Mathematics-II
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Module - 2
Lecture - 20
Functions of Complex Variables Part - 2

Welcome viewers, today we are going to discuss Functions of a Complex Variable. This lecture is in continuation to my earlier lecture on the same topic, where I had discussed function of a complex variable. Their limit continuity, differentiability and analyticity of functions of complex variables.

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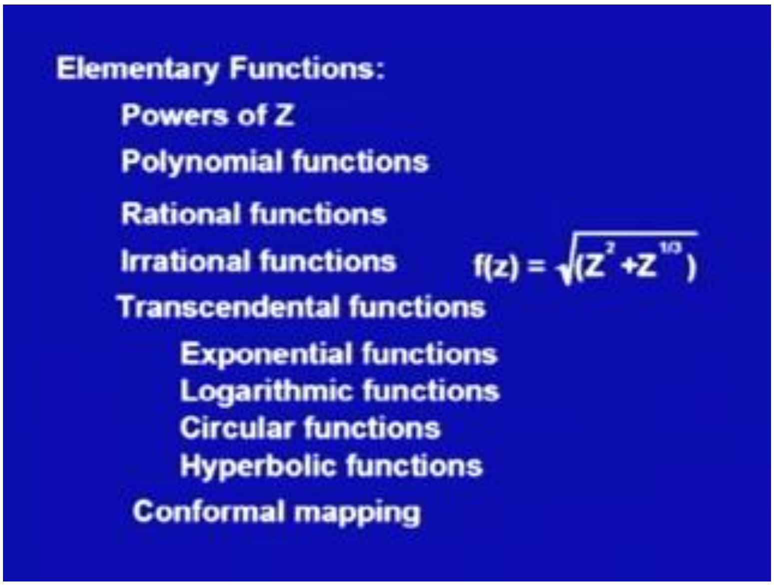

There I have introduced some elementary functions like power of Z, polynomial functions, rational functions, irrational functions and transcendental functions I am going to discus today. In this I will be discussing exponential functions, logarithmic functions, circular functions and hyperbolic functions. I will also be discussing mapping by element mapping by elementary functions of complex variables. And finally, conformal mapping will be discussed in this lecture.

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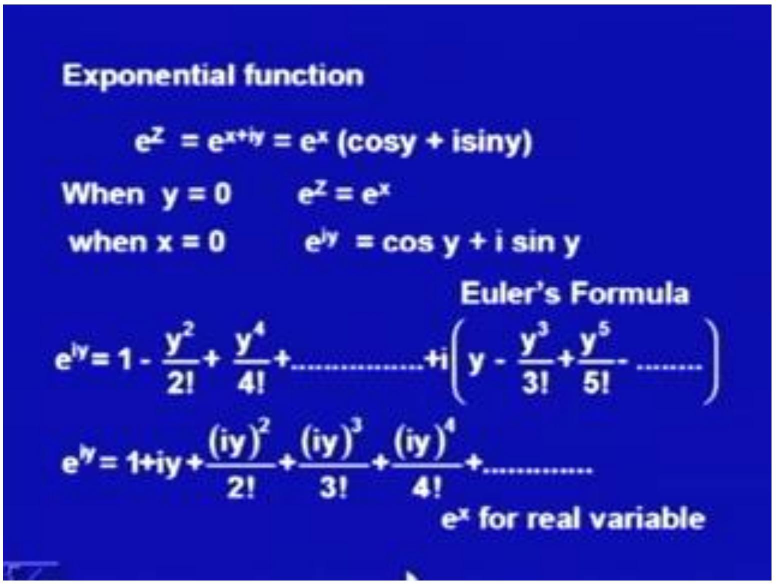

We start with the exponential function, the function e Z, the exponential function e Z. We define it as e x plus i y as e x multiplied by cos y plus i times sin y. We have defined this function in a manner. So that, when Z becomes x it is the usual exponential function of a real variable.

So, when we take y is equal to 0. Then, Z becomes a real variable. And then e Z is a same as ex and when x is equal to 0. Then, this definition gives us e i y is equal to cos y plus i sin y. And this is known as Euler's formula. Since cos y and sin y, they are function of real variable y. So, we can write down cos y as 1 minus y square by factorial 2 plus y 4 of factorial 4 plus and so on. So, this is the cosine series.

