Mathematics - II
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Module - 3
Lecture - 1
Complex Numbers and their Geometrical Representation

My dear young friends, today I will speak on Complex Numbers and their Geometrical Representation.

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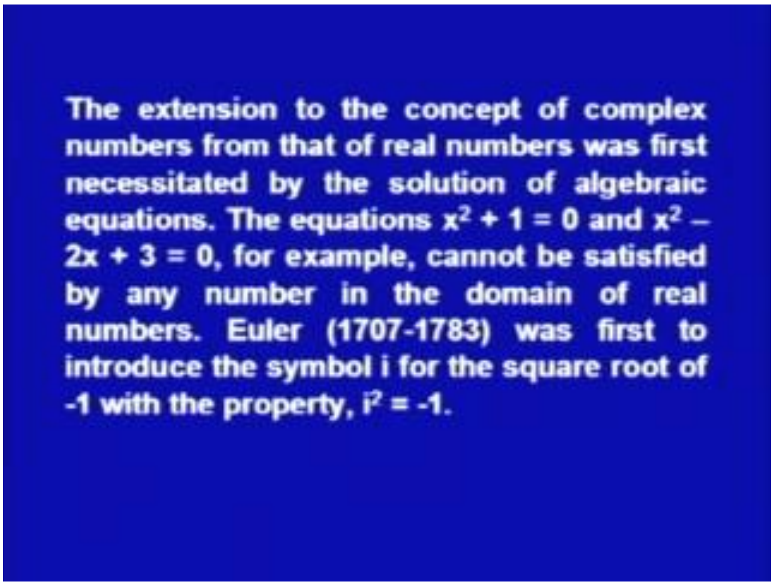

As an introduction to the subject the extension to the concept of complex numbers, from that of real numbers was first necessitated by the solution of algebraic equations. For example, the equations x square plus 1 equal to 0 and x square minus 2 x plus 3 equal to 0 cannot be satisfied by any number in the domain of real numbers. Euler 1707 to 1783 was first introduce the symbol i for the square root of minus 1 with the property, that i square is equal to minus 1.

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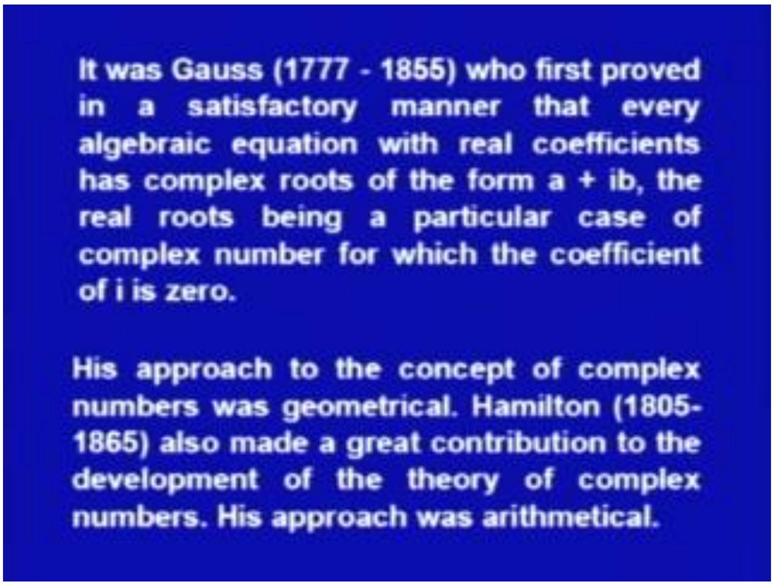

It was Gauss 1777 to 1855, who first proved in a satisfactory manner that every algebraic equation with real coefficients has complex roots of the form a plus i b, the real roots being a particular case of complex number, for which the coefficient of i is zero. His approach to the concept of complex numbers was geometrical, Hamilton 1805 to 1865 also made a great contribution to the development of the theory of complex numbers, his approach was arithmetical.

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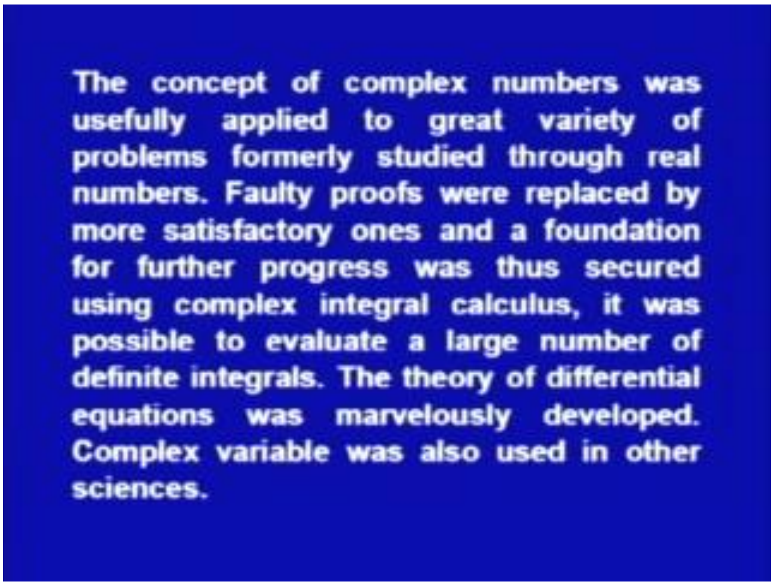

The concept of complex numbers was useful or usefully applied to great variety of problems formerly studied through real numbers. Faulty proofs were replaced by more satisfactory ones and a foundation for further progress was thus secured using complex integral calculus, it was possible to evaluate a large number of definite integrals. The theory of differential equations was marvellously developed complex variable was also used in other sciences.

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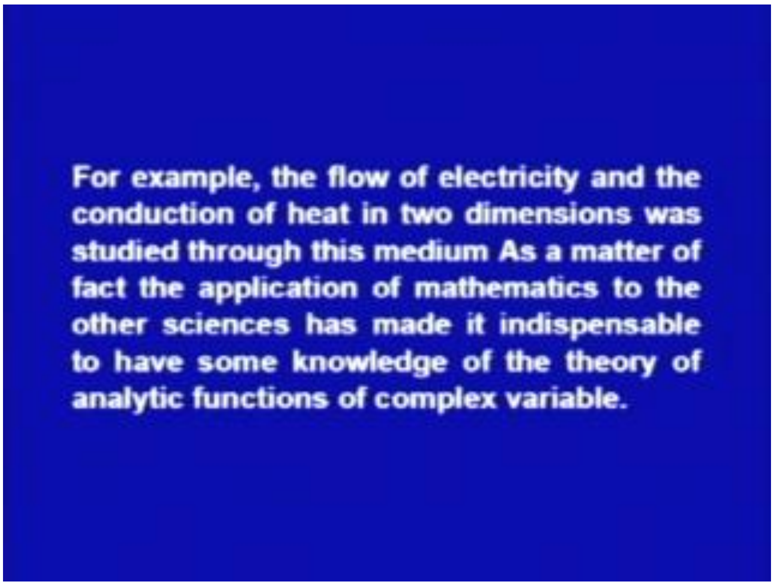

For example, the flow of electricity and the conduction of heat in two dimensions was studied through this medium. As a matter of fact the application of mathematics to the other sciences had made has made it indispensable to have some knowledge of theory of analytic functions of complex variable.

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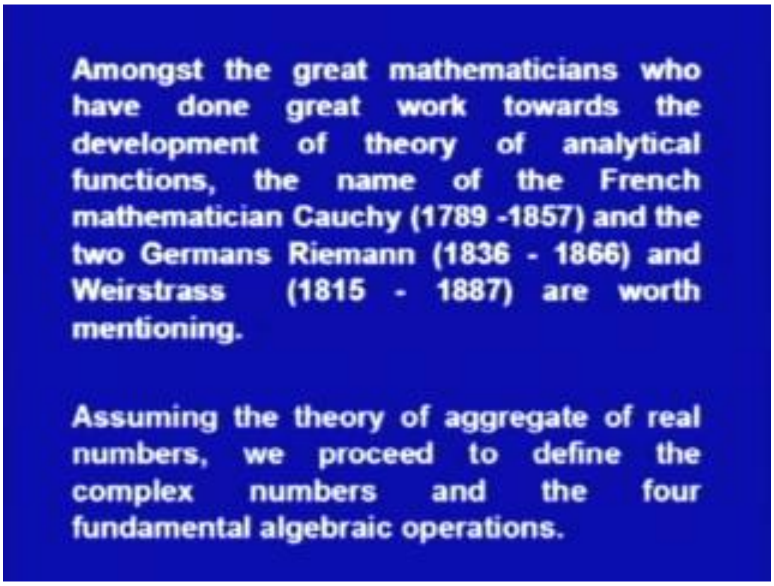

Amongst the great mathematicians who have done great work towards the development of theory of analytical functions, the name of the French mathematician Cauchy 1789 to 1857 and the two Germans Riemann 1836 to 1836 and Weirstrass 1815 to 1887 are worth mentioning. Assuming the theory of aggregate of real numbers, we proceed to define the complex numbers and the four fundamental algebraic operations.

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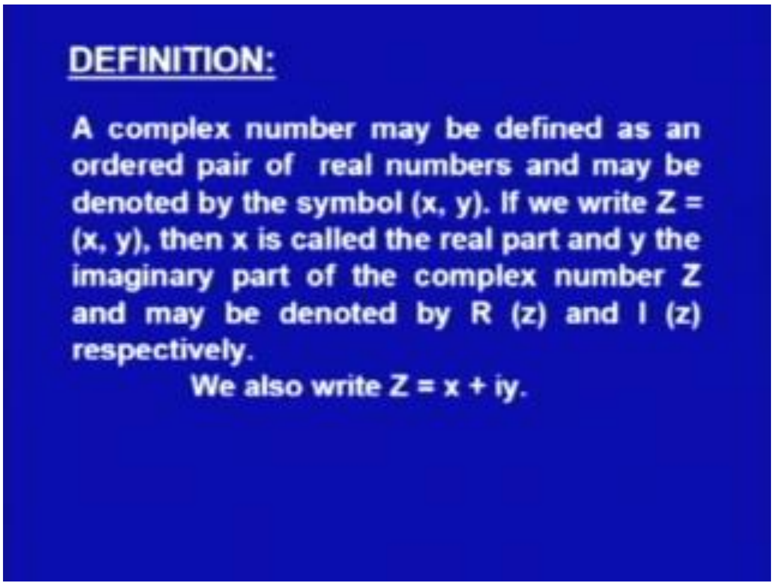

First we take the definition, a complex number may be defined as an ordered pair of real numbers and may be denoted by the symbol x comma y within brackets. If we write Z equal to x comma y, then x is called the real part and y the imaginary part of the complex number which we represent by Z. And may be denoted by R z that is real part of Z and I z, that is imaginary part of Z respectively, we also write Z equal to x plus i y.

