do find out that is the f dash z would be 4 z cube minus 6 z square, it is again derivative I would get 12 z square minus 6.

The formula what I would use is f double dash z naught is 1 upon that is 2 upon for factorial 2 upon 2 pi i that is 1 upon pi i integral along this contour C of f z upon z minus z naught cube d z. Now, here z naught is now my minus i. So, what I would get is from here this integral this function is f z is z to the power 4 minus 3 z square plus 6. So, i would get, this integral as pi i times f double dash z naught that is 12 z square minus 6 evaluated at z is equal to z naught. That is z is equal to minus i. So, when I keep z is equal to minus i I would get here minus 12. And this I would get as minus 6 so minus 18, so I would get the answer as minus 18 pi i. Let us see, one more interesting example.

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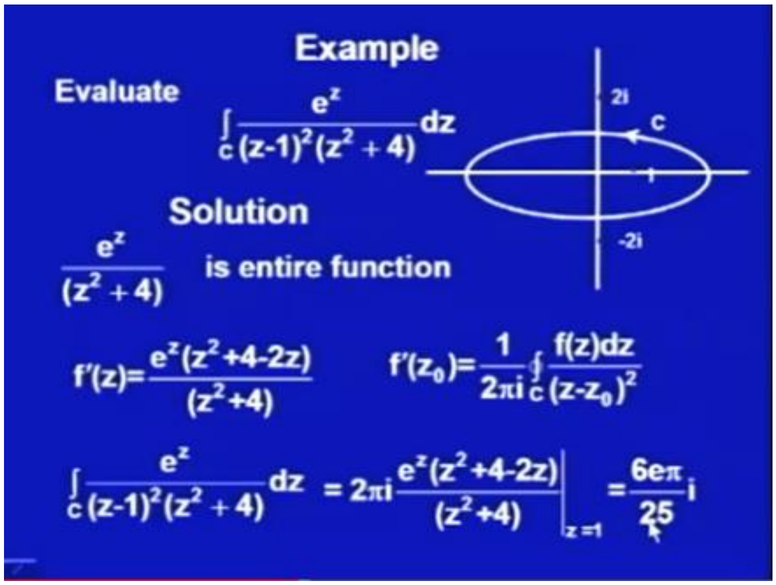

Evaluate the function e to the power z upon z minus 1 whole square into z square plus 4 integral of this function along the contour C, where my contour C is actually an ellipse. Whose this x axis is going from minus 3 to plus 3. And or you could say is that is function, which is having the other points of not analyticity C outside only point, which is where this function is not analytic. That is only one is that interior and the point 2 i and minus 2 i are not interior to the close contour C.

So, I am having this contour C, where 1 is interior, but this 2 i minus 2 i both are outside this one. So, now I will choose this function e to the power z upon z square plus 4. Of course, e to the power z is entire function, but when I take e to the power z upon z square plus 4. The it will not be analytic at plus minus 2 i. But, from given contour, I am finding it out that plus minus 2 i are outside our domain D. So, this function e to the power z upon z square plus 4 is analytic inside and on the contour C.

I would like to use this z minus 1 whole square, so 1 is inside this one. So, I would go with the first derivative f dash z of this one would be e to the power z z square plus 4 minus 2 z into e to the power z upon z square plus 4 whole square. So, f dash z naught using this formula again 2 pi i integral along the contour C f z upon z minus z square d z, where z naught I will take as 1. I would get this integral as e to the power z, z square plus 4 minus 2 z upon z square plus 4 the whole square at evaluated at z is equal to 1. This is your evaluated at 1, you will get it 6 e pi over 25 i.

Now, here what we have done is that we have taken analytic function. On a simple close contour and inside that one, that is we have talked about the derivative of the analytic functions, inside a close contour and on the close contour. Let us, make this result little bit more general. Let us, talk about functions defined by the integrals.