Similarly, we write down the sin series here and it is y minus y cube by factorial 3 plus y 5 by factorial 5 minus and so on. If I combine these terms, then e i y is equal to 1 plus I y plus. This y square is written as i y square by factorial 2, plus i y cube by factorial 3 is coming from this. Then, y 4 by factorial 4 is written as i y rest to power 4 by factorial 4 and so on. So, if I arrange these terms in this particular manner. Then one can see that this is nothing but expansion of e x, where x is replace by i y. So, this is very similar to what we have for real variables.

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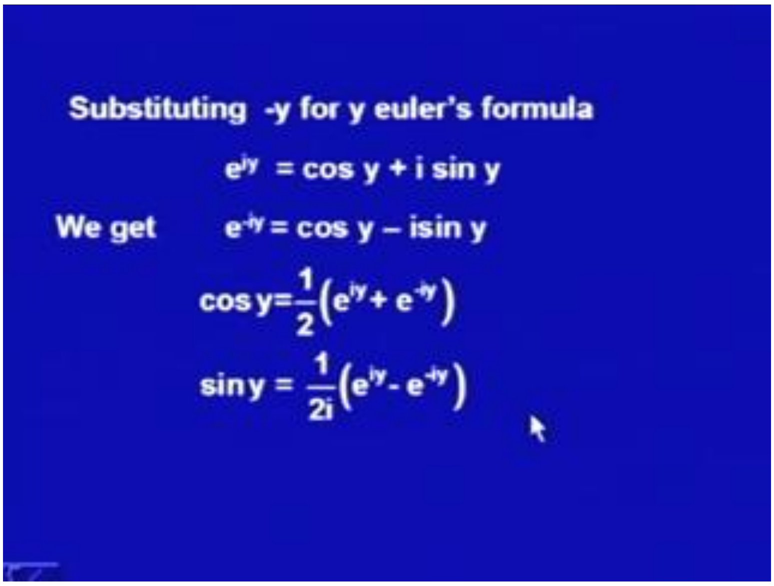

Now, even the same formula I substitute minus y for y. Then, the Euler formula e i y is equal to cos y plus i sin y. And this substitution will given me e minus i y is equal to cos y minus i sin y. Adding these two equations, I will get cos y is equal to half e i y plus e minus i y. So that is how we are relate cosine function of a real variable as this. Similarly, if I subtract, then sin y is obtained as 1 upon 2 i into e i y minus e minus i y.

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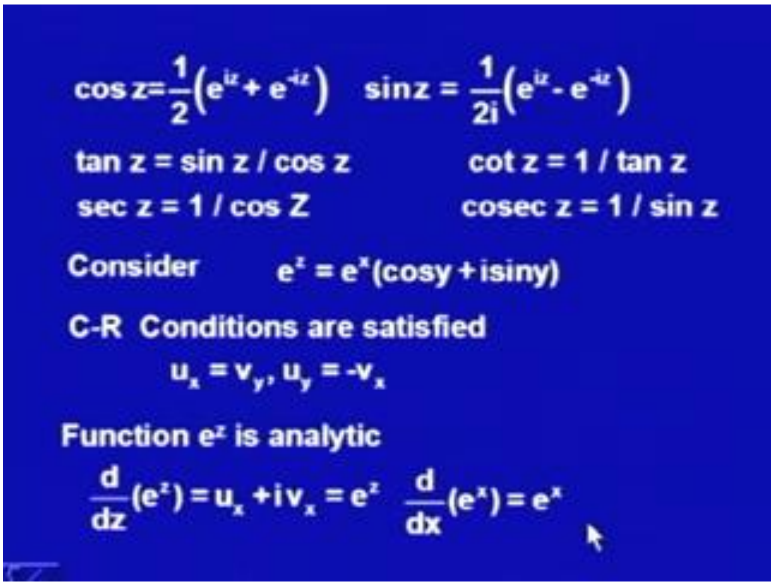

Using this, we can extend this idea to cosine function of a complex variable Z. As half of e i Z plus e minus i Z and sin Z is upon 2 i into e i Z minus e minus of i Z. So that is how we introduced the concept of sin and cosine functions of complex variables Z. This is in terms of exponential functions.

Once we know, what is cosine Z and sin Z? We can write down tangent Z. As sin Z over cosine Z cotangent Z is one upon tangent and so on. Again, if we write e Z is equal to e x multiplied by cos y plus i sin y by the definition of exponential function of Z. Then, one can notice that this function satisfies C-R conditions. Like here u is e x cos y the real part and the imaginary part v is e x sin y. So, if we apply C-R conditions, which can see that u x is equal to v y and u y is equal to minus v x in this case.

