Mathematics - II
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Lecture - 23
Rank of a Matrix

Welcome viewers, today we are going to discuss Rank of a Matrix. The preliminary concepts, which is required for rank of a matrix is the determinants and the sub matrices. So let us have an m by n matrix A.

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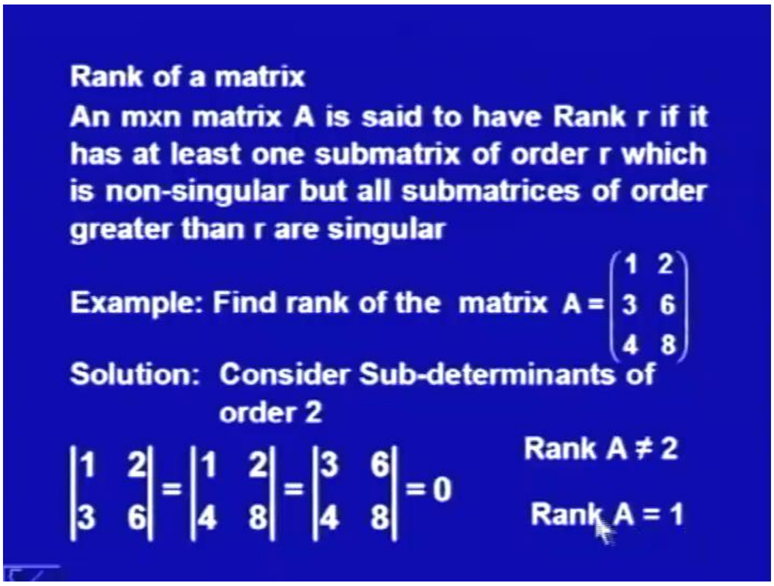

Then r will be it is rank, if this matrix has at least one sub matrix of order r, which is non singular. But, all sub matrices of order greater than r are singular. Now, illustrate this definition, I will take an example. I have a 3 by 2 matrix this. That means 3 rows and 2 columns. And rank of this matrix is to be obtained. Find out the rank, I will first consider all sub matrices of order 2. Matrices of order 3 are not possible for this matrix, so I will start with sub matrices of order 2.

This matrix will have 3 sub matrices of order 2. That is 1 2 3 6 that is the first determinant corresponding to this. The second determinant is corresponding to the sub matrix. Consisting of first row and the third row and the third determinant will be second row 3 6 and 4 8. If you compute the determinant of these matrices, one can notice that, this determinant is 0, 1 into 6 minus 2 into 3, 0. This determinant is also 0, 1 8 minus 8 is equal to 24 minus 24. So, determinant of all these sub matrices they are 0.

So, the determinant at this matrix cannot have rank 2. So let us see what is the rank of this matrix? So we will consider sub matrix of order 1. So, this matrix will have 6 sub matrices of order 1. That is the element itself 1 2 3 6 4 and 8 and their determinants are nonzero. So, we have a determinant of order 1, which is nonzero. And all determinants of order 2 bigger than 1, they are 0. So, we can say, rank of this matrix cannot be 2, but rank comes out to be 1. So, that is how we define, rank of a matrix. Now, one can absorb that, for a matrix which is of order m by n.

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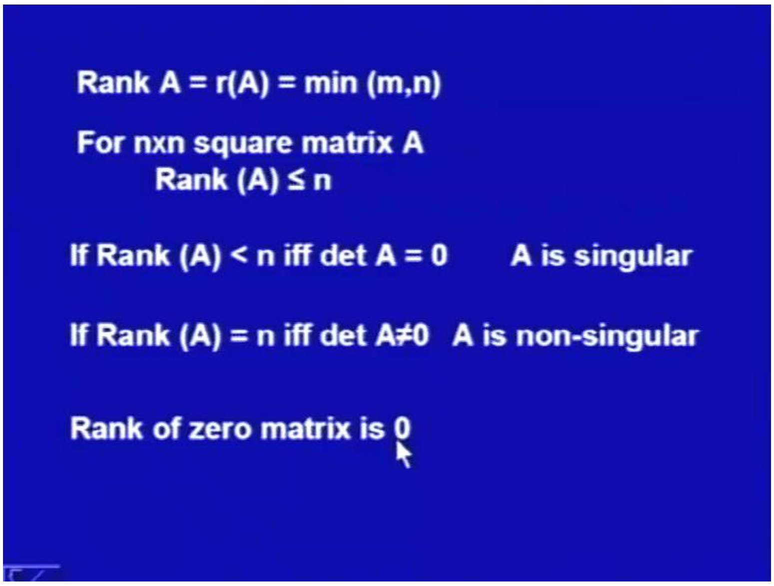

Then, rank A is minimum of m and n, because we have to obtain sub matrices, which are square matrices. So, the minimum of this will be the rank. For n by n square matrix A, rank A is less than equal to n. Rank cannot be bigger than that. So we say, for a square matrix rank A will be less than equal to n. If rank of A is less n that clearly means, the determinant A is 0. This is coming from the very definition of rank. And, that means A is singular.

On the other hand, if rank A is equal to n. Then determinant has to be nonzero. Again this is from the definition. And, this will be now become the definition for non-singular matrix A. So we say A is non-singular, if it is determinant is nonzero. Further, one can observe that, rank of zero matrix is 0, because in that case, all determinants of order 1, they will be 0. So, the rank of zero matrix is 0.

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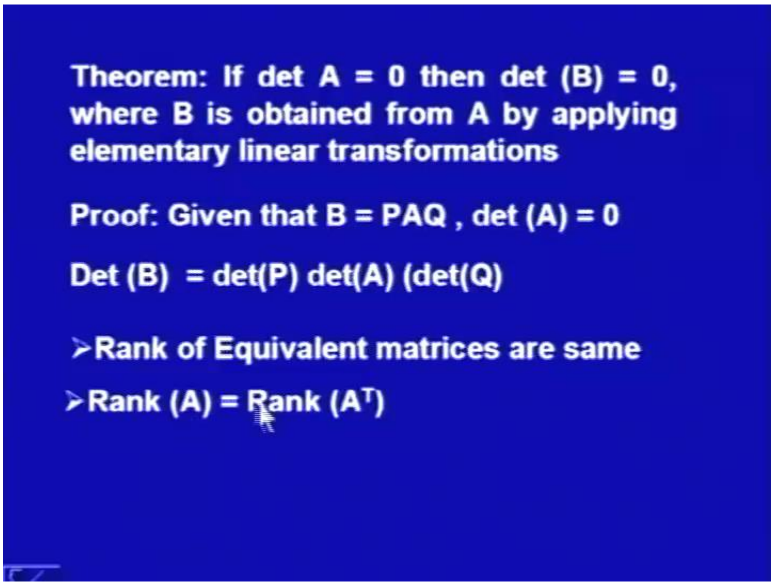

Now, we have a result which says that, if determinant A is equal to 0. Then, determinant B is 0. Then B is obtain from A, the elementary linear transformations. This result we have established in my earlier lectures. To prove this, let us say B is equivalent to A. Why, because B is obtained by applying elementary linear transformation to A. That means, there exist non-singular transformation P and Q. So, that B is equal to PAQ. Further, it has been given the determinant A is equal to 0.

We have to prove, that determinant B is also 0. So to do this, we can find out the determinant of this. So, determinant B on the left hand side is equal to, determinant of PAQ, which can be written as, determent of P into determinant of A and determinant of Q. But, from because they are obtain, they are the elementary linear transformation matrices, determinant P and determinant Q. So they cannot be their determinant cannot be 0. They have to be non-singular.

So, determinant P is not 0, determinant Q is not 0. And what are these determinates, they are simply numbers. So, product of three numbers is 0, if one of them has to be 0. So, determinant B is equal to determent P into determinant A determinant Q. Determinant Q is not 0, determinant P is not 0. So, we have only one alternative. That, is determinant A is equal... That is determinant B is equal to 0. So, product of this means, the determinant B is equal to 0, when determinant A is equal to 0.

So, whenever a matrix is obtained, whenever we apply the any elementary transformation. Then, the obtained matrix will be, will have determinant 0, if the original matrix also has a determinant 0. So, that is the result. And since, rank and determinants are related. So, we can say that, on the basis of this result that rank of equivalent matrices are same. So if, determinant A is equal to 0. Then after applying linear transformation, determinant B is also 0. If it is nonzero, then this also be nonzero. So, rank of equivalent matrices are same.

Further rank of A and rank of it is transpose, they are equal. Because, rows and columns can be, the determinant will not be affected by changing the transpose. So, determinant is not 0. Then, determinant of A transpose will also be not 0. And that is why rank of A is equal to rank of A transpose.

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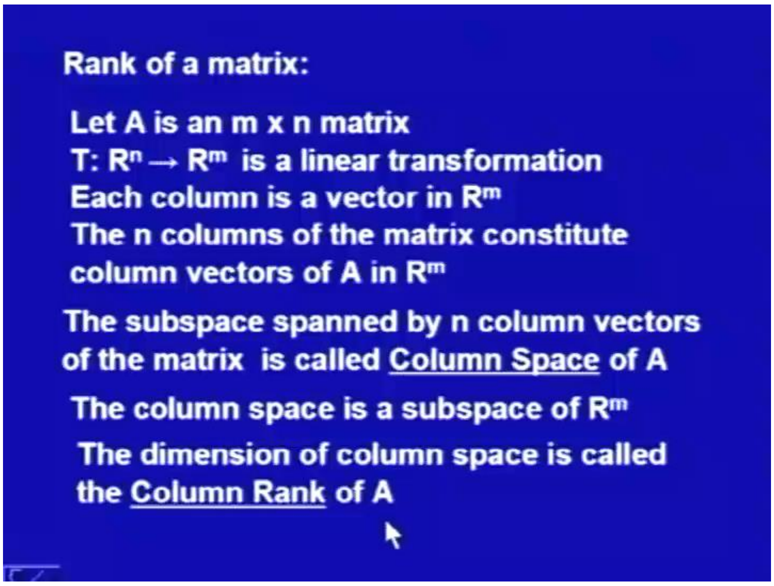

Now, I will try to define rank of a matrix in a slightly different manner. Now, this will do, in the same manner as we have done for linear transformations. So, I want to relay the concepts. So, let us have a matrix A, which is an m by n matrix. And let us now have a linear transformation T, which is from R n into R m. We say that, it is each the column is of vector in R m.

If this is the linear transformation, then each column in A is a vector in R m, then n columns of the matrix. Because, it is an m by n matrix, it has n columns. Then the n columns of the matrix constitute column vectors of A in R m. The idea is, this matrix being m by n matrix. It will have m column vectors in R m. Then, this subspace spanned by these n column vectors of the matrix is called column space of A. This we have proved that, if A represents the matrix of linear transformation. Then, the then that will be a vectors space. So we say we call that vectors space as a column space.

So, the subspace spanned by n column vectors of the matrix is called column space of A. The column space is a subspace of R m, this we have already established. The dimension of column space is called the column rank of A. So, the number of linearly independent column vectors is the column rank of A.