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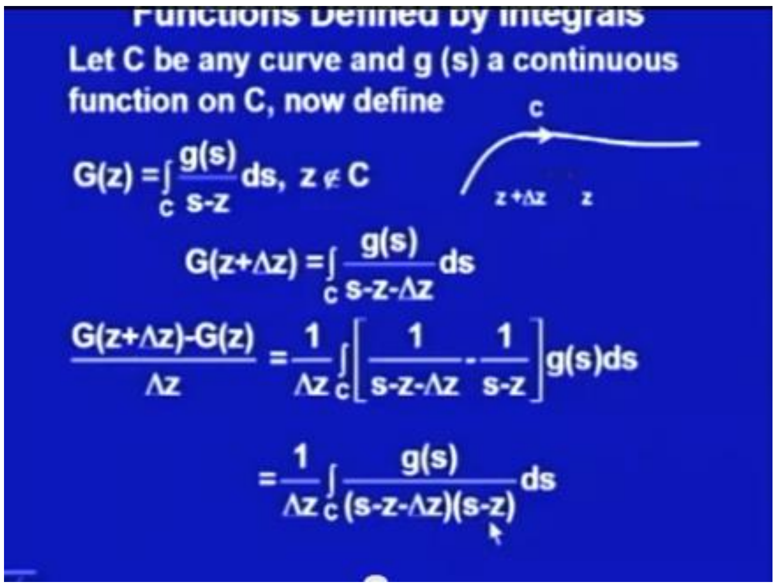

Let C be any curve and g s a continuous function on C, now define a function G capital G as the integral of g s upon s minus z with respect to s on this path C, where this z is not on the path C. So, let us see that is what I am saying is suppose c is any curve. So, I am not talking about simple close curve. It is any curve and the orientation let us say is that is this manner. Then, we are defining and we are having g, which is not an analytic function we are having it as a continuous function on this curve c.

And now, I am defining one more function capital G on capital G at a point z, which is not on C. That is any other point here z, i am taking and I am defining it the function at this point using the function on this path integral of this. So, we are saying is g s upon s minus z ds on this path integral. For any z, which is not in C? Now if I have defined this one now what I would like to say, that this function capital G, which we have defined as the integral of this one. This is actually analytic.

And if this is analytic, it is all order derivatives are existing and they would also be analytic. So, we are now moving a little bit further not on any simple close curve and not starting with this function f to be analytic in the domain, we are staring with any continuous function g. So, for that let us have to another point say in the small neighborhood of this z, as z naught plus delta z.

So, by this definition, which we had made at this point also I could this point the function would be again the integral along this path of g s upon s minus z minus delta z. So, this would be, because z this z plus delta z is also outside this path c. So, it is g s upon s minus z minus delta z ds along this path c. Now, what I will again go with the first definition of the derivatives. And I will show that is d G z plus delta z minus G z it is difference divided by the delta z should go to some limit.

And that limit must be the derivative of this function. So, that limit we will find out again the form of integral. So, let us first move this the difference of function G, at z plus delta z minus that function G at z divided by delta z. So, this 1 upon delta z is as such. This difference of these two integrals, so I would write it as integral on the path c. This g s is taken common and what is being here is 1 upon s minus z minus delta z minus 1 upon s minus z. Simplify it, I would get s minus z minus s minus z minus delta z. That is would get here delta z upon s minus z minus delta z into s minus z. So, we would get 1 upon delta z g s s minus z minus delta z into s minus z and delta z.

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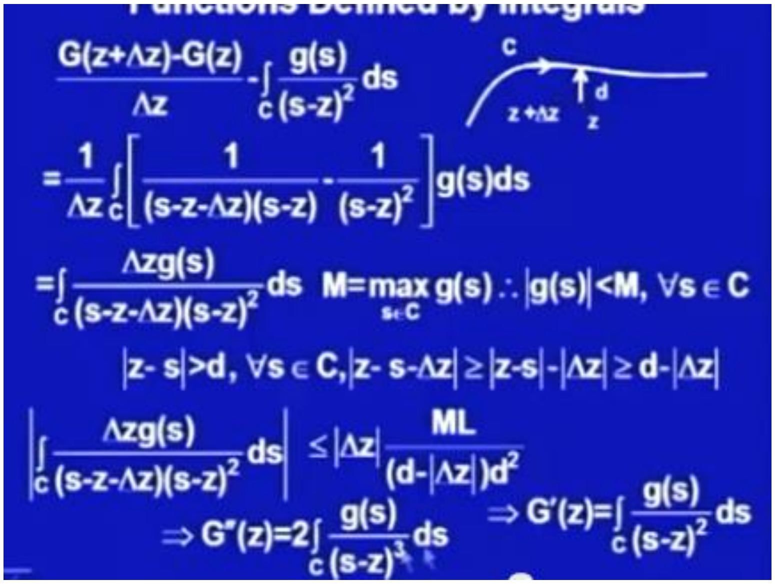

So, that delta z and delta z would cancel it out. And I would get that G f z minus G of z plus delta z minus G z upon delta minus integral c of g s upon s minus z square ds. Now, you see is that is I am just moving in the same manner, as we have moved for the analytic function f, where we have taken is 1 upon 2 pi i factorial n upon 1 upon 2 pi i f z upon z minus z naught to the power n plus 1. So, I am moving the same one, that is only thing is that constant 1 upon 2 pi i is not here.

