Mathematics - II
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Module - 2
Lecture - 21
Taylor Series

Welcome viewers, today we are going to discuss Taylor Series of complex variables, if we have a function f of a real variable x. Then we know that this function can be expressed in power series of x under certain conditions. The condition is, that function f x has derivatives of all orders. And we know that how to express cosine function, sin function and exponential functions in power series of x.

Now, the question is can. We express as these functions, when they become function of complex variables in the form of a power series. So that is what we are going to discuss today. But before we actually discuss Taylor series in complex variables, I will first introduce some basic concepts related to series.

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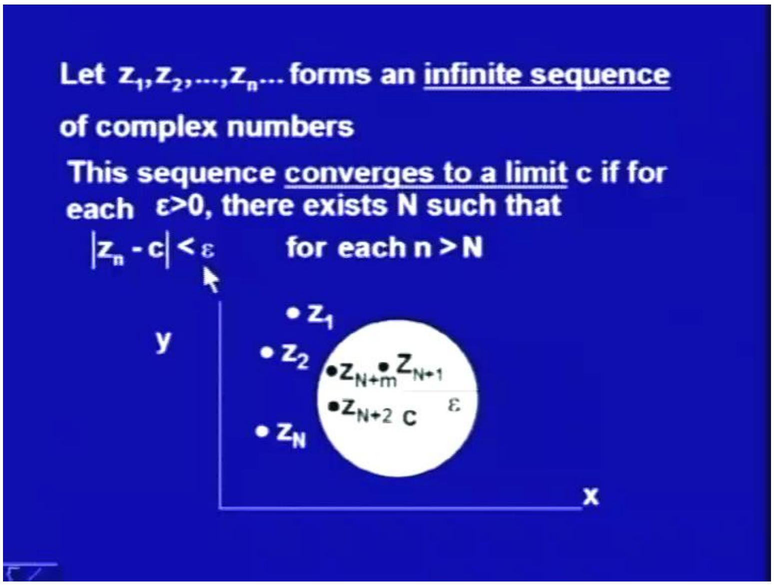

So the start with, let us say Z1, Z2, Zn forms an infinite sequence of complex numbers. Then this sequence converges to a limit c. If for each epsilon positive there exist N such that magnitude of Z n minus c or modulus of Z n minus c is less than epsilon for each n greater than N. This can be explained graphically like on the Z plane we have numbers Z 1, Z 2, Z 3, Z N, Z N plus 1 and so on.

Let us say this is the number c, then we can we draw a circle at c with radius epsilon in such a way that all the points after some N. They all lie in this circle, so we if we can for given epsilon, we can find sufficiently large N. So that after some N all the points will lie in this circle, if this is possible, then we say c is the limit of the sequence Z 1, Z 2, ZN.

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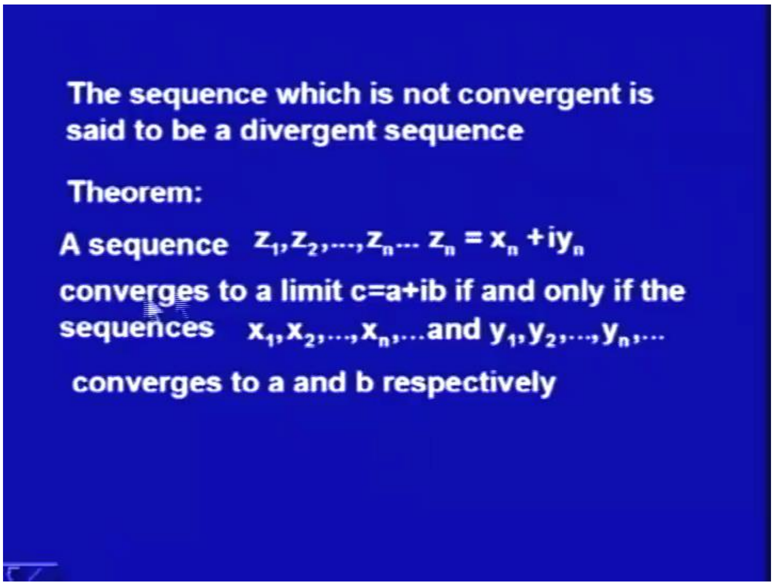

The sequence, which is not convergent is said to be a divergence sequence. Now, we state a theorem, according to this a sequence Z 1, Z 2, Z n which is in Z n can be expressed as X n plus i Y n. If this sequence converges to a limit c is equal to a plus i b. If and only if the sequence is of real parts X 1, X 2, X n and sequence of imaginary parts Y 1, Y 2, Y n converges to the respective parts a and b.

Then, we say the sequence is convergent and it converges to c is equal to a plus i b. The idea is that if we have a complex sequence. Then it can be equivalent to two sequences X 1, X 2, X n of real parts and Y 1, Y 2, Y n of imaginary parts. And if these two sequences converge to a plus i b a and b respectively. Then the sequence of complex numbers will converge to c is equal to a plus i b.

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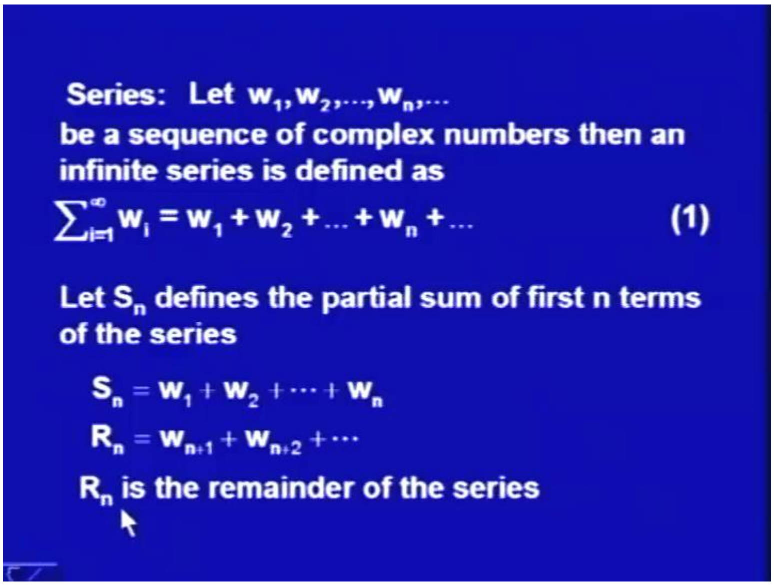

I am not proving this result, further we define series. So let W 1, W 2, W n be a sequence of complex number. Then an infinite series is defined as summation i is equal to 1 to infinity W i is equal to W 1 plus W 2 plus W n and so on. So, this is a series. So let S n defines, the partial sum of first n terms of the series; that means, we have S n is equal to W1 plus W 2 plus W n. Then R n is sum of remaining terms starting from W n plus 1 plus W n plus 2 and so on. Then R n is called the remainder of the series.

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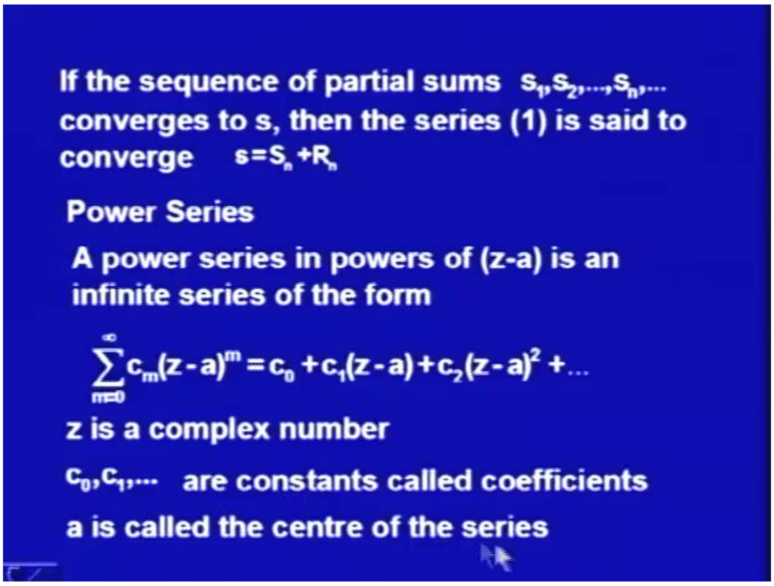

And, if the sequence of partial sums S 1, S 2, S n converges to S. Then the series 1 said to converge. So, this s is equal to S n plus R n. Power series is defined as a series in powers of Z minus a. It is an infinite series of the form summation m is equal to 0 to infinity C m Z minus a raise to power m and in expanded form we write the series as c naught plus c 1 Z minus a plus C 2 Z minus a square plus and so on.

We call it a power series, because here each term is power of sum power of Z minus a. In this case Z is a complex number, C naught C 1 etcetera these are constants called coefficients of the power series and a is called the centre of the series. This a is called centre of the series. When a is equal to 0. Then this series will be a power series in powers of Z only.

