Mathematics - II

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Lecture - 4
Cauchy's Integral Formula

Welcome to the lecture series on complex analysis for undergraduate students. Today's lecture is on Cauchy's integral formula. Last lecture, we have done the Cauchy integral theorem. The main consequence of Cauchy integral theorem is Cauchy integral formula. This formula is very helpful in evaluating the integral. But, equally important is this formula in having an important role improving certain results of the analysis. It helps improving, that analytic functions have derivatives of all order.

It helps improving the Taylor's expansion of any function. Moreover, it is helpful or it gives the proof of fundamental theorem of algebra. So let us see, what is Cauchy integral formula? But, to find out the Cauchy integral formula. First we have to extend our Cauchy integral theorem to multiply connected domains. We have done the Cauchy integral theorem for simply connected domains. We have to extend it to the multiply connected domains.

But, for the multiply connected domains, we require it to prove that it is only on the boundary of the multiply connected domain. While Cauchy integral theorem was holding true for any close simple close contour of the simple domain. For multiply connected domain will stick only to the boundary. Let us see the Cauchy integral theorem for multiply connected domain.

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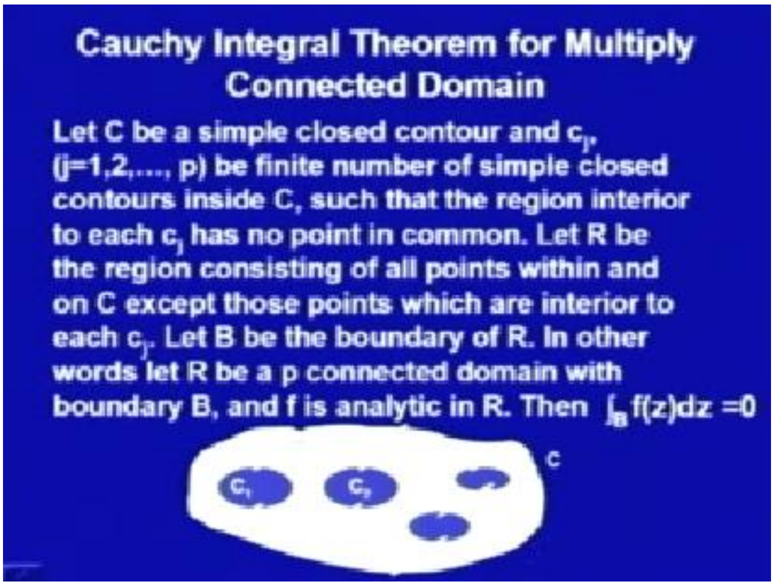

The statement let C be a simple close contour and C j for j is equal to 1 to p be finite number of simple close contours inside C. Such that the region interior to each C j has no point in common. And let R be the region consisting of all points within and on C. Except those points which are interior to each C j. And let B be the boundary of this region R. In other words what we are saying is that let R be a p connected domain with boundary B. And f is analytic in R. Then integral of f z along this boundary B is 0.

What we are trying to say is let us see with the help of a figure. This is a domain multiply connected domain you see. We do have here for example these 4 holes, so this is a 4 connected domain let us say. What we are having is one boundary outer boundary C. And then we do have, so we do have a simple closed contour C, and another contour Cj's, finite number of simple close contour Cj's. Here I have taken them elliptic one they could be of many form.

But, close contours Cj's, such that each of these Cj's does not have any point in common. And moreover, they are interior to this interior of this contour C. Then, we say is that the boundary will consist of this C, C 1, C 2 and so on, all this C p. So, if I take up the boundary in sense, that all the interior points. Interior points means, the interior points of these each of these C j curves that we have to take out. So, that it is a multiply connected.

So, we take that this orientation of the outer boundary C is the counter clockwise, while is for all the inner a curves Cj’s it is clockwise. So, that the interior points are all left to the boundary. Then, we say is that if we take the boundary in this sense, then along this complete boundary that is this plus C 1 plus C 2 and so on. This would be integral of analytical function would be 0.

This is what is extension of your mean Cauchy theorem, which says is that along any simple close contour in simply connected domain integral of f z along that contour C is 0, we are having it on the boundary. Let us go to the proof of this for proving this we would see, it one by one that is first we will go with the doubly connected domain.

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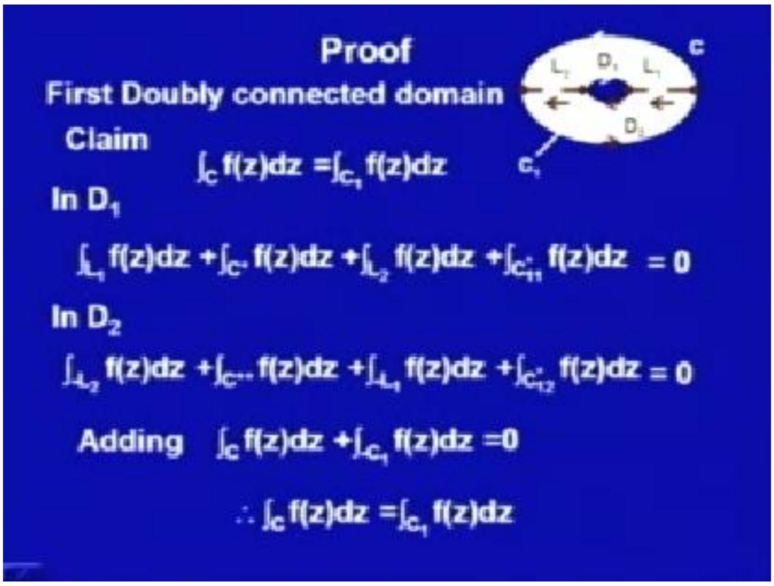

Let us see, this is a figure of doubly connected domain. We have this outer boundary C and this inner boundary C 1. Now, for proving this in the doubly connected domain, what we will do is we will first make this divide this domain into two parts. The divide is with these through these two cuts. One is the L 1, another is L 2. So, in this one or what we do says we want to prove that integral along this boundary is 0. So, now if I do take our simple definition of any closed contour the orientation is always counter clockwise.

Then, what we say is this integral interior along this boundary C plus integral along this boundary C 1 in this manner is 0. What it says is we want to claim, that integral along this boundary C is same as the integral along this boundary again in the counter clock manner C 1 is same. This is what we want to show. So, we have divided this into two domains now. So, now see, the upper domain from this part this is D 1 and this lower one.

Now, you see this upper one, this is now a simply connected domain, this is a simply connected domain. Similarly, this is also a simply connected domain. So, what we have done, we have divided this doubly connected domain into two simply connected domains D 1 and D 2. Now, in simple connected domain the Cauchy integral theorems holds. So, let us see how we are going to do this in D 1, the simple close contour I am taking the boundary of this domain.