So, we just want that is this in derivative of this function must be this function. So, let us just take this one. This one we had find out that this was also integral along the path c of g s upon s minus z minus delta z into s minus z. So now, if I do write it out. I would get it is 1 upon delta z path integral along the path 1 upon s minus z minus delta z into s minus z minus s minus z whole square g s ds.

Again simplify it, what we would get, we would get delta z g s upon s minus z minus delta z s minus z square ds. That say is that now. The difference between these two is, this integral what we have to now show. We have to show that as delta z approaches to 0. This integral must approach to 0. So, that I could say is that the limit of this function is as delta approaches to 0 is this function or this integral. So, that we would establish, that the derivative of this G capital G is this function.

And so, this is analytic at any point z. So, now let us move to this one. Here, what we will say, we have taken that g is continuous on this path c. Now, let us assume, because this is some fixed path. Let us assume that for some s on this path c. My g would be attain the maximum value or you could say is M is the maximum of all these points on the path c. So, then g s would be less than this capital M for all s on this path c. Moreover, let us just take the shortest distance from this z to this path c as the d.

What it says is for every s on this c my the difference from the s to z should be greater than d. Since, I have taken this is the shortest distance from z 2 this path c. So, z minus s must be the absolute value of the distance between z and s, must be greater than d for all s in the c. Now, if I take s plus delta z. s plus delta z is in a small neighborhood of z that says is the distance of this s z minus s minus delta z. That would be again I am using the simple absolute inequalities, would be greater than or equal to absolute value of z minus s delta z.

Since, z minus s for every s on the path c is greater than d. So, it should be greater than or equal to d minus mod absolute value of delta z. Now, what we have got from here let us see. We have got that for all c, this g s is less than bounded by this number M. The denominator s minus z square this is less than this greater than or equal to d square or we could says 1 upon s minus z square. That is bounded by 1 upon d square, and 1 upon s minus z minus delta z that is bounded by 1 upon d minus mod of delta z.

So, what we have got this complete function that is complete integrant. This is bounded by some constant for all s on the c. Now, I will again use my ML inequality. What it is say is, that absolute value of this integral delta z g s upon s minus z minus delta z s minus z whole square ds, should be less than or equal to mod of delta z ML upon d minus delta z d square. So, we do have that my L is nothing but, the length of this path. Whatever, be this length of this path that has to be some finite number. M is also some finite number.

Now, d is the minimum distance from this path from point z 2 his one. So, we are getting is that this is also some fixed numbers. Now, as delta z approaches to 0, this whole right hand side, this can be made arbitrarily a small. While what is the left hand side. Our left hand side is your this integral absolute value of this integral. This cannot be negative this is a positive so a positive quantity can be made a smaller than an a quantity, which can be made arbitrarily small. That is it can be, it should be equal to 0.

So, what we have got this integral was nothing but, the difference of this integral is nothing but, the difference of these two things. That is what I am getting is absolute value of this difference is can be made 0 or in other words, what we had got from the definition of the derivative. That G dash z, which is nothing but, G of z plus delta z minus del G z upon delta z as delta z approaches to 0 is integral along the path c g s s minus z square ds. So now, what we have got, rather than working on a analytic function.

We had worked on first thing on any path c. Then, I had worked on any function, which is continuous only. And if, I could define a function capital G, such that is integral along that path of the function g s upon s minus z. Then, we are saying is that function is analytic or rather we had shown that at any point z we could define, that we could find out its derivative is existing. Now, since this derivative is existing and this z, I have taken arbitrarily it says is that in whole of this one wherever this my function g is continuous.

I would get that this derivative of capital G would be existing that says is capital G analytic. And in the similar manner now, now capital this g dash z this is again in the form of integral of some function. So, I could say is the function g s upon s minus z upon s minus z you could get. So, again we would having or we just go like that one. And we would be move that is this function would again be analytic. And it is derivative can be given two times integral g s upon s minus z cube d s.

So, what we have now shown, rather than just having this for analytic functions. Now, we had started with a function small g which is just continuous. Now, let us put to the, a good use of this result, what we have obtained or what we want to say from result. Let us, go back to our derivative of analytic function.