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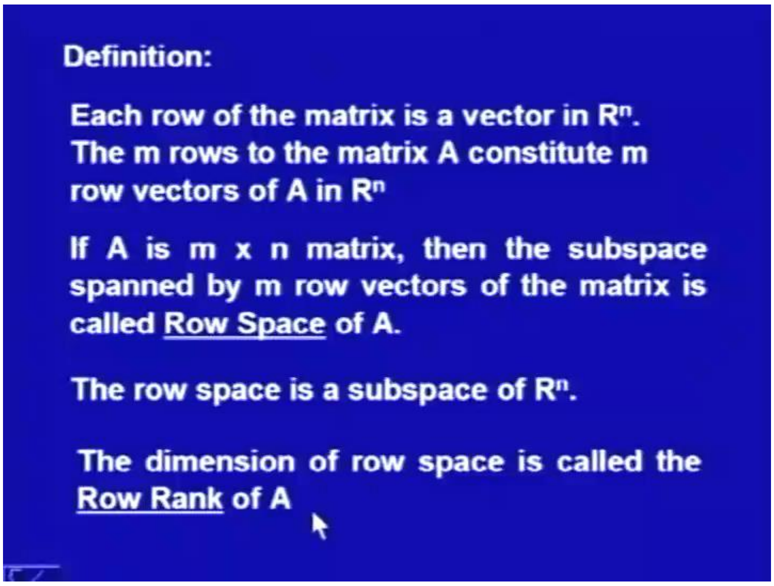

Similarly, we define row rank. So, each row of a matrix is a vector in R n for the given transformation. Then, the m rows of the matrix A which is m by n. Constitute m rows of vector A in R n. And that means, if A is m by n matrix. Then, the subspace spanned by these m row vectors of the matrix is called the row space of A. And the row space is a subspace of R n. This we have proved earlier. And finally, the dimension of row space is called the row rank of A.

So, we have a matrix A, column vectors constitute the column space. And it will have a row rank, the number of linearly independent vectors, linearly independent column vectors. And it will have a row space, generated by the row vectors of the matrix. And the row rank is the number linearly independent vectors. Linearly independent row vectors in B.

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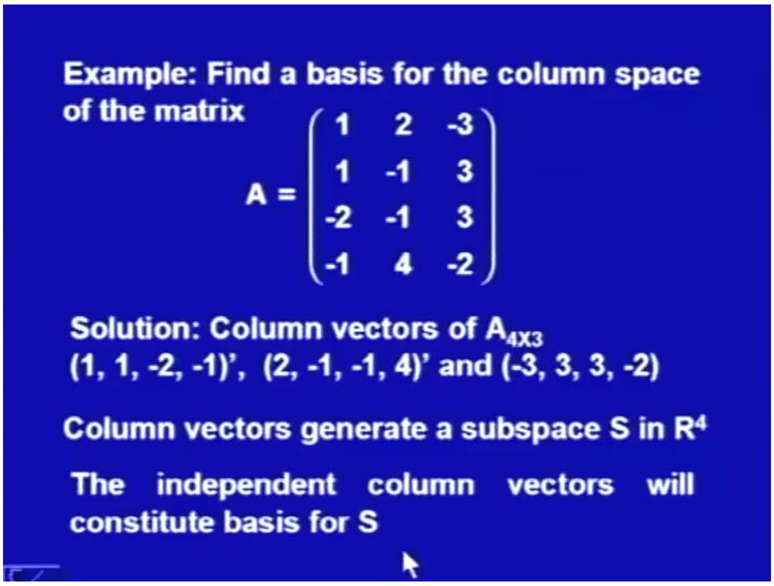

Now, on the basis of this, let us find the basis for column space of this matrix. So, I am just trying to illustrate what I have just described. So, we have a matrix. It has four rows and three columns. So, we have to find out the basis for the column space. Now, this has three column vectors. First vector is 1 1 minus 2 minus 1. Second one is 2 minus 1 minus 1, 4, minus 3, 3, 3 and minus 2. This will generate a vector space. So, what will be the, it is basis. So, let us find out the basis for this matrix.

So, we have these four column vectors. I am denoting these column vectors in this particular manner. This is just to save spaces. So, 1, 1, minus 2, minus 1, they are not written vertically. But, they are written horizontally. And this represent that, it is the transpose of this row vector. Transpose will make it a column vector. So, we have a three column vectors, as I have illustrated. Now, we have to find out, it is basis. So we say, column vectors generate a subspace S in R 4. This n is 4, so it will generate a subspace S in R 4. So, what we have to do is, we have to find out the independent column vectors that will constitute basis for S.

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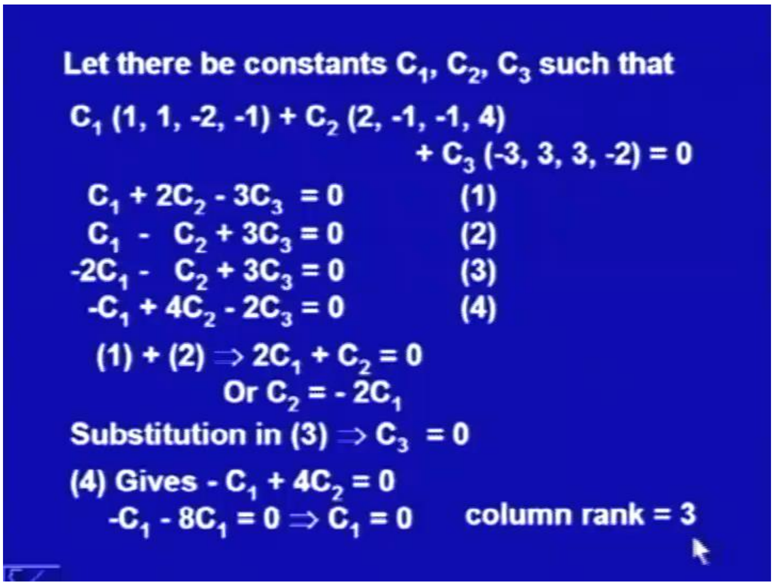

For this purpose, let us assume three constant, C 1, C 2, C 3 such that, C 1 first vector plus C 2 multiplied by second vector plus C 3 multiplied by third vector is 0. That means, the linear combination of these three vectors is 0. So, let us try to find out the solution of this equation. So, this equation actually gives us four equations. So, if we compare the first element from this. So, it is C 1, plus twice C 2, minus 3 times C 3. That is the first equation.

The second equation is C 1, minus C 2 and plus 3 C 3, that is the second equation. So, this may be form four equations. Now, these four equations have to be solved for C 1, C 2 and C 3. So, if we add first two equations 1 and 2. Then this will be simply, 2 C 1 plus C 2, this will cancel out, is equal to 0. And this means C 2 is equal to minus 2 times C 1. Similarly, when we substitute this solution in 3 this equation 3, will have C 2 is equal to minus 2 C 1. So, this will get cancelled with this, what we have is simply, C 3 is equal to 0.

So, this is what we obtained from the third equation. And 4, equation 4 gives, if you substitute in 4 C 3 is equal to 0. This will give me minus C 1 plus 4 C 2 is equal to 0. And if I substitute this here, this minus C 1, minus 8 C 1 is equal to 0. That gives me C 1 is equal to 0. So, C 1 is equal to 0, C 2 is equal to 0, C 3equal to 0. So, this system will have a 0 solution.

So, what we have obtained is we have a linear combination equal to 0. And this is possible only when C 1, C 2 and C 3 are constants. And that means, these three vectors are linearly independent. That means, they will have the column rank as 3. So, the given matrix has column rank 3.

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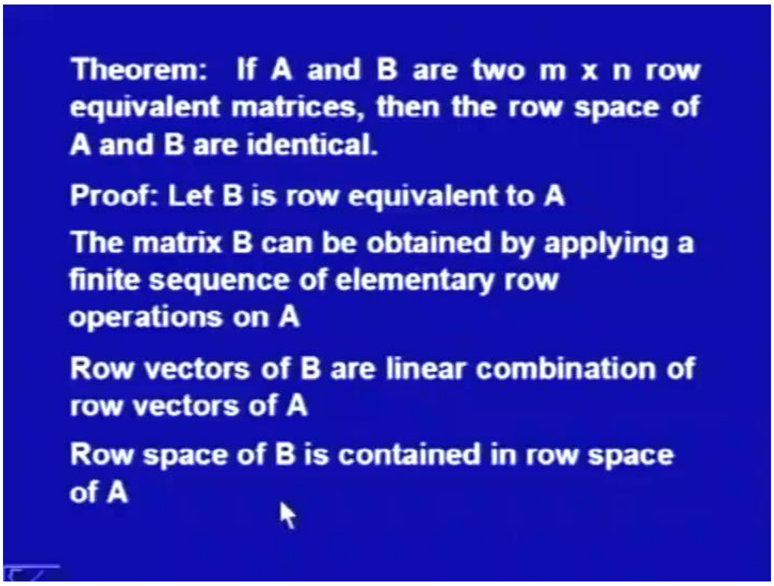

Then the next results relates, the matrices A and B, which are row equivalent matrices. Then, the row space of A and B are identical. That is, if we have two matrices A and B, which are of same order. And they are row equivalent matrices then the row space they are identical. Because they are, let us try to prove this. Let B is row equivalent to A, that is been given to us. The matrix B can be obtained by applying a finite sequence of elementary row operations on A. That is the meaning of A equivalent to B or A and B are two equivalent matrices.

So, the matrix B can be obtained by applying a finite sequence of elementary row operations on A. Then, row vectors of B are linear combination of row vectors of A. Then, row space of B is contained in row space of A. Now, if they are linear combinations. That means, the determinants will be, if one determinant of B is equal to 0 of a sub matrix of B having determinant 0. Then the same will happen to the other matrix also. And that is why, their ranks will be same.

As, ranks will be same, that means, their row spaces will be identical. So, row vectors of B are linear combination of row vectors of A. And row space of B is contained in row space of A.

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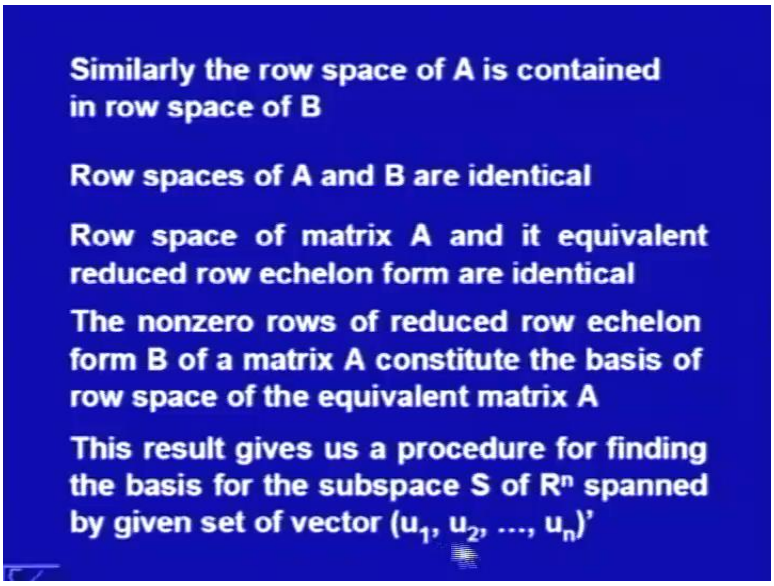

Now, we say that, row space of A is contained in row space of B. Same way we can prove this, the way we have done for the other part. And since, A is contained in B and B is contained in A. So, row space of A and B are identical. So, what is the result? Row space of matrix A and it is equivalent reduced row echelon form are identical. So, I have started with any matrix B. But, then we can apply it for equivalent reduced row echelon form.