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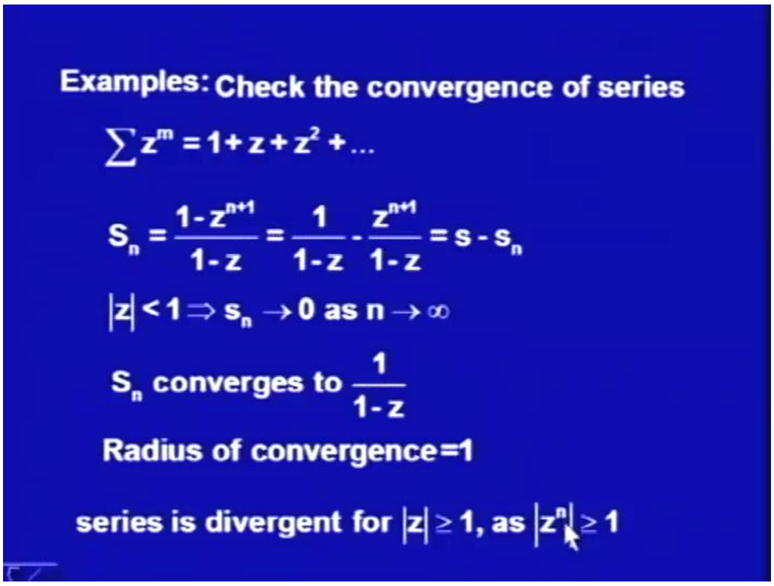

Now in this example, I will check the convergence of this series 1 plus Z plus Z square and so on. One can notice that it is geometric see geometric series. So, if I consider the first n times of this geometric progression them. S n is equal to 1 minus Z raise to power n plus 1 over 1 minus Z. This is the well known formula for a sum of n terms of the geometric series.

Then, this can be written as 1 upon 1 minus Z minus Z n plus 1 over 1 minus Z I write this as S and this as S n. Now, when Z is less than 1 then this term this term will tend to 0 as n tends to infinity. Because as we increase powers this term will becomes smaller and smaller. So, this will tend to 0 and that simply means that S n converges to 1 upon 1 over 1 minus Z.

So, we can say that this geometric progression will converge to 1 over 1 minus Z provided mod Z is less than 1. In this case, radius of convergence is 1 and this series is divergent for mod Z greater than equal to 1 as Z raise to power n will become larger and larger as Z as n increases. So, the series is divergent for mod Z greater than equal to 1.

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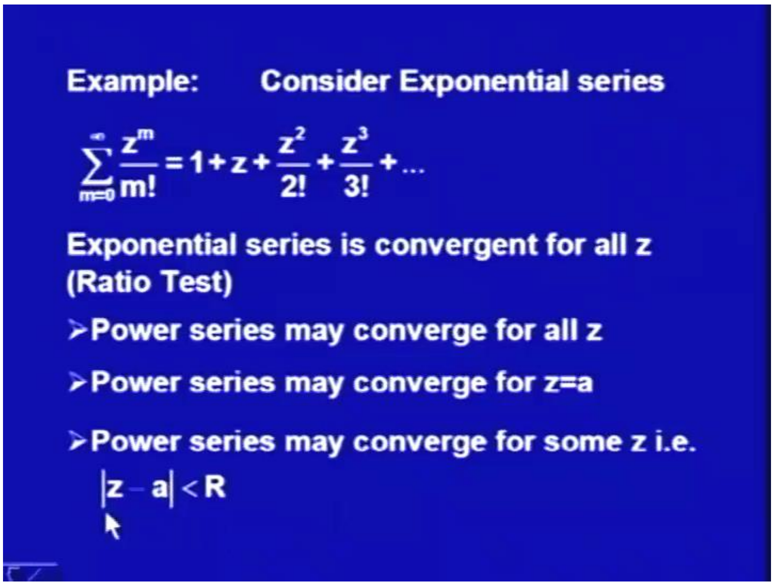

In another example, if we consider the exponential series, which is Z raise to power n divided by factorial m is infinite series. Or this is equal to 1 plus Z plus Z square by factorial 2 plus Z cube of factorial 3 plus and so on. Then this exponential series is convergent for all Z and this can be proved by the ratio test.

So, we have seen that the power series may converge for all Z. The other possibility may be that given power series may converge for some values of Z is equal to a. And in the third case power series may converge for some values of Z. But for others, it may not converge. Like, we can say that power series is convergent in the region when Z minus a is less than R. But for others, it may not converge, so these are different possibilities.

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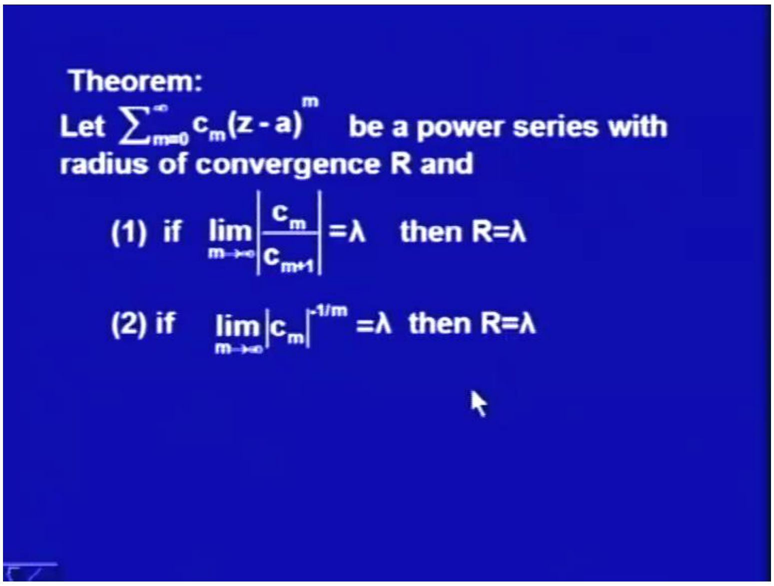

So this theorem says that if we have a power series at centre at a with radius of convergence R. And the series is given as the series is such that limit as n tending to infinity of C m over C m plus 1 is equal to lambda. Then radius of convergence of the series is lambda. Also, if C m rises to power minus 1 by n the limit n tending to infinity is lambda. Then again the radius of convergence is lambda; I am not giving the proof of these results.

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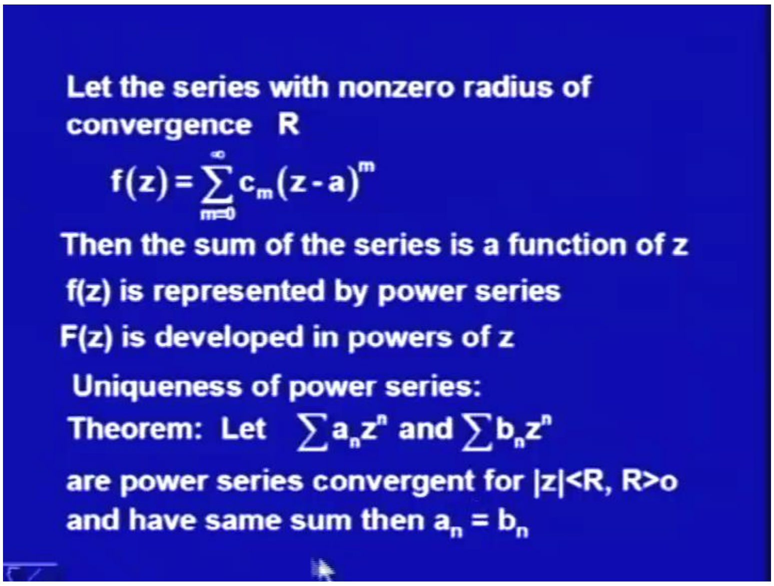

Next, let the series with nonzero radius of convergence R. Let us say it is represented as this. Then the sum of the series is a function of Z. So, this series converges and this will be a function of Z. So, we say the sum of the series is a function of Z.

Now, we say f Z is represented by power series. So, if we can write f Z this way, then we say that f Z is represented by a power series. Or F z is developing powers of Z such type of representation is unique and uniqueness of power series is established. In the form of the theorem is suggest that summation a and Z n plus summation b n Z n.

Let us say we have two different series these are power series, which are convergent for this region mod z less than R a R being positive. Then they have same sum that is say this also represent F Z and this also represent F Z, then an and b n will be equal. So, if there are two power series representing the same function? Then that series is to be unique.

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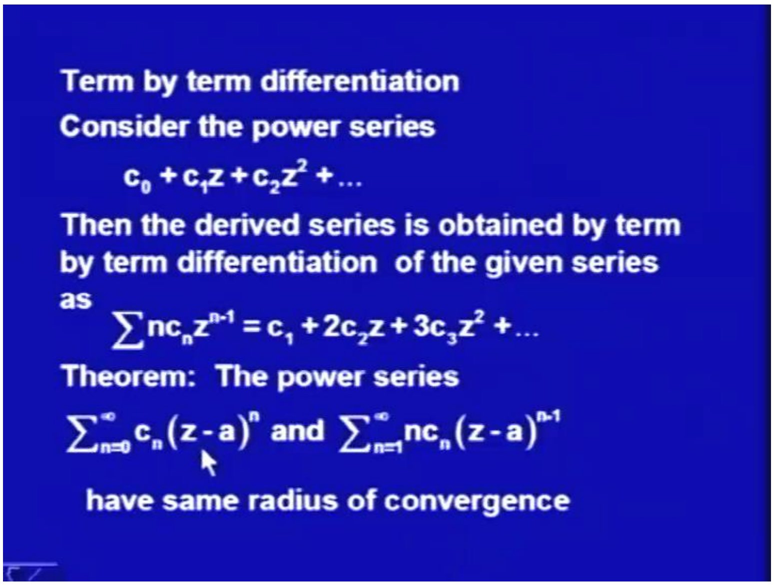

Is one more result again, we are not giving the proof of the theorem. And according to this, if we have a power series C not plus C 1 Z plus C 2 Z square then the derived series, which is obtained by term differentiation of the given series as this. We if we differentiate this term by term. Then it is C 1 plus twice C 2 Z plus thrice C 3 Z square and so on or we can write in the compressed form as this.