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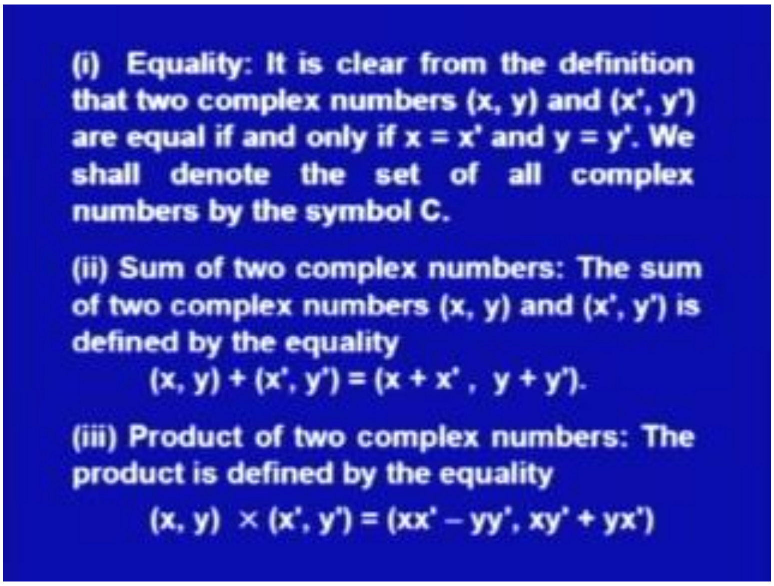

The first operation that is equality, it is clear from the definition that two complex numbers x comma y and x dash comma y dash are equal if and only if x equal to x dash and y equal to y dash, that is their real and imaginary parts are respectively equal. We shall denote the set of all complex numbers by the symbol C, the second operation sum of two complex numbers, the sum of two complex numbers x comma y and x dash comma y dash is defined by the equality x comma y plus x dash comma y dash equal to x plus x dash comma y plus y dash.

That is their real and imaginary parts are added in the new complex number which is the sum of two complex numbers. The third operation that is the product of two complex numbers, the product is defined by the equality x comma y into x dash comma y dash equal to x x dash minus y y dash comma x y dash plus y x dash. That is if we take two complex numbers or the ample say 2 comma 3 and 4 comma y, then their sum will be 2 plus 4 and comma 3 plus 5. Similarly, in the product 2 comma 3, that is 2 plus 3 i into 4 plus 5 i, that will be equal to 2 into 4 minus 3 into 5 and comma 2 into 5 and plus 3 into 4.

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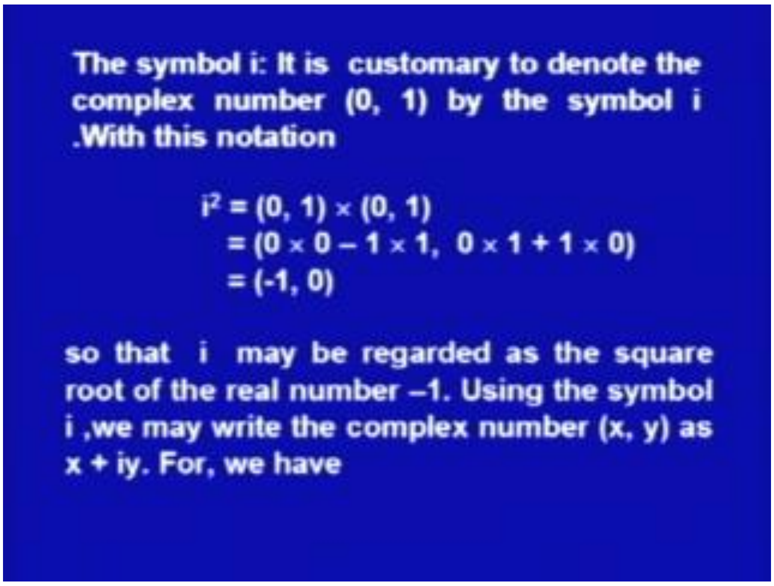

The symbol i, it is customary to denote the complex number (0, 1) by the symbol i with this notation i square will be equal to 0 comma 1 into 0 comma 1, that is 0 plus i 1 into 0 plus i 1 equal to by the property. Or operation of the multiplication it will be equal to 0 into 0 minus 1 into 1, that is the real part and 0 into 1 plus 1 into 0 that is the imaginary part and this will be equal to minus 1.

As the real part and comma 0 that is the imaginary part, so that I may be regarded as the square root of the real number minus 1, using the symbol i we may write the complex number x comma y as x plus i y also.

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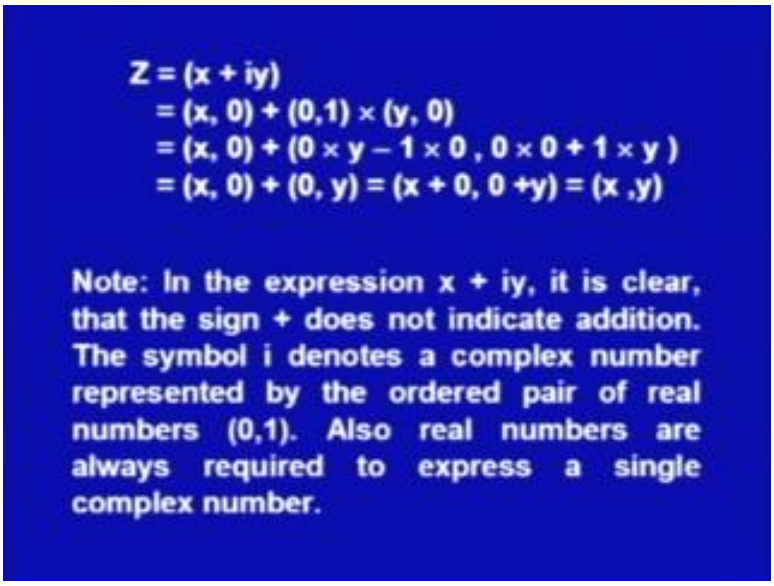

For we have, we can write Z equal to x plus i y and x is x comma 0 that is x plus i 0 and plus i, we can write as 0 comma 1 that is 0 plus i into 1 into y is y comma 0 that is y plus 0 into i. This will be equal to x comma 0 that is x plus i into 0 plus, when the product of these two of complex numbers will be 0 into y minus 1 into 0 as the real part and 0 into 0 plus 1 into y as imaginary part.

This will be again equal to x comma 0, that is x plus i 0 plus 0 comma y and this sum of two complex numbers will be equal to the respective sum of the real and imaginary parts, that is x plus 0 comma 0 plus y and this will be equal to x comma y that is x plus i y again. So, we note here that in the expression x plus i y, it is clear that the sign plus does not indicate addition, the symbol i denotes a complex number represented by the ordered pair of real numbers 0 comma 1, that is 0 plus i into 1, also real numbers are always required to express a single complex number.

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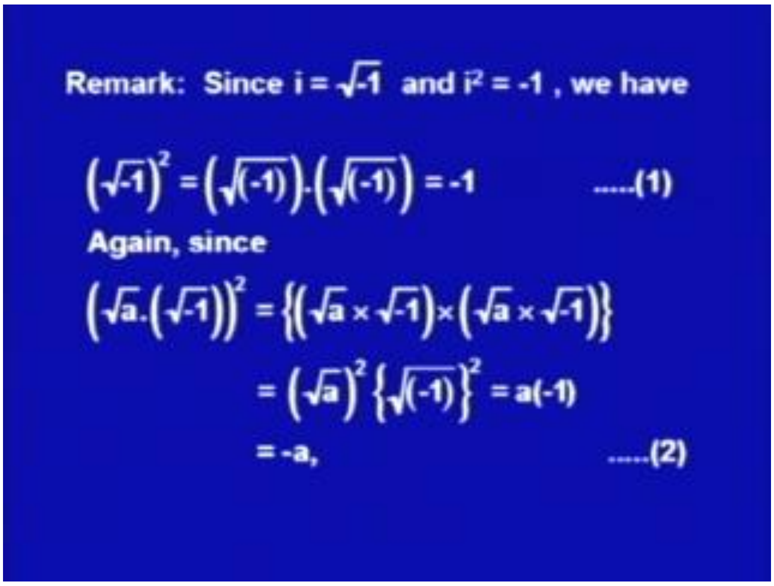

And we can remark here, since i equal to under root minus 1 and i square equal to minus 1, we have under root minus 1 whole square equal to the product of under root minus 1 and under root minus 1 that is equal to minus 1. Again since the square of root a into under root minus 1 equal to under root a into under root minus 1 into again root a into root minus 1, this will be equal to the square of root a into square of root minus 1, that is equal to a into minus 1 that is minus a.

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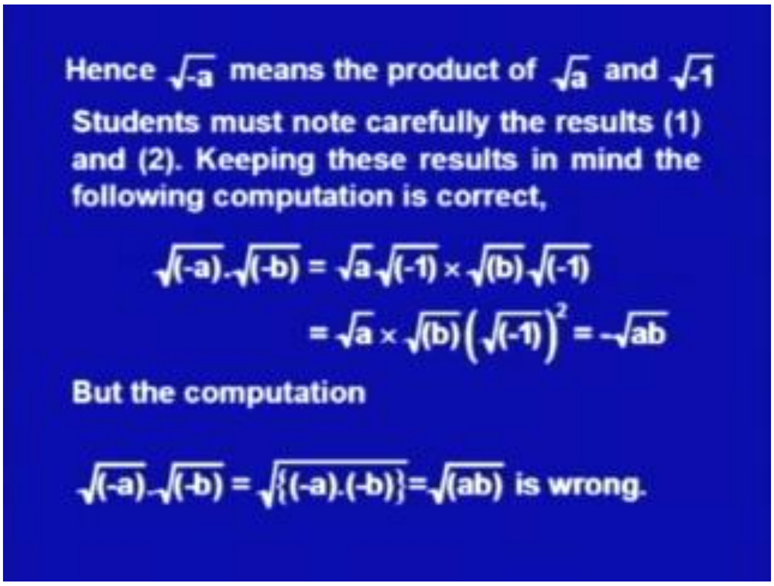

Hence root minus a means the product of root a and root minus 1, students must note carefully that the result 1 and 2, keeping these results in mind the following computation is correct. That is the product of root minus a into root minus b will be equal to root a into root minus 1 into root b into root minus 1, which will be equal to root a into root b into a square of root minus 1 and this will be equal to minus root a into b. But, the computation root minus a into root minus b equal to under root minus a into minus b equal to under root a into b which is wrong.