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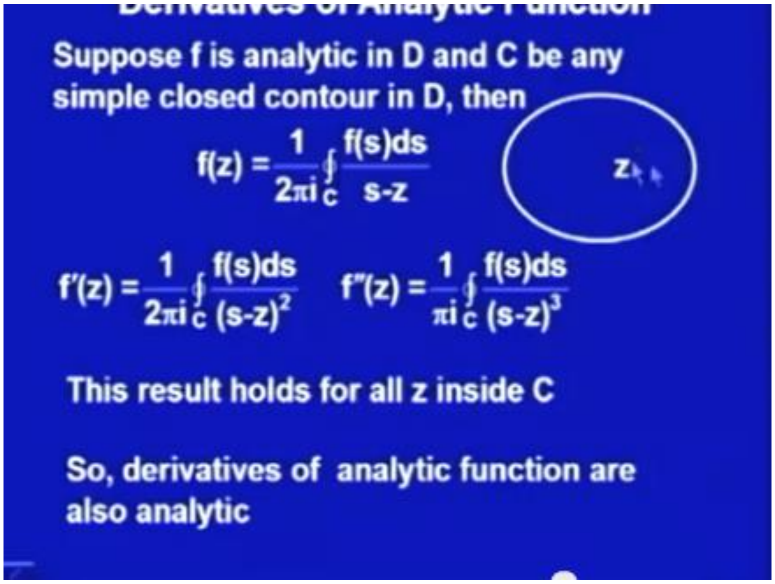

Suppose, f is analytic in D and C be any simple contour a close contour in D, then we do know that f z 1 upon 2 pi i integral along this close contour c f s upon s minus z d s. How we had find it out this is what is your Cauchy integral formula, which say is, where z is interior point of this close contour C. We do know this one. Now, here this a small f. If I replace with the that our small g in the previous result. What I would get that, this f z would be analytic. This is what, we are using now. This is a simple close contour C.

And this is a point z, which is interior to this C. Then by Cauchy integral formula we do know that f z I can write as integral 1 upon 2 pi integral along this contour C of f s upon s minus z d s, where z is interior point of this one. Now, this f if replace with g. That is any continuous function not analytic only, any continuous function. Then, what do we know is that f z over here. Just as there now that would be my this is f z would be the capital G z of the previous results, which just now I had obtained.

So, we would get it that is, this f z would be analytic. So now, what we are trying to say. That this formula is holding for f analytic. Now, if this constant this is the only thing is that is the constant. If this constant I take as in any constant. And this f I replace with a continuous function only. Then, I would get that this f the same f is it all right is analytic at z. And since, this z is any arbitrary interior point of this contour. I would get that this in whole of interior this function f z would be analytic.

And it is derivative would satisfy the same conditions that is we could the same formula we could find out these derivatives. Now, what we have actually got. We have got that rather than taking f to be analytic. If I start f to be continuous only. Still, I could prove that, because what I would have this my this, this is any contour and here is that the d i would take this smallest distance from z to that contour d, that is all.

So, we could get that all this results would be holding true my f z would be analytic, it will have possess all the derivatives all orders derivatives. And those derivatives will also be analytic. Now, what we have got from here. We have actually proved one important result you see.

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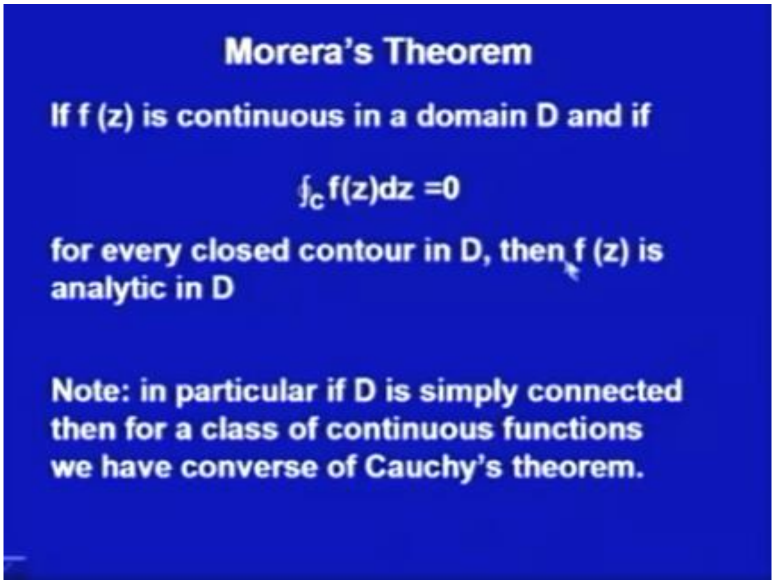