Then, this is a power series for the function f dash Z. So, if we write in the form of theorem. Then summation C n z minus a raise to power n and summation n C n Z minus a raise to power n minus 1 and varies from 1 to infinity in this case and 0 to infinity in this case. Then they have the same radius of convergence and they converge to if these converge to f Z. Then this will converge to f dash Z.

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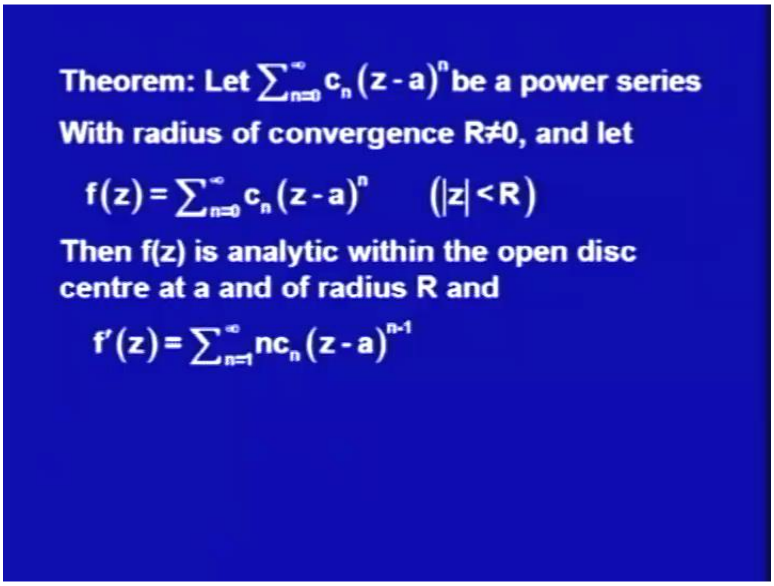

Next, if we have a power series with radius of convergence positive or not equal to 0. And let, f Z is represented as this whenever mod Z is less than R or the radius of convergence of the series is R. Then f Z is analytic within the open disc centre at a radius a center at a and of radius R.

And, f dash Z is equal to summation n is equal to 1 to infinity n C n Z minus a raise to power n minus 1. This means, that if this power series is represented as f Z and f Z is analytic. Then its derivative will also be represented as a power series and it is analytic not only this. But, if you differentiate this series term by term then this will represent the function f dash Z. So far we have discussed some concepts related to sequence series power series and convergence. I have described these concepts angle given. You some results in the form of theorems, but I have not prove those theorems.

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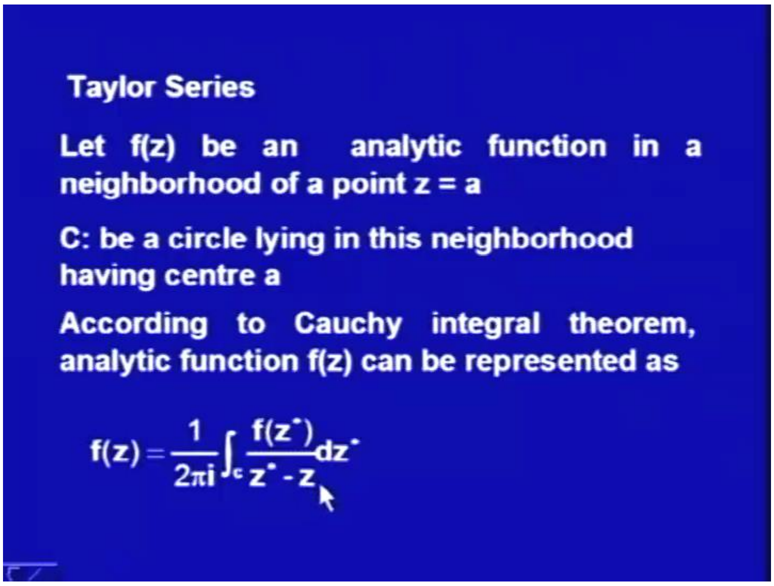

And, now wearer in a position to discuss Taylor series to develop Taylor series let us consider an analytic function f Z. By this I mean to say that f Z is analytic in the neighborhood of a point Z is equal to a. Let us say, C be a circle lying in this neighborhood having centre at a. According to Cauchy's integral theorem analytic function, f Z can be represented as f of Z is equal to 1 upon 2 pi i integral over the close curve C of f Z star divided by Z star minus Z into d Z star; that means, we are integrating this integral over the curve C.

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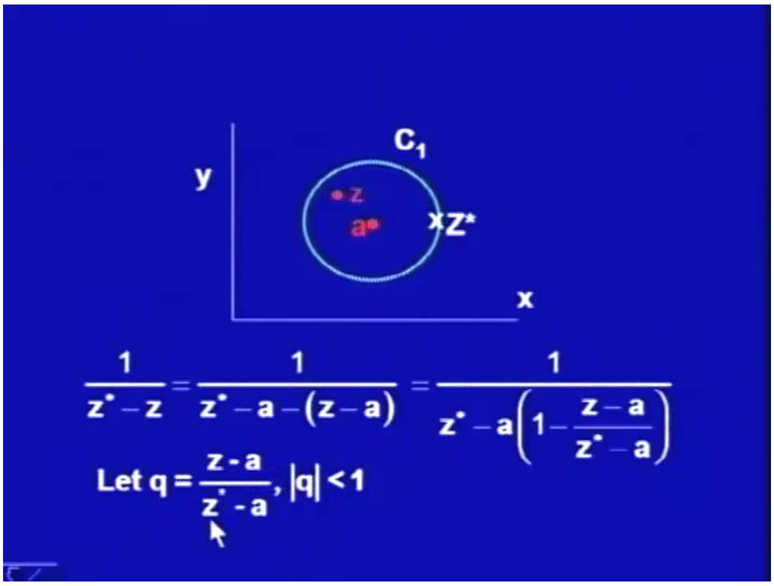

Let me explain this, I have in the Z plane the function f Z, which is analytic in a domain. And in this domain, I have a point a and at a I draw a circle, I call this circle is C 1. And

Z is a point inside this circle and Z star is a point inside Z star is a point on the circle C 1. Then one upon Z star minus Z is equal to 1 upon Z star minus a minus Z minus a. So, I have added and subtracted a here and by this from this, I can write one upon Z star minus a into 1 minus Z minus a divided by Z star minus a.

So, what I am doing is? I am taking Z star minus a outside this parenthesis. Now here, this is a point Z inside the circle and Z star is on the boundary; that means, this distance Z minus a is less than the distance Z star minus a. So, if I denote q as Z minus a divided by Z star minus a. Then I know that modulus of q is less than 1 and if it is less than 1.

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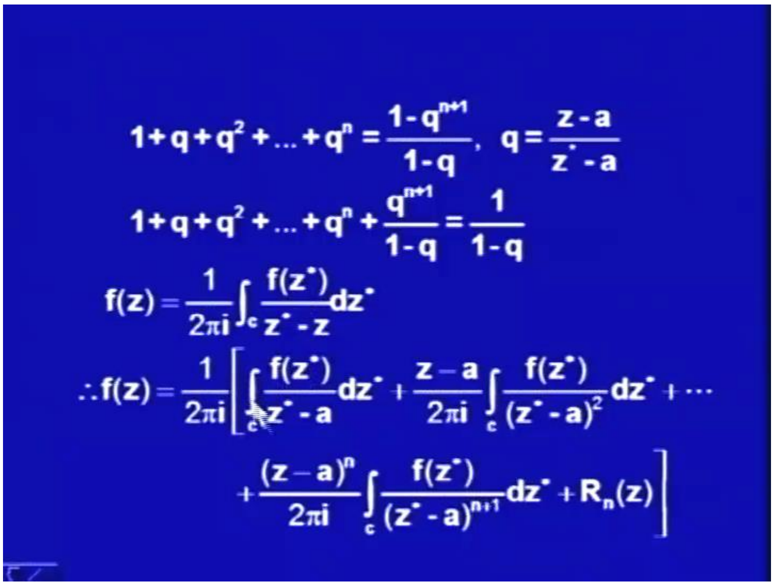

Then, this is a geometric series and in fact these are the first n terms of geometric series 1 plus q plus q square plus q n. And we know that this sum is equal to 1 minus q raise to power n plus 1 over 1 minus q, q being Z minus a divided by Z star minus a. So, if I take this term on this side. Then it is 1 plus q plus q square plus q n plus this term. I am taking on the other side, so this is equal to 1 over 1 minus q.

Now with this simplification, we come back to the integral 1 upon 2 pi i integral f Z star over Z star minus Z d Z star representing f of Z. And we write it as 1 upon 2 pi i f Z stat Z star minus Z into Z star minus a d Z star. And this time corresponds to this 1 plus z minus a into 2 pi i integral f Z star Z star minus a whole square d Z square d Z star. This will correspond to the second term. And the last term will be this, so we have f z represented as this.