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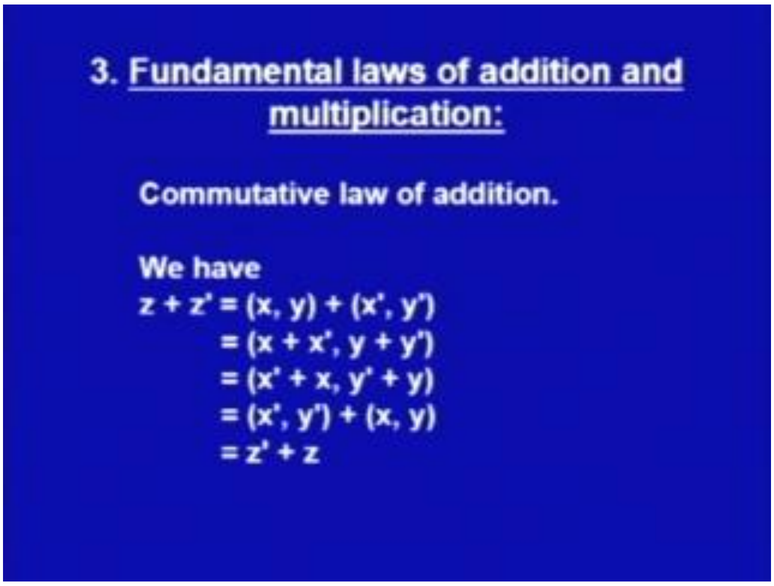

Then the third point is fundamental laws of addition and multiplication about the complex numbers, the first one is commutative law of addition. We have z plus z dash that is the sum of two complex numbers equal to z we can write x comma y or x plus i y plus z dash, we can write x dash comma y dash or x dash plus i y dash. This will be equal to the complex number having the real parts as x plus x dash and imaginary part as y plus y dash.

Now, x plus x dash and y plus y dash we can commute, because there is commutative law of addition about the real numbers, so we can write the real and imaginary parts by commutating as x dash plus x and y dash plus y. This will be equal to the complex number x dash comma y dash or x dash plus i y dash plus x comma y or x plus i y, that is the sum of the numbers z dash plus z, so this is the commutative law of addition.

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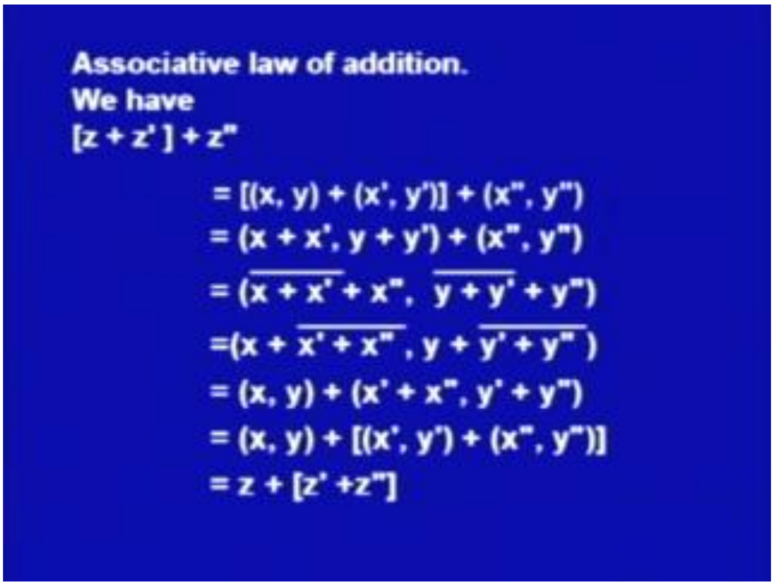

Similarly, the associative law of addition we can write all that z plus z dash plus z double dash the third complex number. So, if you write these complex numbers as x comma y that is x plus i y plus x dash plus i y dash and then, add or plus x double dash comma y double dash that is x double dash plus i y dash. This whole will be equal to the sum of the real parts and imaginary parts of the first two complex numbers.

And then, the third complex number x double dash comma y double dash, this whole will be equal to x plus x dash plus x double dash. Now, we take the addition of all these two complex numbers, so real part is x plus x dash plus x double dash and the imaginary part will be y plus y dash and plus y double dash, this whole will be again equal to x plus x dash plus x double dash.

And the imaginary part will be y plus y dash plus y double dash, this whole again will be equal to or can be equal to x comma y that is the complex number x plus i y plus the complex number x dash plus x double dash comma y dash plus y double dash. And this whole again will be equal to the complex number x comma y or x plus i y plus the complex number x dash comma y dash, that is x dash plus i y dash plus the complex number x double dash comma y double dash or x double dash plus i y double dash, which we can write in terms of the complex numbers z plus z dash plus z double dash.

So, the associative law of a addition that we can associate the three numbers in addition, that is we take the addition of the first two numbers, and then add the third number to them or we take the addition of the last two numbers z dash and z double dash and add to z, so they are equal or simply if you add the second number to the first and then, the third number to the sum, so they all will be equal, so that is the associative law of addition.

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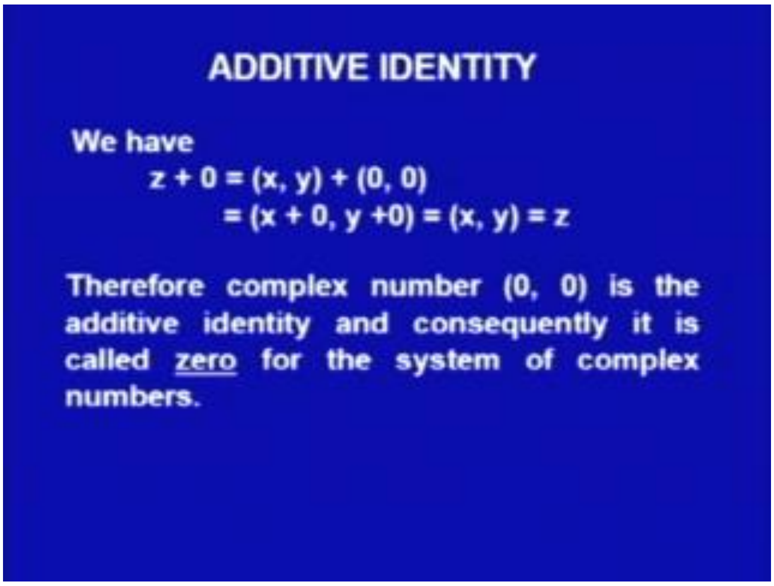

Next we take additive identity, we have z plus 0 if you add the two, so this is equal to the z we can write x comma y or x plus i y and 0 we can write 0, comma 0 or 0 plus i 0, this sum will be equal to, the sum of the two numbers x plus 0 comma y plus 0 that is x plus 0 and y plus 0 are real and imaginary parts. And that is equal to x comma y that is x plus i y and which is equal to z, so after adding 0 we get the same number, so this 0 is called the additive identity. And this 0 the complex number which you have defined, so this is defined as the additive identity and this is called 0 for the system of the complex numbers.

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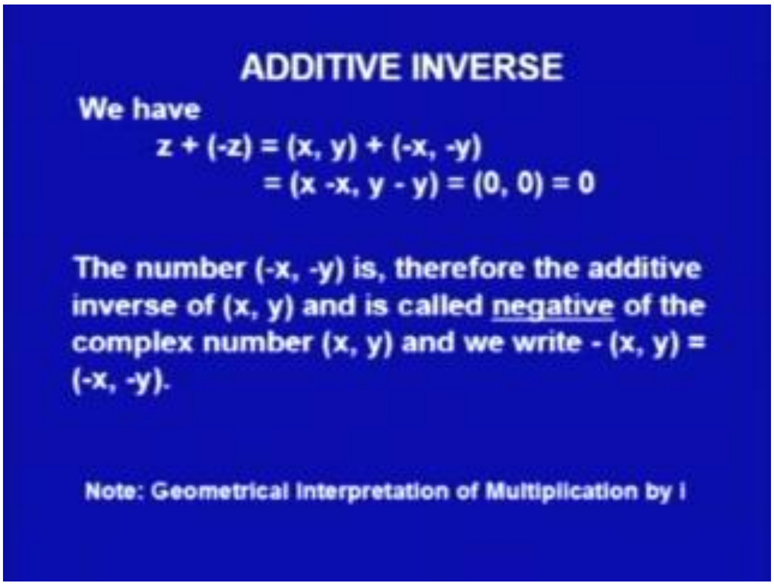

Then we define the additive inverse, inverse of z with respect to addition, that is z plus minus z if you see this is equal to the number z means x comma y, that is x plus i y plus minus z means minus x comma minus y over minus x plus i into i times minus y. And this addition is equal to the addition of the two complex numbers with real and imaginary parts as x minus x and y minus y, that is 0 comma 0, that is 0 plus i 0 that is the complex number 0.

So, after adding minus z to z we get 0 and therefore, the number minus x minus y or minus x plus i times minus y is called the additive inverse of the complex number x comma y or x plus i y. And this is also called the negative of the complex number x comma y or x plus i y and we write this as minus of x comma y that is minus x comma minus y that is minus x plus i times minus y, so this is the negative of the number z. Here we can note that the geometrical significance of multiplication of a number z by i.