If f z is continuous in a domain D. And if the ((Refer Time: 39:11)) integral of f z along any close contour C is 0. For every close contour C in D. Then f z is analytic in D. Now, you see is that is, what we are trying to say. Say if, my D is simply connected domain, then, this result is you could treat it as the converse of Cauchy theorem. How we are going to say it like that one. You see, just now we had shown that is a, if this is happening is that for every close contour c in D, if this is happening.

What it says is that, this integral. This integral of this function f z d z is independent of path. Because, for every close contour C this is 0, that says is whatever be this path c. This integral is 0, that says is the integral of this function f z is independent path. If it is independent of path, then by Cauchy theorem we do know that it must possess some anti-derivative that anti-derivative should be indefinite integral of this f z that is capital f z. Or in other words, then f z would be the derivative of that capital f z, capital f z is certainly analytic, because that is anti-derivative of this one. So now, we are having is capital f z in the form of integral of function f z, which is analytic. So, it is all derivatives would be analytic. And hence, it would go ahead so what we say is that f z is analytic. This is what, we have that is if f z is continuous and this is happening. Then f z is analytic in D. This is what is we have proved this result is known as Morera's theorem. Now, let us move one more application of this Cauchy integral formula, Cauchy's inequality.

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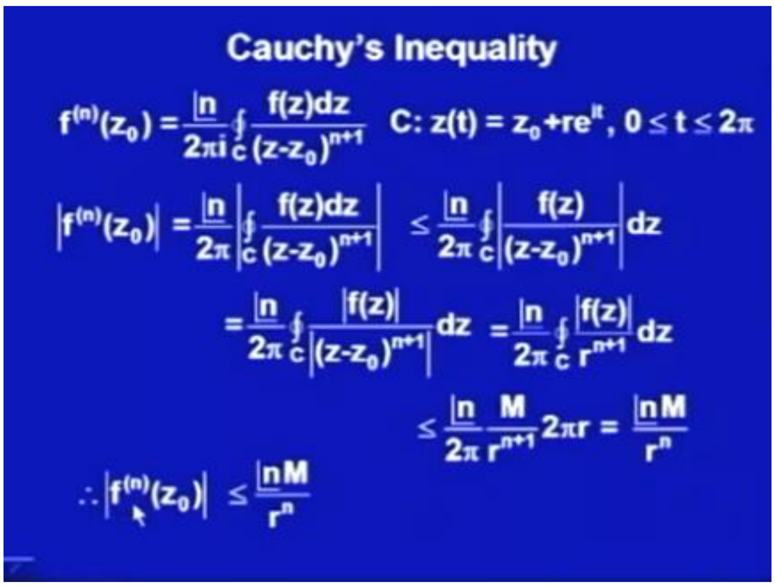

From the analyticity of this the function f z, we have find it out that the derivative of nth derivative of f or z naught can be given as by the formula of factorial n upon 2 pi i integral along the close contour C of f z upon z minus z naught to the power n plus 1. Now, let us assume this C is any close contour let us see is that this and a circle centered at z naught with the radius as r. So, this is C is my z t is z naught plus r e to the power i t, where t is ranging from 0 to 2 pi.

This is a parametric representation of the a circle centered at z naught with radius as r. Now, the absolute value of the nth derivative of z naught, would be absolute value of this one. So, this factorial n upon 2 pi i that is, it is absolute value would be factorial n upon 2 pi. And then multiplied with the absolute value of this integral. That is integral along the path c f z upon z minus z naught to the power n plus 1. Now, I have taken this path C as the circle centered at z naught.

What it says is z minus z naught would be r times e to the power i t. And f z is any function, so I would simply write it out. As this is less than or equal integral of the absolute this is now we are going with the simple result you have done in the real analysis also. That absolute value of the integral is less than or equal to the integral of the absolute value. Now, this absolute value if I do write, this would be absolute value of f z upon absolute value of z minus z naught to the power n plus 1.