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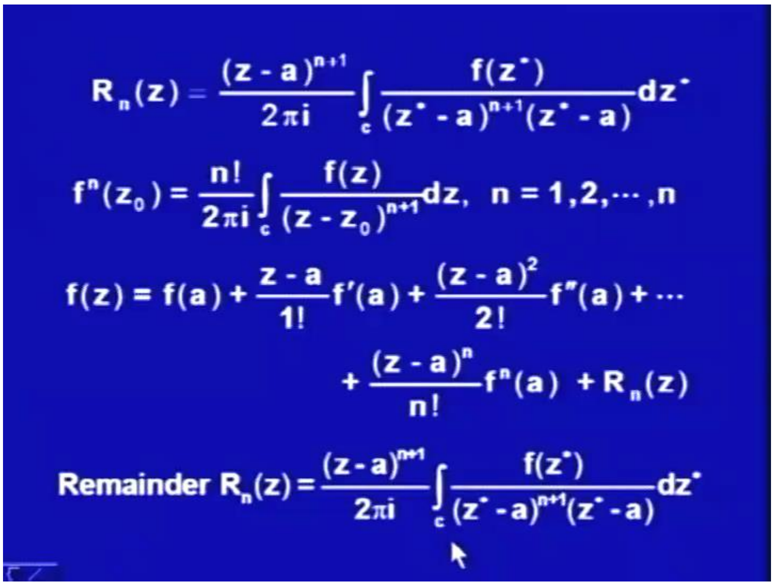

Here R n Z is equal to Z minus a raise to power n plus 1 upon 2 pi i integral f Z star Z star minus a raise to power n plus 1 into Z star minus a d Z star. And we know from the our formulae for derivatives f n Z naught is equal to factorial n over 2 pi i integral f Z over Z minus Z naught raise to power n plus 1 d Z.

Now, this is a known formula for n is equal to 1 2 to n. So, n'th derivative will be given by this integral over the closed curve C. Using this one can write our expression as f of Z is equal to f of a plus Z minus a by factorial 1, f dash a plus Z minus a square by factorial 2 as multiplied by f double dash a and so on.

And, the last term is Z minus a raise power n divided by factorial n f n a plus. The remainder term R n Z. Now, the remainder term R n Z, which is z minus a raise to power n plus 1 over 2 pi i f Z star divided by Z star minus a n plus 1 into Z star minus a d Z star is integrated over the closed curve C.

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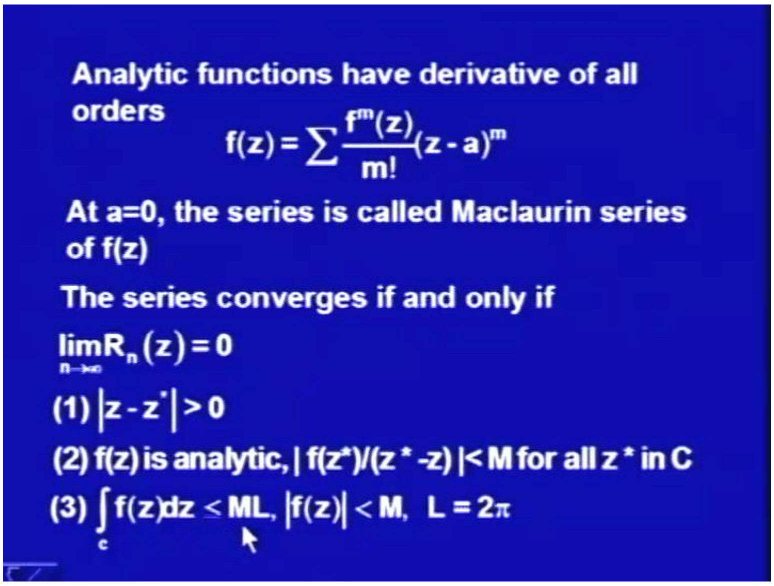

This can be simplified and may be shown that this tends to 0 as n tends to infinity. And with this the analytic function f Z is represented as summation f m Z over factorial, m into Z minus a raise to power m at a is equal to 0, this series called Maclaurin series of f z. The series converges if and only if limit R n z tends to 0 as n tends to infinity.

So this is, we have to be proved and we can notice that Z minus Z star modulus is positive. And f Z is analytic; that means, f Z star over Z minus Z star is less than M for all Z star in C. And in third case, is that integral f Z d Z is always less than equal to M times L, where f modulus of f Z is less than M and L is the length of the closed curve C. C being a circle. So, L is equal to 2 pi if it is a unit circle, then L is equal to 2 pi.

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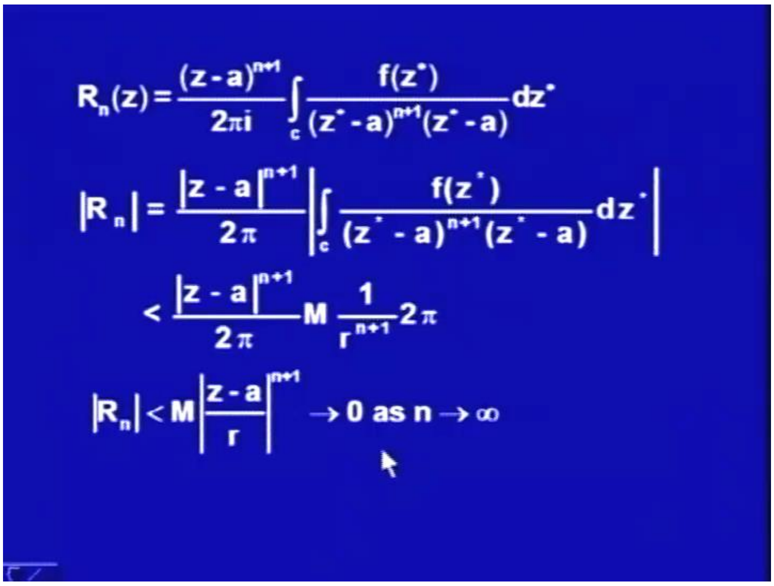

Using these facts, we simplify the expression for R n Z, which is given as this. So, let us consider the modulus of R n Z. And this is modulus of this value divided by modulus of 2 pi i, which is 2 pi and modulus of this integral. And this is written as M times 1 upon r n plus 1 into 2 pi. And this simplifies to modulus R n is less than M times Z minus a divided by r raise to power n plus 1 and this tends to 0 as n tends to infinity. Because, Z minus a divided by r is less than 1. And as n increases, this power increases. And this is less than 1. So, this will tend to 0 as n tends to infinity. So, this proves that R n tends to 0 as n tends to infinity. And that is why we have the Taylor series representation.

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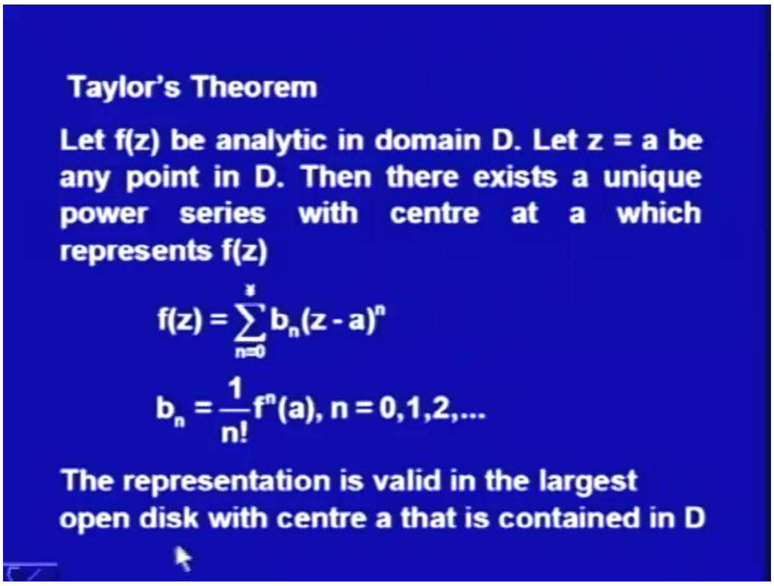

Now will have formally state, the Taylor's theorem. According to this like f Z be analytic in domain D. Let Z is equal to a be any point in D. Then there exists a unique power series with centre at a, which represents f Z. So, f Z is represented as n is equal to 0 to infinity b n Z minus a raise to power n, where the coefficients b n are given by 1 upon factorial n multiplied by f n a. The n'th derivative of f Z evaluated at centre Z is equal to a. And this is true for all values of n 0 1 2 and so on. So, this is the Taylor's series and the representation is valid in the largest open disk with centre a, that is contained in the domain D.

