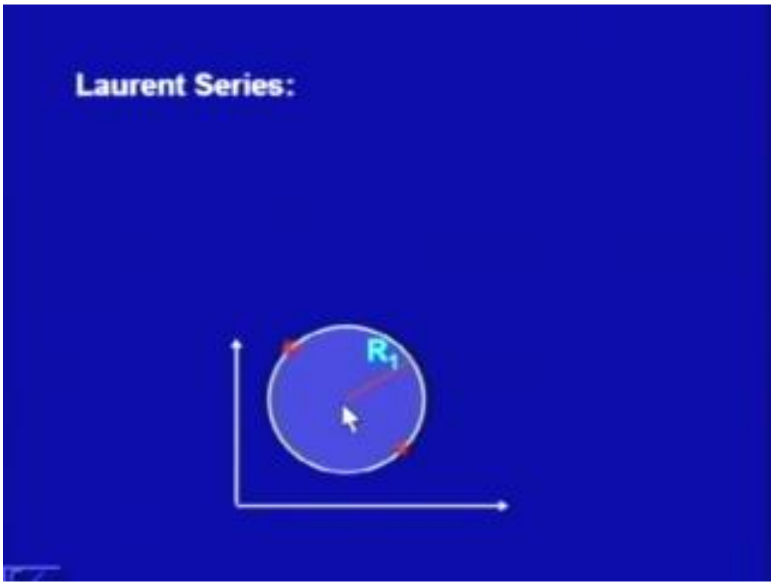
Mathematics - II
Professor Sunita Gakkhar
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture - 22
Laurent Series

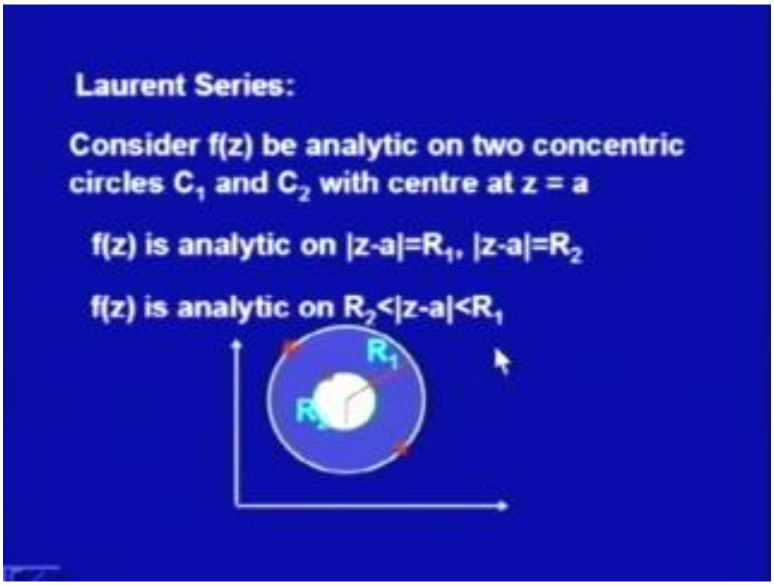
A welcome viewer, today's topic is Laurent Series. Viewers, we have learnt that, a function f of a complex variable z can be represented as a power series, when the function is analytic in a domain.

(Refer Slide Time: 00:48)



Let us say d, in the domain d. We can draw a circle of radius R 1 centred at say z is equal to a. Then, we represent this function f of z in powers of z minus a. And we can say that, this series converges in and it is radius of convergence is R. And in this circle, there is no point where function is not analytic. But what happens, if the function is analytic at some points in this domain. Can we represent still, this function f z as a power series? So, this is the question we are going to answer today. Now, let us say we have a function f of z, which is analytic in two concentric circles c 1 and c 2, with centre at z is equal to a.

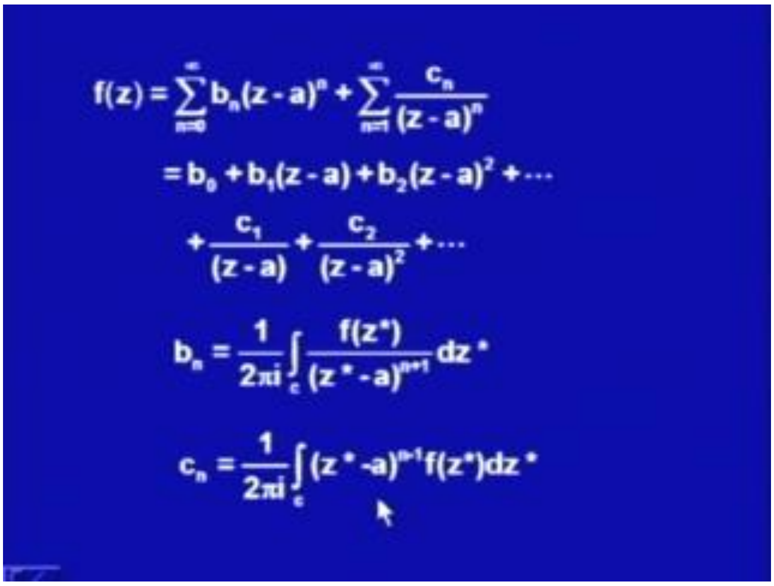
(Refer Slide Time: 01:40)



The idea here is that this function f z may not be analytic at some points, inside the bigger circle of radius R 1. Then, all those points can be enclosed in the circle smallest circle of radius R 2. And then it is the function will be analytic, throughout in this annular region. That is why we say, if f z is analytic on two circles c 1 and c 2 with centre at z is equal to a.

And in other words, we can write it as, that function is analytic on z minus a is equal to R 1. It is a circle of radius R 1 centre at z is equal to a. And another concentric circle, centred at the same point a and of radius R 2. And together with this, f z is analytic inside this annular region.

(Refer Slide Time: 02:48)

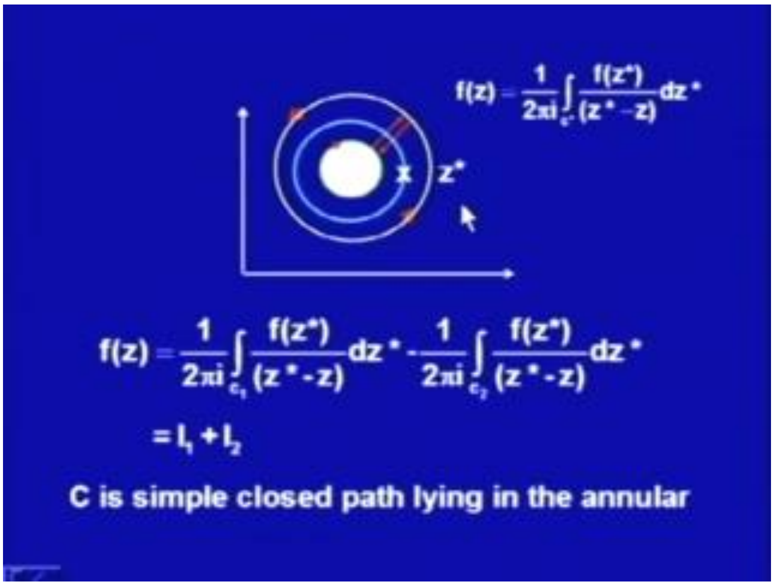


Then, f z can be represented as sum of these two series. The first series is in powers of z minus a and power n, this n is positive. While, in the second series, it is in powers of z minus a, n being negative. So, first series n varies from 0 to infinity. While, in the second series n varies from 1 to infinity. In the first series z minus a appear in the numerator. And here, it appears in the denominator.

And with this, in the expanded form the function f z can be expressed as b nought plus b 1 z minus a, plus b 2 z minus a square plus, and so on. And the second series, will look like as c 1 divided by z minus a, plus c 2 divided by z minus a square, and so on. Here the coefficients b n. They are expressed as 1 upon 2 pi i integral over c, a close curve and the integrand is f z star divided by z star minus a raise to power n plus 1 into d z star.

Here, z star is a point on this close curve c. While, the coefficients c n in the second series. They are obtained as 1 upon 2 pi i integral over the close contour. And the integrand is z star minus a, raise to power n minus 1 f z star d z star. Here again, this point z star will lie on the close curve c.

(Refer Slide Time: 04:38)



Now, to prove this result. We consider the function f z as analytic inside this annular region. So, we consider a close curve c c star s. This outer circle in the anticlockwise sense and in a circle in the clockwise sense. And then we join these two circles by cut. And there are two lines parallel to each other. The one going in the forward direction, another in the backward direction.

And accordingly, the c star will be, if we start from this point, will look like a close curve staring from this point, moving along this line. Then, moving along the outer circle in anticlockwise direction. Then, coming back to this point along this line, but opposite to this. And then moving along the inner circle in clockwise direction. So, this makes the curve c star.

It is a closed curve. This being a cut and function is analytic throughout in this region. With this definition to c star, we can apply cos integral formula. And according to this, the analytic function f z in this domain is 1 upon 2 pi i integral over c star. And the integrand is f z star divided by z star minus z d z star, where z is any point inside this region, where f z star is analytic. And it is inside the closed curve c star.

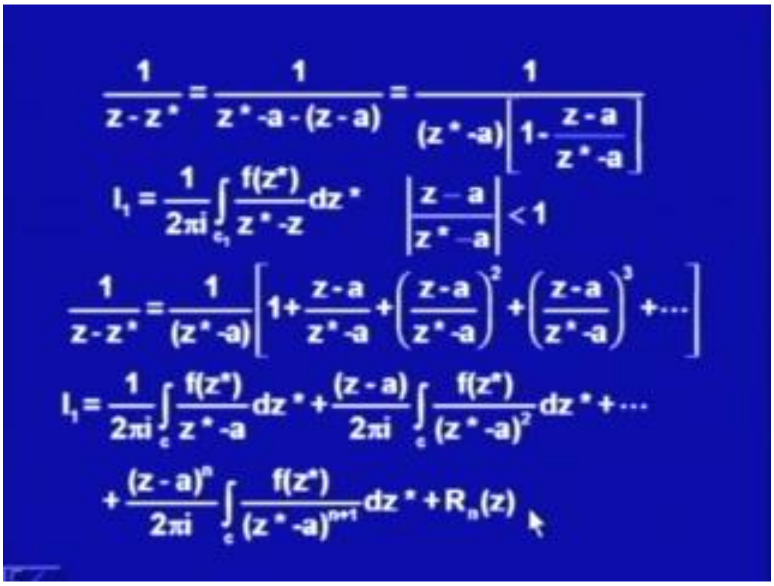
To simplify this expression, we consider c star, to be consisting of the four curves, c 1 these two straight lines and c 2. So, expressing this c star in these four curves will have four such integrals, but integral corresponding to these two straight lines. They will cancel out each other, because the orientation of the curve is just opposite. So, we will be having only two integrals. One along this outer curve in anticlockwise direction. And another along this inner circle in clockwise direction.