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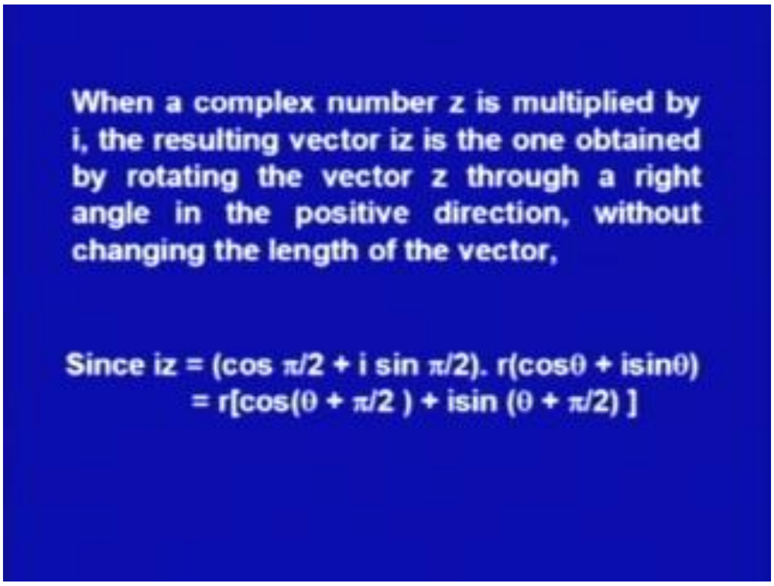

So, when we multiply a complex number by i that is i z, this is the one obtained by rotating the vector z through a right angle in the positive direction without changing the length of the vector. And we can explain here, because if you take i z, so i we can write cosine pi by 2 plus i sin pi by 2, because equals sin pi by 2 is 0, so this is i into 1 and into z that is r cos theta plus i sin theta and this will be equal to r into cosine 0 plus pi by 2 plus i sin theta plus pi by 2.

So, this means the theta in cosine and sin have become theta plus pi by 2, so the number will be rotated through an angle of pi by 2 with the same length of same vector, then we define the commutative law of multiplication.

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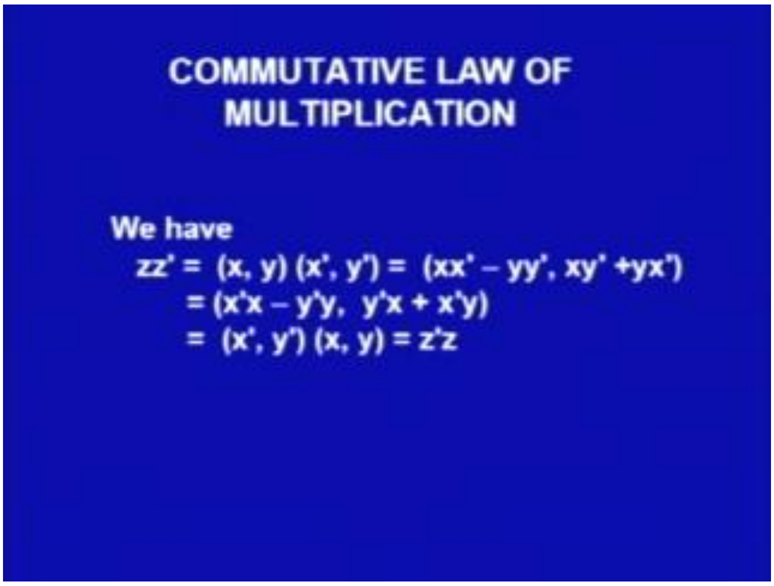

So, if you take z into z dash that is x comma y that is x plus i y and into x dash comma y dash that is x dash plus i y dash, this product of the complex number as the definition as may defined earlier can be written as x x dash minus y y dash as the real part. And x y dash plus y x dash as the imaginary part of the product number, and this will be equal to we can apply the commutative law of multiplication of the real numbers.

And write this number equal to x dash x minus y dash y as the real part and y dash into x plus x dash into y as the imaginary part. And which we can write equal to the complex number x dash comma y dash, that is x dash plus i y dash into the complex number x comma y that is x plus i y and this product is again equal to z dash into z. So, the multiplication has been commutated here and z into z dash equal to z dash into z, this is the commutative law multiplication of two complex numbers.

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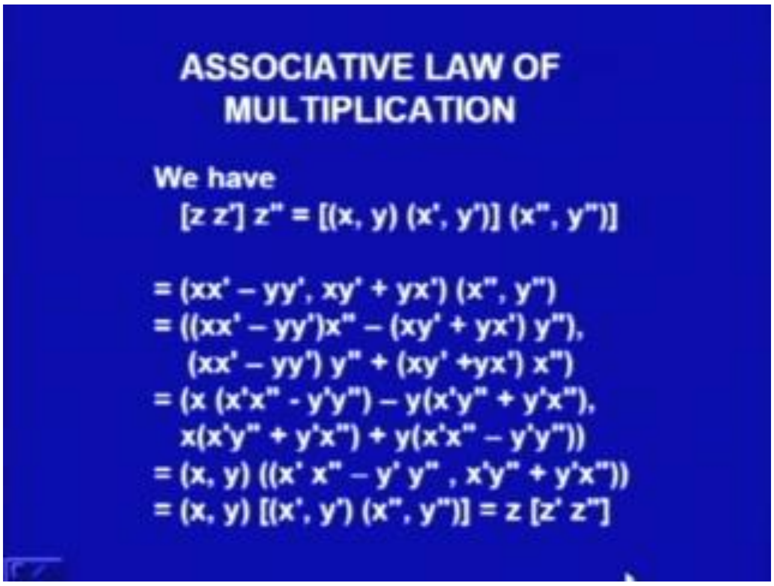

We shall now take associative law of multiplication, if we have three numbers z, z dash and z double dash, so if we make the product of z and z dash first that is x comma y that is x plus i y and x dash comma y dash that is x dash plus i y dash. And into the third number that is x double dash comma y double dash that is x double dash plus i y double dash, so we make the product of the first two numbers by the definition of the multiplication of two complex numbers.

The real part will be x x dash minus y y dash and in the real part and x y dash plus y x dash as imaginary part and into the third number. Now, we make this multiplication again x x dash minus y y dash into x double dash minus x y dash plus y x dash into y double dash as the real part of the product number. And the imaginary part will be x x dash minus y y dash into y double dash plus x y dash plus y x dash into x double dash.

Now, here we apply the law of associative with respect to the complex number, associative law of multiplication with respect to the real numbers, and we make the associative law here. So, the associate like this x into x dash into x double dash minus y dash y double dash minus y into x dash y double dash plus y dash into x double dash as the real part. And the imaginary part will be x into x dash into y double dash plus y dash into x double plus y into x dash x double dash minus y dash into y double dash, which we can write equal to the complex number x comma y that is x plus i y into x dash into x double dash minus y dash y double dash as the real part and the imaginary part will be x dash y double dash plus y dash x double dash, which we can write equal to x comma y as before. And now we can write this complex number equal to x dash comma y dash that is x dash plus i y dash into x double dash into y double dash and this is equal to z into the product of z dash z double dash.

So, this is the associative law of multiplication, we can also make it z into z dash into z double dash, so that is we can multiply z by z dash and the product by z double dash, so all will be equal, so that is the associative law of multiplication.

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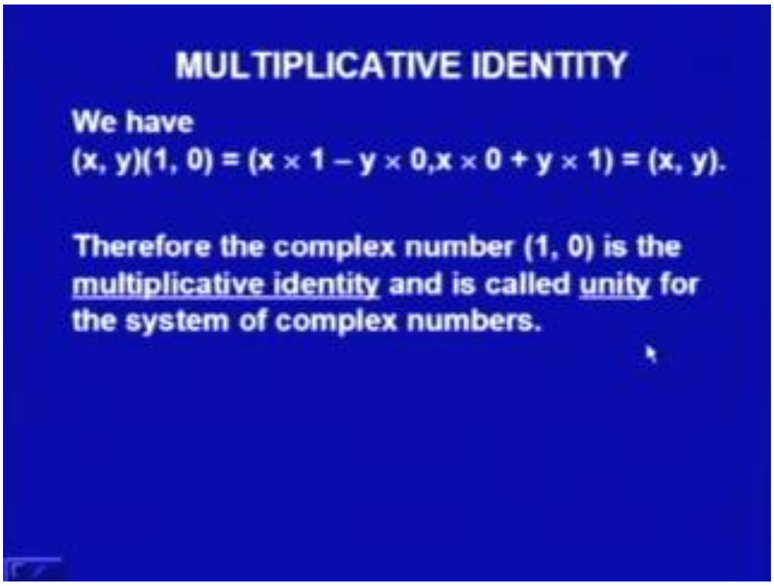

Now, we define the multiplicative identity for complex numbers, so if we have the product x comma y that is x plus i y into 1 comma 0 that is 1 plus i into 0, this product by the definition of the product of the complex numbers will be x into 1 minus y into 0 that is the real part. And the imaginary part is x into 0 plus y into 1 this will be equal to after simplifying x comma y that is x plus i y, so after multiplying x y by b y 1 0 we get the same number. So, we can say therefore, the complex number 1 comma 0 that is 1 plus i into 0 is the multiplicative identity and is called the unity for the system of complex numbers.

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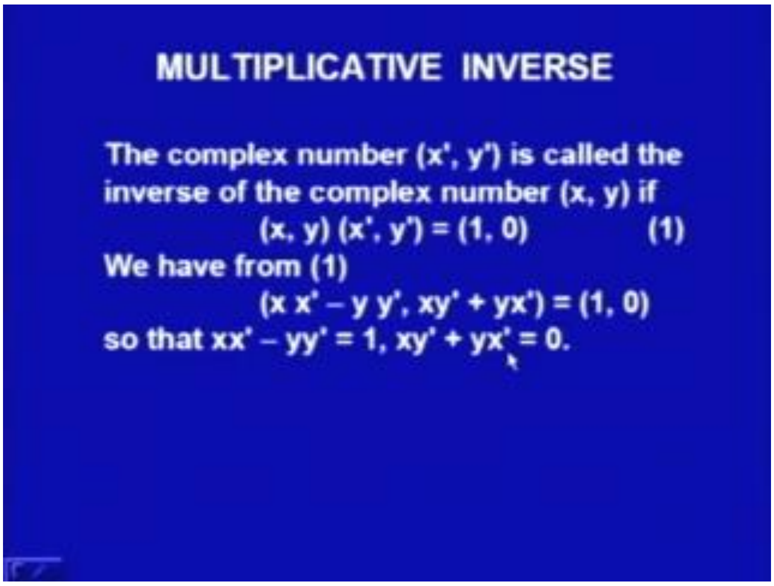

Now, we define multiplicative inverse for the complex number, the complex number x dash comma y dash that is x dash plus i y dash is called the inverse of the complex number x comma y that is x plus i y. If the product x comma y that is x plus i y into x dash comma y dash that is x dash plus i y dash equal to the multiplicative identity 1 comma 0 that is 1 plus i 0.

Now, we can have the left hand side the product of two complex number x y and x dash y dash as by definition x into x dash minus y y dash as the real part and x y dash plus y x dash as the imaginary part. And the right hand side is equal to 1 comma 0 that is 1 plus i 0, so from this we must have x x dash minus y y dash equal to 1 that is the real part and x y dash plus y x dash equal to 0. The real and imaginary parts from both sides must be equal, so we get these expressions.