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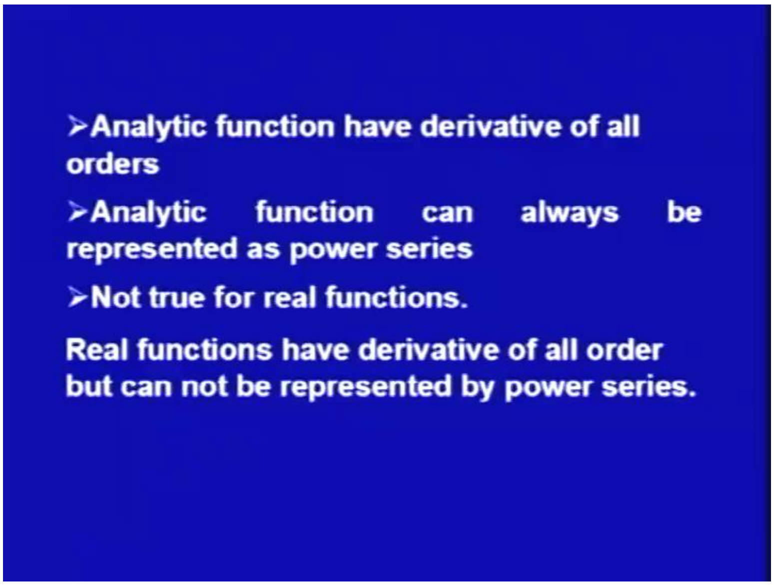

By this I mean to say that analytic functions have derivatives of all orders that is, why we can represented as a Taylor series. Because, we have to calculate different b n's and b n's they are evaluated in terms of derivatives. Analytic function can always be represented as power series, because they have derivatives of all orders. But, this is not true for real functions. Real functions have derivatives of all orders, but cannot be represented by power series. So that is the basic difference between function of a complex variable and function of a real variable.

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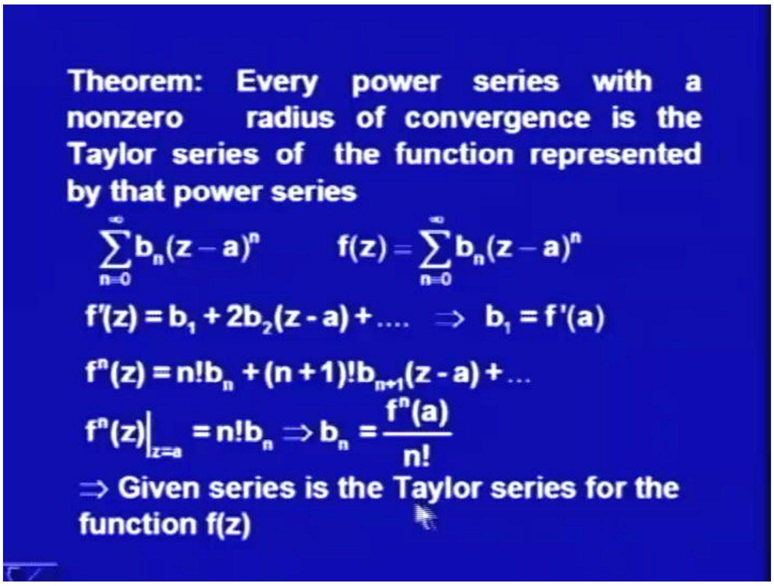

Next if every power series with the nonzero radius of convergence is a Taylor series of some function represented by that power series. So, let us say was have a power series b n Z minus a raise to power n n is equal to 0 to infinity. And let us say it represents a function f Z. Then f Z is f Z is the Taylor series for them. This is the Taylor series for the function f of Z.

And in this case, f dash Z is equal to b 1 plus twice b 2 Z minus a plus and so on implies that b 1 is equal to f dash a. So if you differentiate this, we will get b 1 is equal to f dash a and f n Z is equal to the n th derivative of Z is equal to factorial n over b n pus n plus 1 factorial b n plus 1 Z minus a plus and so on.

So, this is mainly differentiated this series n time. And if you put z is equal to a. Then all these terms will vanish and will have b n is equal to f n Z over factorial n and by this. I mean to say that if we can express if this series is equal to f Z. Then this is nothing but the Taylor series. So, it is a unique representation. So this is, what we have, so given series is a Taylor series for the function f of Z.

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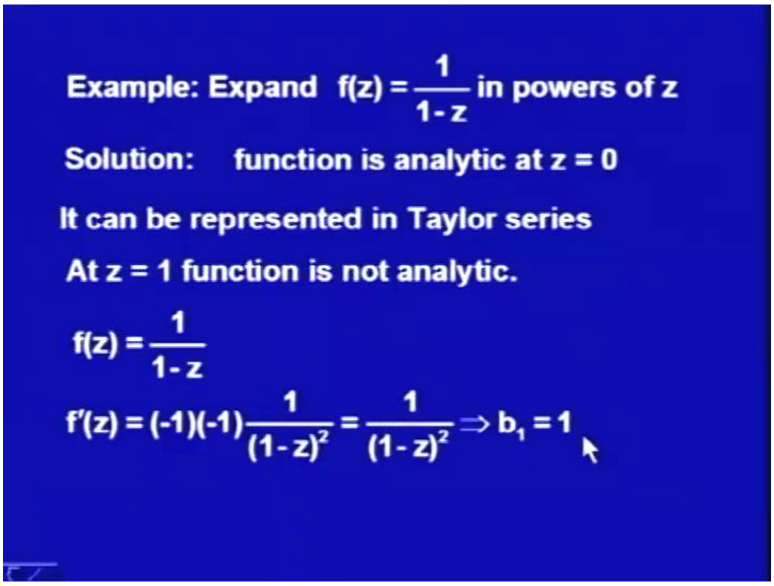

So, let us illustrate with some examples. So, we have we given a function 1 over 1 minus Z and we have to expand it in powers of Z. So, we know that this function is analytic at Z is equal to 0. If we have to express this is powers of Z. Then it should be analytic at Z is equal to 0. So, it can be represented in Taylor series.

At Z is equal to 1 function is not analytic it is not define at Z is equal to 1. So, there is no question of analyticity at Z is equal to 1. Then if you write f Z is equal to 1 over 1 minus

Z. Then f dash Z comes out to be minus 1 into minus 1 multiplied by 1 over 1 minus Z square. So, we evaluate the derivative at Z is equal to 0 and b 1 comes out to be 1.

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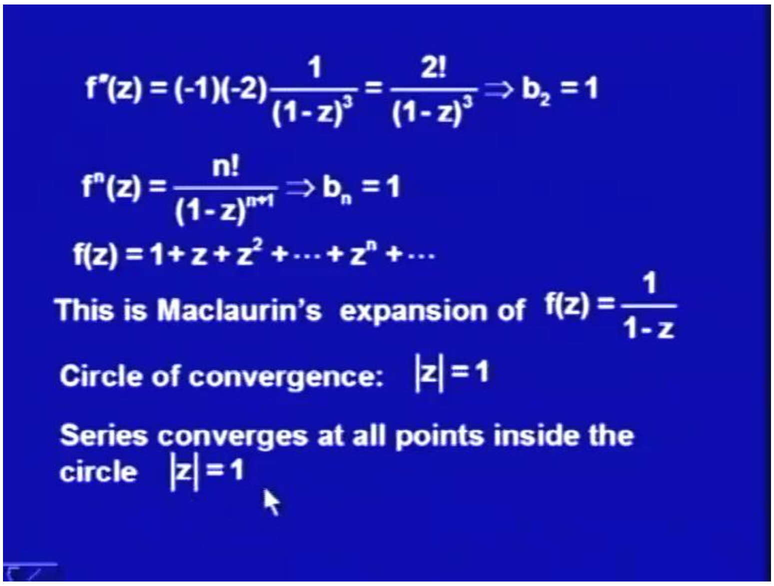

Similarly, for this second coefficient b 2, we have to evaluate the second derivative at Z is equal to 0. So, evaluate the second derivative it comes out to be minus 1 into minus 2 into 1 over 1 minus Z cube. And this simplify to factorial 2 over 1 minus Z cube evaluated at Z is equal to 0 this gives me b 2 is equal to 1.

Similarly, we can calculate f n Z is equal to factorial n over 1 minus Z raise to power n plus 1. This is n'th derivative and evaluated at Z is equal to 0. This gives me b n is equal to 1. So, if we write down the coefficients values of these coefficients. Then f Z is equal to 1 plus Z plus Z square plus Z n and so on. So, this is the power series or we call it as Taylor series of the function 1 over 1 minus Z, this is the Taylor series.

Further, we have seen that this is this series converges to 1 over 1 minus Z. Because, this is geometric progression geometric series, so this converges to 1 over 1 minus Z. So, this is the unique representation it is 1 over 1 minus Z and its power series is this, its Taylor series is 1 plus Z plus Z square plus Z rise to power n and so on.

So, this is unique representation this series, which we expand in in powers of z; that means the center is 0, this is called as Maclaurin's expansion or Maclaurin series for the function f Z is equal to 1 1 over 1 minus Z. The circle of convergence is modulus Z is equal to 1. Because of maximum distance, from the similar point of the function similar point is the point, where function is not analytic.

And, we are seen that this function is not analytic at Z is equal to 1. It is analytic at z is equal to 0. But, not analytic at z is equal to 1. So, the maximum distance is 1. So, circle of convergence is modulus z is equal to 1. So, series converges at all points inside this circle mod Z is equal to 1.