And because of this, will have a negative sign here. This is positive direction is taken as anticlockwise direction and the negative direction is taken as this clockwise direction. So, will represent f z in these two integrals. And accordingly, we write f z as sum of these two integrals I 1 plus I 2. Now, we consider c is simple closed path lying along this annular and z is a point on this.

Now, we will not consider these two integrals along this curve or this curve. Rather we will consider these integrals along this curve. The idea is, you can contract you can shrink this outer circle to this, since there is no point, where function is analytic. So, integral along this curve is the same as integral on this curve. And similarly, you can expand this inner circle come to this circle.

And since there are no points, where this function ceases to be analytic. So, you can expand this to this curve c. I call this curve as c. So now onwards, we evaluate these 2 integrals on this curve c.

(Refer Slide Time: 08:38)



Now, to evaluate these 2 integrals, i will write 1 upon z minus z star s, 1 upon z star minus a minus z minus a by adding and subtracting a in the denominator. And then z star minus a is taken outside, and will have 1 minus z minus a divided by z star minus a.

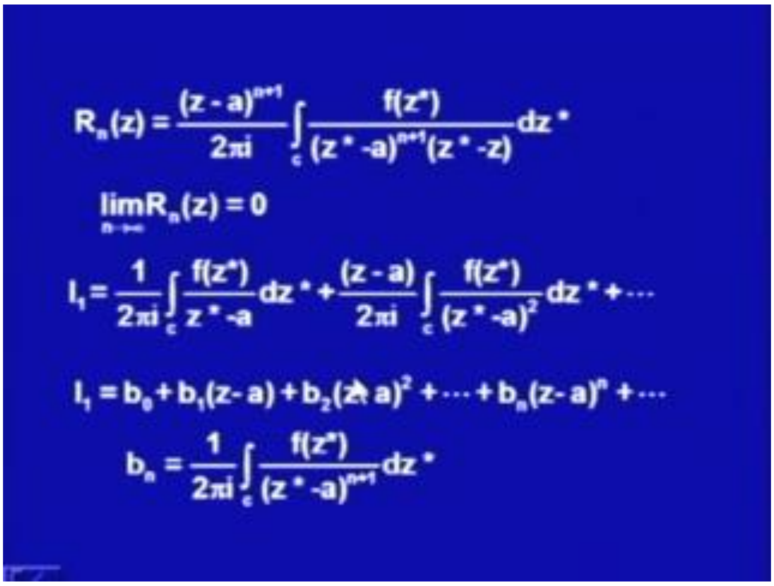
Now, on the integral I 1, where we are integrating over the outer curve c 1. One may notice that, z minus a divided by z star minus a, it is magnitude is less than 1.

Because, z lies in the annular part and z star lies on the outer boundary, so z star will be far away than z. So, z minus a over z star minus a modulus is less than 1. So, if this is the case, then the integrand 1 upon z minus z star can be expanded in the, can be expanded by the geometrical series, as 1 over z star minus a into 1 plus z minus a over z star minus a, plus z minus a divided by z star minus a whole square and so on.

So, here we have expanded this function in powers of z minus a divided by z star minus a, z minus a over z star minus a modulus being less than 1. So, this series is convergent. So, we are writing this integral. This part of the integrand 1 upon z minus z star s this. So, this will be substituted in the integral I 1. So, we multiply this series by f z star integrate over the curve c 1 and then multiply by 1 upon 2 pi i.

This we do for each and every term. So, first term will give me 1 upon 2 pi i. And integrand is f z star over z star minus a d z star integrated over c, plus z minus a into 2 pi i. And then the integrand is f z star, z star minus a whole square d z star and so on. Here, we have assumed that, when we take integral of this, then we can actually integrating it term wise. So, it is this integral is an infinite series. And when we integrate, then integrand can be in a integrated term wise. Of course, we have not proved this so far, but at the moment we are taking it for granted. So, if we express, I 1 in this form. Then, these are the first terms.

(Refer Slide Time: 11:47)

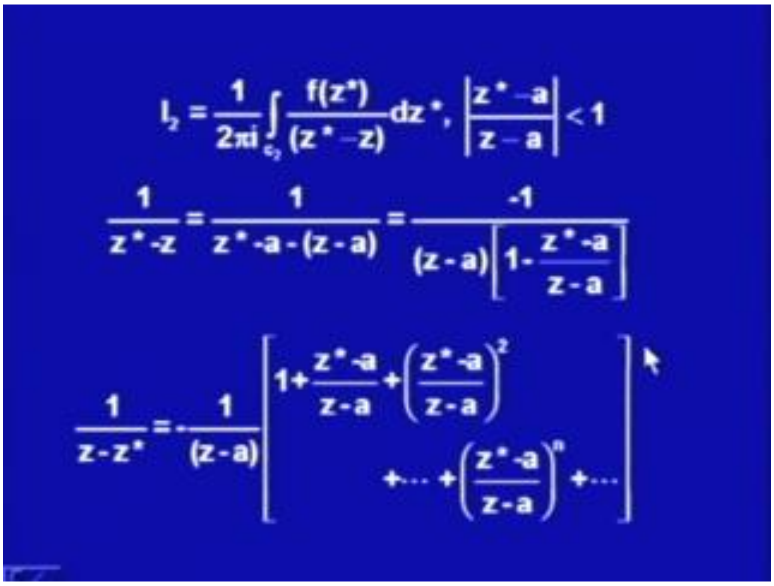


And then after n terms remaining terms are represented as R n z, where R n z is z minus a raise to the power n plus 1 over 2 pi i, integral c f z star divided by z star minus a raise to the power n plus 1, into z star minus z into d z star. Now, as we have done in the case of Taylor series. This expression will tend to 0 as n will tend to infinity.

And accordingly, I 1 will be 1 upon 2 pi i, into f z star over z star minus a d z star, plus z minus a divided by 2 pi i integral c f z star divided by z star minus a whole square d z star and so on. So, in this way, if we express this coefficient as b nought and this the second coefficient of, the coefficient in the second term is z minus a is this one and we represent it as b 2 and so on.

So, I 1 will be b nought plus b 1 z in z minus a, plus b 2 into z minus a square, and the nth term being b n into z minus a raise to the power n and so on. So, this is an infinite series, R n z being tending to 0. Here the general term b n can be given by the formula, 1 upon 2 pi i integral over the curve c and f z star divided by z star minus a into n plus 1 d z star. Here, I am writing the contour integral over c. Because, I am replacing c 1 by c, the expression is given earlier.

(Refer Slide Time: 13:34)



Then, we have to evaluate I 2. Now to evaluate I 2, which is 1 upon 2 pi i integral over c 2 f z star over z star minus z d star, d z star. And here, I will consider z star minus a over z minus a magnitude is less than 1. Because, z this c 2 is the inner curve and z star minus a will be smaller than the point in the annulus region z minus a. So, z star minus a divided by z minus a modulus is less than 1.

So, assuming this, I can expand 1 upon z star minus z the part of this integrand. As 1 upon z star minus a minus z minus a and then z minus a can be taken out. And this will be minus 1 of 1 minus z star minus a divided by z minus a. So, this minus will be observed with I 2.

And accordingly, if we expand it in geometric progression, we will have 1 upon z minus a into 1 plus z star minus a divided by z minus a plus z star minus a divided by z minus a whole square and so on. And, nth term is z star minus a divided by z minus a raise to the power n. And this series is a geometric series, which will be convergent under this given condition.

(Refer Slide Time: 15:21)

$$I_2 = \frac{1}{2\pi i}\int_{c_2} \frac{f(z^*)}{z^* - z}\,dz^*$$

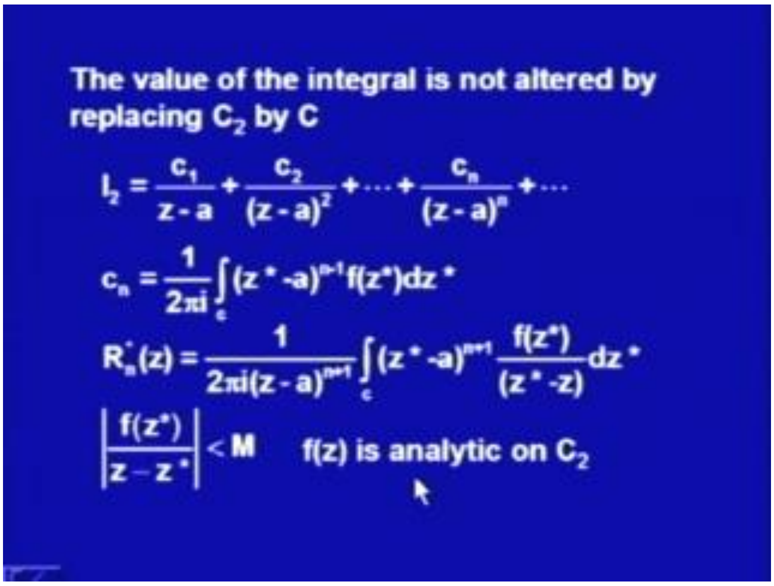
$$= \frac{1}{2\pi i}\left[\frac{1}{z-a}\int_c f(z^*)\,dz^* + \frac{1}{(z-a)^2}\int_c (z^* - a) f(z^*)\,dz^* + \cdots + \frac{1}{(z-a)^{n+1}}\int_c (z^* - a)^n f(z^*)\,dz^* + R_n^*(z)\right]$$

$$R_n^*(z) = \frac{1}{2\pi i(z-a)^{n+1}}\int_c (z^* - a)^{n+1}\frac{f(z^*)}{(z^* - z)}\,dz^*$$

So, once we have done this, then we can again move. We can again proceed in the similar manner. That is, I multiply that expression by 1 upon by 2 pi i and f z star. And then integrate over the curve c 2. And, if you proceed in this particular manner will have I 2 is equal to 1 upon 2 pi i, and 1 upon z minus a will be taken out. And will have f z star d z star as the first term, plus in the second term.