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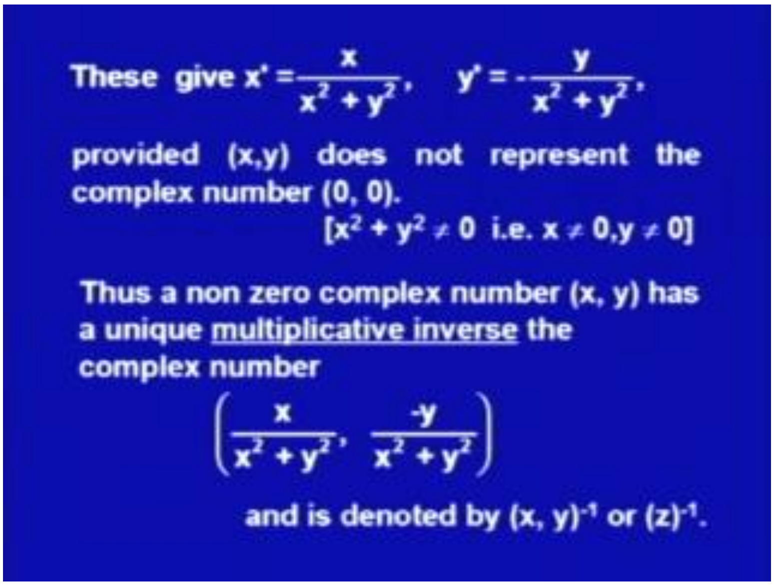

And if you simplify those expressions over x dash and y dash we get x dash equal to x by x square plus y square and y dash equal to minus y by x square plus y square and therefore, of course here the denominator that is x square plus y square it should not be 0. That is this x comma y, the complex number x plus i y does not represent the complex number 0 comma 0, so that is x square plus y square this should not be 0, means that the complex number 0 comma 0 should not be 0.

And from this or from x square plus y square that is this should not be 0 that is x and y both should not be 0, so this means a non 0 complex number x comma y or x plus i y has a unique multiplicative inverse of the complex number. So, this is the multiplicative inverse x by x square plus y square and minus y by x square plus y square, so this complex number is the multiplicative inverse to the number x plus i y or x comma y. And this we denote as x comma y power minus 1, therefore x plus i y power minus 1 or z to the power minus 1 or may be 1 by z.

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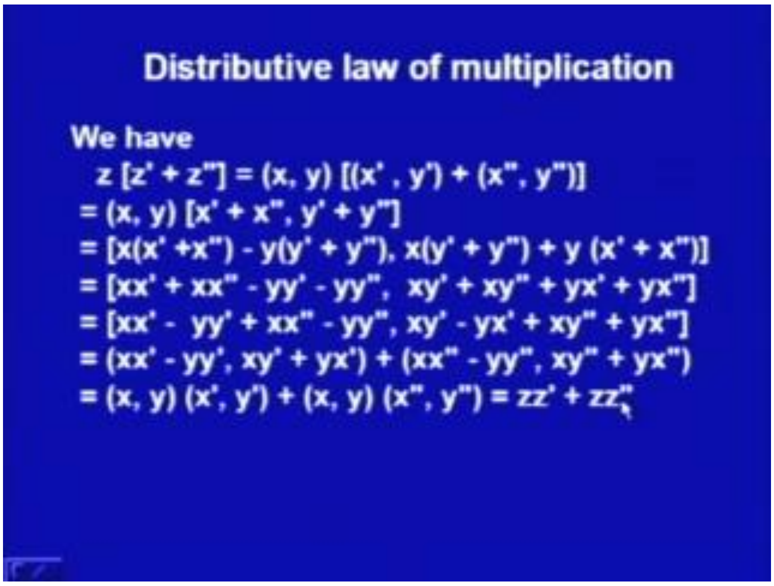

So, we now take the distributive law of multiplication of complex numbers, if we have three numbers again z, z dash and z double dash, then we start with the z dash plus z double dash into z. So, we write x comma y that is x plus i y whole z, x dash comma y dash that is x dash plus i y dash whole z dash plus x double dash comma y double dash that is x double dash plus i y double dash whole z double dash.

So, by addition the addition of the two numbers z dash plus z double dash will be x dash plus x double dash and y dash plus y double dash as the real and imaginary parts and x into x comma y is outside. Now, we take the multiplication x comma y with this complex number by the definition again, so x into x dash plus x double dash minus y into y dash plus y double dash as the real part; and x into y dash plus y double dash plus y into x dash plus x double dash as the imaginary part.

Now, we can simplify it further, the next step x x dash plus x x double dash minus y into y dash minus y into y double dash as the real part and x y dash plus x y double dash plus y x dash plus y x double dash as the imaginary part. Now, again we can rearrange the above expression as x x dash minus y y dash plus x x dash x x double dash minus y y double dash as the real part and x y dash minus y x dash plus x y double dash plus y x double dash as the imaginary part.

We again write it in terms of the real and imaginary parts x x dash minus y y dash real part comma, the imaginary part x y dash plus y x dash, then plus x x double dash minus y y double dash as the real part and x y double dash plus y x double dash as the imaginary part of the complex number. The first product we can write x comma y that is x plus i y into x dash comma y dash that is x dash plus i y dash that is the product of the two complex numbers.

And the second we can write the product of two complex numbers x comma y that is x plus i y and the complex number x double dash comma y double dash that is x double dash plus i y double dash. So, that will be equal to this product is z into z dash and plus z into z double dash, so the distributive law of z into z dash plus z double dash is equal to z z dash plus z z double dash, this is the distributive law of multiplication of the complex numbers. Now, after this we shall take and define the difference and the division of two complex numbers.

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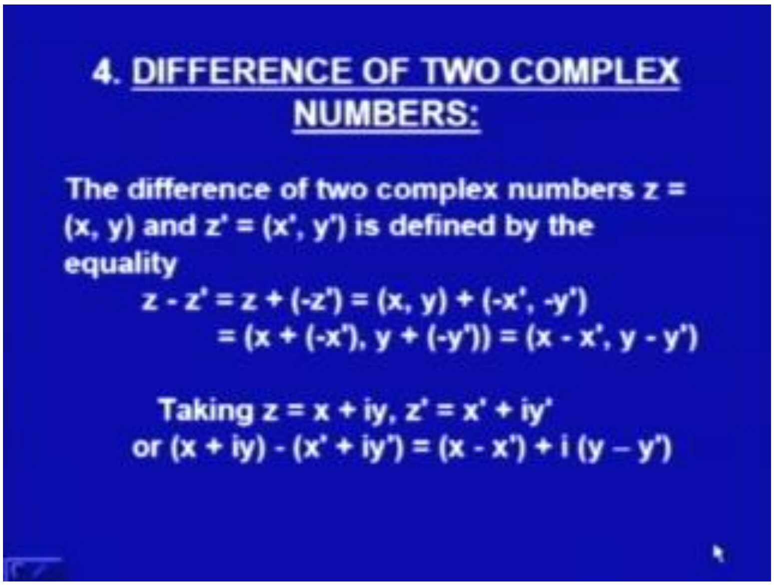

The difference of two complex numbers z equal to x comma y or x plus i y and z dash equal to x dash comma y dash that is x dash plus i y dash is defined by the equality that z minus z dash equal to z, if we add z and minus z dash, that is z plus minus z dash equal to we write x comma y x plus i y for z. And for minus z we write minus x dash comma minus y dash or minus x dash plus i times minus y dash.

And this will be equal to if you take the sum of the two complex numbers by definition and so we add x and minus x dash and as the real part and y and minus y dash, we add them as a real part as imaginary part. This is equal to x minus x dash as the real part and y minus y dash of the imaginary part of the number, so if you take z equal to x plus i y as I explained just now.

And similarly z dash we take x dash plus i y dash and if you take the difference of the two, then this will also be x minus x dash plus i y minus y dash with this summation notation both the complex numbers. So, this is the difference between or difference of the two number z and z dash.

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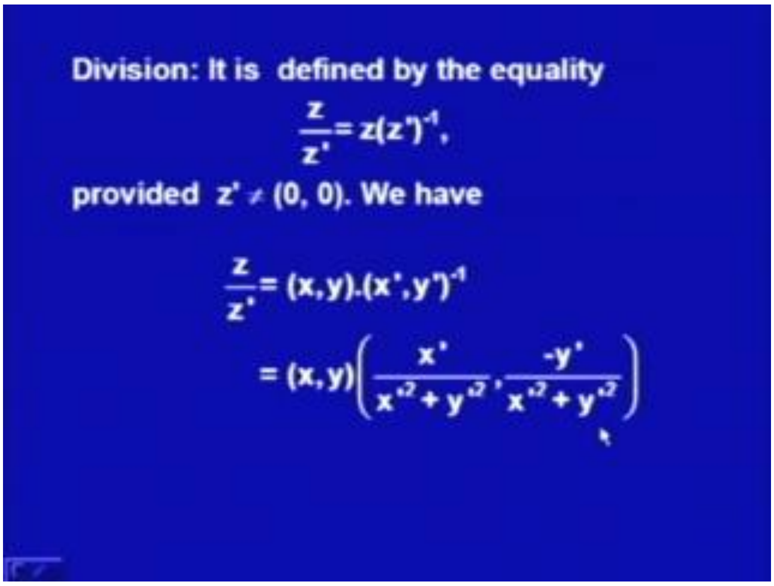

So, now we shall define the division of two complex numbers say z and z dash, so we define this division by z by z dash equal to z, we can take z dash in the numerator as z dash inverse power minus 1. And of course, here because z dash has come in the numerator, so z dash will not be equal to 0 complex number that is 0 comma 0 and 0 plus i 0.

So, we can have the expression z by z dash equal to substituting the values of z and z dash as x comma y or x plus i y for z and x dash comma y dash that is x dash plus i y dash for z dash and it is power minus 1. And this product will be equal to x comma y that is x plus i y is a complex number as such and the inverse as we have defined earlier, we can write this inverse equal to x dash by x dash square plus y dash square comma or plus i times minus y dash by x dash square plus y dash square.