This is of course, so we would start this boundary like this one. L 1 the upper part of the C then L 2 and then this upper part of C 1. This is a close contour in the simply connected domain D. So, the Cauchy integral theorem holds true. What it says is integral along this close contour should be 0. So, let us write it out L 1 f z d z plus C is star is this upper part of C, plus L 2 f z d z plus integral along this upper part of C 1. That is I am writing it C 1 1 star in this clockwise manner. So, this is should be 0.

Similarly, if I just go in D 2. Now, in D 2 the orientation of this boundary I would take in this manner, where this red orientation we had shown. I would start from L 2 that is from this point L 2. Then this portion of C, then L 1 and then we go in this manner so you will complete the complete boundary. So, in D 2, if I write, L 2 that is it should be minus L 2, because L 2 we have taken this direction. So, when I am taking the integral along this direction it should be minus of L 2.

So, minus of L 2 f z plus C star, star that is the orientation is again the counter clockwise of the C. The lower half position of the C’s, that is C double star f z d z, plus minus of L 1. Because, L 1 is in this direction, so this is minus L 1 and then C 1 2 is star that lower portion of the C 1 curve in this direction. This should also be 0, because D 2 is also simply connected domain. Now, if I add these two, what I would get you see I am adding this boundary and I am adding this boundary.

So see, once I am adding L 2 from this direction, then I am adding the L 2 in the negative direction. Since, f z is analytic and it is simply connected domain we do know that it is independent of path. So, integral along L 2 and minus L 2, that should be negative of each other or that you could say is, that is since reversal one, it will give the same one.

So, I would get this L 1 and this minus L 1. And this L 2 and minus L 2, they will get it cancel it out.

Then the C star and C double star. C star is this oriented curve. And C double star is this oriented curve. If I add this up what I do get complete C. Similarly, C 1 1 is star is this oriented curve while a C 1 to star is this oriented curve. So, when I just add it up I would get, this as C 1 in the in the clockwise manner. That is in the negative sense you could say. So, if I write it out in the sense reversal form, that is if I change the orientation it should be minus of C 1.

That is, this minus of C 1 this is what is the clockwise manner in counter clockwise manner I would call it C 1. So, if I add it up, what I would get C f z d z plus minus of C 1 f z d z is equal to 0. So, from here if I just write it out it should be C integral along the contour C of f z d z is same as minus of integral along the contour of minus C 1 f z d z. So, we call it equal to that using the sense reversal one. That is, it should be equal to the contour integral along the contour C 1 f z d z.

So, what we have proved, that this is true. And if I C is here, this is what the theorem says is that a complete boundary. The boundary is C and C 1. So, this is the complete boundary what we ((Refer Time: 10:37)) integral along the complete boundary B of the function f z d z is 0. And this says is that, integral is same or rather you could say is that is again we have established the independence of path, on the close contours in multiply connected domain.

That is, if either I take this path or I take this path. The integral along both this contours would remain same for multiply connected domain for analytic function f z. Now, this is we have done for the doubly connected. Let us extend this to the triply connected and then it would be clear that is it could be and it will be an equally connected domain.

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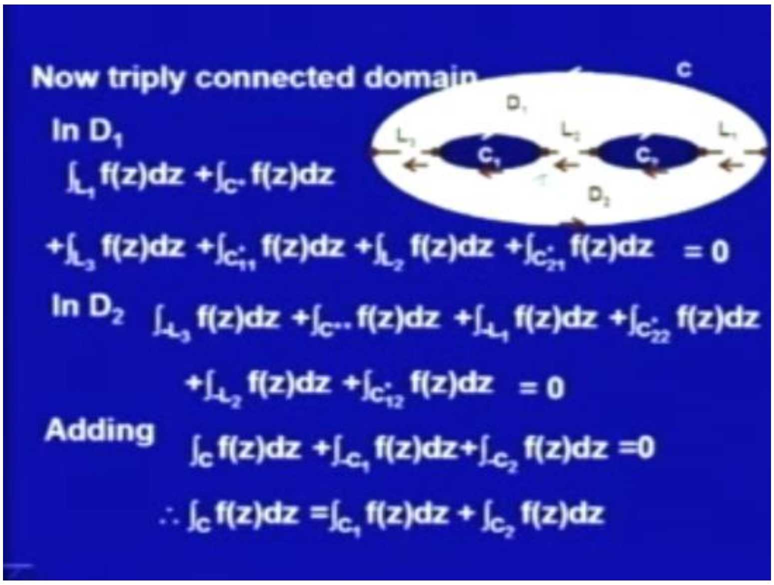

So, let us see, this figure for the triply connected. So, we do have two holes here or you could say the three boundaries. So, we do have this is the triply connected domain. This particular shape of the domain, I have taken in this manner. You can take in any other manner. But, in the sense that you so could divide it into two simply connected domains, that is how we have to just think about. And that is possible for every triply connected domain.

So, we have coming us this triply connected domain. We are introducing here three cuts L 1, L 2 and L 3. By these three cuts, you see I could divide it into two domains. One is this upper domain, you see from this side I am just telling in L 1, L 3, then this upper portion of C 1, then L 2 then upper portion of C 2. Like this, this is upper domain this is a simply connected domain. Similarly, I do have another one this L 1. low portion of C and then this L 3, then this lower position of C 1, then L 2 then lower position of C 2. We are having this is another domain D 2. Both are simply connected domain.

So, again in the in the manner of in the doubly connected domain, I have done. We would just go in the similar manner for here. So, let us first consider this domain D 1. Here, the simple closed curve I would take starting from here this complete boundary. So, what we do have L 1. Then upper position of C, this is C star I am denoting so integral on this of the analytic function f. So, C star f z d z then L 3 .Then upper portion of C 1 in the clock manner, that is negative sense.

So, integral C 1 1 star, C 1 1 star is this directive path. Then L 2, and then we would have C 2 1 star. So, we are having this complete boundary L 1 C star, L 3 C 1 1 star, L 2 and C 2 1 star along this the integral of f z d z, that should be 0. Because, this is a simply connected domain and this is a close contour in the simply connected domain. So, it should be 0 according to the Cauchy integral theorem. Now, in the similar manner, I take the boundary of the D 2. Now, for the boundary of the D 2, I would start from here.

We just go this L 3 in the negative direction. Then the lower portion of C in the counter clockwise manner, then L 1 again in this negative direction, because L 1 we have defined the direction as from this side to this side. Then C 2 in the negative sense again. L 2 in the negative sense and C 1 again this negative sense. So, let us just write it out in D 2, we do get it minus L 3 C star, star that is the lower portion of C. Then minus L 1 lower portion of C 2, that is we are writing C 2 2 star. Then minus L 2 then lower portion of C 1 that we are writing C 1 2 star.