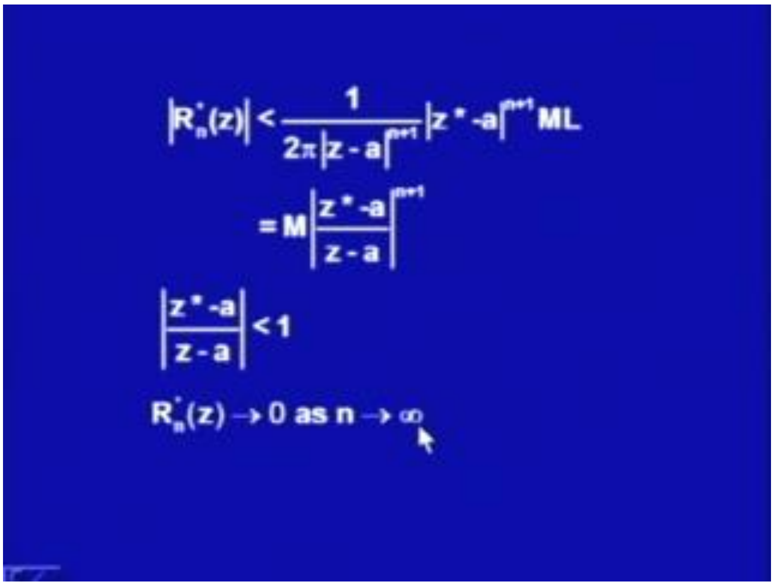
We can take, 1 upon z minus a square, outside the integral sign. Because, we are integrating, with respect to z star and this does not involve z star. So, this can be taken out and will have integrand of the second term is z star minus a into f z star d z star. And again, we have replaced this curve c 2 by c. So, this way we can have general term as this. And the remaining term are observed, in the remainder R n star z. The reminder term R n star z is written as this.

(Refer Slide Time: 16:32)



Now, the value of the integral is not, altered by replacing c 2 by c. This I have already explained. And then I 2 will be represented as, c 1 divided by z minus a, plus c 2 divided by z minus a square and nth term being z n, c n divided by z minus a raise to the power n and so on where the coefficients c 1, c 2, cn's. They are given by this general formula, that is c n is equal to 1 upon 2 pi i, integral over c. And, the integrand is z star minus a raise to the power n minus 1, if the value l and f z star, d z star. So, this is my c n and then R n star z. As we have expressed it as this. And from here, one may notice that f z star divided by z minus z star modulus is less than M, f z being analytic on the curve c 2.

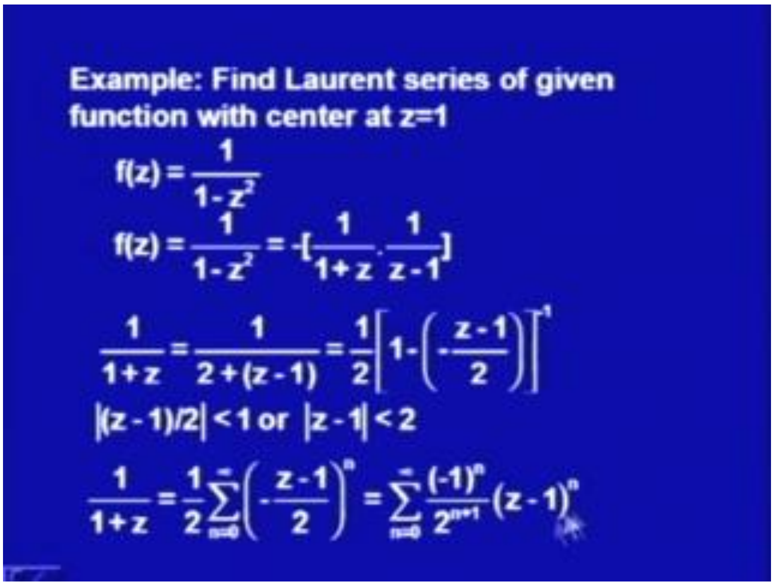
(Refer Slide Time: 17:36)



And modulus of R n star z will be less than modulus of 1 upon 2 pi i, which is 2 pi. And then z modulus of z minus a raise to power n plus 1 multiplied by z star minus a raise to power n plus 1 into M times L. And L happens to be 2 pi. So, we will have this integral is M times z star minus a divided by z minus a raise to power n plus 1. And since this term is less than 1, this modulus is less than 1.

So, as n tending to infinity, this term will become smaller and smaller and ultimately. R n star z will tend to 0 as n tending to infinity. With this, we have been able to represent the function f z in powers of z minus a. And in this series, we have both positive powers of z minus a, as well as negative powers of z minus a. So, let us illustrate this, with the help of an example.

(Refer Slide Time: 18:47)

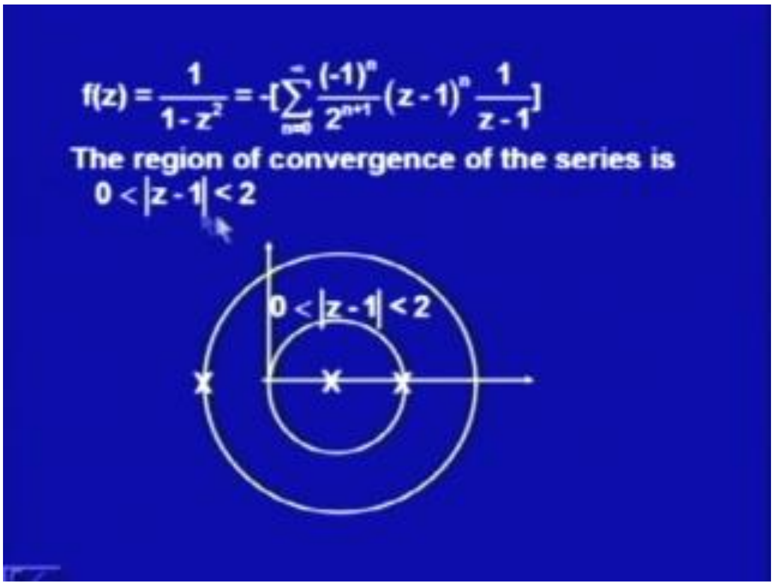


So here, we have to find the Laurent series for the given function f z is equal to 1 upon 1 minus z square with centre at z is equal to 1. One may notice that this function f z has this function f z is not analytic at z is equal to 1. We can express this function f z in the form 1 upon 1 plus z into 1 upon z minus 1. And I have taken minus, outside to make it 1 minus z square. So, this function is not analytic at z is equal to 1.

As well as at z is equal to minus 1. But, since we have to find Laurent expansion about z is equal to 1. So, we are concerned about, it is analyticity, at z is equal to 1. Now, this is already in powers of z minus 1. So, if we express 1 upon 1 plus z in powers of z minus 1. Then, we are through. So, to express this an powers of z minus 1, we rewrite 1 upon 1 plus z as 1 divided by 2 plus z minus 1. And then two can take out and will have 1 minus z minus 1 divided by 2 raise to the power minus 1.

And assuming z minus 1 divided by 2 is less than 1, or z minus 1 modulus is less than 2. Then, this term will be, less than it is modulus will be less than 1. And then we will be able to expand it in power series, in powers of z minus 1. So, to do this, we write, 1 upon 1 plus z is equal to half of minus minus times z minus 1 divided by 2 raise to the power n, n takes values from 0 to infinity. So we have expressed this, in compact form. And this way, we will simplify it to be n is equal to 0 to infinity summation minus 1 raise to the power n 2 divided by n plus 1, and here we have z minus 1 raise to the power n.

(Refer Slide Time: 20:56)

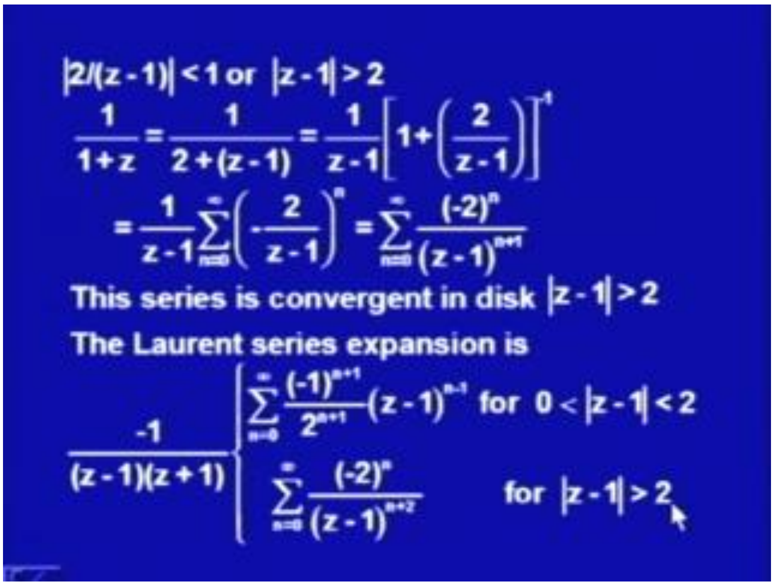


And this gives me, the f z representation of 1 upon 1 minus z square s minus summation n is equal to 0 to infinity minus 1 raise to the power of n, 2 raise to the power n plus 1 in the denominator. And z minus 1 raise to the power n, multiplied by 1 upon z minus 1. So, this is in powers of z minus 1, the region of convergence of this series is modulus z minus 1, lying between, 0 to 2.

Because, there is another point which is at a distance 2 from z minus 1, where function ceases to be analytic and that point is minus 1. Therefore, we consider modulus of z minus 1 is less than 2 modulus of z minus 1 is greater than 0 is will take care. That z is equal to 1 is not included in the annulus, where the function needs to be analytic. And only then Laurent series will be applicable.