Now, z minus z naught to the power n plus 1 from here, I would get r to the power n plus 1 into e to the power i t to the power n plus 1. So, what we would get factorial n upon 2 pi integral along the path C of absolute value of f z upon, absolute value of this what is the absolute value of this one. r times e to the power i t, it is upon absolute value is r only so it is r to the power n plus 1. Now, this r I have chosen some number.

So, f is now we are talking about analytic functions. f is analytic that says is I could find out some number M such that f z is less than M, on this whole contour ((Refer Time: 44:03)) C. So, this is bounded on this contour C. And by using this ML inequality I could write it out this constant as such this integral must be less than or equal to M upon r to the power n plus 1 into the length of this path, length of this path is that is the perimeter of this circle, which is 2 pi r.

What we are getting is factorial n times M upon r to the power n. So finally, what we have got the result. We have got that absolute value of the nth derivative of f at z naught is bounded by number factorial n into M upon r to the power n. Where M is your bound of f. And r is we have taken a arbitrary number or you could say is the radius for this circle around the point z naught. Now, what it says, this inequality we would says that is how it is going to help. This is going to help to get one more important result that is known as Liouville’s theorem.

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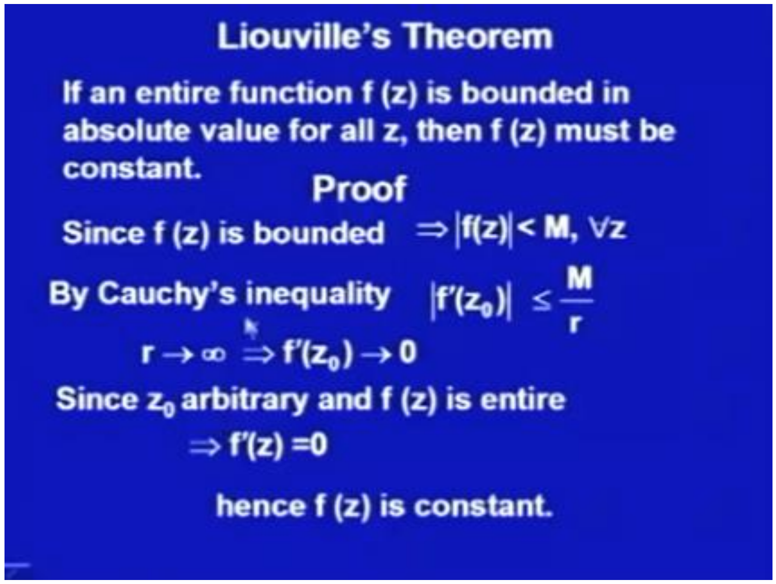

If a n entire function f z is bounded in absolute value for all z. Then, f z must be constant. What we are saying is in other words, that an entire function can be bounded if and only if that is a constant. That is in a region if it is bounded then it must be a constant. Let us see, the proof of this one. We will use this Cauchy inequality. f z is given as bounded that says is f z should be less than M for all z in that reason. By Cauchy inequality, what we do get f dash z naught should be bounded M upon r.

Now, as r approaches to infinity, that is if I am taken r large and large. We do get that f dash z naught would approach to 0. Since, that r if you do remember we have taken the circle around point z naught. And that is that is arbitrary that is radius was arbitrary. So I can make a very large circle. So, it says is that it should approach to 0, because we are saying it is bounded and this is an entire function. So, we can take a very large circle.

So, as r increasing I should get this is 0, approaching to 0. Or rather you could says that the derivative of f z would be 0. Because, this is at z naught, so I can use it at any point z. That says is my function has if the derivative is 0 for all z, then my function has to be constant. So, since z naught is arbitrary and f z is entire this says is f dash z should be 0 for all z. Hence, f z is a constant. What it says is that, an entire function cannot be bounded unless until it is constant. What this theorem is saying, I can use it in a very nice result, let us see.