Now, this is the complete boundary this is a closed curve. So, because of Cauchy integral theorem in the close contour this integral must be 0. So, this whole integral should be 0. Now, if I add these two, what will happen? You see again here, we are having is L 1 integral of f z along the path L 1, as well the integral of f z along the path minus L 1. You see here, we are having is once we are moving this side. And another time we are moving from this side.

So, when we add it up both these two things, I would get it simply 0. That is, whatever be this integral that would give me 0. Similarly, for all this cuts L 2, once we are moving to the positive orientation, another we are moving in the negative orientation. Then for L 3 also in the similar manner, so all these cuts we would get them 0. What is being left is that is these paths we are having is upper half and lower half. For C we do have upper half and then the lower half in the same direction. So, it will add up to give me the complete C.

Similarly, for C 1 we are moving to the upper half in this negative sense. And then the lower half also in the same negative sense. So, this add up will give me the complete boundary C 1 in the negative sense. Similarly, the complete boundary of C 2, we would also get in the negative sense. So, what we would get actually, we would get that integral of f z along the boundary C, plus integral of f z along the boundary C 1 in the negative sense. So, I am writing it as minus C 1.

And plus integral of f z along the boundary C 2 in the negative sense. So, we are writing is minus C 2. You see, is that is I am always using here is negative sense and positive sense. This I have made a convention that is whenever, it is a simple close contour we would call the counter clockwise direction, as the positive sense and clockwise direction as the negative sense, that is how we are writing it out. So, this is equal to 0 that says is this is the complete boundary of the multiply this triply connected domain.

So, we have shown the result over here. But, from here we are getting another formula. That is integral along this outer boundary is some of the integrals along the inner boundaries. So, what we are having is integral along C of f z d z is integral along C 1 f z d z plus along C 2 f z d z. That is what we are saying is, that if it is a in general now we could show that is in similar manner. If it is a P connected domain, then on the complete boundary the integral of analytic function would be 0.

What it says is that the integral along the outer boundary will always be same as the some of the integral along the inner boundaries. This is what is the extension of Cauchy integral theorem in multiply connected domain. Now, let us see how this is helping in evaluation of the integral first and then we will see that Cauchy integral formula. So, let us see one example.

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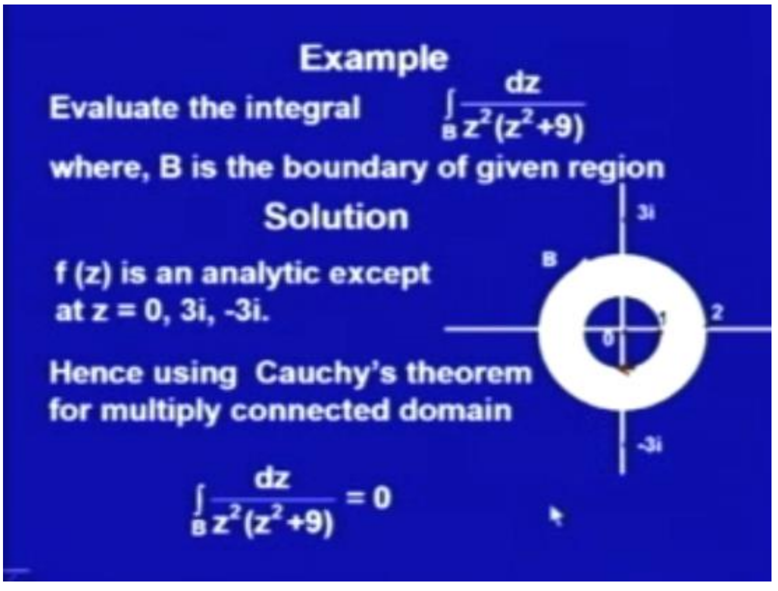

Evaluate this integral d z upon z square into z square plus 9 along the boundary B. Where B is the boundary of this given region that is this is the disk. From z mod z is lying between 1 and 2. This is the disk along this boundary, we have to show that the integral of this function is we have to evaluate. You see, this function is not analytic at z is equal to 0. And z is equal to plus minus 3 i. You see, z is equal to 0 is the origin plus minus 3 i they are here.

The disk and the region on the boundary what we are having is the boundary is of this disk, that is outer boundary and this inner boundary in this negative sense. So, both this points of where this function is not analytic, they are outside this region. What we do conclude is that this function is analytic except at the point 0, 3i and minus 3 i and all these three points are lying outside the given region. So, what we are having is my given function f z is analytic in a doubly connected domain.

That says is that, Cauchy in theorem for multiply connected domain say is that, along the boundary of this doubly connected domain the integral must be 0. Hence, this integral along this boundary should be 0. Now, let us do one more example, here that example is of an importance, just because we you see is that is we would be using that example in finding out many important results. So, let us move to another example.

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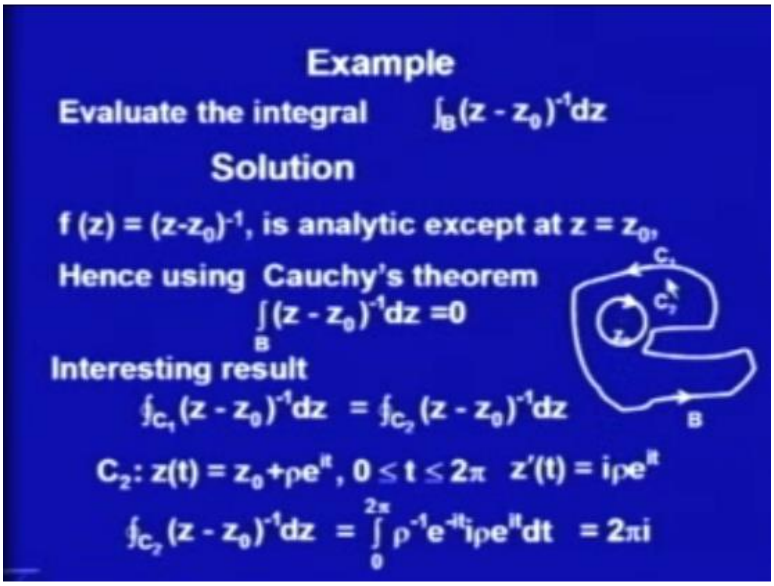

Evaluate this integral let us, simply a one upon z minus z naught or you we are writing it as z minus z naught to the power minus 1 d z, with it is boundary B is of this domain.

You see, again this is the doubly connected domain, you find it out. We are having is this boundary B is completely outer one, as well as this inner one in the negative sense. That says is boundary is now I am taking at the two ((Refer Time: 20:34)) sets. One is this outer boundary that is counter clockwise in the positive sense. Simply, close contour this is C 1 and this inner boundary I am taking C 2. C 2 I will take in the positive sense.