So, the real of convergence of this series is modulus of z minus 1 lying between 0 and 2. That is, what has been shown here, now at z is equal to 1 and z is equal to minus 1. The given function is not analytic. And that is, why we enclose this point by this z minus 1 modulus greater than 0 this circle. And this is the point, where function is not analytic. So, this distance is 2. So, in this annulus region function is a represented by this series. And, this is a region of convergence.

(Refer Slide Time: 22:45)

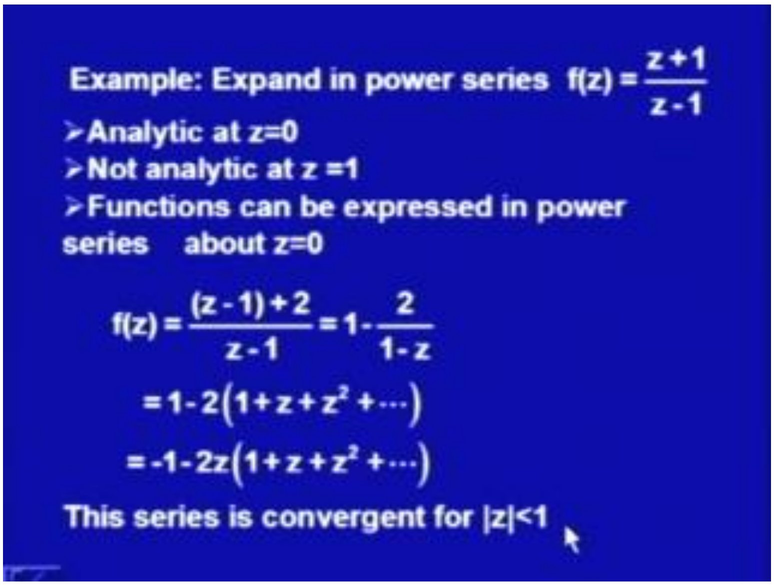


In the outer part, when z minus 1 is greater than 2. There is no singular point outside this. And we can write down to upon z minus 1, modulus is less than 1. And in this region, we can expand 1 over 1 plus z in this form. Here, 2 upon z minus 1 is less than 1. So, we can again expand in power series and will have 1 upon z minus 1.

Summation n is equal to 0 to infinity. And here, we have minus 2 upon z minus 1 raise to the power n. And the series will be minus 2 raise to power n divided by z minus 1 raise to power n plus 1. Summation will take values from 0 to infinity. So, this series is convergent in the disk, when z minus 1 is greater than 2. So, we have one representation inside the z minus 1 less than 2. And this is the representation when z minus 1 is greater than 2.

So, we have two representations and the Laurent expansion. This way is given as minus 1 over z minus 1 multiplied by z plus 1 is represented by this power series in this region z minus 1 lying between 0 to 2 and when modulus of z minus 1 is greater than 2. Then, we got this expansion.

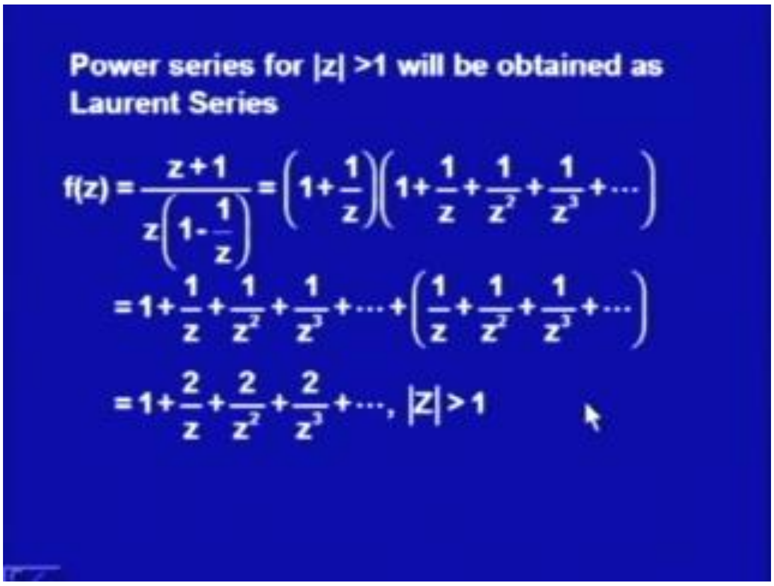
(Refer Slide Time: 24:15)



So this is one example. In the next example, we consider function f z as z plus 1 divided by z minus 1. When we notice that this is analytic at z is equal to 0. It is not analytic at z is equal to 1. And function can be expressed in power series about z is equal to 0. Because, a function is analytic at z is equal to 0. This will be the Taylor series representation of this function.

So, f z will be given by 1 minus 2 over 1 minus z, and this 1 upon 1 minus z can be expressed in powers of z. And we will have this expansion and accordingly, we will have this after simplification. And finally, we will have f z is equal to minus 1 minus 2 z 1 plus z plus z square and so on which will be convergent for modulus z less than 1. Because about z is equal to 0, if we draw circle, then if the radius becomes 1. Then z is equal to 1 will be a singularity lying on the circle. And then series will not be convergence. So we say this series is convergent for modulus z less than 1.

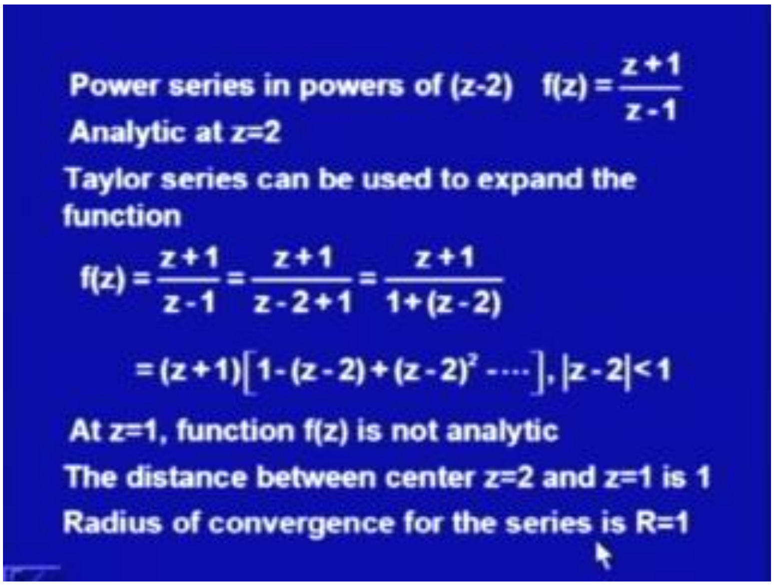
(Refer Slide Time: 25:33)



But, power series for modulus z greater than 1 will be obtained as Laurent series. And for this, we write down the denominator in the form. As this, I have taken z outside. So, it is 1 minus 1 upon z in the denominator. And here, we expand this, this also z can be taken out. So, it is 1 plus1 upon z the numerator and in the denominator is expanded as this. And what we have f z if you multiply these terms will be simplifying to this series.

And further simplification will give me, 1 plus 2 upon z, plus 2 upon z square, plus 2 upon z cube and so on modulus z is greater than 1. And here, you may notice that the terms are appearing in the denominator or that we have negative powers of z. So, it is a Laurent series, Laurent series expansion of the function. But, it is valid for modulus z greater than 1.

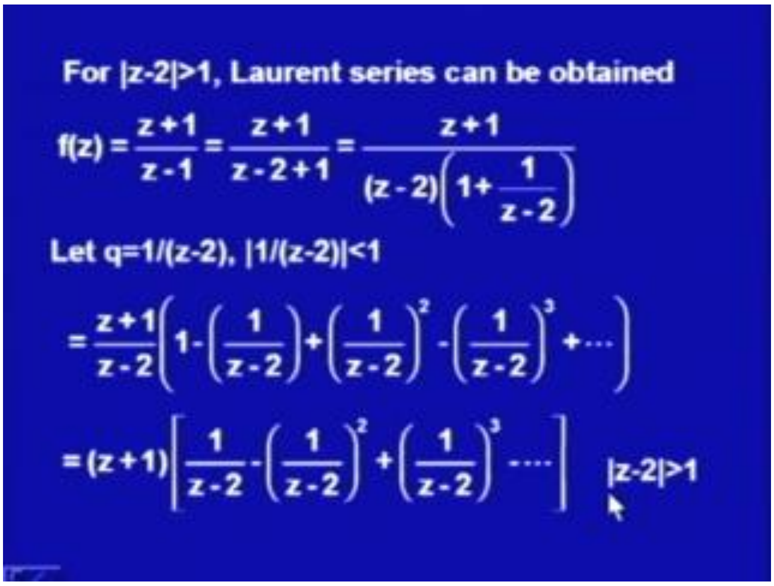
(Refer Slide Time: 26:37)



The power series in if we have to express the same function, but now in powers of z minus 2. Then we may notice that, this function is analytic at z is equal to 2. So, Taylor series can be used to expand this function. And again, we simplify this term, so that we can expand in powers of z minus 2. So, we write down the function in this manner. And then expand the denominator in powers of z minus 2 provided z minus 2 is less than 1.

Otherwise, this will not be convergent. And then at z is equal to 1 function f z is not analytic. So, the distance between centre z is equal to 2. And the point, where function is not analytic is 1. So that is why we say the radius of convergence for the series is R is equal to 1 and we have Taylor series expansion.