And this result is important, because with the help of this result, we can solve system of equations. And we can easily find out the rank of equivalent reduced row echelon matrix. So, to find out the rank of a matrix A. We can find out the rank of a equivalent row reduced echelon form. And we can say that, whatever be the rank of equivalent reduced row echelon form. The same rank will be there for the matrix A. And finding rank of this form will be simpler than finding rank of this matrix.

And that is why, we first we try to reduce matrix A into this form. Find out the rank of this and then we say, the same will be the rank of A. So, this become, this gives us a procedure for finding row space of a matrix A. Now, it says that, the nonzero rows of reduced row echelon form B of a matrix A. Constitute the basis of a row space of the equivalent matrix A. In the echelon form, we will see how many rows are nonzero. And that will be and the nonzero rows will actually constitute the rank of the matrix. That will form a basis and that will give us the rank of the matrix. So, this as I told you, this result gives us a procedure for finding the basis for the subspace S of R n, spanned by the given set of vectors u 1, u 2, u n.

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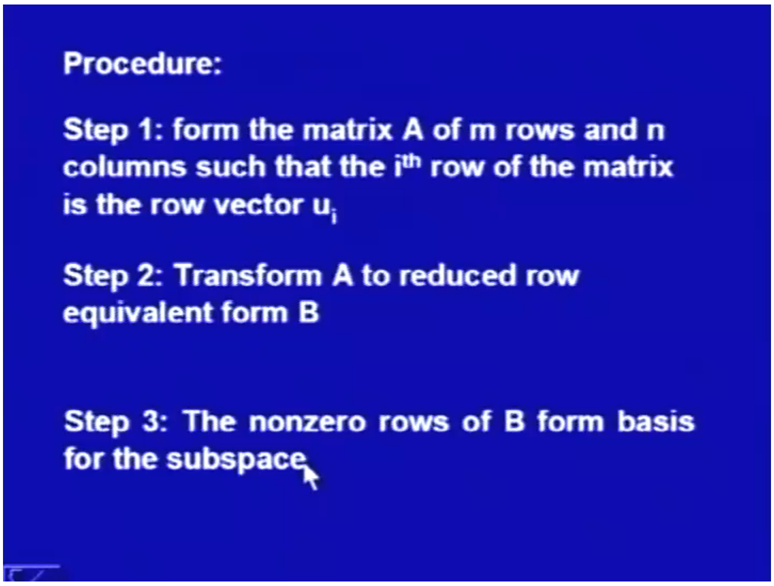

Now, the procedure is, step 1 is form the matrix A of given m rows and n columns. Such that, the ith row of the matrix is the row vector u i, that is the first step. The second step is, transform A to reduced row equivalent form B. Once we have done this, the third step is, the nonzero rows of B form basis for the subspace. So, I have first established this procedure and now after setting this procedure.

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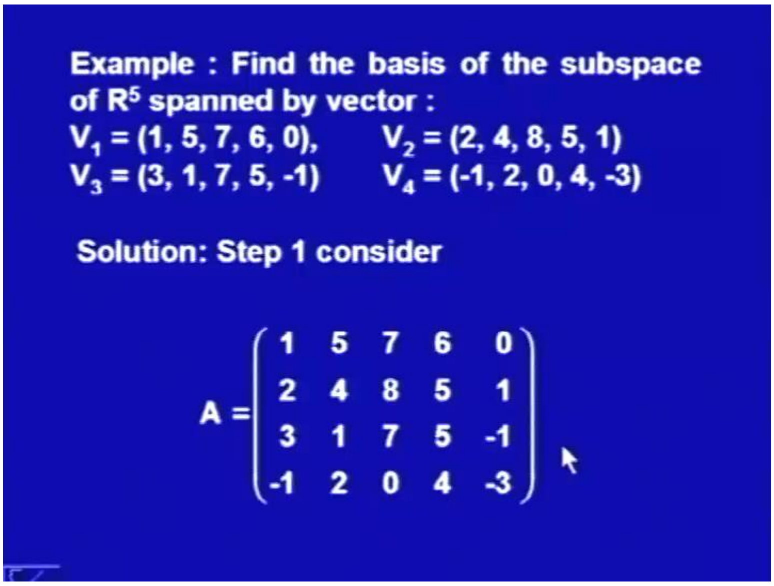

I will illustrate this, with the help of an example, so we have been given four vectors in R 5. We have to find the basis of subspace of R 5 spanned by this, these set of vectors. So, V 1 is 1, 5, 7, 6, 0, V 2, 2, 4, 8, 5, 1, V 3 is 3, 1, 7, 5, minus 1 and while V 4 is minus 1, 2, 0, 4 and minus 3. So, according to the first step, the first vector is written as the row vector of the matrix A. The second vector is written as the second row of the matrix. Third vector forms the third row of the matrix. And fourth vector forms the fourth row of the matrix.

That means, from these given vectors, I will form the matrix A, where these vectors are the, where these vectors are it is rows. Once we have obtained this matrix A. We tried we apply linearly elementary transformation, so that this matrix will be reduced to, reduced row echelon form. And then we will see, how to form it is determinant. So, first we apply linear transformations to reduce is it into, reduced row echelon form.

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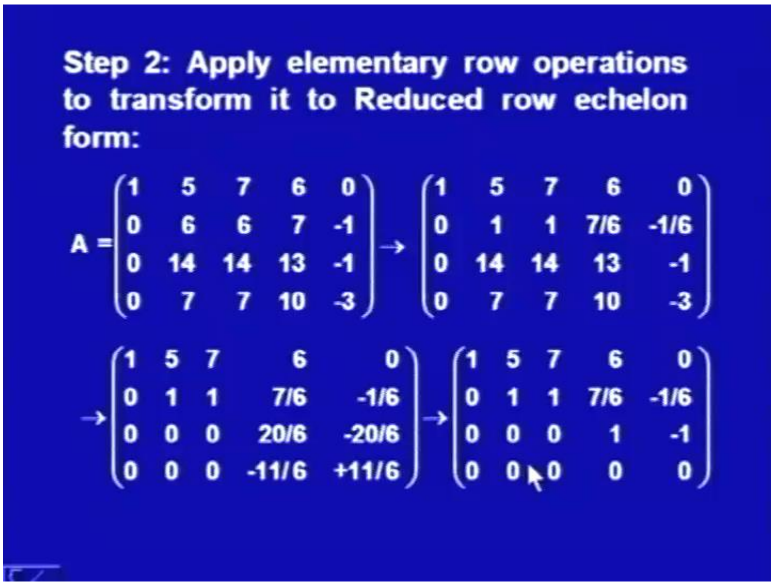

So, I will give you a sequence of steps, to reduce the given matrix A into row echelon form. So, I have the first row, written as it is. And I will first try to make, the first columns 0. So, when I apply linear transformation, then the matrix A will reduce to this form. And from this, we apply a more linear transformations, so that this will further reduced to this form. And then from here will come to this form, again apply a linear transformation, suitable linear transformation, then we will have this matrix. Now, you can notice that this matrix has these the bottom row as 0.

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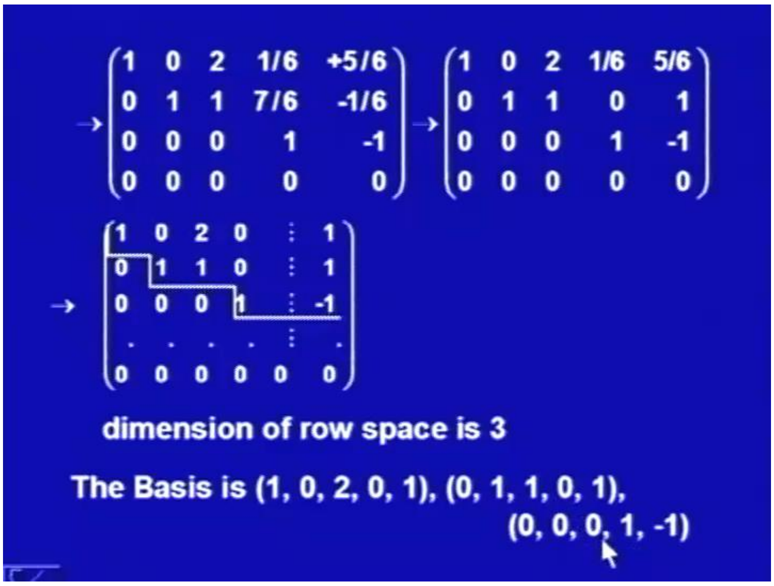

We further, apply a linear transformation to check, whether this row or any other row can become 0. So, we apply these transformations, so this matrix reduces to this. Now, this is the final form. If I try to apply more transformation, then either this will be 0 or this will be 0. So, I can stop here, so we can divide this matrix into four different parts. One part is this, another part is this, third part is this, all 0s and we have a 0. Now, in this we can say that, this is the echelon form, will have 1 here, 1 here and this is 1.

So, we have, this is 0 and this is echelon form and from here. One can notice that, there are three rows, which are nonzero. So, the dimension of row space is 3. So, we have obtained that, the dimension of this matrix, it is row space is 3, not 4. But, what are the basis vectors, that is to be obtained. Now, the first vector for, this is the first vector. It has 1 here, so first vector is the 1, 0, 2, 0, 1, 1, 0, 2, 0, 1. This is the element of the basis vector of given row space. Then this vector 0, 1, 1, 0, 1, this also a basis, this constitute the basis and the third vector will be 0, 0, 0, 1 and minus 1. So, the given matrix A, we generate a row space and the basis for the row space is determined as this. So, these are three vectors, which are linearly independent and they will generate the row space.

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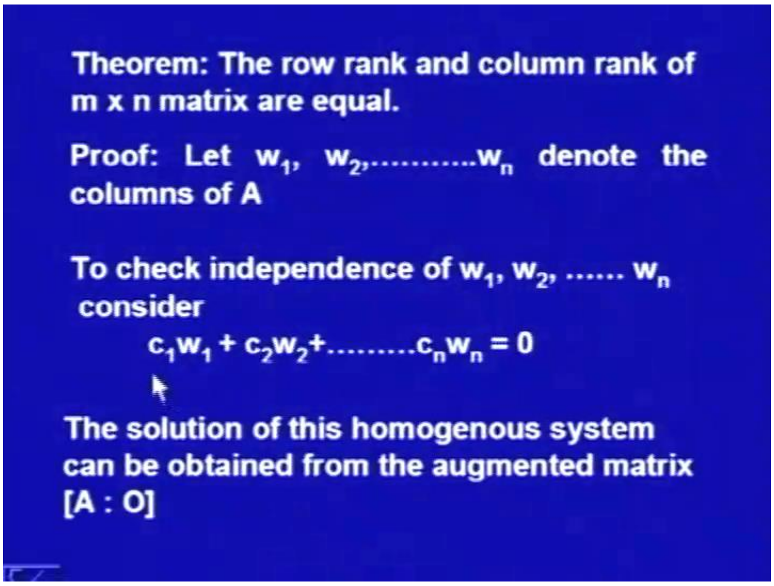

Now, this theorem says, that the row rank and column rank of m by n matrices are identical. Now, to prove this, let us consider w 1, w 2, w n denote the columns of A, to check the independence of these vectors. We consider the linear combination of these vectors C 1 W 1 plus C 2 W 2 plus C n W n is equal to 0. Now, the solution of this homogeneous system, it is a homogeneous system and this can be obtained from the augmented matrix A 0.