So, this is a here for the boundary, I am taking into the negative sense. But, when I will take this close contour C 2, I will take it in the positive sense. This is including z naught, z naught is inside. Now, you see now a function one upon z minus z naught. This would be not analytic only at a point z naught. So, along this part z naught, this there is a circle. See, analytic except at z is equal to z naught. Now, I would use this Cauchy theorem for doubly this again using the doubly connected domain, what I would get integral along this whole boundary should be 0. So, this integral must be 0.

Now, you see as I have divided it into two parts that is the boundary as C 1 and C 2. I will get one interesting result. It simply says that integral of f z, that is 1 upon z minus z naught along this boundary C 1 is same as the integral of f z, that is 1 upon z minus z naught along this circle C 2 is same. Now, let us say, this circle is centered at z naught. And it has some radius rho, so let us see this C 2 I am saying is so a circle z naught plus rho e to the power i t, t is lying between 0 to 2 pi, this is the parametric representation of circle of radius rho centered at z naught.

So, I would like to evaluate this contour integral. We do know that, we can find out this integral using that our basic definition of contour integration. We could the function is 1 upon z minus z naught. This also we would write in the parametric form. And my C 2 is we are having this parametric form. So, we would write as f z t and z dash t d t. And t is ranging from 0 to 2 pi. Let us write it out. So, z dash t would be from here, because z naught is constant, i times p e to the power rho e to the power i t.

Now, evaluate this integral 1 upon z minus z naught from here z is z t is z naught plus rho e to the power i t. So, z minus z naught would be rho e to the power i t. That is 1 upon rho e to the power i t. And z dash t is i times rho e to the power i t. So, what we would get integral 0 to 2 pi rho inverse e to the power minus i t, i times rho e to the power i t d t. Now, all these the terms are getting cancel it out, giving us 1 only this i is there.

So, I am getting is i is the constant, I have to integrate it from 0 to 2 pi. I would get it simply 2 pi i. This is a some interesting a result you see. I was having this function 1 upon z minus z naught. And any contour here I have taken this contour. I had shown that this is equal to the integral along this one now. Here, I have taken this z naught, because this function is not analytic only at z naught. And I have taken a circle centered at z naught of some arbitrary radius rho.

I am getting it this 2 pi I, the integral which says is that this rho is immaterial. And now, since Cauchy theorem says is that integral on the whole boundary from this multiply connected domain should be 0. That says is whatever, be this outer contour C 1. I can always make it equal to the integral along this inner contour C 2. What it says is, that if it is not decimal it is any other close contour is still I would get the integral of 1 upon z minus z naught as 2 pi i. This is very interesting result.

You see, is that how we are going to use it. So, now we are coming to our Cauchy integral formula. You see there, we would be using this result and this is really very important and interesting result. So, the Cauchy integral formula?

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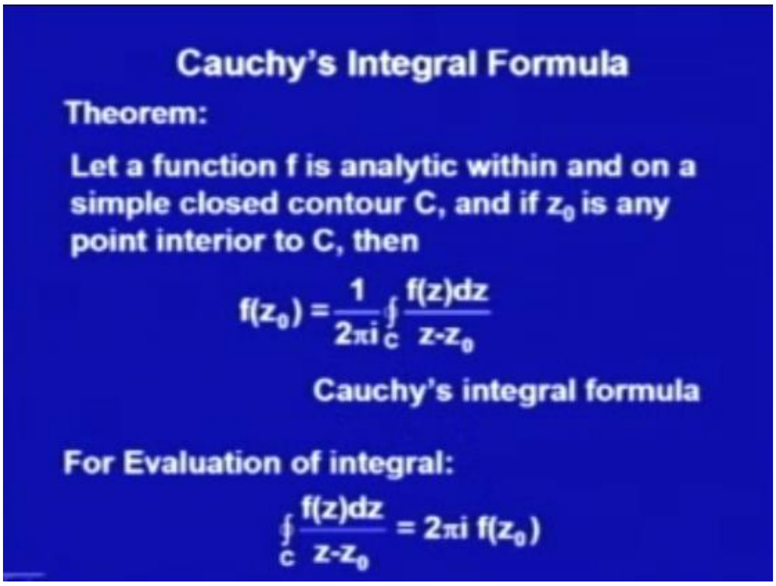

I am writing it as in the form of theorem. Let, the function f is analytic within and on a simple closed contour C. And if z naught is any point interior to C. Then f of z naught is 1 upon 2 pi i integral along the close contour C of f z upon z minus z naught d z. What it is saying is my function f is analytic within and on a simple closed contour C. That is on the contour C it is analytic, as well as interior to the C this function is analytic. And z naught is an interior point of this C.

Then, what we are saying is that the function at z naught is being determined through the function along this Cauchy or this is what result it is a seen this is known as Cauchy integral formula. Now, you see is, this is the as I said is this is helpful in calculating the integrals evaluation of the integrals. As well as, it is helpful improving many results. So, proving the theoretical results, we do use this formula as Cauchy integral formula. For evaluation of integral, we could rewrite this you see as integral along the closed contour C of f z upon z minus z naught d z is 2 pi i times f of z naught.

Now, you see what it is giving, that is not anything new we are starting with that f is analytic. f is analytic all right. But, when we are writing is f upon z minus z naught. And z naught is interior to the C. So, of certainly we are not having the function f z upon z minus z naught analytic on the complete domain, because Cauchy integral theorem says is simply connected domain, the function has to be analytic in the whole of the domain. No we are not having we are having that 1 point z naught, we are having that the function is not analytic.

If it would have been analytic at that point also, then we do not along any simple close contour C, it should have been 0. But, it is not so what we are saying is, that is we are having is it is, this integral is equal to this. So, if I do have in a simply connected domain let us say, a function which is analytic only at one particular point. Then, integral can be obtained as that particular point the function value at that particular point into 2 pi i. So, you see, as that we what is the consequences of this result and how we are using it before moving to that let us first find out this formula. That is why, I have written it in the form of theorem and proof. So, proof means is finding of this formula. So, let us move to the proof of this result.