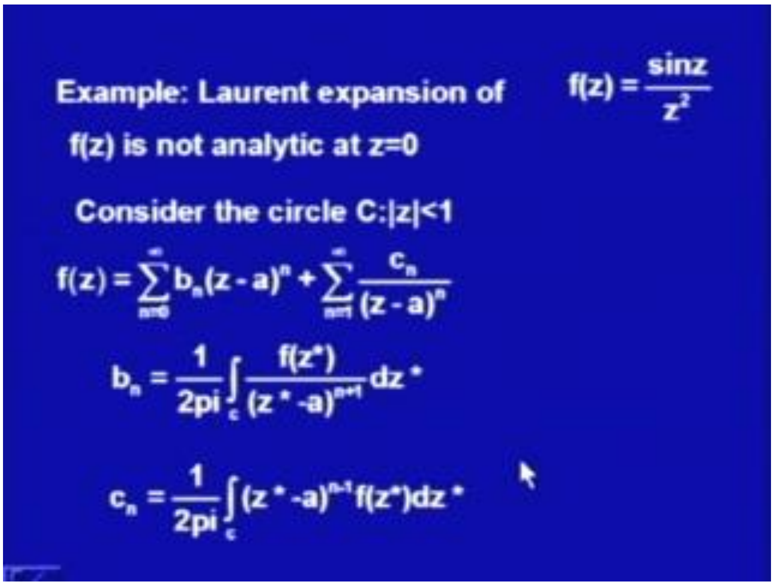
(Refer Slide Time: 27:38)



But, for modulus z minus 2 greater than 1, Laurent series can be obtained. And we have f z as z plus 1 divided by z minus 1. We again rewrite it and since our domain is modulus z minus 2 is greater than 1. So, we will now take z minus 2. Outside and will have 1 upon z minus 2 in the denominator. So simplifying this taking q as 1 upon z minus 2 and modulus of q is less than 1.

And, expanding it, it is z plus 1 divided by z minus 2 in powers of 1 upon z minus 2. This is the geometric series. And simplifying it, it is z plus 1 multiplied by 1 upon z minus 2 minus 1 upon z minus 2 whole square and so on. And the region is z minus 2 is greater than 1, because after that there is no point, where function is not analytic. So this is the region of, convergence for the Laurent series.

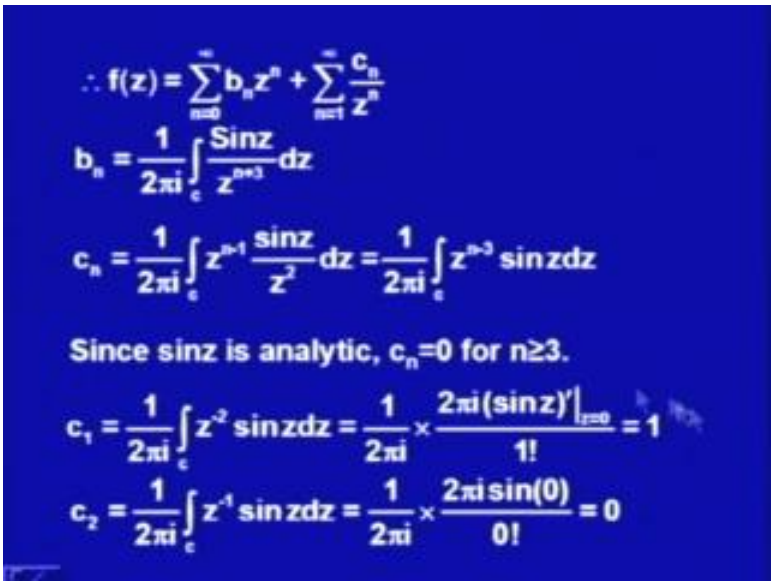
(Refer Slide Time: 28:48)



Now, this example is slightly different. So far, we have function which we can use. We have functions and we are using geometric series expansion for obtaining the Laurent series. But in this example, we will be using and the Laurent series expansion. And the formulae, which we have derived just now so let us, consider f z as sin z upon z square and since f z is not analytic at z is equal to 0.

So, we consider the circle, c modulus z is less than 1. Then in this circle, we can write down f z as b n z minus a raise to the power n, plus summation n is equal to 1 to infinity c n z minus a raise to the power of n. So, this is a negative power of n and this series in positive powers of z minus a. And here we have to consider that mod z is positive. And the formulae for b n and c n, they are obtained as we have already done it.

(Refer Slide Time: 29:54)



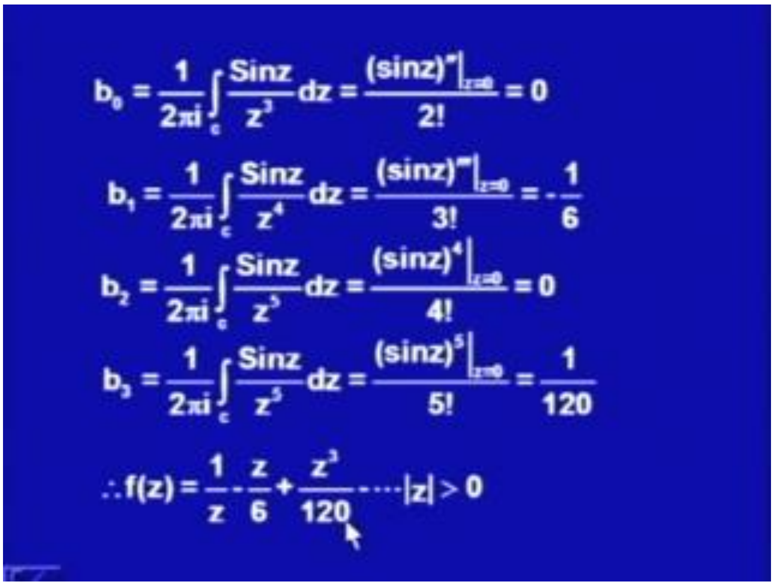
So, let us, use this formulae to obtain, b n and c n. So accordingly, it is 1 upon 2 pi i integral sign z divided by z raise to power n plus 3 dz. So, this is the formula for b n and for c n the formula is 1 upon 2 pi i z into n z raise to the power n minus 1 multiplied by sin z divided by z square. This is by function f z. So this simplifies to integrand, z raise to power n minus 3 sin z d z and 1 upon 2 pi i has to be multiplied.

So this, the formula for b n and c n, since sin z is analytic. So, c n is equal to 0 for n greater than equal to 3. So if n is greater than equal to 3. So these terms will not be appearing. And all cn's will be 0 for n greater than equal to 3. So, we need to calculate only c 1, c 2, c 3 etcetera. They are going to be 0. So c 1 is given by 1 upon 2 pi i z raise to the power minus 2 sin z d z.

And this is equal to 1 upon 2 pi i into 2 pi i sin z derivative divided by factorial 1. So, this is the formula, which I will be making use of. And this derivative is evaluated at z is equal to 0. So, if you simplify, this comes out to be 1. This and this will get cancelled and derivative of sin z is cos z, which is the evaluated z is equal to 0 gives me 1. So, this is c 1.

Similarly, c 2 will be z raise to minus 1 sin z d z. And according to the formula, we have 1 upon 2 pi i into 2 is pi i into sin function evaluated at z is equal to 0 divided by factorial 0. And this comes sin 0 being 0. This comes out to be 0.

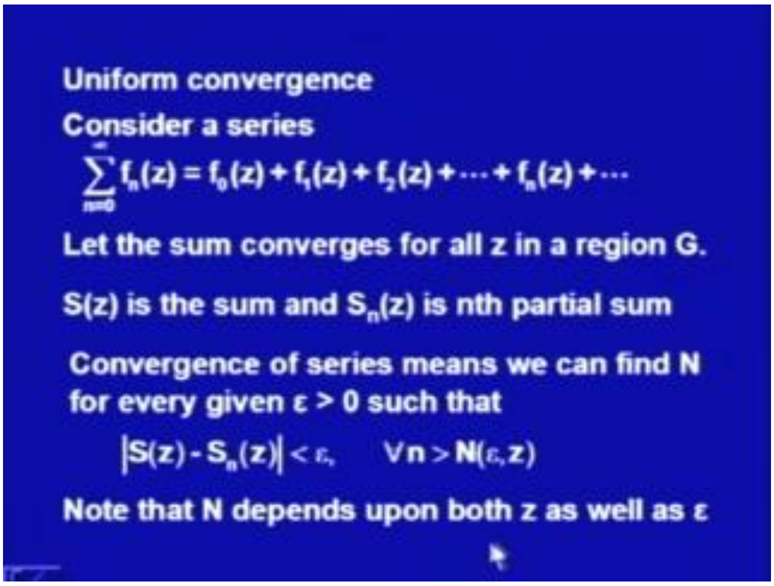
(Refer Slide Time: 31:58)



So, this is c 1 and c 2 ((Refer Time: 31:58)) is 0, so will have z cube over 120 and so on. And it is convergent for modulus z greater than 0. So, this is Laurent series expansion for the given function f z. Now, we come to uniform convergence up to this point. We are we are using number of results. We are term by term multiplying series. We are integrating terms of this series and so on.

Without bothering, whether this is possible or not. But here, I will discuss uniform convergence of the series first. And then we will see that for power series. This term by term differentiation and integration is possible.

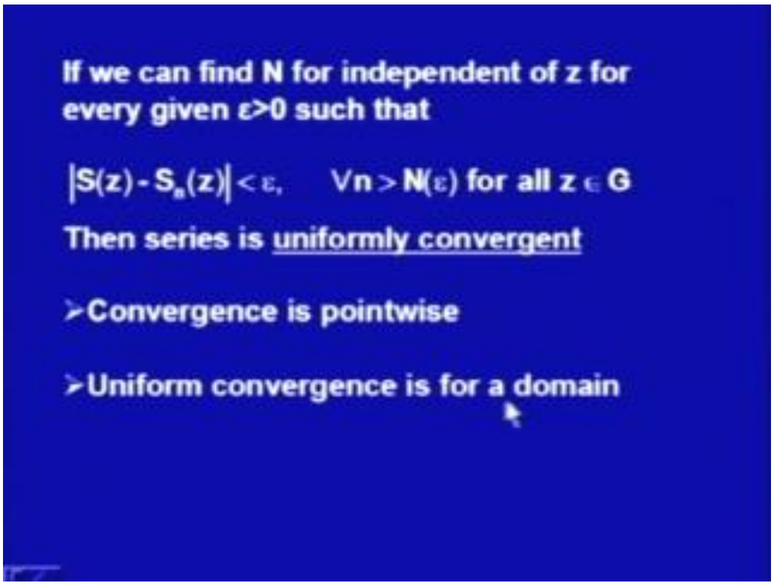
(Refer Slide Time: 32:55)



So, to define uniform convergence, consider a series of functions of z. And so, we have is equal to 0 to infinity. Summation f and z, which is f 0 z plus f 1 z plus f 2 z plus f n z and so on. So this series is considered. Let the some converges for all z in a region G. And S z be the sum of this series. And S n z is the nth partial sum of this series.