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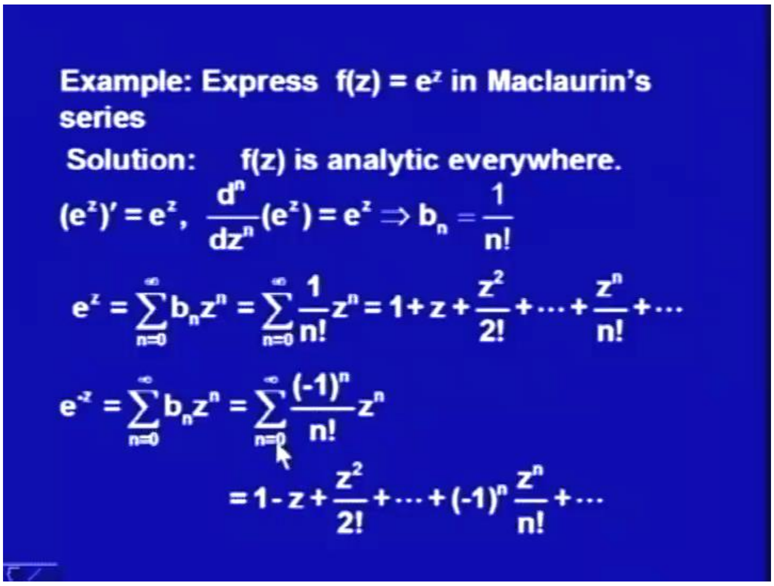

So this, we try to explain the concept of Taylor series for the given function. Now, we try to expand e Z in Maclaurin's series. So, we know that this function e Z is analytic. Everywhere, so we can try to represent this function as a Maclaurin series for this. We need to calculate various derivatives. So, derivative of e Z is e Z.

And in fact, the n'th derivative of e Z is also e Z. So, this gives me that b n is equal to 1 upon factorial n. And this means e Z is represented as summation n is equal to 0 to infinity 1 upon factorial n Z raise to power n or in expanded form. It is 1 plus Z plus Z square by factorial 2 and so on plus Z n over factorial n and so on.

So, the Maclaurin series for the exponential function is given as this. Further, if this is Maclaurin series for the function e raise to power Z, then what will be the Maclaurin series for the function e raise to power minus Z. So, if we replace Z by minus Z here. Then we will get the Maclaurin series for the function e raise to power minus Z. So, doing this proceeding this way, we say that e minus Z is equal to minus 1 raise to power n factorial n divided by factorial n into Z raise to power n. And that gives me 1 minus Z plus Z square by factorial 2 plus minus 1 raise to power n z raise to power n factorial n and so on. So, we have alternating positive and negative terms in this series.

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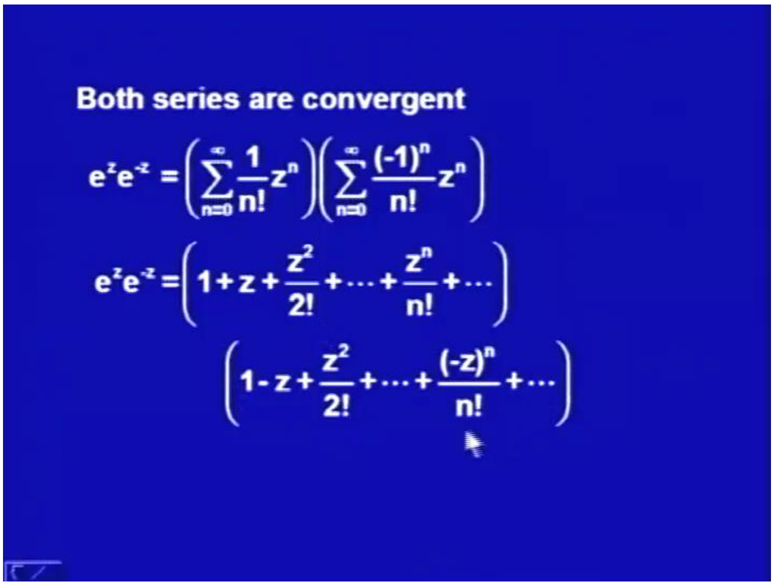

Now, since both the functions e Z and e minus Z are analytic. So, both the series are convergent and we multiply them. Then e Z into e minus Z is equal to summation, n is equal to 0 to infinity 1 over factorial n Z raise to power n. This is the series for e raise to power Z. And this is the series for e raise to power minus Z.

So, if you multiply them, then this is what we have the series in expanded form for e Z this is the series expanded form for e minus Z. If you multiply, then we collect various terms. So, we first multiply one by this plus Z times. This plus Z square by factorial 2 times this and so on. So, this way we multiply the two series and let us then collect various terms.

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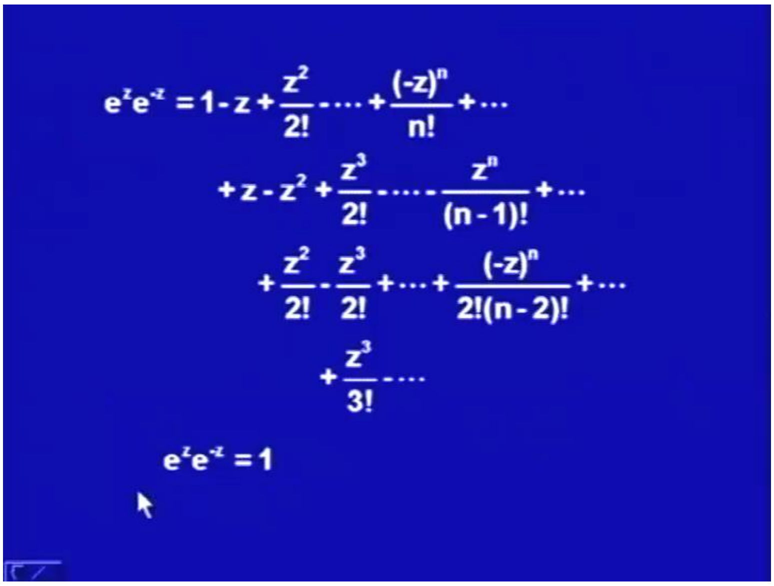

So, preceding in this manner this is the first term. Then Z is multiplied by the second series, so will have Z minus Z square plus Z cube by factorial 2 and so on. Then Z square by factorial 2 is multiplied by this series, this is the next term. Then a next term and so on.

So now you can notice that if is combine these two terms it is 0, so combining the powers of Z square. This is Z square by factorial 2. So, there will be simply Z square and this minus Z square. So, they cancel out. Similarly, if you consider Z cube by a facto Z the next term Z cube. Then this and this term will cancel out. And this term and next term here, will cancel out.

So, this way each and every term each and every power will be 0 and, what we have simply the first term 1. So that proves the result that e Z into e minus Z is equal to 1. So, we can multiply infinite series term by term and result remains ineffective. So, this is the result. We already know that e Z into e minus z is equal to 1. But, by actually multiplying 2 power series this also gives us 1. So, term by term multiplication of power series is possible.

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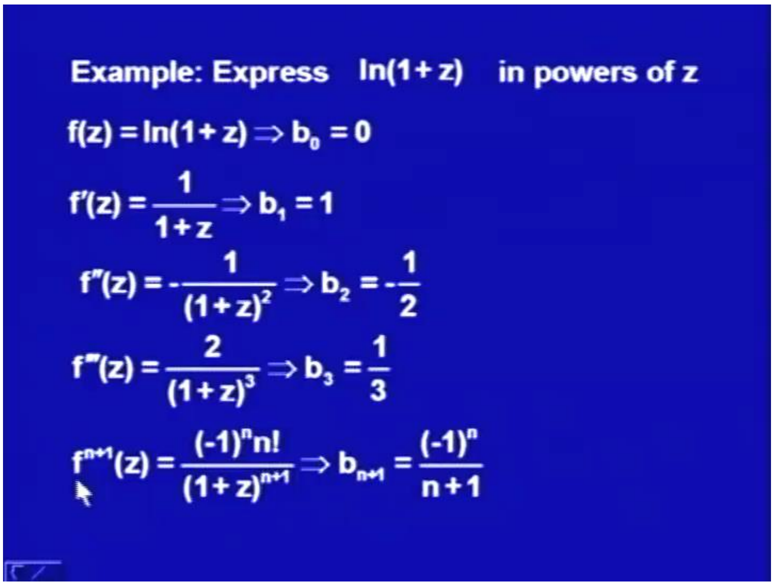

Now in this example, we try to express log 1 plus Z in powers of Z. For this purpose, we consider f Z is log of 1 plus Z differentiating. And then, writing Z is equal to 0 this gives me b naught is equal to this simply writing Z is equal to 0 here. It may give me b naught and this is nothing but 0, because log of 1 is equal to 0. Then differentiating this that is 1 over 1 plus Z and then writing Z is equal to 0 gives me b 1 is equal to 1.

Further we compute the second derivative by differentiating f dash z and this gives me minus 1 over 1 plus Z square and at Z is equal to 0 this gives me f double dash Z is equal to minus 1 giving me b 2 is equal to minus half. Next, we calculate the third derivative as 2 times 1 plus Z cube. And evaluating this derivative at Z is equal to 0 gives me b 3 as 1 by 3 and so on. We can calculate b n plus 1 Z, we calculate b n plus 1 from the n plus 1'th derivative. And this comes out to be minus 1 raise to power 1 divided by n plus 1.