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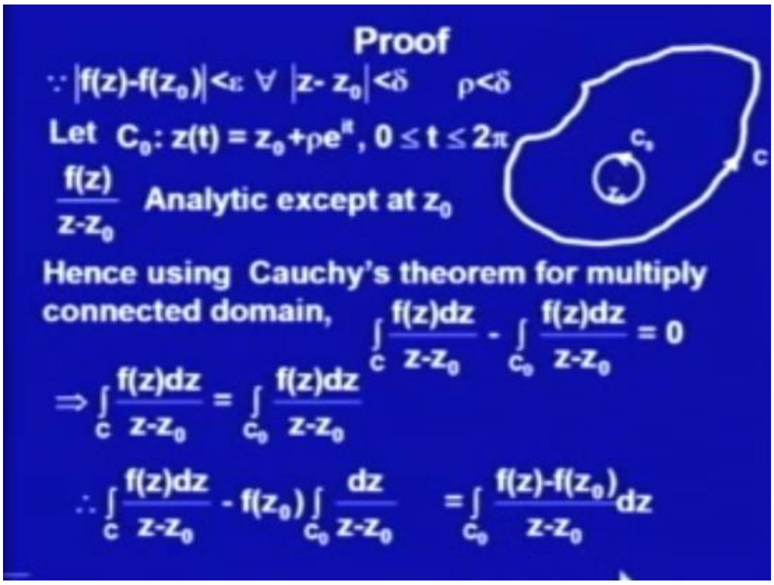

For proving this, we would use multiply connected domain in Cauchy integral theorem. So, let us say, this is a closed contour C. And this is point z naught, which is interior to this contour C. And I do have circle centered at z naught. This circle C naught and this is again positively oriented. This contour C is also positively oriented. So, let us say, start with this one f z is analytic in this whole region. That says, if f z would be continuous. If f z is continuous by definition of continuity, we do know that f z minus z naught, because a function is analytic everywhere.

So, function is analytic at z naught also. It is analytic at z naught means, it is continuous at z naught. Continuous at z naught that says is if I take any small neighborhood of z naught. The difference between the function would always be made arbitrarily small. So, that is what we are writing by the definition of continuity, that f z minus f z naught would be less than epsilon for all z. Lying, in a delta neighborhood of z naught or writing it z minus z naught is less than delta mod of z minus z naught is less than delta.

Now, let us take rho, which is less than delta. And this rho Ii will take the radius of this circuit C naught. So, now, let us define a circle C naught as a centered at z naught with the radius rho. So, this is the parametric form of my circuit z naught plus rho e to the power i t for t lying between 0 and 2 pi. So, let us see, the function f z upon z minus z naught. f z is analytic throw out this region interior to the C. So, f z upon z minus z naught, would be analytic throughout this region except at the point z naught.

Now, if I take this internal circle C naught. Then, this take this out of this C naught interior of C naught out of this region C. Then, what I would get, I would get a multiply connected or doubly connected basically domain. In which this function f z upon z minus z not is analytic. So, using this the Cauchy integral theorem for doubly connected domain. I could say that integral along this complete boundary must be 0.

What it says is that integral along this C minus integral along C naught that is you know, that is if we are taking the boundary. Then it should have been the negative sense. So, we are writing it minus C naught f z d z must be 0. What it says is, it says is that my integral of this boundary C along this contour C of f z upon z minus z naught d z must be same as integral along this contour C naught of f z upon z minus z naught d z. So, we have to evaluate this integral, what we are saying is we have to actually prove our formula.

So rather then evaluating this integral what we will do is you see is that we would do that this integral must be equal to 2 pi i f z naught. So, for writing a f z naught we do write it out like this one. C times this f z naught if Ii do take this z naught is here. So, f z naught that is constant with respect to this contour C naught. So, I can write the integral of 1 upon z minus z naught along this contour C. And f z naught outside very nicely, Ii could write it out, because that is a constant.

So, what now I am writing you see, making a little bit tricky you could say, but it is not actually tricky we are just proving that formula for that we required. What we are writing f z d z upon z minus z naught integration along this contour C, minus f of z naught integral of 1 upon z minus z naught along the contour C naught. This what from here, we do know that this is equal to this. So, this is same as integral along the contour C naught of f z upon z minus z naught. If I write it out substitute over here, what I would get this would be equal to the integral along the contour C naught of f z minus f z naught upon z minus z naught. Now, here I would use for ((Refer Time: 33:31)) inequality that is what a ML inequalities.

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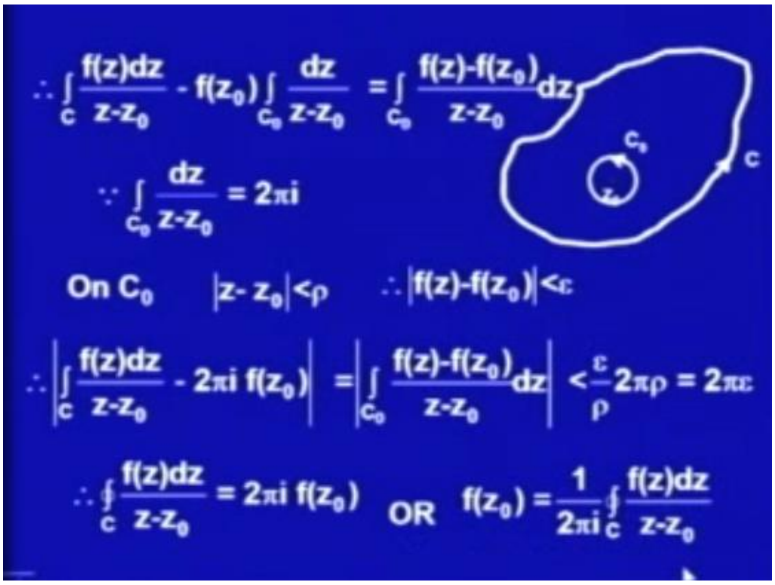

So, what we are ((Refer Time: 33:37)) find it out that, integral of f z d z upon z minus z naught along the contour C minus f z naught integral of 1 upon z minus z naught along the C naught is same as integral of f z minus f z naught upon z minus z naught along the contour C naught of d z. This integral if I do see, you remember that is whatever, be this radius rho, this integral we had find it out as 2 pi i. So, that is helpful, that is you are getting is that is on the left height, we are having this minus 2 pi i f z naught, that is we are nearing to our the formula.

And we could say that this would be equal to this formula, that is 2 pi i f z naught if the right hand side integral goes to 0. Because, the left hand side here, this integral is nothing but, 2 pi i we have all ready done in the example. So, this would be, if I could show that this is 0. That is this integral on the right hand side is 0. Then, we have done. So, let us see, the right hand side integral. On C naught z minus z naught is less than rho.

Because, we have taken this radius as rho and rho we have taken less than delta if you do remember. That says is f z minus f z naught would be less than epsilon. So, now I am going to use this ML inequality on this integral. Integral, so I am writing this complete thing, mod of C integral along the contour C of f z upon z minus d z naught d z, minus 2 pi i f z naught is equal to the modulus of the integral of f z minus z naught f z naught upon z minus z naught along the contour C naught.

Here, I am going to use the ML inequality since, f z minus f z naught is less than epsilon in whole of this contour C naught. So, I do write it out, that this is less than epsilon and z minus z naught is less than rho. So, what we are getting is this whole thing is less than epsilon upon rho. And this length of this contour, that is what we are having is ML inequality, that is less than M and this is the length of the contour, the radius of this perimeter of this circle we do know 2 pi r, r is the here rho. So, this is 2 pi rho, what we have got 2 pi epsilon.