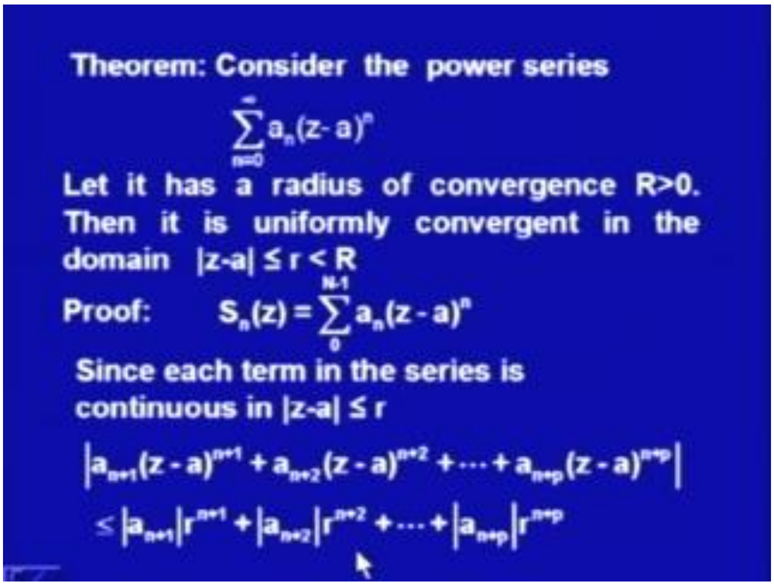
Then convergence of series means that we can find N for every given epsilon greater than 0 such that S z minus S n z is less than epsilon. For all n greater than N, capital N, which will depend upon the epsilon we choose. And the point z at which the convergence of the function is convergence of the series is considered. This is the meaning of convergence. Note that N depends upon both z as well as epsilon.

(Refer Slide Time: 34:05)



However, if we can find capital N which is independent of z for every given epsilon greater than 0 such that the difference between S z and S n z. It modulus is less than epsilon. For all n independent of z, it is a function of epsilon only. Then we say that, the series is uniformly convergent. So accordingly, convergence is point wise that means we say, the series is convergent at z is equal to a. And uniform convergence, is for a domain. So, ((Refer Time: 34:45)) we can find the capital N irrespective of z. In that domain, so uniform convergence is for a domain.

(Refer Slide Time: 34:56)



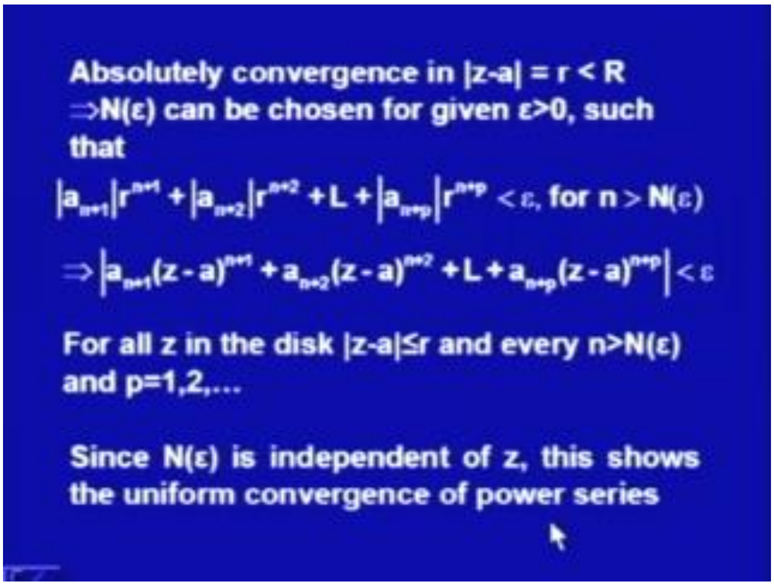
Now, we establish a theorem, regarding power series. We say that if we have this power series a n z minus n, z minus a raise to the power n, n is varying from 0 to infinity. That is a power series centre at z is equal to a. Then, it has a radius of convergence R greater than 0. So, this power series will have a radius of convergence. R greater than zero, if this is true then this is uniformly convergent in the domain z minus a, which is less than equal to r.

For some r, which will be smaller than capital R that is the region of convergence. So, the power series a n z minus a raise to the power of n will always be uniformly convergent in this domain. Now to prove this, let us consider S n z partial sum as a n z minus a raise to the power n. It is a sum of n terms, so it is 0 to N minus 1. So, this is partial sum.

Since each term in the series is continuous in z minus a less than equal to r. Because, each term of this series is a power in z. So, each term will be continuous in this region. So, we can say that a n plus 1 z minus a raise to the power n plus 1 plus a n plus 2 z minus a raise to power n plus 2 and so on. Plus a n plus p multiplied by z minus a raise to power n plus.

So this is sum of p terms after a n this is less than a n plus 1, raise to power r multiplied by r raise to the power n plus 1. This being less than r plus an plus 2 modulus. These may be these coefficients may be imaginary. So, we have to consider modulus here and r raise to the power n plus 2 plus. And the last term is a n plus 2 modulus into r raise to the power n plus p. So this sum is less than sum of these terms. so, the triangular inequality is applied.

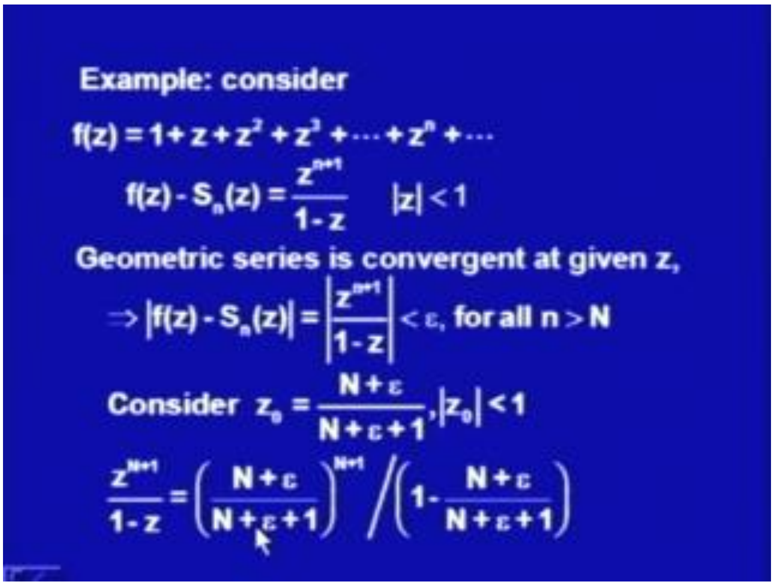
(Refer Slide Time: 37:25)



So, absolute convergence in z minus a is equal to r, which is less than R. So, we can say that, N epsilon can be chosen for given epsilon positive, such that this condition is satisfied. For n greater than N epsilon; that means, we can find epsilon such that this condition is satisfied. We can find n such that this condition is satisfied. So, this is true.

For all z in this disk, z minus a modules less than equal to r and for every n greater than N epsilon and for p is equal to 1 2 and so on. So, we can say that since N epsilon is independent of z. This follows the uniform convergence of power series. So, it does not depend upon z and the series is uniformly convergent. So, power series is uniformly convergent.

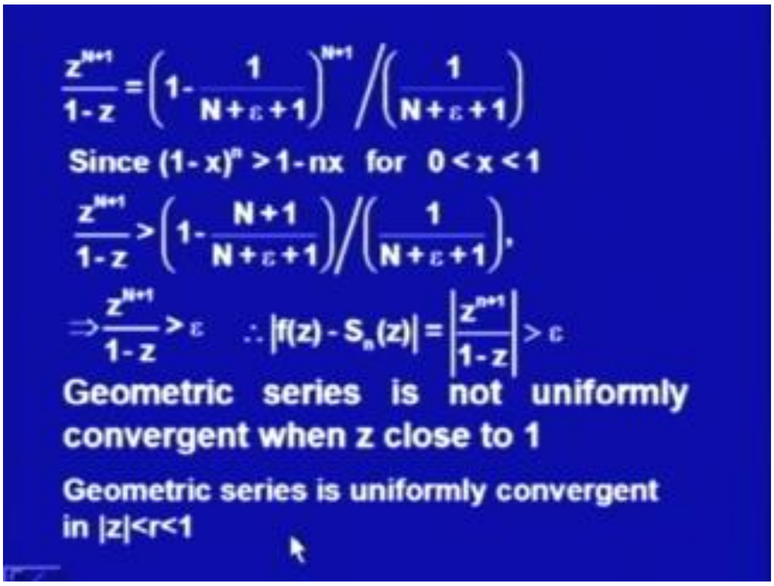
(Refer Slide Time: 38:31)



Now, we consider this power series 1 plus z Plus z square plus z cube. This is geometric series and we know that this f z minus S n z, that is partial sum. The difference between f z and partial sum is z raise to the power n plus 1 over 1 minus z, modulus z is less than 1. This is the result, which we have already established. And the geometric series is convergent at any given z.

So we can say that, modulus of minus S n z is equal to modulus of z raise to the power n plus 1 over 1 minus z less than epsilon. For all n greater than N, so this is for a given z. Now we say that, if we consider z naught as slightly different point. N plus epsilon divided by N plus epsilon plus 1. So, we can notice that still z nought is less than 1. But for this z naught, we can say that z raise to the power n plus 1 over 1 minus z. If you simplify this expression this comes out to be this.