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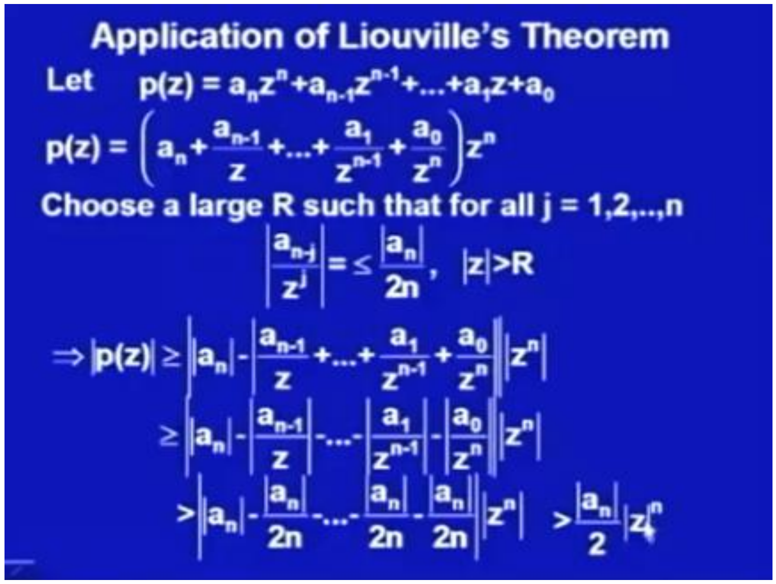

Let, I do have that polynomial of degree n. Let us, write this polynomial as a z to the power of n plus a n minus 1 z to the power n minus 1 and so on plus a n z plus a naught. Of course, a n should not be 0, why that is why we could say this is of degree n. And if I take this rewrite it. I take the common z to the power n. So, I could write it as a n plus a n minus 1 upon z and so on. Since, I have taken this a n to be naught 0. Now, I would like to choose a large R, such that for all j my a n minus j upon z to the power j.

You see here, what I am getting is a n minus 1 upon z here a n minus 2 upon z square and so on. So, a n minus j upon z to the power j is bounded by the absolute value of a n upon 2 n. What we are talking about actually, for all the z lying outside the circle with radius r. So, I am choosing a large R such that, this numbers they are that is I would, if mod z is greater than R, we are getting is that they are being bounded by some number like this one. Of course, why I had chosen a n, because a n I have taken that nonzero.

So, this absolute value I have taken and this n. This n you would find it out that is why I am choosing it a little later, if this is happening then what. Now, I would use simple absolute value of p z, and the inequality involving the absolute values. Since, mod of p z would be mod of this one, so which could we could write mod of this first bracket value into mod of z to the power n. So mod of z to the power n as such I have kept. Now, see this bracketed value.

This is the sum of many values using this inequality of absolute numbers. First I am taking this a n out. So, mod of a n minus mod of a n minus 1 upon z and so on plus a 1 upon z to the power n minus 1 plus a naught upon z to the power n. Again, for this one again I would use the inequality. That inequality what now I would use. Not this one, that I would use mod of x plus y is less than or equal to mod x plus mod of y. That will give me that is this one would be smaller than or equal to mod of this plus mod of this and so on.

And since, it is in the negative sign. It will again be greater than or equal to, so I would get is greater than or equal to mod of a n. And this here this inequality mod of a n minus 1 upon z minus minus so on. mod of a 1 upon z to the power n minus 1 minus a naught mod of a naught upon z to the power n. And this z to the power n is as such and this complete absolute value is outside. Now, from here since, we had assumed that for z greater than R, a n minus j upon z to the power j in absolute value.

This is bounded by a n upon 2 to the power n. That says is if I take the minus sign, this would be greater than or equal to minus of a n upon 2 n. So now, let us substitute this so this should be greater than a n is as such. This should be greater than mod of minus n mod of a n upon 2 n and so on. Everything, I am replacing with the mod of a n upon 2 n. Now, how many times we do have a naught 2 a n minus 1, that is n terms.

So, I am having this n terms like this one. n terms like this one, if I add up this n terms like this, what I would get n times mod of a n upon 2 n that is mod of a n upon 2. So, what I am getting is this is a n minus mod of a n upon 2. Or that is same thing as mod of a n upon 2 mod of mod of a n upon 2. And this is holding true when z greater than R, z is greater than R means is this would be greater than R. So, I am getting is its greater than mod of a n upon 2 and this absolute value of z to the power n, we could write as absolute value of z to the power n. Now, let us see what we have got.