And since, epsilon we have taken arbitrary a small quantity. And this can be made arbitrarily small, what it says is here we are talking about the absolute value. This cannot be negative, we are saying this absolute value can be made as small as possible less than a value, which is as small as possible. What it says is simply that this would be 0, that this is equal to 0.

Hence, by definition of this continuity we do get that integral of f z upon z minus z naught along the contour C is nothing but 2 pi i f z naught where z naught is interior to this point. So, we had established our Cauchy integral formula. Let us see this in other words we could write it out this one, f z naught 1 upon 2 pi i that is rewriting it. Now, let us see, that is how it is helpful in solving certain integral. So, we just go the examples of evaluation of integrals.

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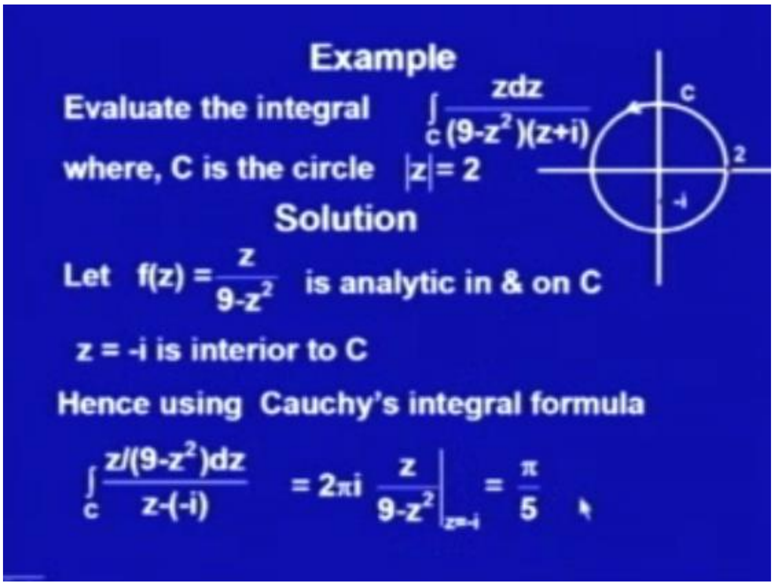

Evaluate, this integral along the contour C if z upon 9 minus z square into z plus i, where C is the circle mod z is equal to 2. That is the C is the circle with the radius 2 center at origin. Now, let us see how we are going to solve it. Let us first see is that is what is my contour. My contour is this, where a circle of radius 2 with center as origin. Now, let us see what is my function? The function I am considering is as z upon 9 minus z square. This function will is to be analytic at z is equal to plus minus 3.

z is equal to plus minus 3, that would be outside your points. Why less, if I take the complete function, that is now that would be f z upon z plus i. z plus I, if i do take that is minus i point. So, this is this function is completely analytic inside and on C. But, z is equal to minus i, where this whole function is not analytic, this minus i is here you see this is 2 and this is my minus i.

So, now what we have got a function f z, which is analytic and a close contour and within it. And z naught that is minus i is any other point, which is interior to the contour C. Then integral along that contour C of f z upon z minus z not can be given as 2 pi i f z naught. So, now we can use it. So, using this Cauchy integral formula, this integral I am writing it the function is f z is z upon 9 minus z square. And here it is z minus of minus i, this should be 2 pi i f z naught. f z is z upon 9 minus z square at z is equal to minus i.

When I keep z is equal to minus i, I would get is i square, so i square is minus 1 I would get a 10. And here it is minus i. So, I am getting is 2 pi i square that is minus i square. So, 2 pi minus i square upon 10 and minus i square 1 so 2 pi upon 10 or pi upon 5. You see is that, how simply I could solve this integral just looking very complex one is that function. So, I could find it out using this Cauchy integral formula, we could find out the solution of this integral. Let us see, some more nice examples.

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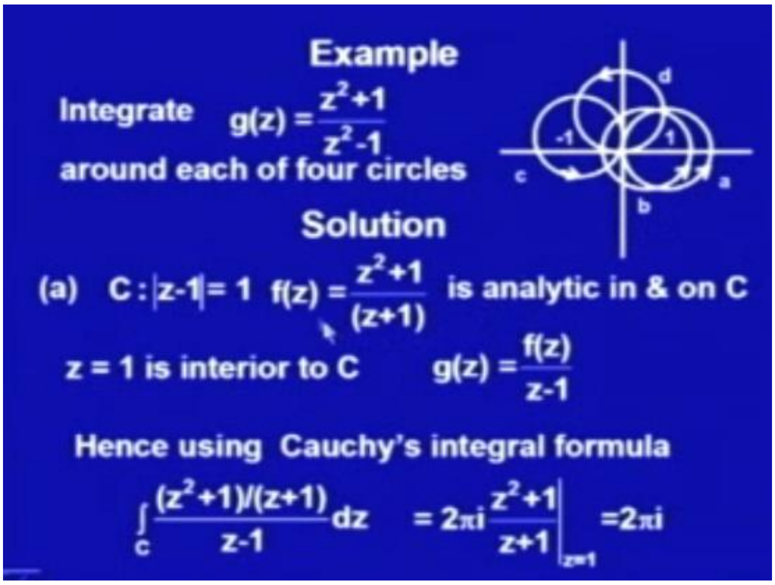

Let us have this example. Integrate this function g z, which is z square plus 1 upon z square minus 1 and each of these four given circles. You see, the four circles, one is this circle a, which is the centered at 1 with the radius as 1. Or rather you could see you find it out that is here all the circles are looking like that is there unit circle if I just compare with this circle. Then, there are there is another circle b which is a shifted once. Then there is another circle c, which is shifted towards the side then there is another circle d.

So, let us start one by one the solution. Let us, first consider the circle a. This circle a is centered at 1 with radius as 1. So, this is my circle is z minus 1 is equal to 1. Now, consider this function f z as z square plus 1 upon z plus 1, because I could write it g z as z square plus 1 upon z minus 1 into z plus 1. So, I have taken the f z as z plus 1. Now, this function is analytic, because this function seems to be analytic only at z is equal to minus 1 and z is equal to minus 1 is outside this contour a.

So, in on this contour a and it is interior this function is completely analytic. And the point z is equal to 1 is inside the circle. So, we do get at that interior to c. So, now we I use the Cauchy integral formula g z as f z upon z minus 1. So, using this Cauchy integral formula, I would get z square plus 1 upon z plus 1 divided by z minus 1, that is f z upon z minus 1 d z .This should be 2 pi i into f z naught what is z naught, z naught is 1 and f z is this.

So, let us write it out, z is z square plus 1 upon z plus 1 at z is equal to 1. So, evaluate it when z is equal to 1, I would get this 1 plus 1 is 2. And this is 1 plus 1 is 2. So, this is canceling out, I would get it 2 pi i. So, we have got along this a contour, we have got this integral of g z as 2 pi i.