(Refer Slide Time: 39:39)

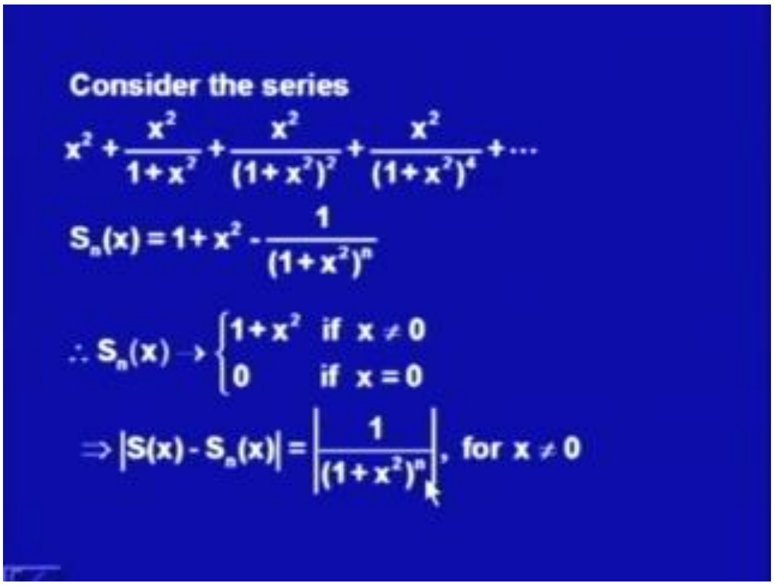


And further simplification, yields this and from here, we may notice that 1 minus x raise to power n is greater than 1 minus n x. When x lies between 0 to 1 and this gives me z raise to the power n plus 1 upon 1 minus z is greater than this expression. Or if we further simplify, we say z raise to the power n plus 1 over 1 minus z is greater than epsilon; that means, the N which we have consider will not work for this specially. When we are when z is very close to 1.

So, f z minus S n z is equal to z raise to the power n plus 1 over 1 minus z greater than epsilon. So geometric series is not uniformly convergent. When z is very close to 1, so there is a neighbourhood, there is a region very close to z, is equal to 1, where this condition is satisfied. And whatever n we have computed. For given x will not work for a different z. So this means that it is not uniformly convergent.

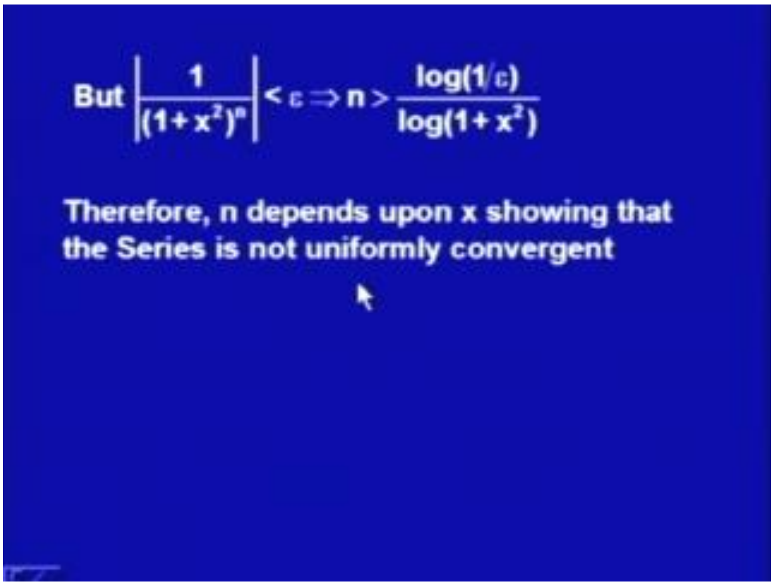
However, if we apply the result of earlier theorem then geometric series is uniformly convergent, in z modulus z less than r. So, there exist some r which will be in which the geometric series will be uniformly convergent. But, it will not be convergent throughout modulus z less than 1. There has to be some r in which this series is uniformly convergent.

(Refer Slide Time: 41:17)



Now, if we consider different series which is x square, plus x square over 1 plus x square plus x square over 1 plus x square whole square and so on. Then for this series, S n x some partial sum is 1 plus x square minus 1 over 1 plus x square raise to the power n. Then we can notice that, S n x tends to, 1 plus x square. If x is not equal to 0 and will tend to 0, if x is equal to 0; that means, S x, that is sum of this infinite series minus the partial sum is equal to 1 over 1 plus x square raise to power n, for x is not equal to 0.

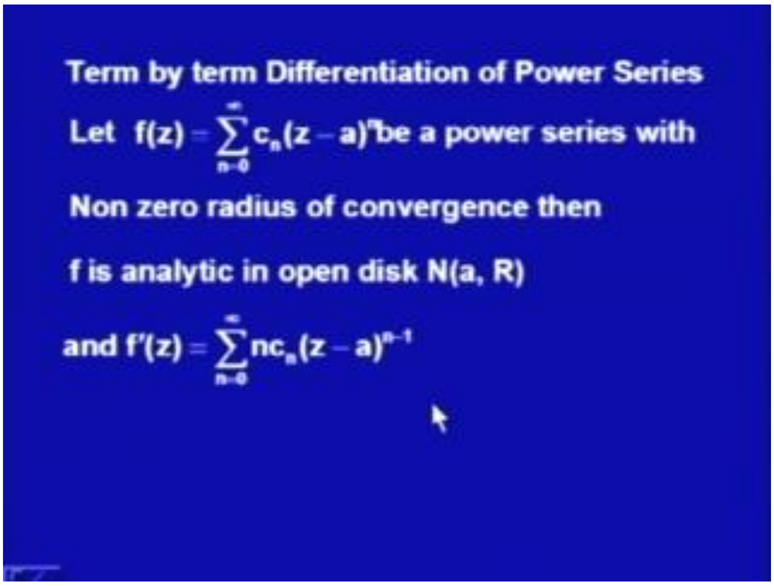
(Refer Slide Time: 42:02)



But we know that 1 over 1 plus x square raise to power n modulus is less than epsilon. If implies that n must be greater than log 1 over epsilon divided by log 1 plus x square. And; that means, this n we can choose for a given epsilon and this depends upon x. And therefore, n depends upon x showing that the series is not uniformly convergent, so this series which is not a power series. So, this is not uniformly convergent.

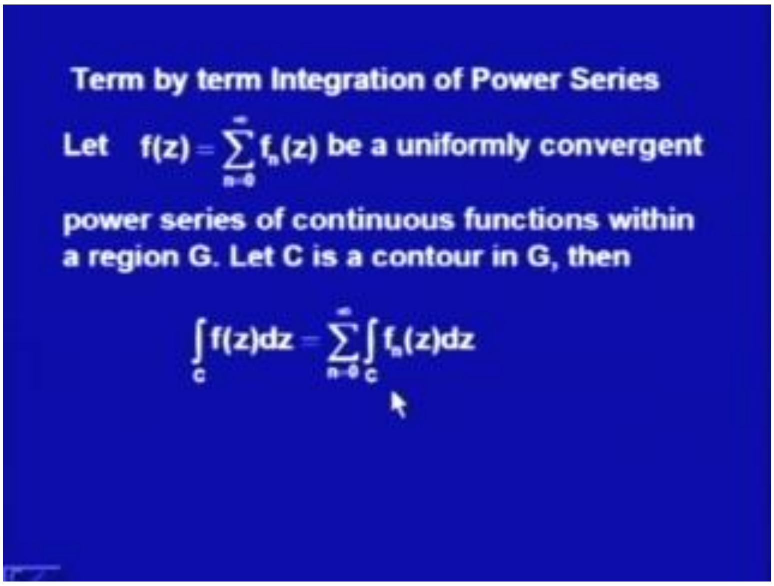
But the earlier series was a power series, it will be uniformly convergent. But, it is not uniformly convergent in a small neighbourhood about z about modulus z is equal to 1. Now, if the series is uniformly convergent. Then, we try a term by term differentiation of Power series.

(Refer Slide Time: 42:58)



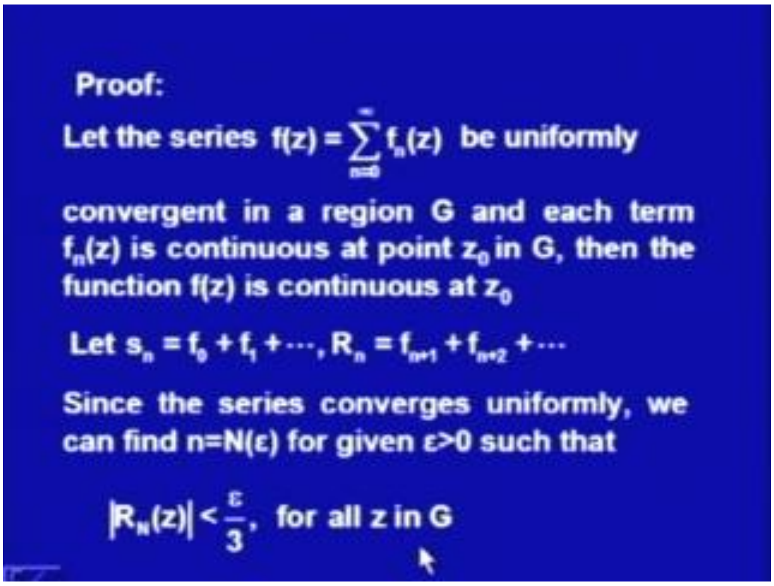
So for this, we have a result, which states that let f z is equal to a power series in powers of z minus a, with non zero radius of convergence. Then, f is analytic in a disk which is centred at a and of radius R. Then, f dash z is equal to summation n is equal to 0 to infinity n times c n into z minus a raise to power n minus 1. So, if we term by term differentiate this series. We will get this and this series is converging to the derivative of this function f of z. So, term by term differentiate is possible.

(Refer Slide Time: 43:50)



Similarly, term by term integration of power series is also possible. Let f z is equal to summation and n is equal to 0 to infinity, f and z be uniformly convergent power series of continuous functions, within a region in G. And, let C is a contour in G. Then, f z d z over integral over c is equal to summation, n is equal to 0 to infinity and integral of f n z d z. So, this is the meaning of term by term integration.