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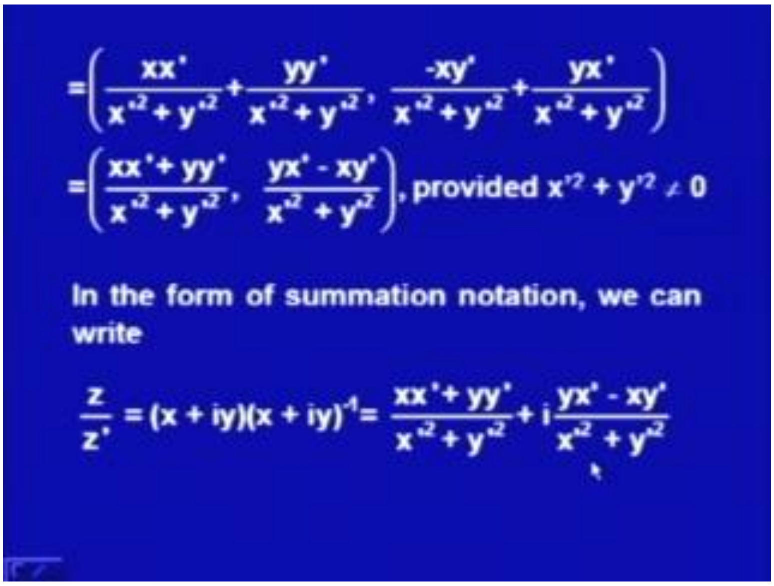

And simplifying as the product of two complex numbers, we get x into x dash divided by x dash square plus y dash square plus y y dash divided by x dash square plus y dash square as the real part. And the imaginary part will be minus x y dash divided by x dash square plus y dash square plus y x dash divided by x dash square plus y dash square, so we can again write this number as x x dash plus y y dash divided by x dash square plus y dash square as the real part.

And y x dash minus x y dash divided by x dash square plus y dash square is the imaginary part of the complex number, so this number is the division of the two complex numbers and the denominator that is x dash square plus y dash square should not be equal to 0. So, if we write or if we use the summation notation over the proof, then again we can write z by z dash equal to x plus i y x plus i y inverse.

Then we can write this as equal to x x dash plus y y dash divided by x dash square plus y dash square plus i into y x dash minus x y dash divided by x dash square plus y dash square, so this is the complex number that we have.

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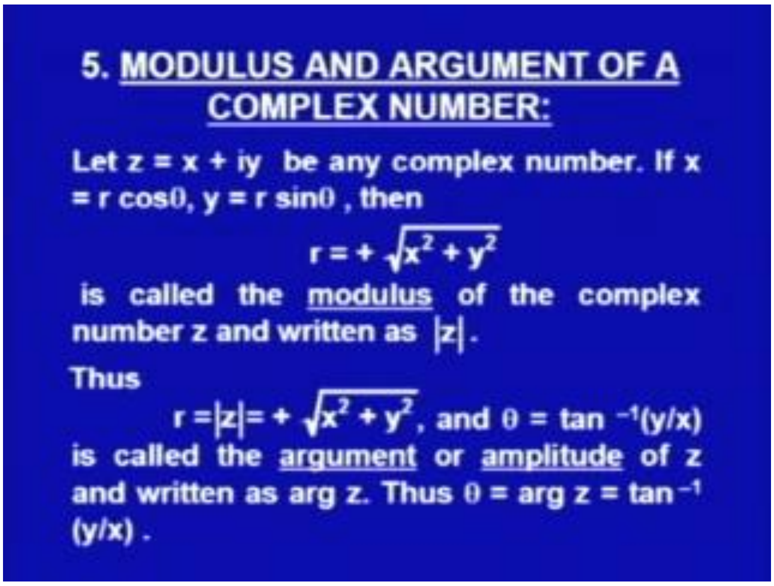

The next property we take modulus and argument of a complex number, now if z equal to x plus i y is any complex number and we take x equal to r cos theta and y as r sin theta. Then, from these two expressions we can calculate easily that r is equal to the positive value of under root x square plus y square, the some of the squares of real and imaginary parts of the complex number.

Now, this r is called the modulus of the complex number z and written as modulus z that is two vertical lines on both sides of z, so this means r equal to modulus z equal to the positive value of under root x square plus y square. And theta is, if you calculate theta from these expression x equal to r cos theta and y equal to r sin theta, so tangent theta comes out to the y by x, therefore theta equal to tangent inverse y by x. This theta is called the argument or amplitude of the complex number z and in short it is written as a r g, that is r z, so therefore theta equal to argument z equal to tangent inverse y by x.

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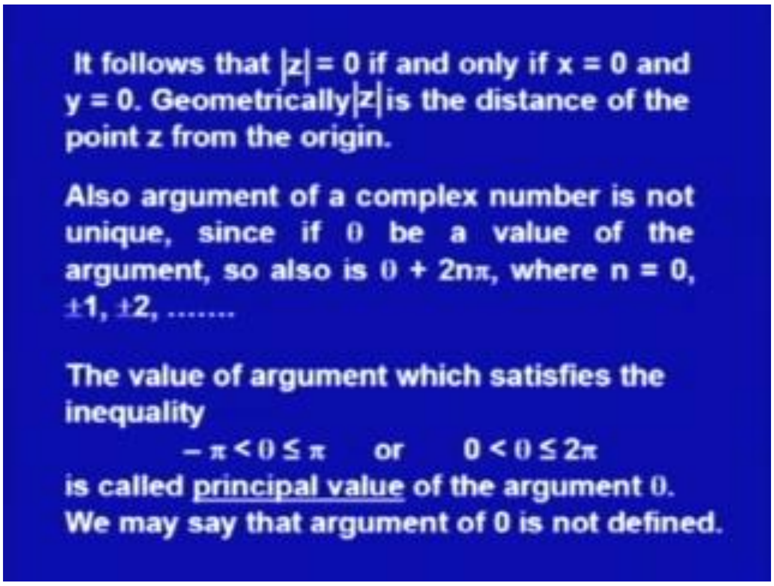

So, from this it follows that modulus z equal to 0 if and only if the real part x and the imaginary part y both are 0, and geometrically modulus z is the distance of the point z from the origin. Also argument of a complex number is not unique, since if 0 be the value of the argument or theta be the value of the argument, so also is theta plus 2 n pi where n is equal to 0 plus minus 1 plus minus 2 etcetera, so this is not unique, so we have many values.

The value of the argument theta we satisfies the inequality that is theta is greater than minus pi or less than equal to pi or theta is greater than 0 and less than equal to 2 pi, so this value of theta is called the principal value of the argument theta, we may say that argument of 0 is not defined.

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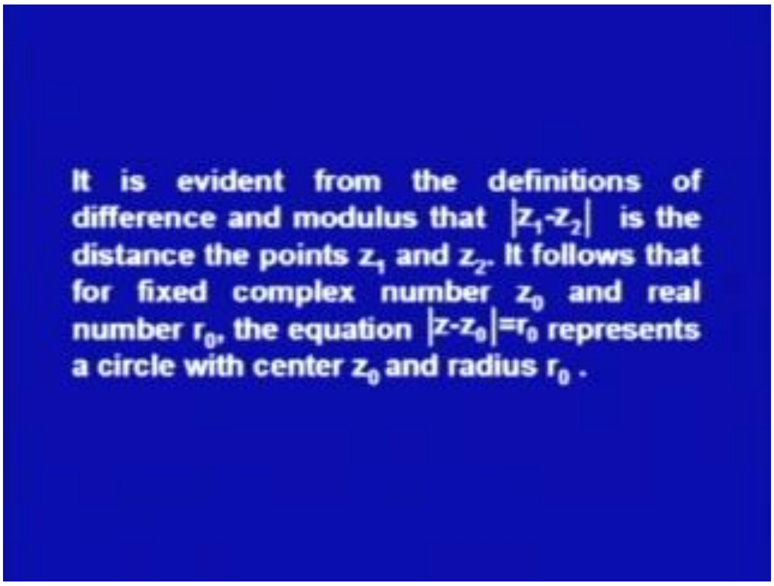

It is evident from the definitions of the difference and modulus that z 1 minus z 2 modulus is the distance the between the points z 1 and z 2, it follows that for fixed complex number z 0 and real number r 0 the equation z minus z 0 modulus of this equal to r 0 represents a circle with centre z 0 and radius equal to r 0.

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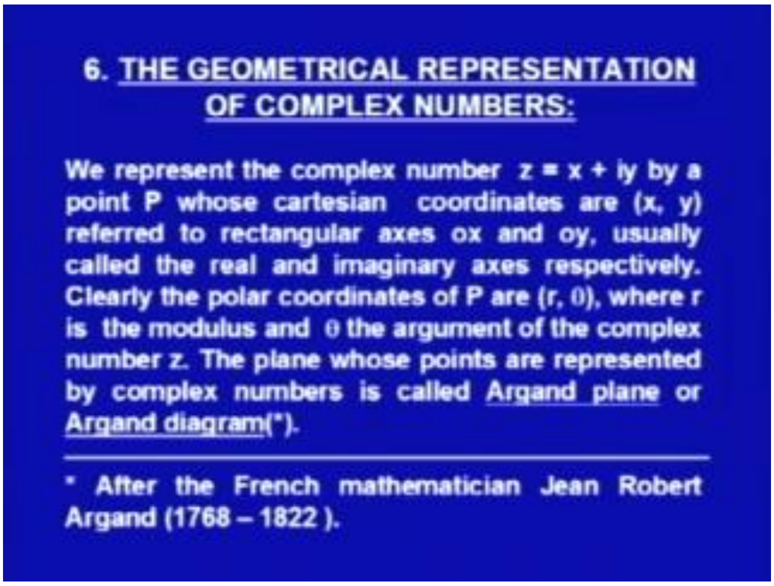

Next we take the geometrical representation of complex numbers, how we can define or we can represent geometrically the complex numbers. We represent the complex number z by x comma y or x plus i y by a 0 whose Cartesian coordinates are x and y referred to the rectangular axis o x and o y, usually called the real and imaginary axes respectively. Clearly the polar coordinates of P are r and theta, where r is the modulus and theta is the argument of the complex number z. The plane whose points are represented by complex numbers is called Argand plane or Argand diagram after the name of the French mathematician Jean Robert Argand 1768 to 1822.