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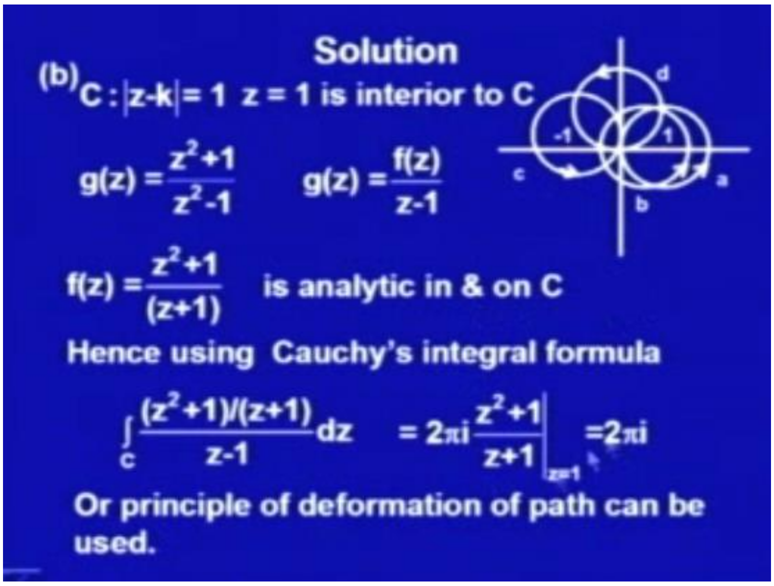

Let us move to this b 1, b is as I said is that is looking that is all the circles have the radius 1. So, radius is 1 is all right. But, the center is now is not very clear which point. So, let us say, it is any number k. So, the circle would be z minus k is equal to 1. And now, z is equal to 1 is interior to that circle also you will see, b if from 10 z is equal to 1 is the integral. So, again I would break my g z in the similar manner. That is z square plus 1 upon z square minus 1, I would write it as z square plus 1 upon z plus 1.

That is on that 1 this circle is completely analytic because z is equal to minus 1 is still outside this point p. So, g z I would consider this 1. And I would get it f z as this. So, analytic on C, so we do get it again using this similar manner 2 pi i z square plus 1 upon z plus 1 at z is equal to 1 that is same as 2 pi i. Now, you see, here what I have done is I have used this Cauchy integral formula again we could get it very simple you see how. If I consider, this function z square plus 1 upon z plus 1.

So, still I am not considering this minus 1 point in all this reason this function would be, which is excluding this minus 1. This function would be analytic. And, we are talking about any close contour. So, integral along a close contour of analytic function in a simply connected domain is independent of path. Or along any contour this is independent of path, that says is that this integral along this b, which should be also same as the integral along a.

You see, is that is why it is not 0, because we are having this 1 is interior. So, we are using it the deformation of path or independence of path you could say. Now, that is close contour that is Cauchy integral theorem for the close contour we are not using, rather we are using is the independence of path. And we are getting it rather a bit same as to pi i, so it is not astonishing that is we are getting that same kind of thing it has to be actually this one. And if I just go by this 1, you see here I could not apply this d let us, come one by one. So, let us now move for the C. So, this is what. I was explaining you. So, now, move to the C 1.

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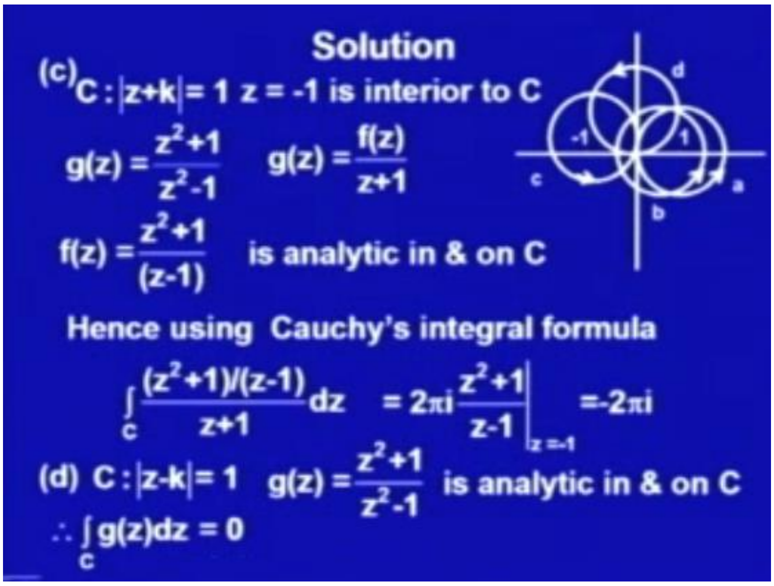

C is again any z minus k that is a rather this center is somewhere on the negative side. So, let us write it z plus k something. This k is certainly different from that whatever, I have taken for the b. And this is some other constant. But the radius is looking is that is it is 1, so this is 1 and minus 1 is interior to this 1. So now, and plus 1 is outside. So, now you see, is that is how I am going to choose my f z. f z I would choose such that z plus 1 I will take outside because minus 1 is interior. So, I will take z minus, minus 1.

And f z, if I do take that would be now my z square plus 1 upon z minus 1. And this would seems to be analytic only at z is equal to 1. Now, that z is equal to 1 is outside this region. So, again all the results all the conditions of Cauchy integral formula are satisfying. So, hence using this Cauchy integral formula, we do get is that or integral of z square plus 1 upon z minus 1 divided by z plus 1 along this contour C must be 2 pi i f z naught. What is z not here is minus 1 and f z is this.

So, 2 pi i z f z is z square plus 1 upon z minus 1 evaluated at z is equal to minus 1. So, when I keep z is equal to minus 1, the denominator I would get minus 2 in the numerator I would get plus 2. So, I would get it minus 1. So, it is minus 2 pi i. So, now we have got the interior along this contour C is minus 2 pi i. Now, let us see for the d. And you see that what I have set for the d. If I see the d and my function d z as z square plus 1 and z square upon z square minus 1.

This function seems to be analytic only at two points at z is equal to plus 1 and minus 1. And if I see this contour d, both the points minus 1 and plus 1 are outside this contour. So, if I take this contour c and it is interior in this domain my function g z is analytic. So, using this Cauchy integral theorem and the simply connected domain. You simply get it that integral along this contour g z d z should be 0. So, what I have, I am do have is some k, which is on the positive and again.

This k is a constant, which is changing the values it is not going to be the same one for all. Here, the function is analytic in and on the C. So, it is using the Cauchy integral theorem. We do says, that integral along this curve d that is C here we are denoting this contour is 0. So, we have done this one. Now, we can actually extend this Cauchy integral formula for multiply connected domain as well you see how we are doing it, so Cauchy integral formula for doubly connected domain.