And, that simply means that the function is analytic, because this condition satisfied everywhere. And on the basis of this, we can say that d by d z of e Z exists. And it is computed as u x plus i v x. And substituting u x and v x from this, we can see that this derivative is equal to e Z. So, we have a result that derivative of e Z is e Z. Now, this is also in line with function of a real variable, where derivative of e x is equal to e x.

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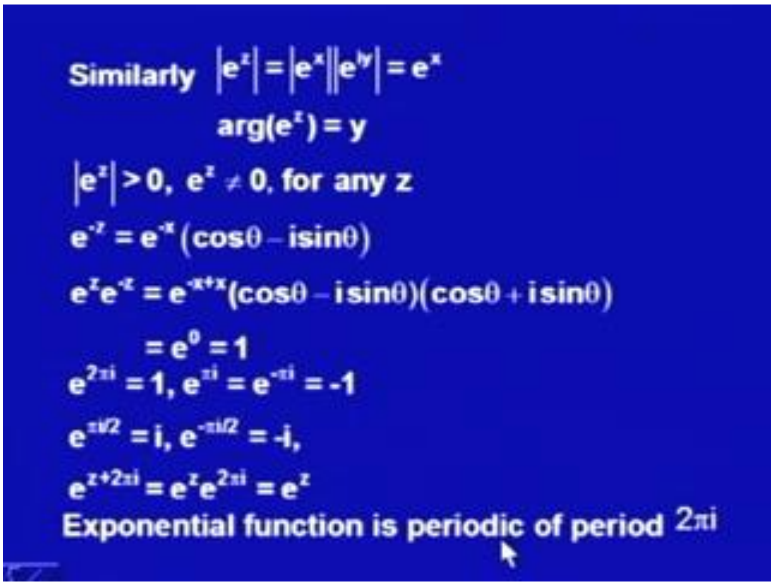

Similarly, the magnitude of e Z is magnitude e x plus i y. And this is written as e x into e i y, so their magnitudes are multiplied. The magnitude of e i y is cos y plus i sin y magnitude. And this is equal to 1. And that give us magnitude of e Z is equal to e x. Similarly, the argument of e Z is equal to y.

One can also notice that magnitude of e Z is always positive and e Z is not 0 for any value of Z. We can define e minus Z as e minus x multiplied by cos theta minus i sin theta. What I have done is? I have replace Z by minus Z. So, this x becomes minus x and due to minus y, this minus y will appear. So, we have e minus Z is equal to e minus x into cos theta minus i sin theta.

If, we can multiply e Z and e minus Z, this is equal to e minus x plus x multiplied by cos theta minus i sin theta plus cos theta plus i sin theta multiply the two. It is cos square theta minus i square sin square theta. And because minus i square is minus 1, So, we will cos square plus sin square theta from this product and ultimately will have e 0, which is 1. Now, this is also in line with real variables.

Further, if we say e raised to power 2 pi i. So, real part is 0, the complex part y is 2 pi. So, if I write y is equal to 2 pi in our Euler formula, then this will be 1. And e raised to power per pi i and e minus pi i both are equal to minus 1. One can also get e i pi by 2 as i and e minus pi i by 2 is minus i.

Further, e Z plus 2 pi i is equal to e raised to power z multiplied by e 2 pi i. And since e 2 pi i, I have already prove it to 1. So, I can say e Z plus 2 pi is equal to e Z. And from here, one can notice that exponential function is periodic function of period 2 pi i, this period is imaginary. So, in this sense exponential function is periodic.

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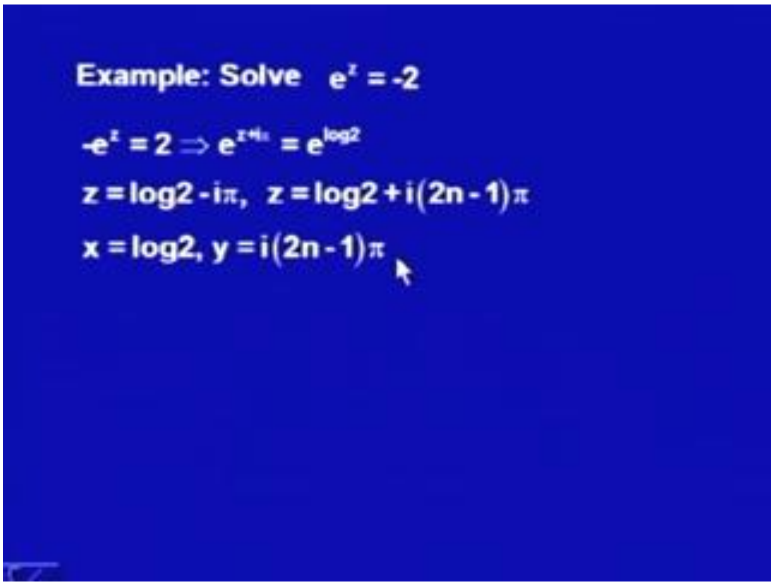