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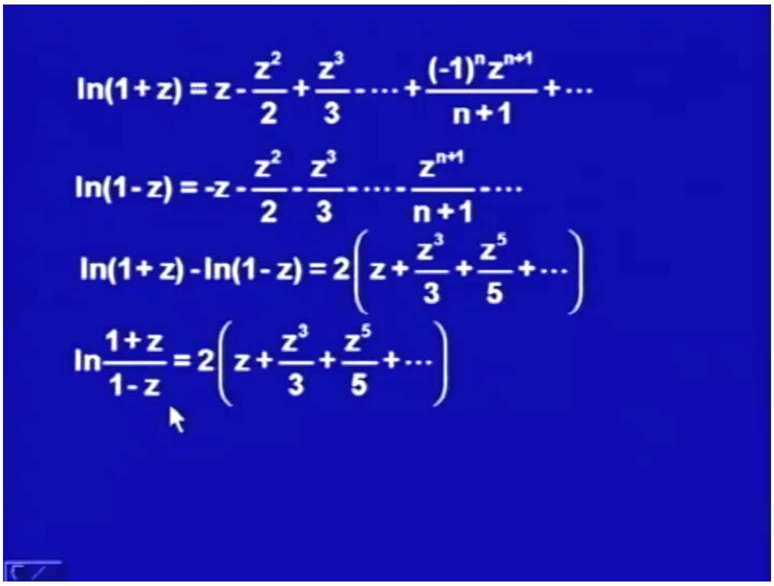

Substituting these values, I will get l n of 1 plus Z is Z minus Z square by factor Z square by 2 plus Z cube by 3 and minus 1 raise to power n Z n plus 1 over and plus 1 and so on. So that will be In 1 plus Z expansion. So, this is the Taylor series and since the centre is 0, we can call it as Maclaurin series.

Similarly, from this, we can derive Taylor series for the function l n of 1 minus z by replacing Z by minus Z in this series, so will have the series with all negative terms. So, if I subtract the two series that is l n of 1 plus Z minus l n of 1 minus Z is 2 times Z plus Z cube by 3 plus Z5 by 5 and so on. And this expression can also be recognized In of 1 plus Z over 1 minus Z is equal to 2 times Z plus Z cube by 3 plus Z 5 by 5 plus and so on.

So in this case, I have expressed l n 1 plus Z over 1 minus Z, this function as a power series in Z. So, this is Maclaurin series for this function In 1 plus Z over 1 minus Z. So, I am not calculating in the power series of this function from the basic principle. But, I am applying results for l n for power series of l n 1 plus Z and power series of l n 1 minus Z and I get this result.

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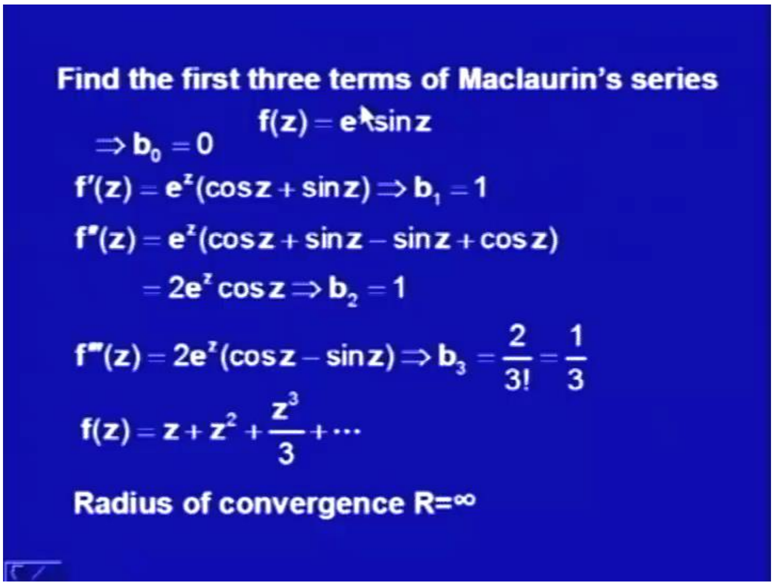

Now in the next example, we find the first three terms of Maclaurin series for the function e Z sin Z. Now in this case, we have to expand this function about Z is equal to 0. So, if you put z is equal to 0. Then f of Z will be 0 and; that means, b naught is equal to 0. We calculate various derivatives of this function and for first derivative it is e Z cos Z plus sin Z that gives me b 1 is equal to 1.

Differentiating this again give me, the second derivative and it is e z into cos z plus sin Z minus e Z times derivative of cos Z plus sin Z. That is minus sin Z plus cos Z and putting Z is equal to 0 gives me two raise 2 into e Z cos Z. That implies that b 2 comes b 2 is 1. Then f triple point Z is computed as this by simply differentiating this. And that gives me b 3 is equal to 2 by a factorial 3 minus 1 by 3.

And simplifying, we will get this Taylor series for the function f Z is equal to e Z sin Z. So, this series has radius of convergence R is equal to infinity. Because, there is no singular point for this function this function is analytic everywhere. Because, e Z function is analytic everywhere sin Z function is analytic everywhere. So, the product will become analytic everywhere. So, radius of convergence will be infinity or we will call such a function as entire function.

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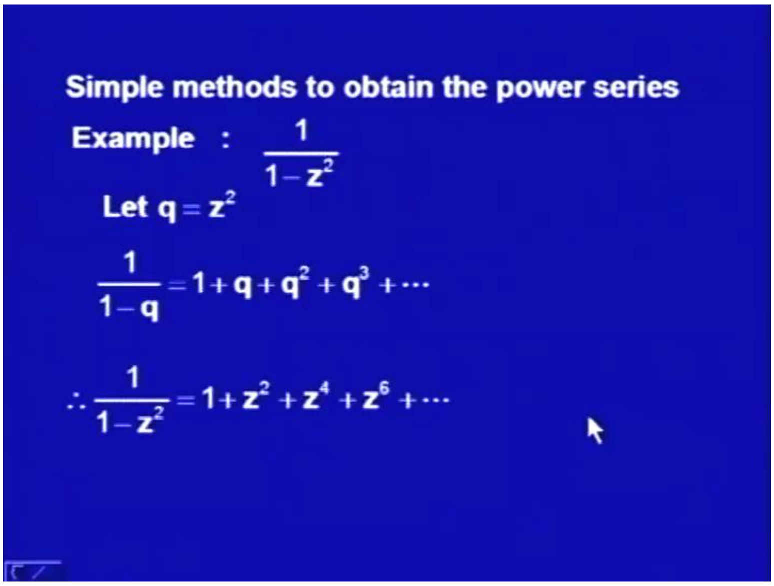

We have number of functions. So far, we have obtained there power series by applying them formulae by calculating derivatives. But now, we develop some simpler methods for obtaining power series. We do not have to always find the derivatives. And then, express it in a power series, because, there is a uniqueness of power series. So, it is not necessary that we always obtain derivatives finding a derivative is a difficult task.

So, to simplify the matters, we try to express a given function in power series and, because of uniqueness whatever power series. We obtain that will nothing but the Taylor series for Z is equal to a and Maclaurin series, when Z is equal to 0 is the centre. So for this, let us consider q as Z square. So, we can write down this function as 1 over 1 minus q. And we know, this can be expressed as 1 plus q plus q square plus q cube and so on.

So, substituting Z q is equal to Z square. We have power series for 1 over 1 minus Z square. And it comes out to be 1 plus Z square plus Z 4 plus Z 6 and so on. So, we do not have to calculate derivatives from this geometric progression. We can find out the power series for some functions we can use this technique.

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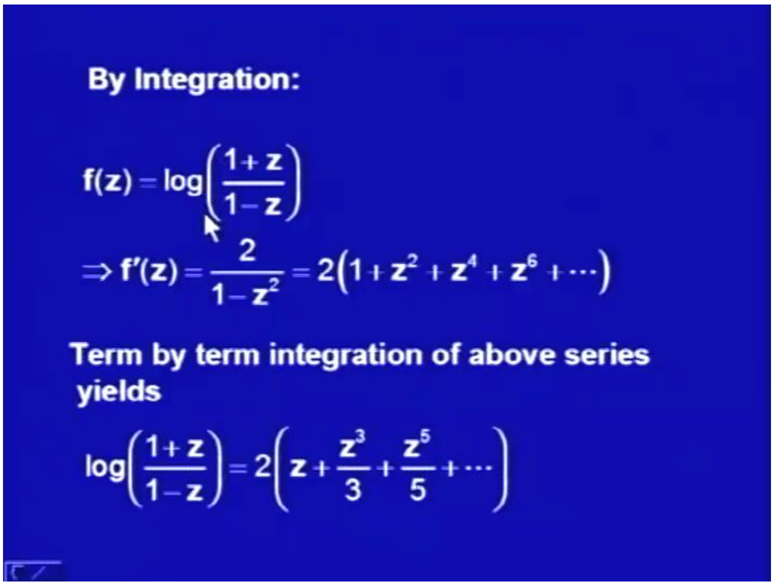

In another example, we calculate various coefficients of a we derive at Taylor series by using integration. So, let us consider f Z is equal to log 1 plus Z over 1 minus Z then when we differentiate this. Then this comes out to be f dash Z comes out to b 2 over 1 minus Z square. So we know that 1 over 1 minus Z square, can be expanded in this form. So, we write down 2 over 1 minus Z square as 2 into 1 plus Z square plus Z raise to power 4 plus Z raise to power six and so on this is my f dash Z.

So, from this Taylor series I can obtain f Z by integrating this series. And then, I apply term by term integration on the series. So, integration gives me f of Z which is nothing but log of 1 plus Z over 1 minus Z. So on left hand side will have log 1 plus Z over 1 minus Z. And when we integrated right hand side it is 2 times Z plus Z cube by 3 plus Z 5 by5 and so on. So, this is log of 1 plus Z over 1 minus Z. Now, we have obtained power series representation for this function as this. But in my earlier example, I have obtained the Taylor series for this function using the usual formulae given by Taylor series theorem and one can notice that both are equivalent.