This method we have already discussed, so I will just apply this method to solve this system. Because, to establish the linear independence of these vectors, I have to find out the constant C 1, C 2, C n if all these concepts come out be 0. Then the vector, these vectors W 1, W 2, W n will be linearly independent. Otherwise, they will be dependent. So, I have to solve this and this is the method which I applied, so let us consider.

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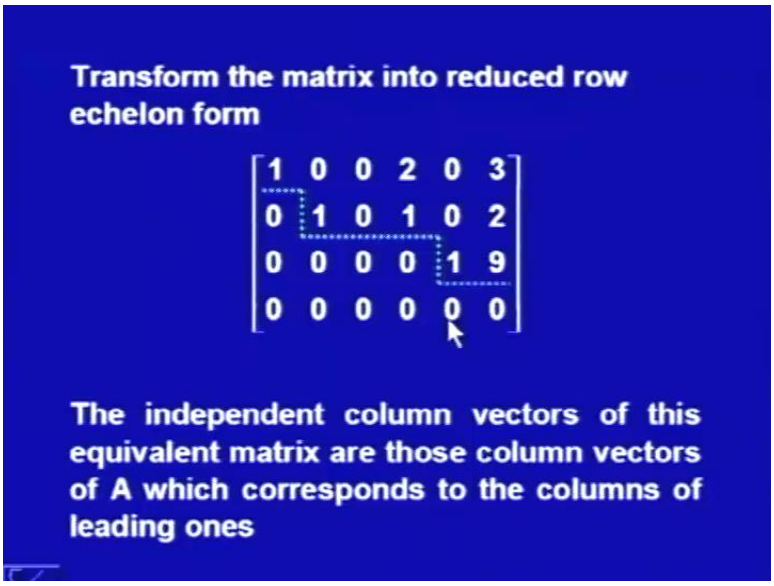

Now, we try to transform the matrix into reduced row echelon form. Before, I proceed further, I like to introduce the meaning of reduced row echelon form. Although we have done it earlier, but for the sake of the viewers, I will just define, what do we mean by echelon form. So, this particular matrix is in reduced row echelon form. Because, the once they appear in this matrix. They, actually form an echelon, a staircase pattern is obtained that is why we call it, an echelon form.

We call it reduced row echelon form, because, every row will have a leading 1. This, row has a leading 1, this row has a leading 1. And the third row has leading 1. The leading 1 means, if this particular row is having leading 1, then all other elements in this column will be 0. So, this is leading 1, all other elements are 0. So, in this case, this is 0 and we have this. So, we try to reduce the given matrix in this particular form. Now, this matrix has some number of rows as 0. And rest of the rows will have this pattern.

So, now, the elements in this matrix will have this staircase pattern. Once, we have obtained a row echelon form. We try to find out the independent column vectors of this equivalent matrix. These are those column vectors of A, which corresponds to the columns of leading once that means, if this is the echelon form. Then this column corresponding to this leading 1, row and this column in which, we have this leading 1 and this column. So this vector, this vector and this vector, they are linearly independent vectors.

All, this vector, this vector and this vector, they are linear combinations. So, we have only this vector, this vector and this vector, which are linearly independent. So, we have three rows three leading once. So, three vectors will form the set of linearly independent vectors. And that is why we say, row rank and column rank are equal.

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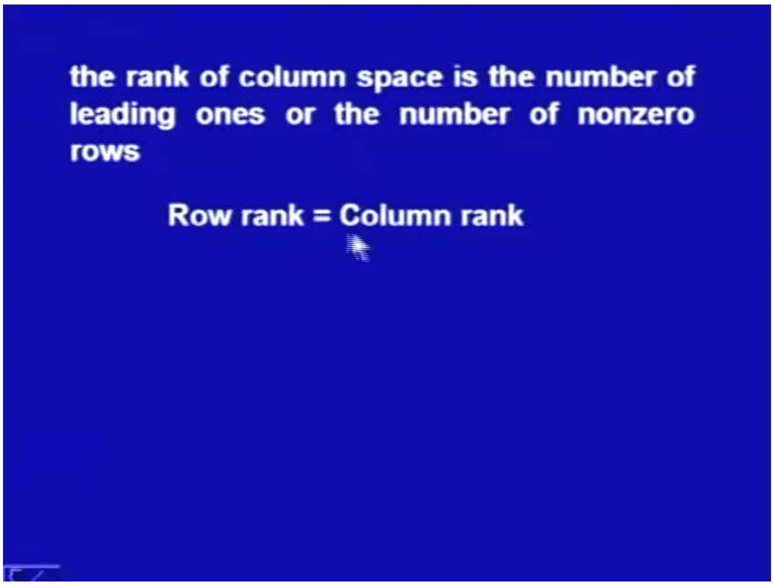

So, the rank of column space is the number of leading ones or the number of nonzero rows. And this proves that row rank is equal to column rank.

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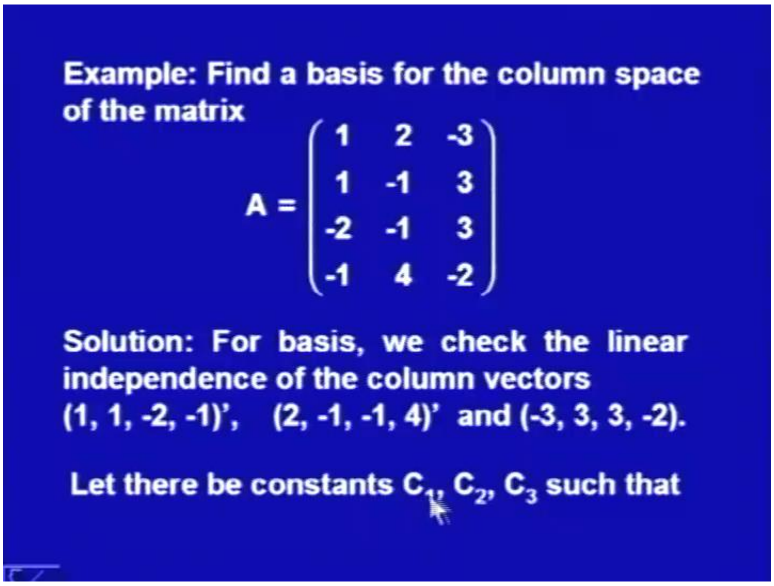

Now, in this example, we try to find a basis for the column space of the matrices, in earlier example. We have obtained basis for the row space. Now, we try to find the basis for the column space of the given matrix A as this. Now, given this matrix A, the column vectors 1, 1, minus 2, 1, 2, minus 1, minus 1, 4 minus, 3, 3, 3 and minus 2. They are arranged, they are considered and let we have C 1, C 2, C 3 arbitrary constants.

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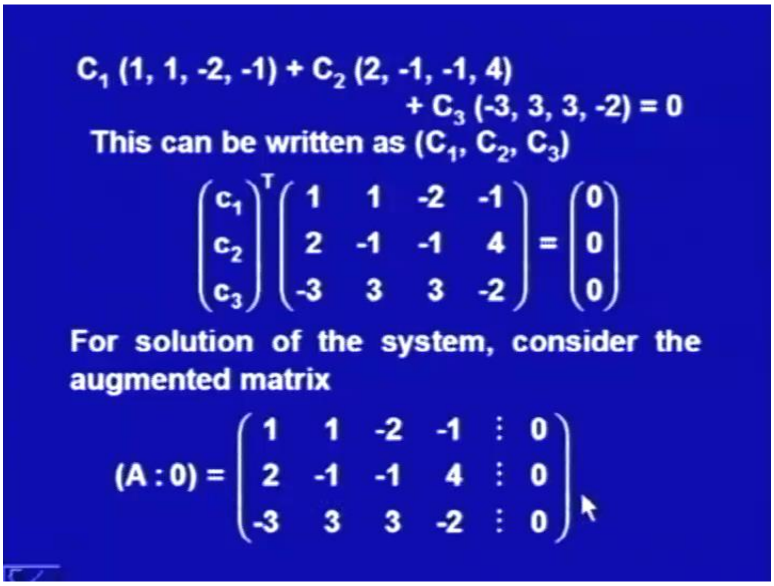

This linear combination, C 1 first vector, plus C 2 second vector, plus C 3 third vector is equal to 0. Now, this system can be written in the form, as this matrix and from here. We can obtain the augmented matrix as this. Now, we try to reduce this augmented matrix into row reduced echelon form.

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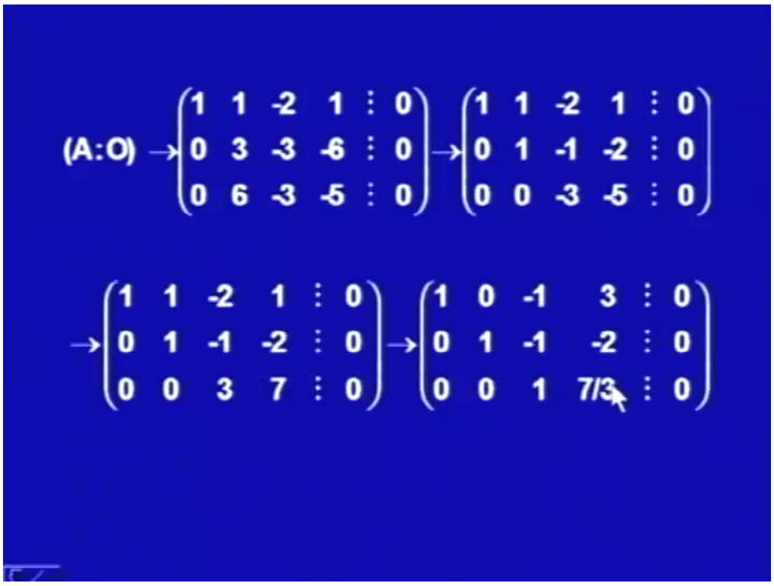

So, we start with A 0, it is this given matrix. We apply linear transformations, first we try to make this element. Simplify, this is 0, so we have this reduced form. Further, when we apply a more linear transformations then from here. We can go to this matrix from here we can go to this matrix.