Now, let us solve this equation e Z is equal to minus 2. One can write e Z is equal to minus 2, as minus of e Z is equal to 2. And minus of e Z can be written as e Z plus i Z. And two, I am writing as e raised to power log 2, 2 is a real number. So, this can also be written as e rest to power log 2, because exponential and log are inverse functions.

So from here, if I compare then Z is equal to log 2 minus i pi. Or in general, it is Z is equal to log 2, plus 2 n pi i I have added, because it is a periodic function. So, it is i times 2 n minus 1 pi and this gives me x is equal to log 2 is a real part. And y is equal to i 2 n minus 1 pi. And n can take value 0, 1, 2 n and so on.

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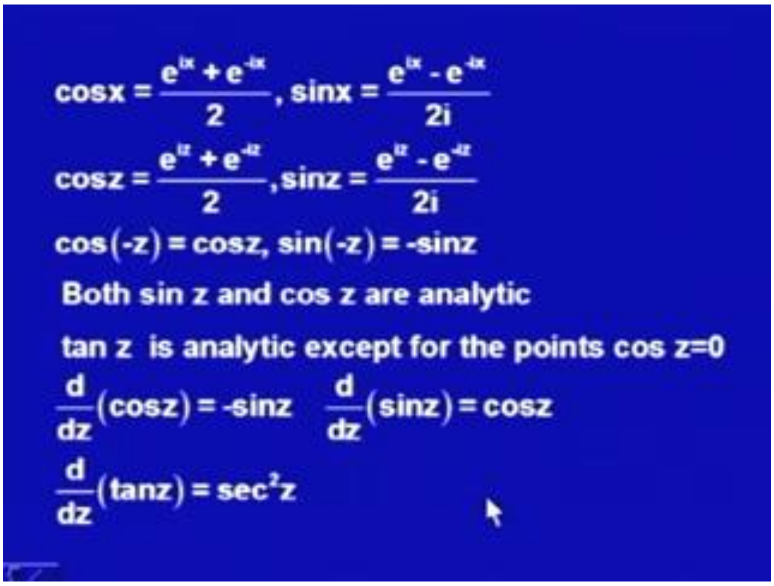

So, this is a solution of given equation, we will continue with this definition cos x is equal to e ix plus e minus ix by 2, sin x is equal to e ix minus e minus ix by 2 i. And cos Z is equal to e i Z plus e minus i Z by 2, just an extension for this and from here sin Z is equal to e i Z minus e minus i Z over 2 i.

From these formulae, one can easily notice that cos of minus Z is a same as cos Z and sin of minus Z is equal to minus sin Z from here, you can notice? And since cosine and sin are expressed in terms of e i Z and e minus i Z. And since, we have already said that e i Z and e minus i Z are analytic functions. So, their sum will also be analytic and divided by 2 is also analytic in the same sin Z will be analytic.

So we conclude, that both sin Z and cos Z are analytic functions of Z. Further, tan Z is analytic except for the points, where cos Z becomes 0. Because, at those points tangent Z will not be defined. So, d by d Z of cos Z is equal to minus of sin Z. One can easily differentiate this and that comes out to be minus sin Z. And using this formula d by d Z of sin Z comes out to be cos Z.

And, one can again write down tangent Z. As sin Z upon cos Z and with some simplifications and manipulations, one can obtain d by d Z of tangent Z is equal to sec square Z. Here, one may noticed that these formulae's are very similar to, what we have for real variable. Because, we know d by d Z of cos x is equal to minus sin x, d by dx of sin is equal to cos x d by dx of tangent x is equal to sec square x. So, derivative of sin cosine functions of a complex variable are the same as. What we had earlier for real variables? This is not all other formulae can also be proved in the same line.