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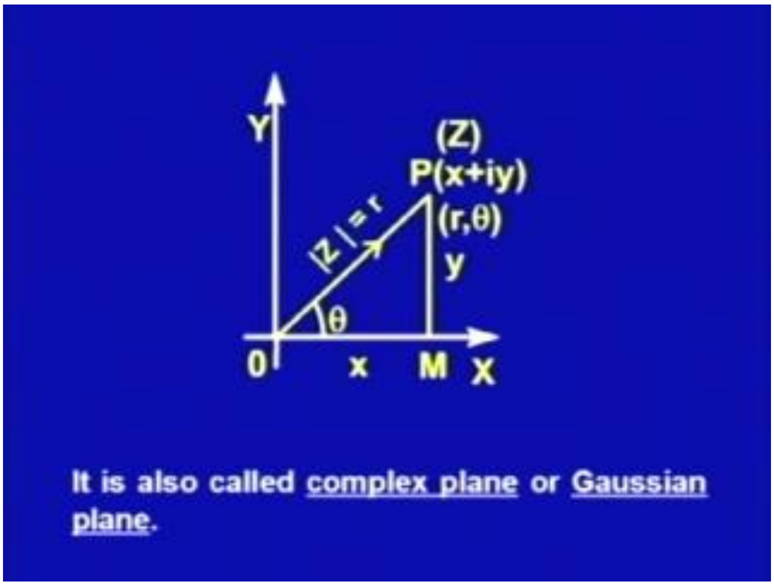

So, on the figure along the x axis we take distance of the 0, P x plus i y or Z, so the distance OM along X axis is x and the distance of P along Y axis is y and r is the distance OP that is modulus z equal to r and theta is the angle P O M that is tangent inverse y by x. So, r is the modulus and theta is the argument of the complex number, so the complex number can be represented on the Argand diagram, in terms of x and y the real and imaginary parts. And in terms of the modulus and argument r and theta, this two dimensional plane is also known as a complex plane or Gaussian plane.

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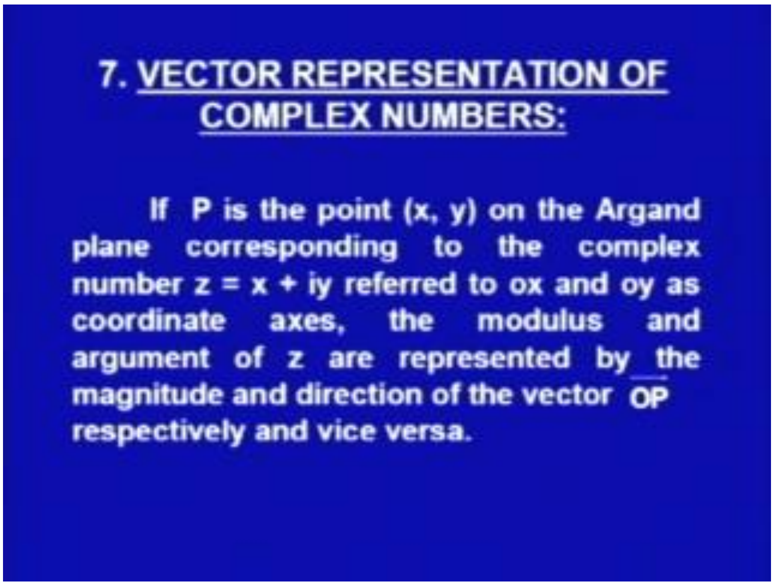

Then we can take the vector representation of the complex numbers, so if P is the point x y on the Argand plane and Argand diagram corresponding to the complex number z equal to x plus i y or x comma y referred to ox and o y as coordinate axes. The modulus and argument of z are represented by the magnitude and direction of the vector OP respectively and vice versa.

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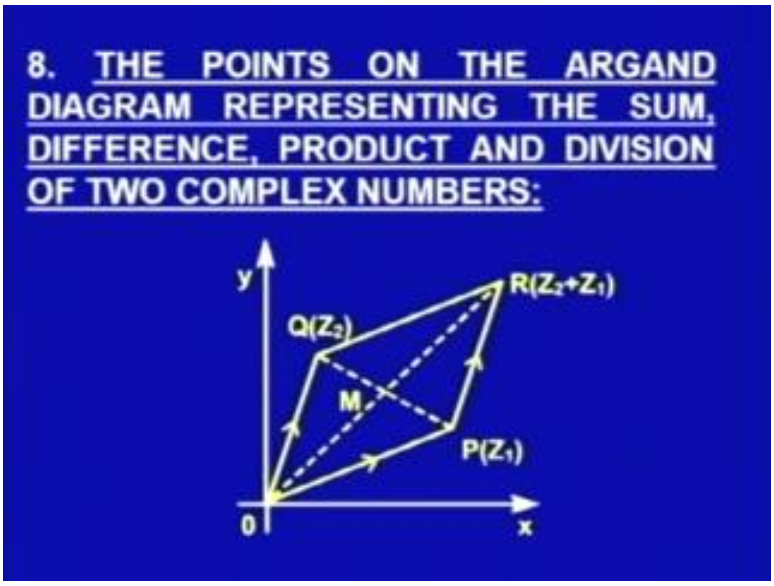

Next we take the points on the Argand diagram which will represent the sum difference product and division of any two complex numbers. So, in the Argand diagram if we take z 1, this is represented by the point P or we can say the vector OP and this point Q represents the complex number z 2, so that OQ represents the distance it is modulus z 2. So, if we complete the parallelogram OPRQ with OP and OQ as the two sides OR will be the diagonal. So, this point R will represent the sum of the complex numbers z 1 and z 2 that is z 1 plus z 2,

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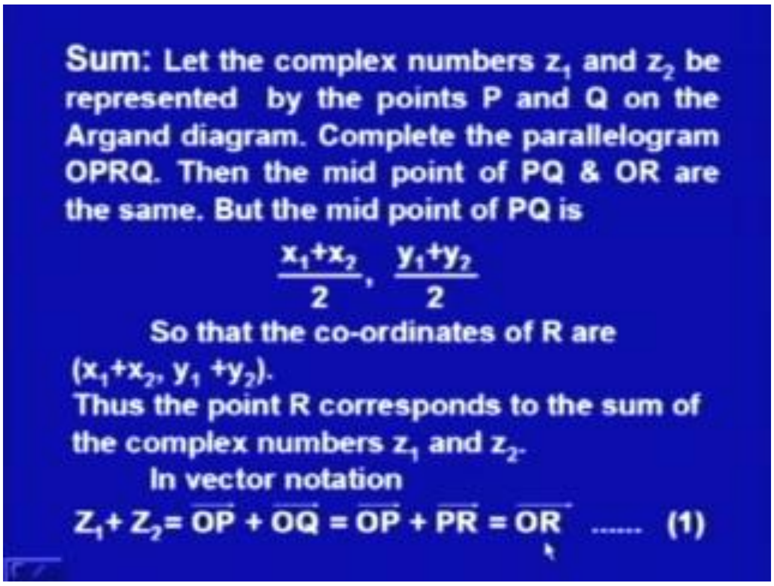

So, here as an explanation the complex numbers z 1 and z 2, they are represented by the points P and Q on the Argand diagram, as I explained the compete the parallelogram OPRQ. Then the midpoint of PQ and OR are the same, because if we refer to the diagram, then the midpoint M is the midpoint of so this M is the midpoint of PQ and OR, that is x 1 plus x 2 by 2 and y 1 plus y 2 by 2, so these are the coordinates of the midpoint M

The coordinates of the point R will be x, therefore because this is M is the midpoint and R is the point on the other accessibility, so the coordinates of the point R are x 1 plus x 2 and y 1 y 2. So, therefore, the point R corresponds to the sum of the complex numbers Z 1 and Z 2, because x 1 plus x 2 is the real part of the sum and y 1 plus y 2 is the imaginary part of the sum Z 1 and Z 2, so Z 1 plus Z 2 is represented by the point R in the figure.

So, in vector notation we can write this if you refer to the figure Z 1 plus Z 2 is equal to the vector OP plus vector OQ and this will be OP plus PR, because OQ vector is equal to PR and OP plus PR is O vector OR.

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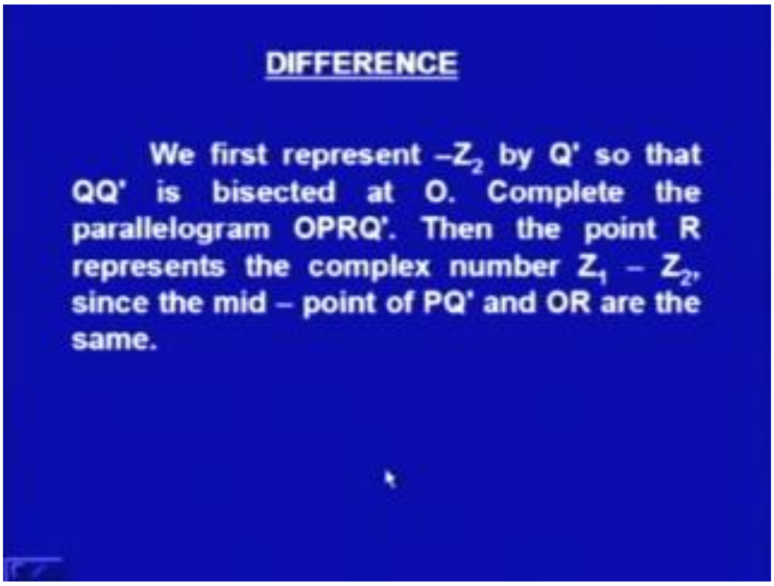

Now, we take the difference of any two complex numbers on the Argand diagram or geometrical representation we take, so we first represent minus Z 2 by Q dash on the other side of Z 2, so that QQ dash is bisected at the point O. We complete the parallelogram OPRQ dash, then the point R represents the complex number Z 1 minus Z 2, since the or as the midpoint PQ dash and OR are the same as the last time.

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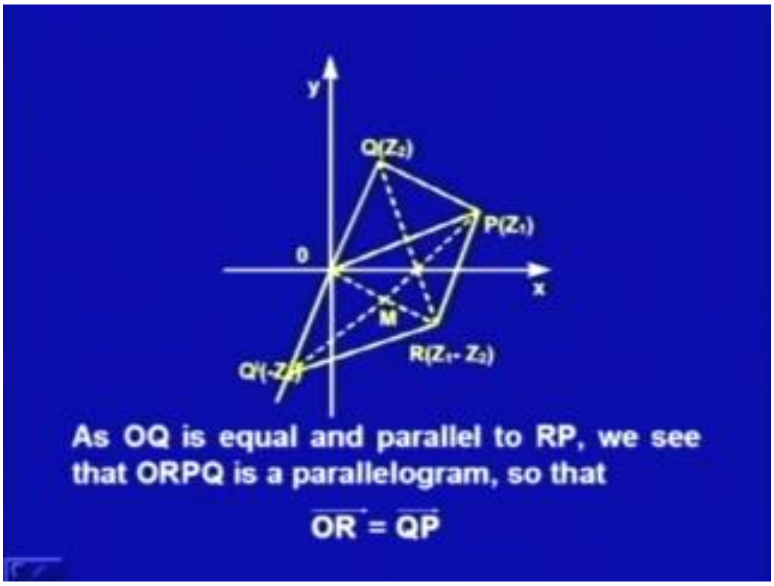

So, we can refer this figure, this Q is represented by the complex number Z 2 and this Q dash represents minus Z 2, so this point R, if you complete the parallelogram OPRQ dash then this point R represents the number Z 1 minus Z 2, the difference of the two. Where Z 1 is represented by the point P and Z 2 is represented by the point Q on the other diagram and the number Z 1 minus Z 2 will be represented by the point R when we complete the parallelogram. So, as explained earlier OQ is equal and parallel to RP we see that ORPQ is a parallelogram, so that OR vector is equal to vector QP.