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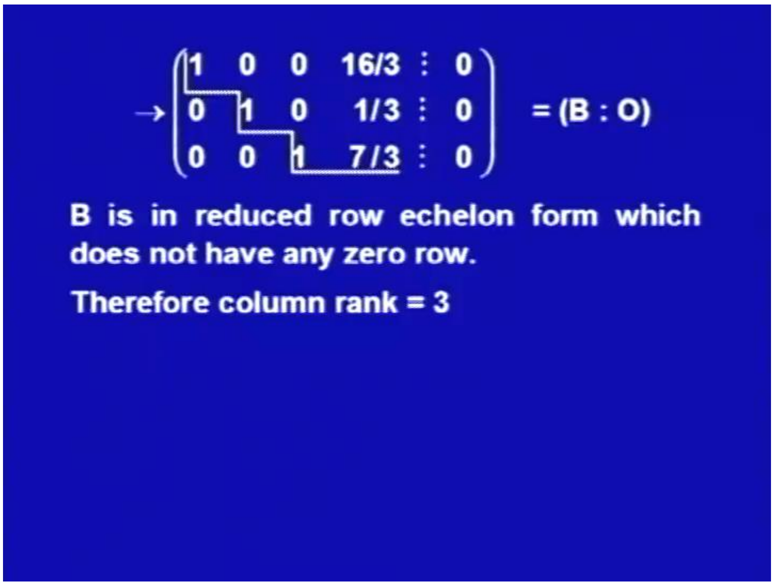

And finally will have this matrix, so we have now B matrix and this B matrix consisting of these identity matrix and these elements. So, we have now row reduced form, as this consisting of three rows and these are the column vectors. And these are the leading rows with leading elements. So, this vector, this vector and this vector, they constitute the basis. So, B is reduced row echelon form, which does not have any 0 row and that means the column rank is 3.

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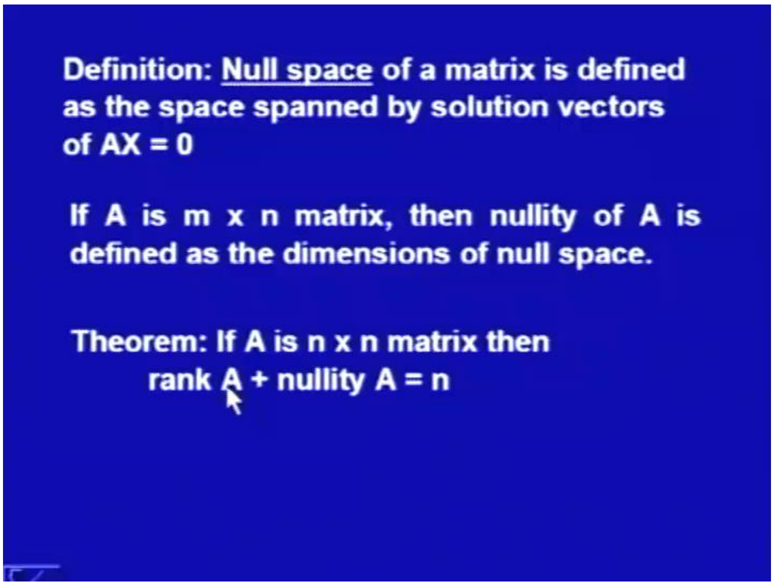

Now, we discuss null space of a matrix. So, to first we will define what do we mean by, null space of a matrix. It is defined as the space spanned by the solution vectors of the system A X is equal to 0. So, if A is an m by n matrix, then nullity of A is defined as the dimensions of null space. Now, on the basis of this, we can say that, if A is an m by n matrix. Then rank A plus nullity A is equal to n. Actually this result, we have started, we have already established, when we have a linear transformation T.

But, now we have, since we have already established the association between the linear transformation T and the matrix A. We have similar, result for the matrix A, but only difference is that A is an n by m matrix. So, if you have an m by n matrix. Then rank A plus nullity A is equal to n.

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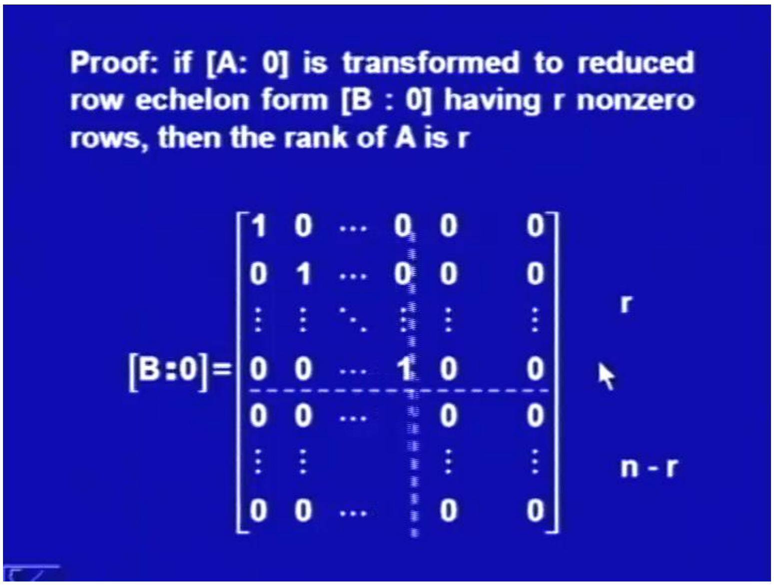

Now, to prove this, we start with the augmented matrix A 0. And we transform it to reduced row echelon form. B 0 having r nonzero rows then the, the definition of the rank, we say the rank the matrix will be having rank r. So, we have a matrix A. When, it is transformed to echelon form consisting of r rows then it will look like as this matrix. So, transform matrix B will look like as this.

One can notice that, this matrix has four parts. We have r rows, here which are nonzero. We have n minus r rows, which are 0. Then these are the two parts and other two parts are this. So, this is the identity matrix here, we have null matrix here and null matrices here. So, given the matrix A is transformed to this matrix B. So, this is identity matrix having of order r. So, we say that this is how we have divided.

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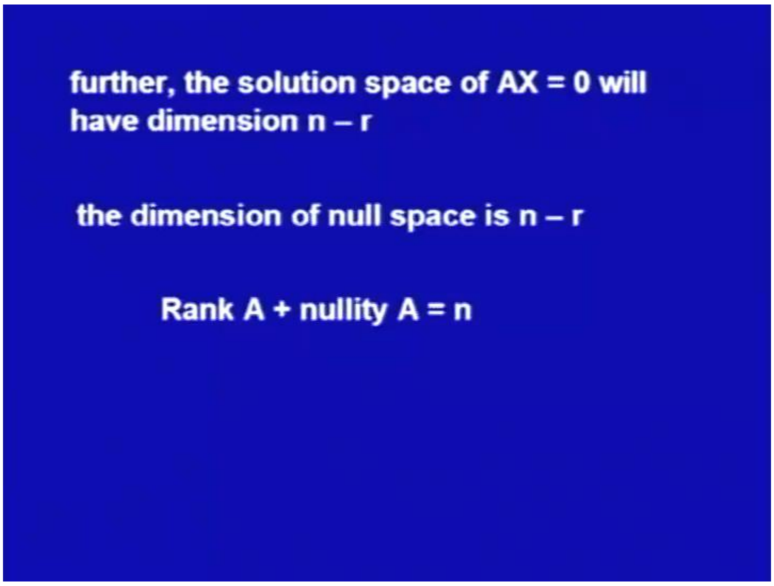

Then, we say that the solution space of A X is equal to 0, will have dimension n minus r. Because, the vectors will be represented, in terms of these n minus r vectors. So, the dimension of null spaces n minus r. And then rank A plus nullity A is equal to n, rank A is r, nullity A is n minus r. So, this is equal to n, so we have define rank of matrix A.

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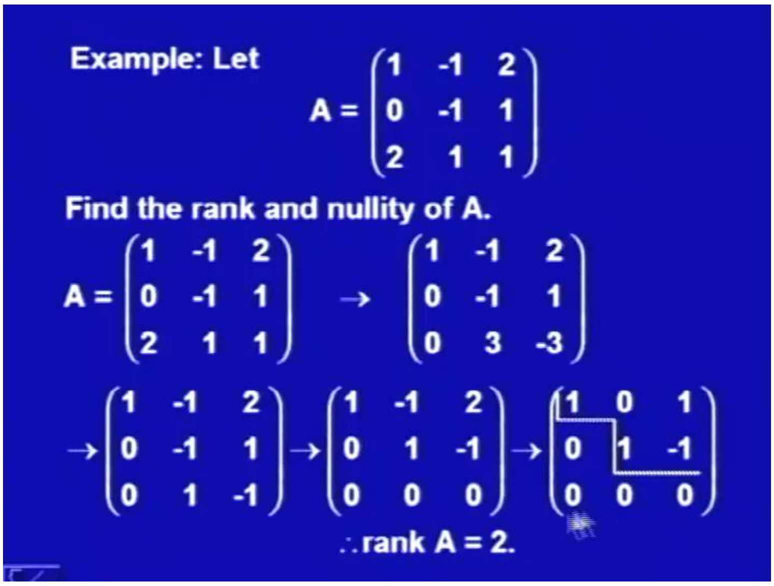

The nullity of A and then we have established rank A plus nullity A is equal to n. Now, we illustrate these concepts with the help of an example. So, let us have a 3 by 3 matrix A as this matrix. Then, we have to find the rank and nullity of this matrix. So, I will start with the matrix A and try to reduce it into row echelon form. So, it is 1, minus 1, 2, 0, minus 1, 1 2, 1 and 1, I try to make this 0. So, this matrix is transformed to this and by applying suitable linear transformations.

Then, elementary transformation, when applied to this will reduce this matrix to this. And further, this will be reduced to this and from here, will go to this matrix. Now, one can notice that, this is reduced to row echelon form, like this, this has leading row. This is another leading row and this third row is 0. So, we have echelon form, this is i r that is i 2, this case and this is 1 minus 1, they are nonzero here. And then we have a 0 row and that means the rank of A is equal to 2. That means, what we have done is, this we can very easily see that, we have a second order determinant, which is nonzero. And third order determinant will be 0, because of this 0 row. So, rank of this matrix is 2, now.

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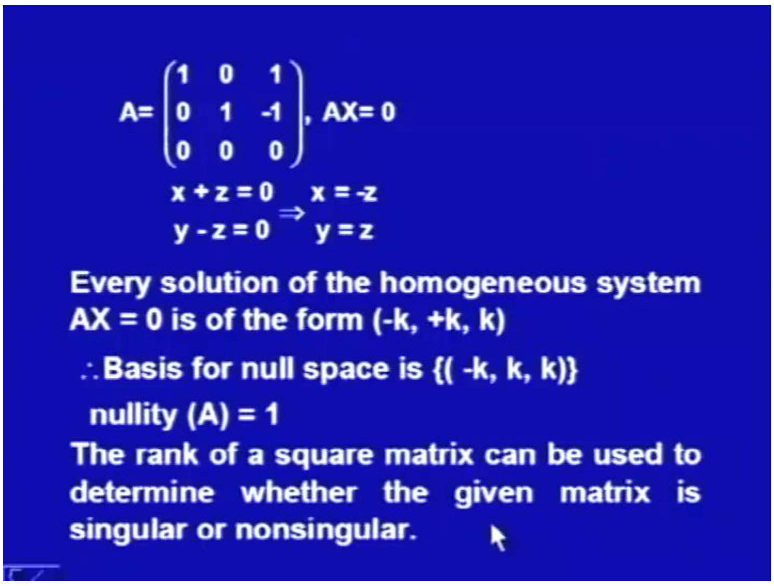

To find the nullity, we have to solve this A X is equal to 0. So, what are the solutions for X is equal to 0. So, what we can say is, for this equation will not contribute anything, because this is identically satisfied. So, this row will not contribute towards the solutions. So, we will have only this and this row. So, will have x plus z is equal to 0. And y minus z is equal to 0. And this gives me, x is equal to minus z and y is equal to z, where x y z, these are the elements of this column vector X.