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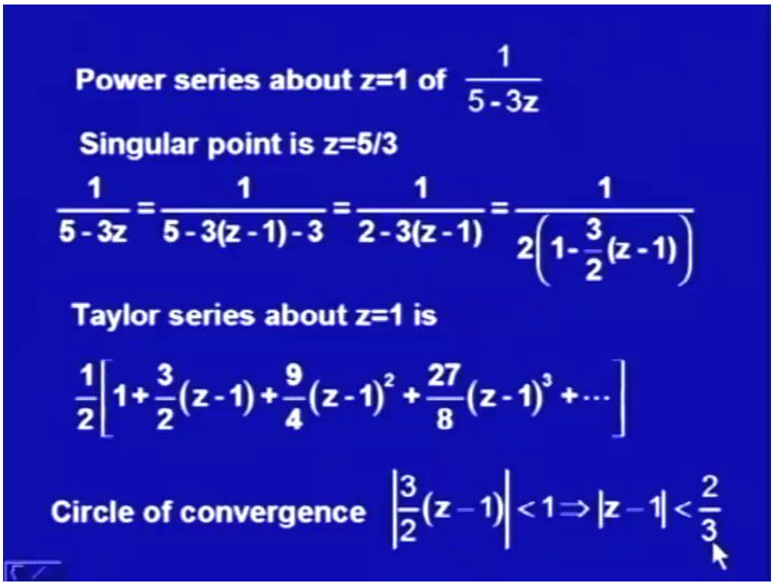

In this example, we are trying to obtain power series of the function 1 upon 5 minus 3 Z about Z is equal to 1. So far, we are considering power series of functions about Z is equal to 0. Now in this case, one can notice that this function is not analytic. Then 5 minus 3 z equal to 0 or we say that Z is equal to5 by 3 is a singular point for this function. But, at Z is equal to 1 this function is analytic. So, we can expand this function about ZZ is equal to 1 or in powers of Z minus 1.

So to expand this, I write the function 1 upon5 minus 3 Z as 1 upon 5 minus 3 Z minus 1. So, I have added and subtracted 3 here. And this from here, we can write down the series as 1 upon 2 minus 3 Z minus 1 or if I take two outside. Then it is 1 minus 3 by 2 into Z minus 1. So, Taylor series about Z is equal to 1 is you can expand this as a geometric progression. And this comes out to be 1 by 2 of 1 plus 3 by 2 into Z minus 1 plus 9 by 4 into Z minus 1 square plus 3 by 2 whole cube into Z minus 1 whole cube plus and so on.

Now, this series will be convergent provided. This term is less than 1. So, the circle of convergence for this series is given as 3 by 2 into Z minus 1 is less than 1. Or we can say modulus of Z minus 1 is less than 2 by 3. So, the radius of convergence is 2 by 3. But, the centre is at 1.

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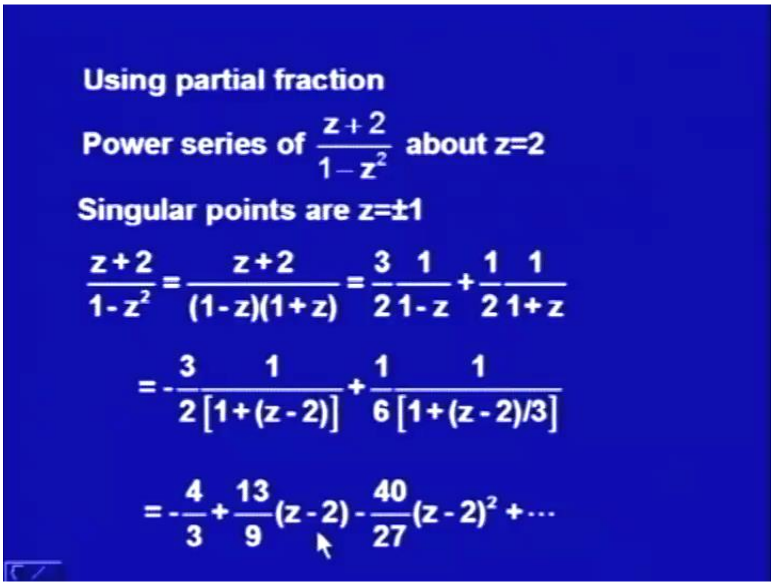

Now this is in this example, we will be using partial fractions to obtain power series of given function. So, illustrate this I consider f Z as 2 Z plus 2 over 1 minus Z square. And it is to be expanded about Z is equal to 2. So, singular points of this function are Z is equal to plus minus 1. So, about Z is equal to 2 we can expand this function. Because, it is analytic at Z is equal to 2. So, to write down this function Z plus 2 1 minus Z square, I expand it as Z plus 2 over 1 minus Z into 1 plus Z.

And then, I write down the partial fractions. So, partial fractions of this will be 3 by 2 into 1 over 1 minus Z plus 1 by 2 over 1 by 2 into 1 over 1 plus Z. So now we can write down power series for 1 over 1 minus Z and I write down power series for 1 over 1 plus Z. Here and that will and simplifying this that will give me power series for z plus 2 over 1 minus Z square in powers of Z. But, this is required for Z powers of Z minus 2.

So, we have to rewrite this, so 3 by 2 into 1 upon 1 plus Z minus 2. So, I write this expression as 1 plus Z minus 2. This is equivalent to 1 minus Z and this 1 plus Z is written as 1 plus Z minus 2 by 3. So, this we simplify and expanding this. We will have minus4 by 3 plus 13 by 9 Z minus 2 minus 14 by 27 Z minus 2 square and so on. So, this series will be obtained when you expand this in powers of Z minus 2. And expand this in powers of Z minus 2 raise to of Z minus 2 divided by 3 and simplify we get this series.

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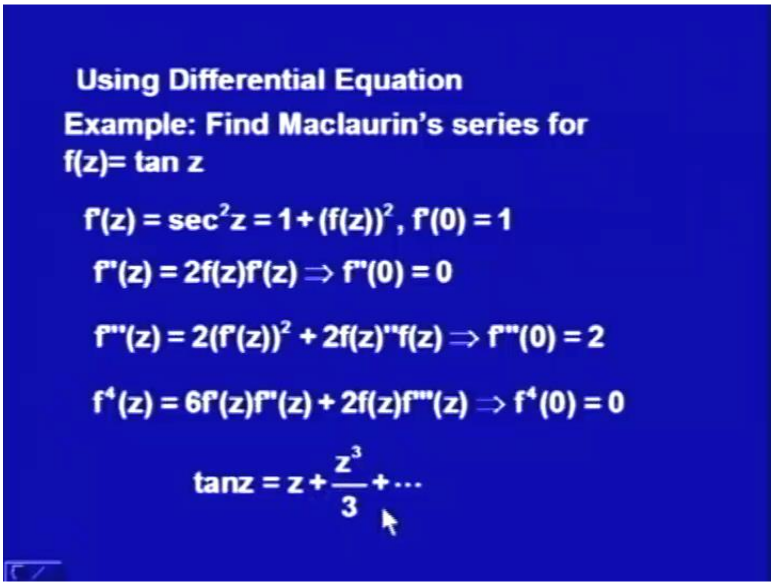

There is another method for obtaining Taylor series that is using differential equations. To illustrate this we consider Maclaurin's series for the function tangent Z. For this, f Z is taken as sec square Z and sec square Z is nothing but 1 plus f Z square, because f Z is tangent Z. So, we can say that f dash Z is equal to f dash 0 is equal to 1. So, if f z is tangent z then f dash Z is sec square Z and f Z being 0 at Z is equal to 0. So, f dash 0 is 1.

Now, differentiating this again, we have f double dash Z. And now, we differentiate from here, it is 2 times f Z into f dash Z and since f dash 0 is 1 and f 0 is 0. So, f double dash Z is 0. Further differentiating it again, I am starting from here. I am differentiating this f triple dash Z is equal to 2 f dash Z square plus 2 f Z f double dash f Z. And from here, we can find out, we can write down that f triple prime 0 is equal to 2 this is 0.

So this comes out to be 2, f double prime 0 is 0, so this is 0 and will have 2. From here, we can find out the fourth derivative and 4th derivative is computed like this and putting Z f 4 0 is equal to this gives me f 4 0 as 0. And substituting these values will have tangents at is equal to Z plus Z cube 3 and so on. After taking number of examples, let me summarize what we have done today.

We have started with some basic concepts of sequence series and convergence. Then examination have discussed that series can be added series can be differentiated a and integrated terms wise they can be multiplied term wise. Of course, I have not given you the proof of these things. Then we have developed a analytic function is a power series.

We have discussed the convergence of the power series and uniqueness of this Taylor series.

And then, we have taken various examples to illustrate, how to find Taylor series for a given function f of Z. We have number of methods for obtaining this Taylor series one is by applying the Taylor’s theorem, where we calculate various coefficients by calculating derivatives. Or we can use some alternative methods; I have discuss number of them to arrive at a Taylor series. This side due to the fact that Taylor series representation or power series representation is unique. So if any case, if in any way we can represent a function in a power series that will be the Taylor series for the function that is all for today is lecture.

Thank you.