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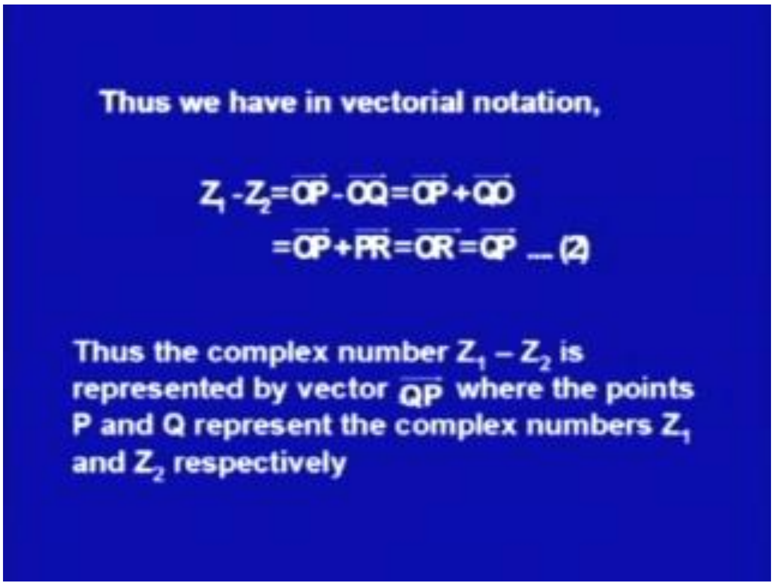

So, in terms of the vectorial notation Z 1 minus Z 2 will be vector OP minus vector OQ and that we can write again as vector OP and vector QO and this will be equal to vector OP plus vector PR and this will be vector OR that is QP vector. So, the difference of the complex numbers Z 1 minus Z 2 is represented by the vector QP where the points P and Q represent the complex numbers Z 1 and Z 2 respectively on the Argand diagram.

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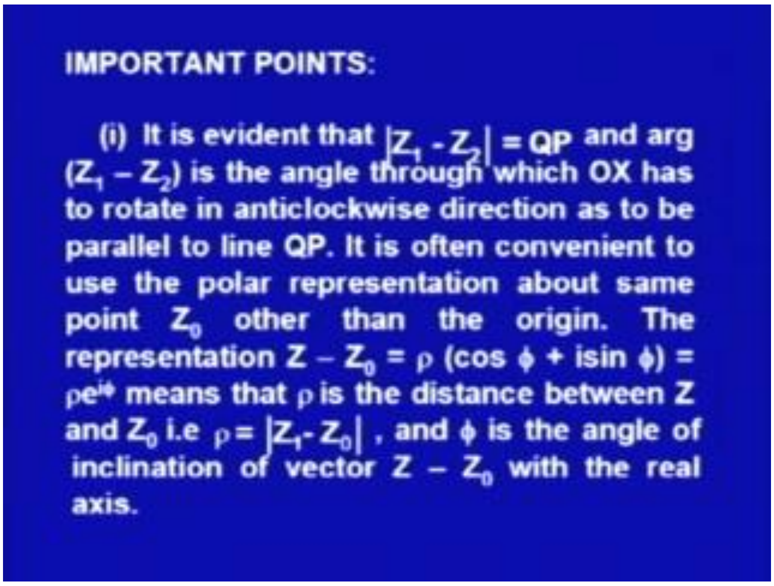

So, here there are important points, that to it is clear that Z 1 minus Z 2 modulus is QP and argument of this Z 1 minus Z 2 is the angle through which OX has to rotate in anticlockwise direction as to be parallel to line QP. It is often convenient to use the polar representation about the same point Z 0 other than the origin, the representation Z minus Z 0 equal to rho into cosine phi plus i sin phi or rho into e to the power i phi. Means that rho is the distance between Z and Z 0, that is rho is equal to modulus Z 1 minus Z 0 and phi is the angle of inclination of vector Z 1 minus Z 0 with the real axis.

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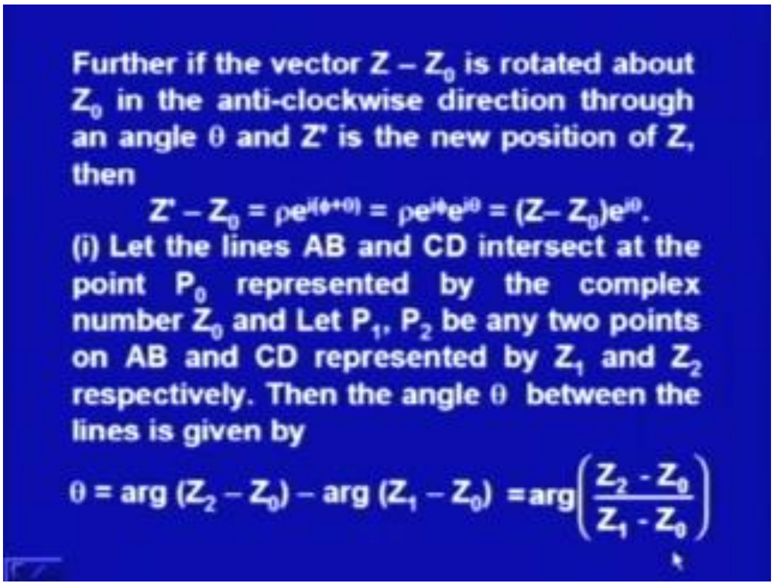

Further, if the vector Z minus Z 0 is rotated about Z 0 in the anticlockwise direction through an angle theta and Z dash is the new position of Z. Then Z dash minus Z 0 that will be equal to rho e to the power i phi plus theta, which we can write as a rho e to the power i phi into e to the power i theta and that will be Z minus Z 0 into e to the power i theta. If the line AB and CD intersect at the point Z 0 represented by the complex number Z 0 and if P 1 and P 2 be any two points on AB and CD represented by Z 1 and Z 2 respectively. Then the angle theta between the lines is given by argument Z 2 minus Z 0 minus argument Z 1 minus Z 0 equal to argument of Z 2 minus Z 0 by Z 1 minus Z 0

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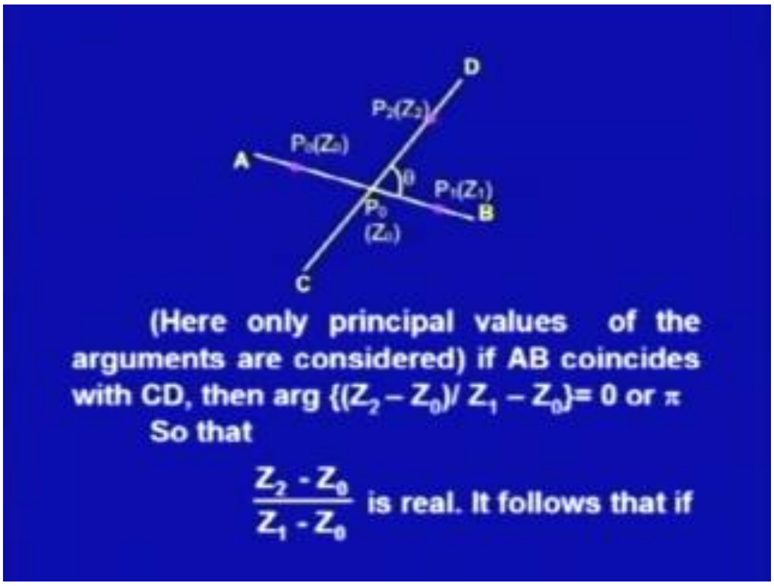

So, from this diagram we can have the angles between the points, so here we have taken only principle values of the arguments and if AB coincides with the CD, then argument of Z 2 minus Z 0 divided by Z 1 minus Z 0 will be 0, because the two lines will be coinciding with each other. So, it will be 0 or pi, so that Z 2 minus Z 0 by Z 1 minus Z 0 this complex number is will be real in that case.

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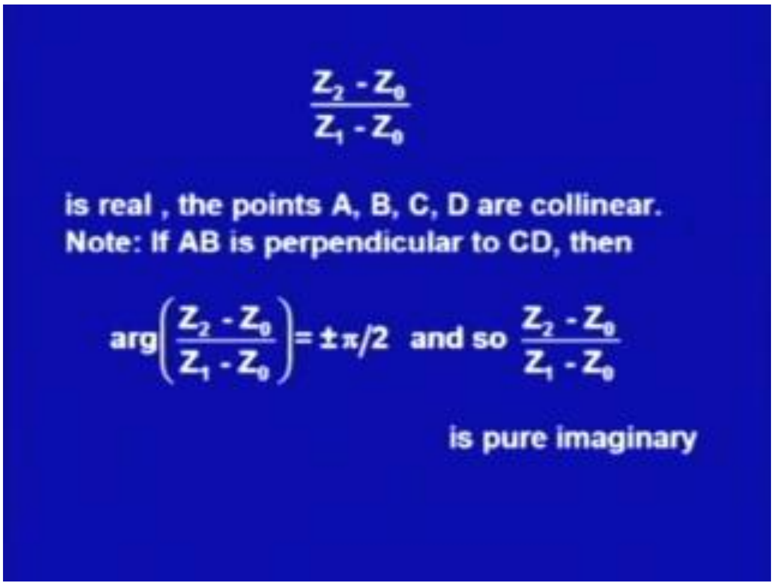

And it will follow that this is real the points A, B, C and D will be collinear that is they will lie in the same line. If the line AB is perpendicular to the line CD, then the angle or argument Z 2 minus Z 0 by Z 1 minus Z 0 will be plus minus pi by 2, and so the complex number Z 2 minus Z 0 by Z 1 minus Z 0 will be in that case purely imaginary. So, today we have defined the complex numbers, we have taken their properties fundamental operations, some definitions and their geometrical representations through graphs etcetera. And next time we shall continue with the properties of the complex numbers.

Thank you for your patient hearing.