So, X is equal to minus z and y is equal to z that means, we can write down the first vector in the first component in terms of third. And second component in terms of third.

And we can assign any value to the third component z. So, every solution of the homogeneous system A X is equal to 0 is of this form. So, you change the value k and you will be having this k and minus k. The second component and the first component so we say the solution of this equation will be of this form. So, this will be the null space.

So, the basis for null space is this. Every vector which is a solution of this will be generated by this vector. So, this forms a basis for null space. And since, this is constitute of one vector, that is why we say nullity is 1. And for this example, we have rank is equal to 2, nullity is equal to 1 and the order of the matrix is m by n or 3 by 3. So, 2 plus 1 is equal to 3. Now, the rank of a square matrix can be used to determine, whether the given matrix is singular or non-singular. If nullity is equal to, if suppose, if some example nullity of A comes out to be 0. The rank will be equal to the size of this matrix A. That so we can use this result to know the whether a given matrix is a singular matrix or a non-singular matrix.

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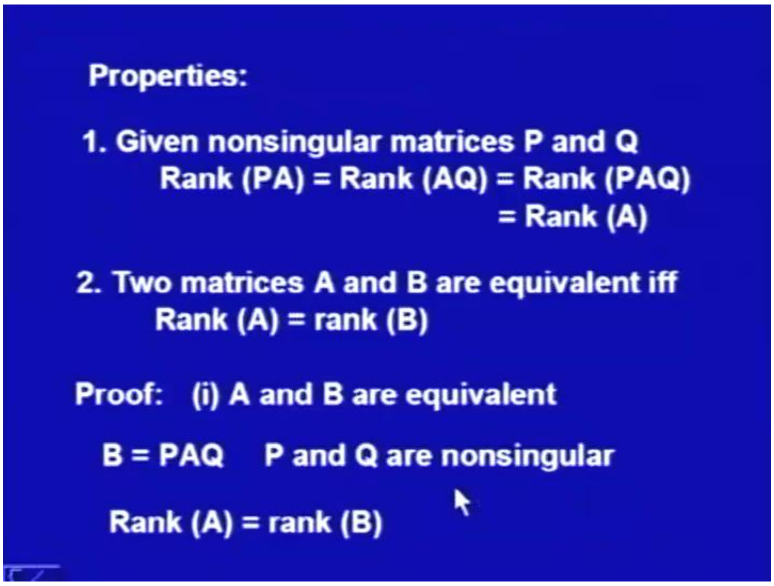

Now, we will establish certain properties of rank. So, first property is, if a non-singular matrix. Matrices, P and Q are given, then rank of P A is equal to rank of A Q and is equal to rank of PAQ is equal to rank of A. So, rank this matrix A being given to us, P is non-singular. So, P into A, will have the same rank as rank of A. Similarly, Q for the given matrix Q, rank of A Q will be the same as rank of A. Also, rank of PAQ is equal to rank of A.

Further, two matrices A and B are equivalent. If and only if, rank A is equal to rank B. So, proof is simple. We first assume that A and B are equivalent. And that means, there exist non similar transformations P and Q. So, that we can write down B as PAQ, that means, there are some elementary transformations, which can be applied to A, that will give me B. And similarly, we have some linear transformations, set of linear transformation. So, that we can have B is equal to PAQ. So, the two matrices are equivalent. If B is equal to PAQ, P and Q are non-singular. Then, rank of A is equal to rank of B. Just, because P and Q are non-singular.

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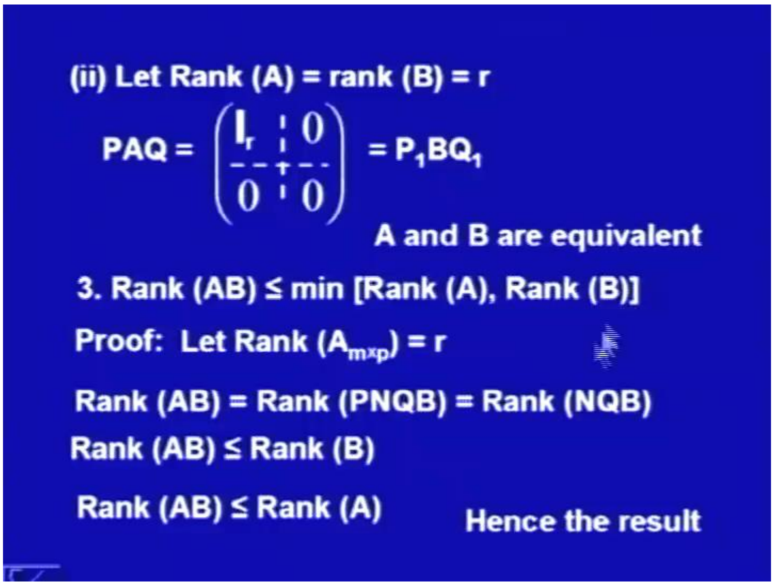

Otherwise, if rank of A is equal to rank of B is equal r. Then, rank of PAQ, the two matrices will be similar. So, let us try to do this, let us say, rank of A is equal to rank B is equal to r. That means, you can apply linear transformations to the matrix A. So that, it can be reduced to row echelon form and since it is rank is equal to r. This matrix will be obtained as this and rank B is also r. So, this can also be reduced by series of linear transformations. So, B will be equivalent to this row reduced echelon form.

So, when you apply linear transformations to B, it will be reduced to P 1 B Q 1 form. And when you apply the linear transformation to A, it will be reduced to PAQ. And since, both of them are equal. So, rank is r, so if we can say PAQ is equal to P 1 B Q 1. So, A and B are equivalent. Now, the next property says that, rank of A B, the product of two matrices is less than equal to rank. Minimum of rank of A and rank of B, this is an important property. So, to prove this, let rank of A, which is an m by p matrix is r.

Then, rank of A B is equal to, rank of A matrix is written as PNQ, where N is a echelon form. So, after applying linear transformation, A can be reduced to PNQ. So, A can be written as PNQ. So, I am substituting PNQ for A. So, rank of PNQ into B, is equal to rank of NQB. Because, P is a non-singular transformation, PNQ are non-singular transformation. So, that will not affect, so that is why rank of A B is equal to rank of NQB and rank NQB.

Now, rank of A B is less than equal to rank of B and from here. Rank of A B is less than equal to rank of A. Similarly, when constitute this and now, we can combine these two results. And that result gives me, that rank of A B is less than equal to minimum of rank of A and minimum of rank of B.

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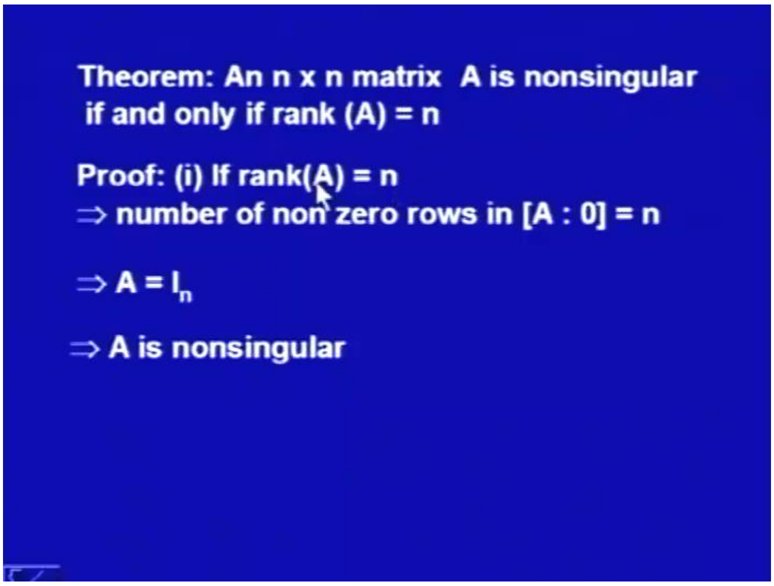

Now, we have an important result, it says that, if n by n is a non-singular matrix. Then, rank of A is equal to n. So, if n by n is a non-singular matrix. This is possible if and only if, rank of A is equal to n. So, this has two parts again. If rank is equal to rank of A is equal to n. We have to prove that, it is non-singular and if it is non-singular. Then rank of A is equal to n. So, we start with the first part, if rank of A is equal to n that means, there exists number of nonzero rows in A in and is augmented matrix is equal to n.

That is how we have define the rank of A. So, the number of nonzero rows in this matrix is equal to n. Then A is I n. So, we can reduce this matrix into echelon form. So, A will be I n, that is the meaning of rank of A is equal to n. And if A is equal to I n, then it is determinant is not 0. So, A is going to be non-singular. So, if we start with rank is equal to A, rank of A is equal to n, then A is non-singular.

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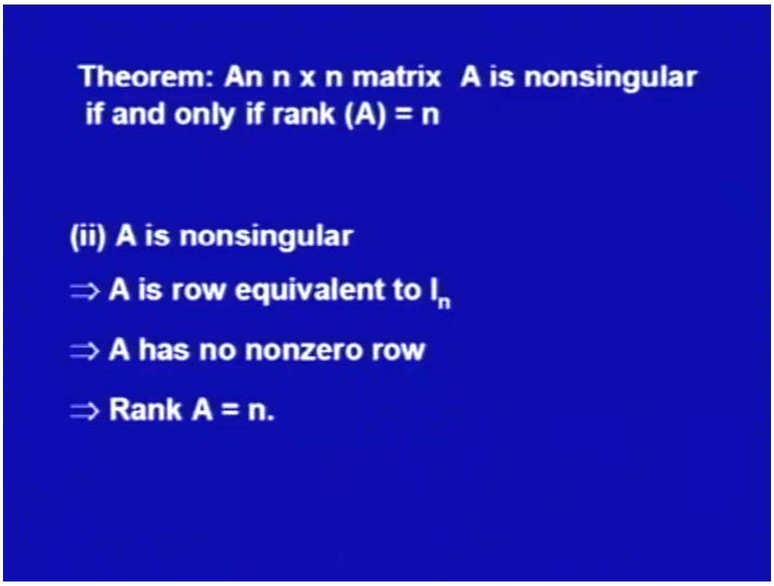

In the second part, we have to prove that, if A is non-singular, then rank of A will be n. So, to prove this, so A is non-singular. So, A is row equivalent to I n. And since, A is row equivalent I n. A has no nonzero rows, A is n by n matrix and it is row equivalent to I n. So, it does not have any nonzero rows. And that means, rank of A is equal to n.