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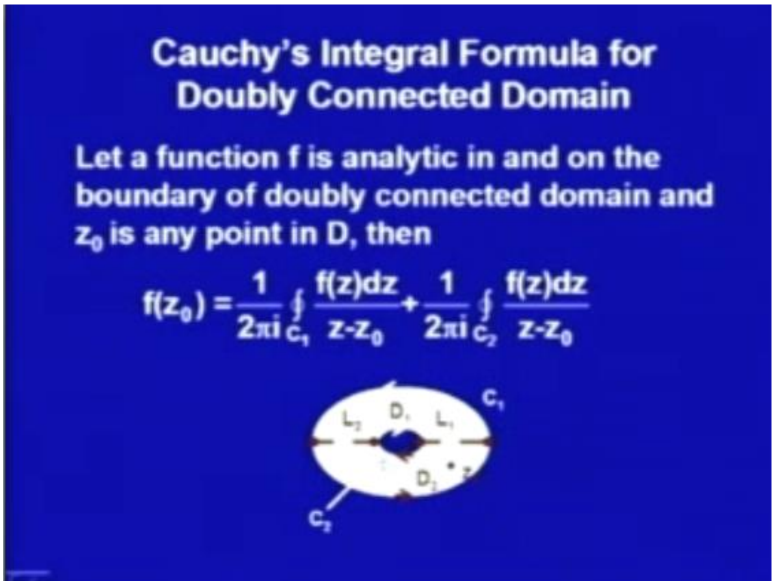

Let a function f is analytic in and on the boundary of a doubly connected domain. And z naught is end point in D. So, what we are having is my function f we are having a doubly connected domain D. And my function is analytic throughout that domain D including the boundary. And z naught is an interior point of the domain D. Then we say f of z naught as 1 upon 2 pi i integral along the boundary C 1. You see is that is because we are talking about the doubly connected domain. So, we will have two boundaries. So, one boundary is C 1 and another boundary is C 2.

So, C 1 f z upon z minus z naught d z plus 1 upon 2 pi i integral along the boundary C 2 f z of upon z minus z naught with respect to z. You see, it is a doubly connected domain have equal obtain this formula. We could get this formula, if this doubly connected domain, what we could do is we could divide it again into two simply connected domains. And in simply connected domain we will use the Cauchy integral formula and Cauchy integral theorem. Let us see how we are going to do it.

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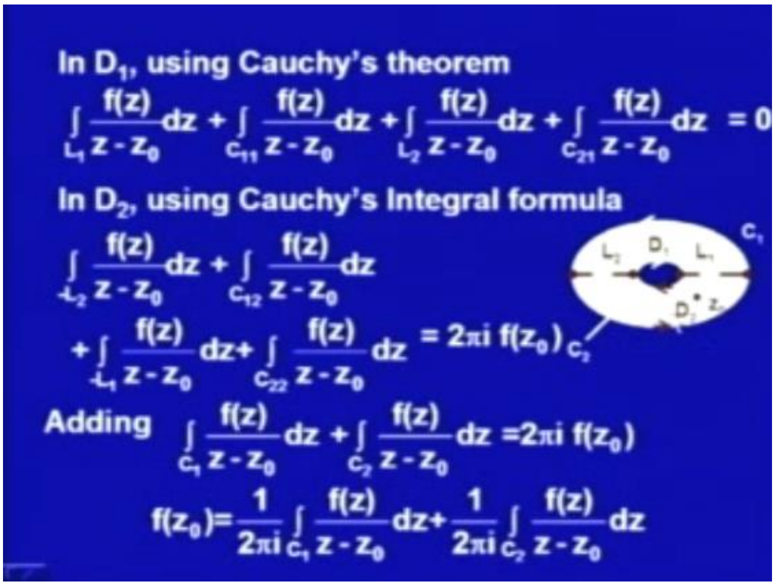

So, let us find out this one. This is my doubly connected domain, which we had cut into open into two simply connected domains D 1 and D 2. And z 2, I have taken in the domain D 2. You cannot take z 2 anywhere that is on the D 1 or D 2. Or if you are getting is z your z 1 is somewhere here. You can cross this a little bit different manner. So, that it should be in one of the domains D 1 and D 2. So, let us start in the domain D 1.

Domain D 1 my this simply connected. And if I take the boundary of this domain, I would be getting it close curve on a simply connected domain. So, using the Cauchy integral theorem, that should be 0. So, in the D 1 using the Cauchy theorem, we would get the integral of L 1 f z upon z minus z naught. Because, the function f z upon z minus z naught would be analytic throughout this region. Because, of the function f z upon z minus z naught is not analytic only at the point z naught.

So, this is analytic, so along this boundary L 1 then C 1 1 and then L 2 and then this C 1 2 this should be 0. In domain D 2 I would use the Cauchy integral theorem. Or this Cauchy integral formula I will take the direction as again from this side on the boundary. That is from L 2 side, negative of the L 2 side. So, minus L 2 of f z upon z minus z naught d z plus C 2 that is you we will get it a C 2 1 ((Refer Time: 51:10)) in the C 1 2, that is this is C 1 2, and then minus of L 1 and then C 2 2.

This is simply connected domain z naught is interior to this close contour. So, using the Cauchy integral formula, it should be 2 pi i f z naught. Now, add these two. If I do add you find it out that the integral along this L 1 and minus L 1. The integrant is same f z upon z minus z naught. And it is analytic on this whole region L 1. So, it is just the sense reversal 1 and we do get if that it is 0. Similarly, with L 2 and minus L 2 it would be 0. When we are talking about C 1 1 and C 1 2, they are in the same orientation.

So, they should add it up and similarly C 2 1 and C 2 2 they would also add up. In this orientation, so that should be minus C 2. So, what I do get is C 1 f z, z minus z naught d z plus C 2 f z, z minus z naught d z is equal to 2 pi i f z naught. And that, gives me the formula that f z naught is 1 upon 2 pi i integral along the boundary C 1 of f z upon z minus z naught, plus 1 upon 2 pi i integral along the boundary C 2 of f z upon z minus z naught.

So, similarly we could, if you do have a more P connected domain you can extend this formula. And this can be used for finding out the integral that is evaluation of integral. Examples are not doing over here. I am leaving it for you to do this kind of examples, you have now learnt that is Cauchy integral formula, we had learnt that is how to obtain it. And we had also learnt to use it in evaluation of many integrals. So, today we would I would finish up my lecture over here.

But, as I said is this Cauchy integral formula is very crucial or playing very main role improving certain results of analysis. So, next lecture we would go with those thing. So, today, we had learn that is one important consequence of Cauchy integral theorem, as the Cauchy integral formula and it is use in evaluation of many integrals. That is all for today.

Thank you.