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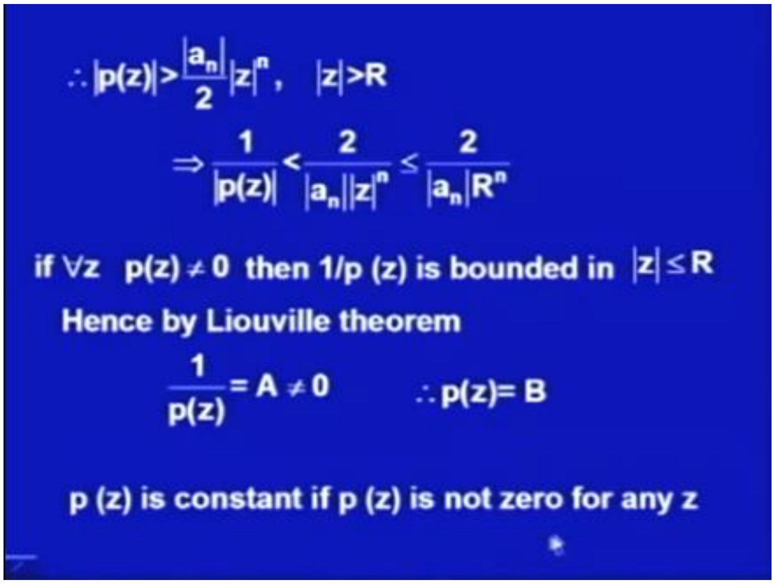

We have got that p z is greater than or equal to absolute value of a n upon 2 into modulus of z to the power n for all z greater than R. Now, suppose my this p z is naught 0 for any z in this whole region. Then, what I would get is that 1 upon p z is less than 2 upon mod of a n into mod of z to the power n. So, this is because mod of z is greater than R. So, in this region it should be 2 upon mod of a n into R to the power n. So now, if what it simply says is that, if for all z p z is naught 0.

Then, 1 upon p z is bounded in the region mod z less than or equal to R, because p z is bounded above by in the region mod z greater than R. So, 1 upon p z is bounded in the reason mod z less than or equal to R. Hence, by Liouville's theorem, Liouville's theorem says is if the function is entire and bounded. Then it must be constant, so the function p z is naught 0. So, 1 upon p z is entire function and bounded. So, it must be some constant a, which is naught 0. Hence, p z is B, because we have taken that p z is naught 0 for any z.

So, whatever 1 upon p z that constant cannot be 0. So, what it says is that p z would be 1 upon A that is B some other constant. Now, what we have got the result that if, p z is naught 0 for any z. Then it is a constant. So, what we have got p z is constant. If p z is naught 0 for any z. And this gives us the actually the fundamental theorem of algebra, which says is,

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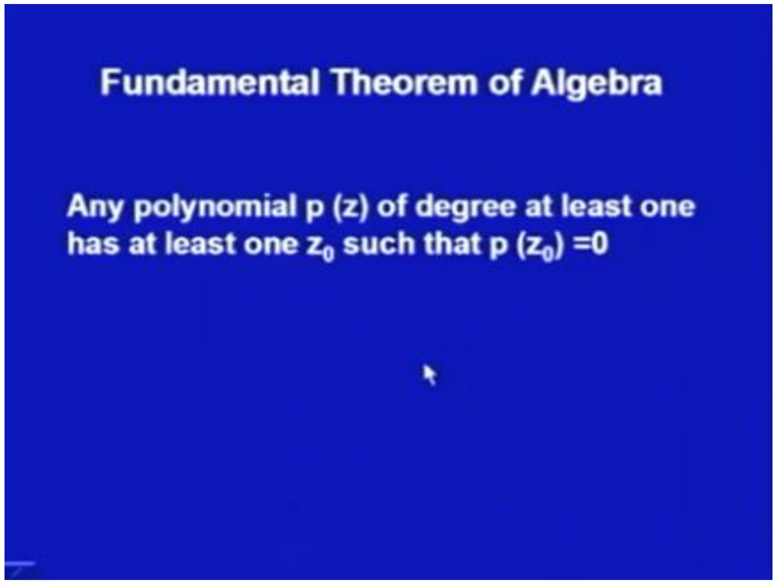

Any polynomial p z of degree at least 1 has at least 1 z naught such that p z naught would be 0. Because, the proof you could see is by contradiction as in the previous one we have slides we have done, that you could treat it as proof with the contradiction. That is you can assume that p of z naught, if there is no z such that p of z is 0. Then, we do get that p z would cannot be a polynomial of degree n. Then it must be a constant.

So, this is the fundamental theorem, which says is that for any polynomial of degree n and it must have at least n at most n roots, that is what you have done in the algebra, but not the proof the proof is here. So, we have got one important result, over here also. So, we had learnt, that how this Cauchy integral formula is playing magic not only evaluation of the integral. But, in the proof of certain basic or very a important results in the analysis and algebra. So, we had learnt today the Cauchy integral formula its application that is all.

Thank you.