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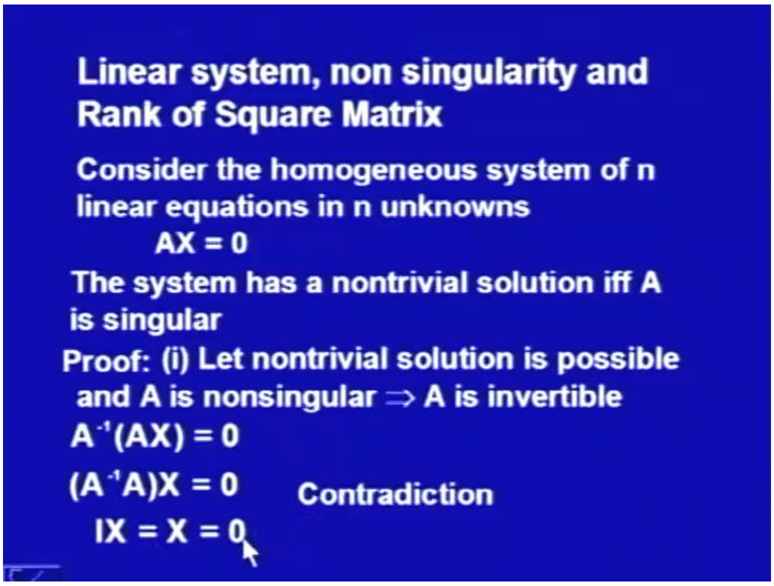

Now, we will try to relate linear systems, non singularity and rank of a square matrix. So, we consider a homogeneous system of n linear equations in n unknowns. Then, we can write down this system in the form A X is equal to 0, where A is an n by n matrix. And X is a column vector in R n and this is identity in R n. Then, the system has a nontrivial solution. If and only if A is singular, so this system will have a 0 solution. Because, when X is equal to 0, A 0 is equal to 0.

But, the problem is, when this will have a nontrivial solution. So, we say, the system will have a non trivial solution. If and only if A is equal to, A is singular or determinant A is equal to 0. So, to prove this, we first assume that, the non trivial solution is possible for the system. And A is singular and since A is singular, so A is invertible. Since, A is invertible, so we can write down we can multiply it by A inverse. So, A inverse A X is equal to 0. A inverse A X is equal to 0 and A inverse A is identity.

So, I X is equal to X, so X is equal to 0. So, we have started with unknown trivial solution. We assume that A is non-singular. But, the finally arrive at a contradiction, that X is equal 0. And that means, our assumption that A is non-singular is not correct.

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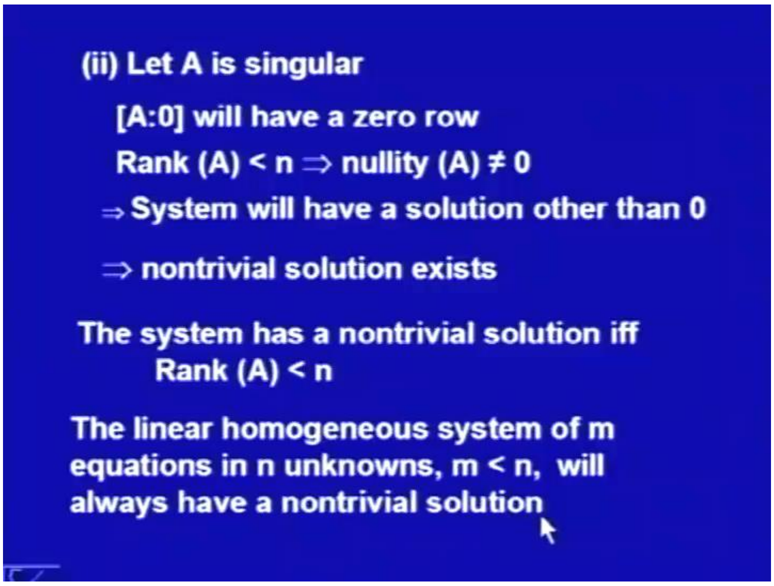

And that means, it is non singular, now the second part, we say A is singular. If it A is singular, that means this augmented matrix will have a 0 row. And if it have a 0 row, then rank of A is less than n and this implies the nullity of A is not 0, it will be 1. And that means, the system will have a solution other than 0. That is a basic definition of nullity, will have a solution, which is 0. So, this proof nullity A is not 0. So, will have a solution which is nonzero, but A X is equal to 0.

The system will have a solution other than the 0, and that means a non trivial solution is possible. And that proves that the system has a nontrivial solution. If and only if rank A is less than n. And the linear homogeneous system of m equations in n unknowns, when m is less than n will always have a nontrivial solution, the next thing is...

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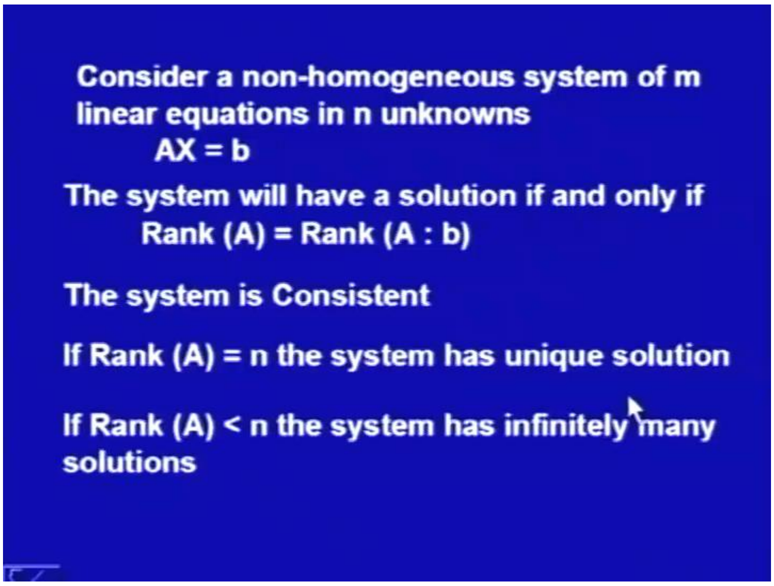

We consider a non homogeneous system of m equations in n unknowns. So, we have between the system in the form A X is equal to b, where A is m by n matrix, m equations in n unknowns. That is why a is m by n and b is a column vector. This system will have a solution, if and only if rank of A is equal to rank of augmented matrix A b. Now, if such a thing will happen, then we say the system is consistent, if it is not, then system is not consistent. If rank of A is equal to n, then the system has a unique solution. And if rank A is less than n, the system has infinitely many solutions.

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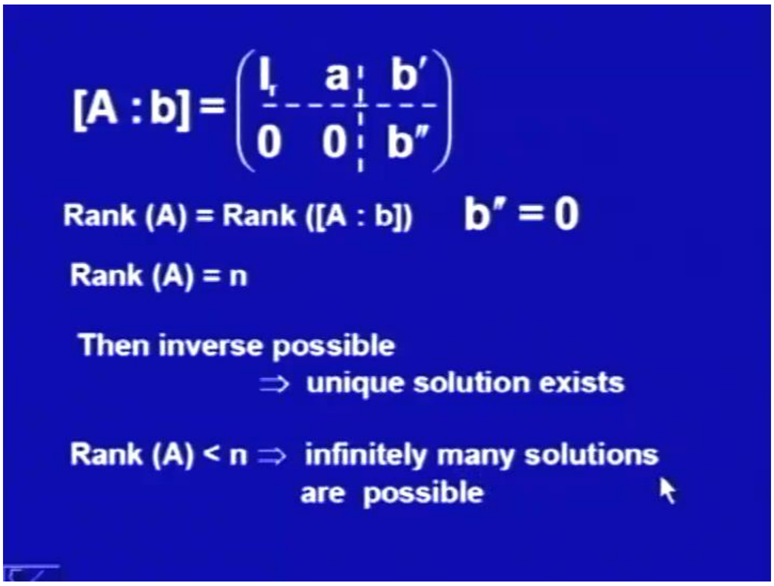

Now, let us see this augmented matrix A b, this can be written as I r, a, b dash, 0, 0, b double dash. That means, you can transform this matrix by using elementary transformation into this form. So, r being the rank of this matrix, these are 0 rows. Now here, if b double dash, if rank A is equal to rank of A b. If that is possible, if rank of A b, if the rank of A is equal to rank of A b, that means B double dash is equal to 0. If b double dash is not 0, then this is not possible.

And if b double dash is 0, then their consistent, this is solution is possible at b double dash is not 0. Then, this condition will not be satisfied and that means, 0 is equal to some nonzero value on this side. So, that is not feasible and that is why, the system will be inconsistent. Now, rank A is equal to n, then the inverse is possible and then the inverse is possible. Then, we can have a unique solution and when rank A is less than n. Then, we will have infinitely many solutions, for the given system.

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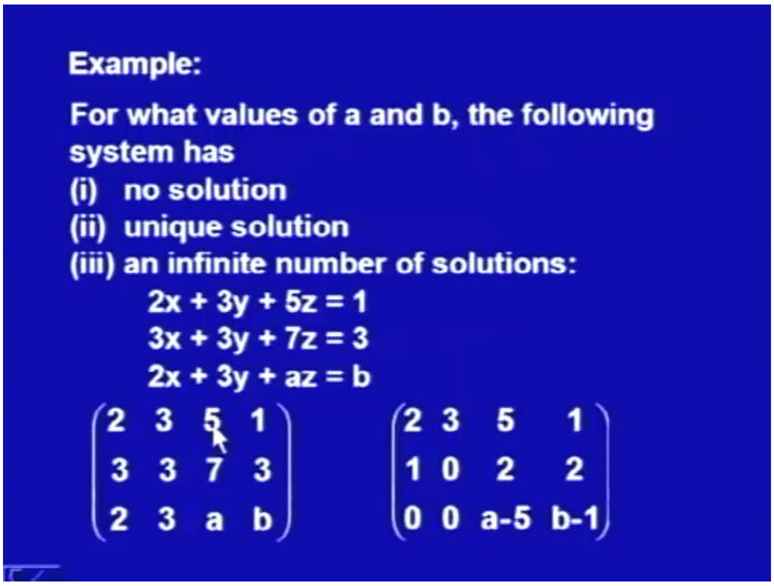

Now, let us illustrate this with the help of an example. I have been given a set of a system of equations, three equations in three unknowns x, y and z. You, have to find out the values of a and b for which, this system has no solution, have a unique solution or an infinitely many solutions. So, we use the results, which we have discussed just now one by one. So, to start with I will consider the augmented matrix 2 3 5, 3 3 7, 2 3 a, that is the coefficient matrix.