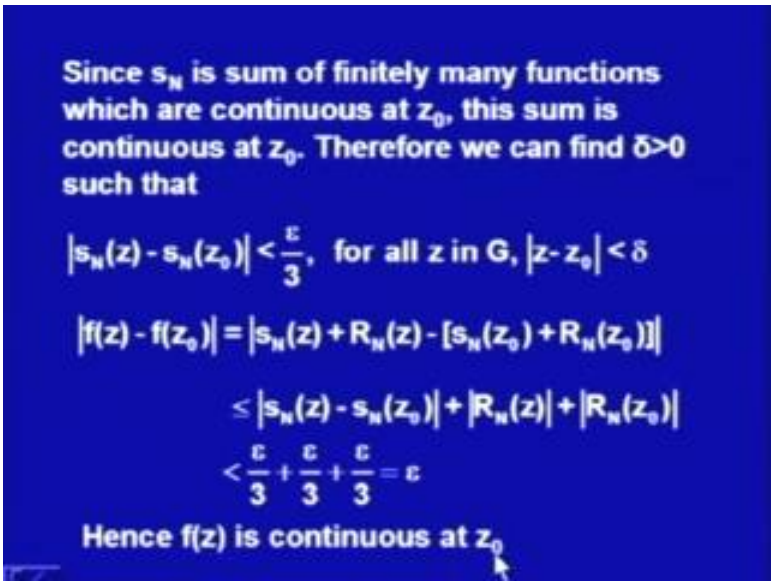
(Refer Slide Time: 44:25)



Now, let us prove this. Let the series f z is equal to summation n is equal to 0 to infinity f n z be uniformly convergent in a region G. And each term f n z is continuous at point z naught in G. Then, the function f z is continuous at z naught. So, we first prove this result. And then we will prove the, term by term integrability. So let us consider, S n is equal to f nought plus f 1 and so on. That is sum of first n terms and remainder term is f n plus 1, plus fn plus 2 and so on.

Since, the series converges uniformly that is been given to us. We can find n which is a function of epsilon. For given epsilon greater than 0 such that R N z modulus of this is less than epsilon by 3. For all z in G, so I have taken arbitrary value as epsilon by 3. So this is by definition of convergence.

(Refer Slide Time: 45:30)

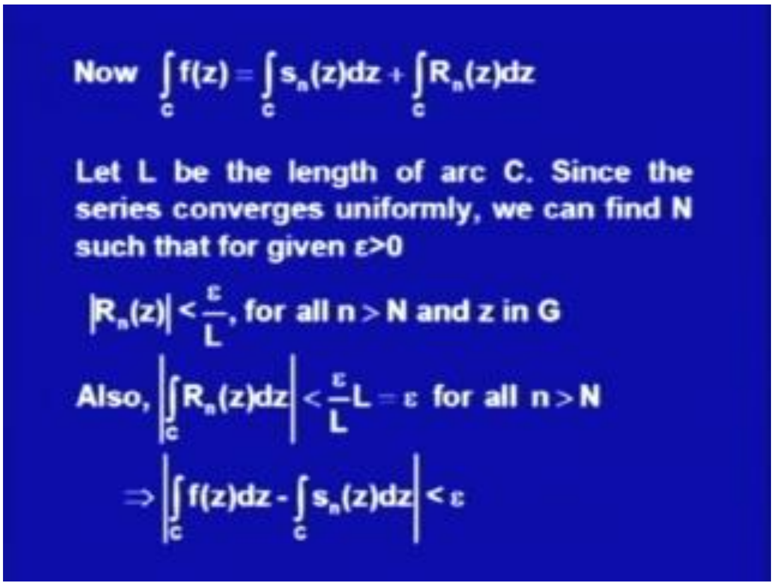


And, since S N is sum of finitely many functions which are continuous at z naught, this sum is continuous at z naught. Therefore, we can find delta greater than 0 such that S n z minus S n z naught. So, this is S N is the valuated at z. And this is at z naught. This can be made smaller than epsilon by 3. For all z in G provided we can find delta such that z minus z naught is less than delta.

So, for this, in this, this, this difference will be less than epsilon by 3. This is coming from the continuity. And, therefore, we can write down f z minus f z naught as S N z plus R N z. This is for f z and for f z not I write S N z nought plus R N z naught. So, this will be less than S N z minus S N z naught plus R N z modulus plus R n z naught modulus. Now this modulus is taken as less than epsilon by 3.

And this is uniformly convergent. So, it is again taken epsilon by 3 and for z naught also. We have also, we this also is less than epsilon by 3. And that makes f z minus f z not is less than epsilon. On this proves that the difference is as f z minus f z naught is less than epsilon and this proves that f z is continuous at z naught.

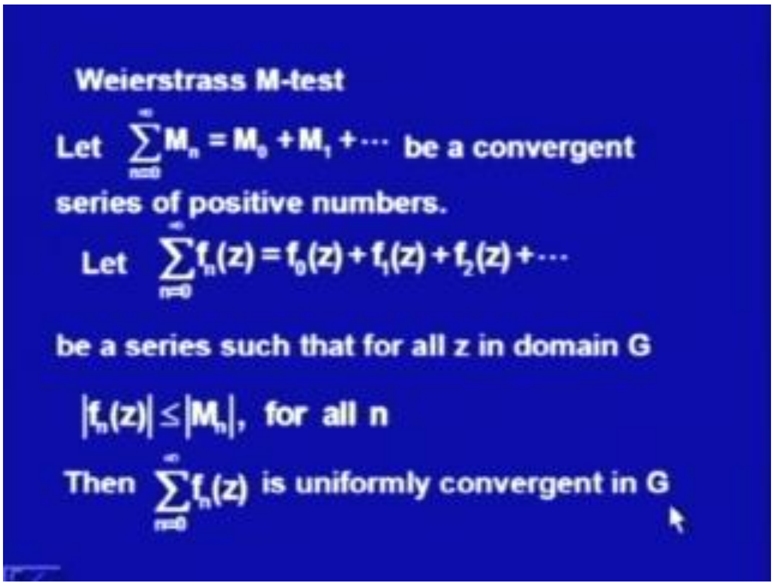
(Refer Slide Time: 47:01)



Once this is done, we write down integral f z over integral over the curve c is equal to S n z d z over c plus R n z d z over c, so integration over c. So this, we can write, let L be the length of arc C. Since, the series converges uniformly. We can find N such that for given epsilon greater than 0. R n z is less than epsilon by L. For all n greater N and z in G. This is due to uniformly convergence which is being given to us.

So, R n z d z integral and then taken its modulus will be less than epsilon by L. This, we have taken and this is length of the curve c is L. So, this is less than epsilon for all n greater than N and this implies. That integral f z d z over the curve c minus S n z d z integral over c is less than epsilon. This proves our result that term by term integration is possible. On similar lines we can prove the result which we have already stated. But have not given the proof that is the term by term differentiation of the series. So, we can go about this proof and result can be reproduced. Now, we will discuss Weierstrass M-test for Uniform convergence.

(Refer Slide Time: 48:37)

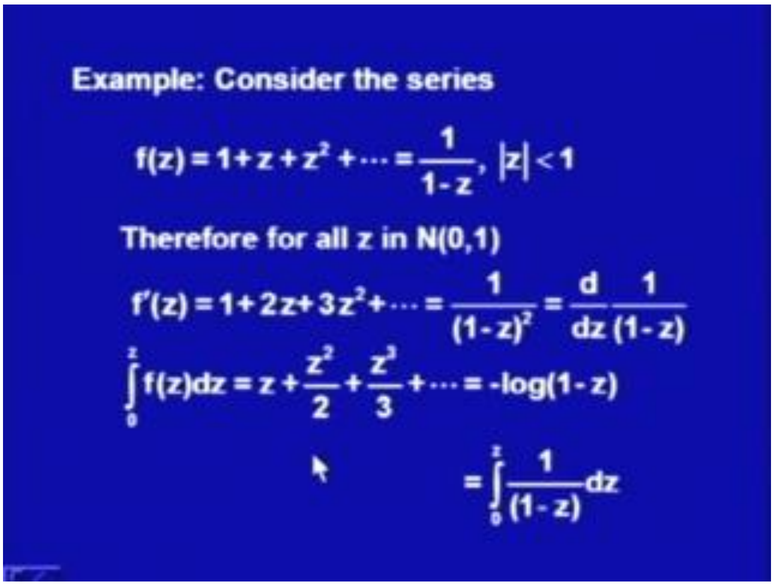


So let us consider, a series of positive terms summation n is equal to 0 to infinity M n. So, we have M naught, plus M 1 plus M 2 series. We have convergent series then this series, of functions f n z n varies from 0 to infinity. That is f naught z plus f 1 z, plus f 2 z, and so on. So, this is again, this is another series.

Now, we say that this series be a series such that for all z in a domain G satisfying this condition. That is f n z modulus is less than equal to M ((Refer Time: 49:17)), for all n. The idea is term by term. This term is smaller than this, f 1 z is smaller than M 1 and so on. So the term by term, if you compare then terms of this series are smaller than terms of this series. So if that is the case, then if this series convergent.

If this series is convergent, then this series will also be convergent. So accordingly, summation n is equal to 0 to infinity, f n z is uniformly convergent in G. So, if this is possible. Then the series is uniformly convergent in G. So this result can be used to establish the uniform convergence of given series. So, what we have to do is, we have to find out corresponding M naught, M 1 etcetera, and if the resulting series is convergent, then this series is uniformly convergent. Once the series is uniformly convergent then term by term differentiation, integration etcetera are possible.

(Refer Slide Time: 50:19)



So let us, apply this. To this geometric series f z is equal to 1 plus z plus z square and so on. And we know that this is 1 minus z, when modulus z is less than 1. So therefore, for all z in this interval in this neighbourhood centred at z is equal to 0 and radius 1. We can find f z is equal to f dash z is equal to 1 plus 2 z plus 3 z square. And we know that, this series is nothing but 1 upon 1 minus z whole square.

And we know that this is nothing but d by d z of 1 upon 1 minus z. So, the idea is this series is being uniformly convergent. And if we term by term differentiate this series, and we get this sum of this series is this. We know from we can see that, this is converging to this series. And we also know that, this is nothing but derivative of this. So this establishes the term by term differentiation of uniformly convergent series.

Similarly, if we integrate this series term by term then on the right hand side will have this series. And we know that, this series is nothing but the series for minus log 1 minus z. And if you take it is integral it is nothing but 0 to z 1 upon 1 minus z. So this also establishes term by term integration for the given function f of z.

Viewers, today we have discussed Laurent series representation of a function. If the function is analytic in a domain then we can represent this function in power series of z by Taylor series. But, if the function is not analytic at certain points in the domain then we can always enclose those points by a circle of some suitable radius. Then, in the annulus region, if the function remains analytic then we can represent this function by

Laurent series. So this is what we have done today and apart from this. We have discussed uniform convergence and term by term differentiation and integration of the series.

Thank you.