And the matrix b is 1 3 b, I try to reduce, I try to apply linear transformations and reduce this matrix into this form. So, what I have done is, I have subtracted this element, this row from this. This is the linear operation I am applying, 3, minus 2 is 1, 3minus 3 0, 7 minus 5 is 2, 3minus 1 is 2. And I am subtracting third row from the first row that gives me 0, 0, a minus 5 and b minus 1.

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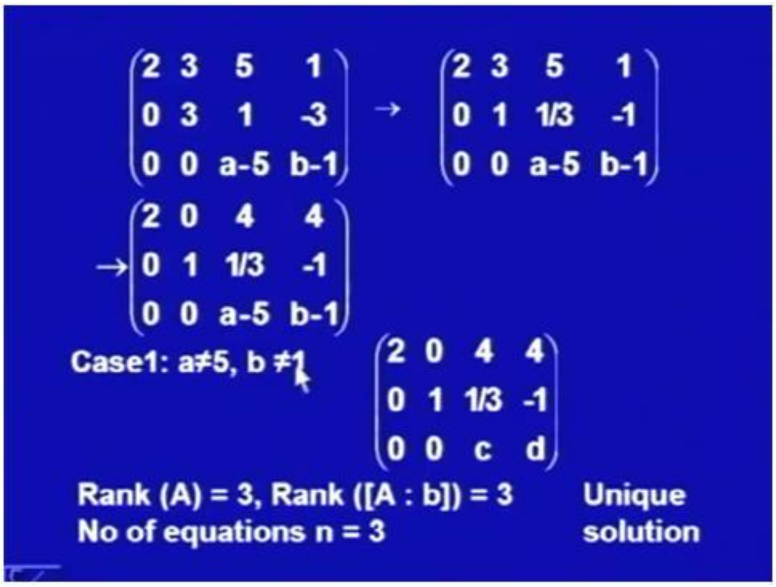

So, if I start with this and apply a linear transformation again then this can be reduced in this form. So, what I have done is, I have divided the second row by 3. So, it is 0, 1, 1 by 3 and minus 1 and 0, 0 a minus 5 and b minus 1. Further, if I subtract 3 times this row, then this become 2 minus 0 0. This become 0, 3 minus 3 is 0 and this becomes 4 and this elements becomes 4. And then one can notice that, this, you can divide this by 1. So, this will become an identity matrix and if a is equal to 5. So, this row will be 0.

And if b is not 1, then 0 the system will be this system is actually, equivalent to z into 0, is equal to b minus 1, b minus 1 is not 0. So, this equation does not have any meaning and that is why, this will not have any solutions. So, if a is not equal to 5 and b is not equal to 1, that is a is not 5. Let us call it c and b not equal to 1 is d. Then, rank of 3 is equal to rank of A is equal to 3. Because, c is not 0 so rank of this matrix is 3 and rank of this matrix, so b is not 0 will also be 3.

So, whether it is 0 or not, this rank will be 3. So, rank of A is equal to rank of the augmented matrix, the number of equations are 3. So, this is a consistent system and this, also have a unique solution. Because, this is also rank A is equal to rank of A b is equal to the number of equations. So, this system will have a unique solution. But, if a is equal to 5 and b is not equal to 1.

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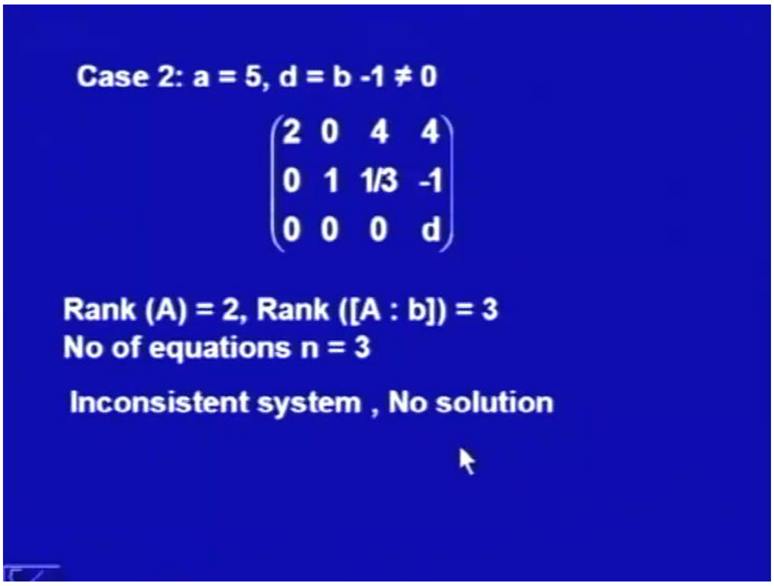

Now, in the second case, we say that, a is equal to 5. And d is equal to b minus 1, is not 0. So in that case, the augmented matrix will become as this and you may notice that, the last line has 0s here and d here. So, that means, that rank of A matrix is not, cannot be 3, because this row is going to be 0. But, the there exist a matrix, we are having rank nonzero. So, the augmented matrix, the rank will rank the augmented matrix may have rank 3. But the matrix A has rank 2.

So, this system is not consistent, rank of A is equal to 2. Rank of it is augmented matrix is 3. And the numbers of equations are given to be 3. So, this has an inconsistent system and it has no solution. Because, z into 0 is equal to d, is not possible. So, we have an inconsistent system in this case, so we can get this by simply checking the rank of A and rank of it is augmented matrix. If they do not match the system will be inconsistent and no solution will be possible in that case. Now, in the third case, if I consider c is equal to a minus 5, which is not 0.

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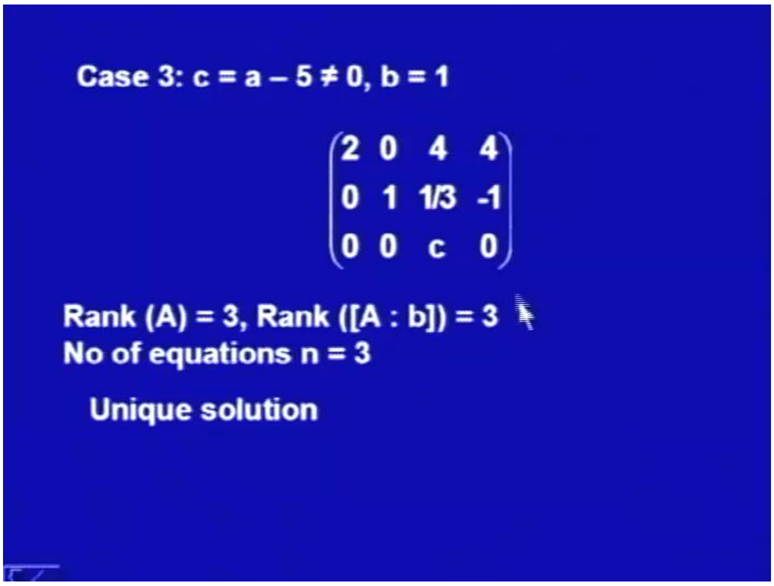

And b is equal to 1 then the augmented matrix will look like as this. Now, in the third row will have c, which is not 0 and that may that means, that this 3 by 3 matrix. A it has determinant nonzero or rank of the matrix A is 3. And rank of A colon b, that is the augmented matrix will also be 3. Because, this is 4 by 4 matrix, is not 4 by 4, determinant is not possible at the most rank will be 3. In this case, so rank of augmented matrix is also 3.

And this is what I have, rank of A is equal to 3, rank of A colon b is equal to 3. And the number of equations, they are 3. So, this will have a unique solution, any difference is that here this z is going to be 0, because this equation is actually equivalent to z into c equal to 0. But, we have unique solution in this case.

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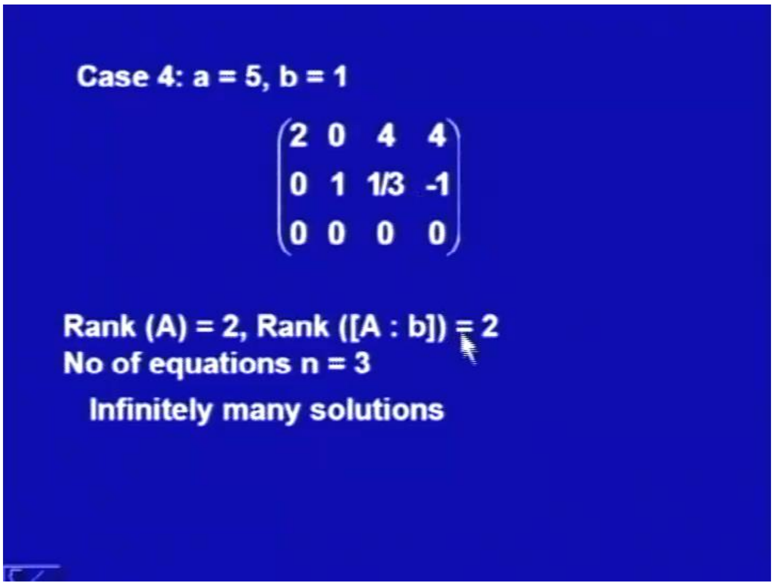

And the last case is, when a is equal to 5 and b is equal to 1. And if this happens, then we have, then our augmented matrix will look like as this. Now, what will happen, this is the coefficient matrix, this coefficient matrix has a row equal to 0. That means, the rank cannot be 3, because this determinant will be 0 rank cannot be 3. And if you consider this is to be 0. Then this augmented matrix also will not have rank equal to 3.. So, the rank of augmented matrix will also be less than 3.

Now, let us see, what is the rank of the coefficient matrix, it is not 3 that is clear. But, this matrix has determinant nonzero. So, rank of this matrix is, this coefficient matrix will be 2. And similarly, if you consider this complete matrix, then the rank of this augmented matrix will also be 2. So, rank of A is equal to 2, rank of A colon b equal to 2. That means, this system may have a solution, it is not inconsistent. It is a consistent system, it will have a solution. But, since the number of equations are 3. And rank of A is less than 3.

So, unique solution is not possible. And in fact, this system will have infinitely many solutions. So, nullity is 1 here, rank is equal to 2. So, there are solutions possible and the third variable can, we can assign arbitrary value to third variable. And still this will be, this third equation will be satisfied. So, what we can do is, we can consider the system as 2 x, plus 0 y plus 4 z is equal to 4. And then we have 0 x, plus y plus 1 by 3 z is equal to minus 1. You can assign arbitrary values to z, this especially still be satisfied.

So, we can say x is equal to 4 minus 4, z and from here. We can say, y is equal to minus y minus 1 by minus 1, minus 1 by 3z. So, you can assign different values to z and we will get different solutions. And that is why we say, that infinitely many solutions are possible. So with this, we have come to an end of this lecture. In this lecture, I have started with the basic concept of rank. And then we have discussed nullity, we have discussed column space. How we have discussed, how to find the columns rank and row rank.

We have discussed number of results, row rank is equal to column rank is equal to rank of the matrix. Then, we have tried to relate this, with the system of equations. When, the system, what should be the rank of the given system of equation. So that, we can have a unique solution, we can we may have infinitely many solutions possible. So, on the basis of this, one can see when the solutions will have, when the system of equations will have a unique solution. When it will have infinitely many solutions or when the solution is not possible, that is it.

Thank you.