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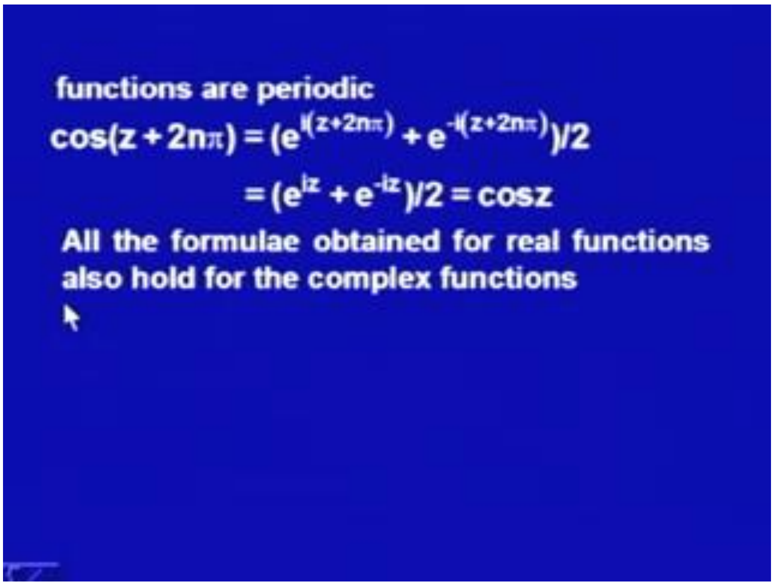

And from the way, we have defined sin function and cosine functions. One can further notice that these functions sin and cosine functions are also periodic function. And they are of period 2 pi, we can check it cos Z plus 2 n pi is equal to e i Z plus 2 n pi plus e minus i times Z plus 2 n pi divided by 2 and since exponential functions are periodic. So, this parts is written as simply e i Z and this part is written as e minus i Z. And this gives me cos Z.

So, cos Z plus 2 n pi, it gives me cos Z. This proves that cos Z is periodic function. You can similarly prove sin Z is also periodic function. And in fact, all the formulae which we have obtain in a trigonometry using real variable. These formulae can be developed for complex variables also.

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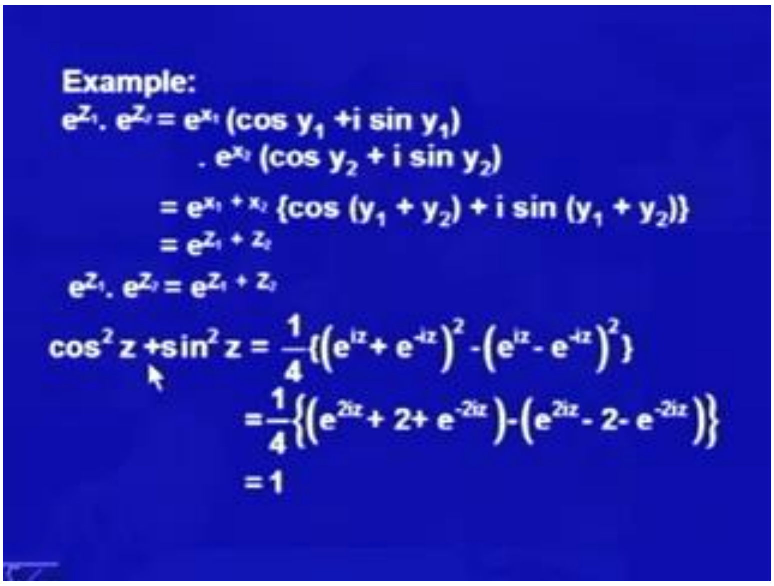

I will take some examples, let me consider this product e Z 1 into e Z 2. I write down e Z 1 as ex 1 cos y 1 plus i sin y 1 by the definition of e Z 1. Similarly, e Z 2 is written as ex one cos y 2 plus i sin y 2. If I multiply these two terms, I will have ex 1 into ex 2 is common. And these, this is multiplied by this and cos y 1 cos y 2 minus sin y 1 sin y 2 is the real part. And this is nothing but cos y 1 plus y 2.

Plus i times, if I collect the imaginary parts in the product. It will be sin y 1 into cos y 2 and cos y 1 into sin y 2. This is nothing but sin y 1 plus y 2. So, this is written as a e x 1 plus x 2 plus i times y 1 plus y 2. And that gives me e Z 1 plus z 2. So, this formula is also very similar to what we have for real variables, ex 1 multiplied by ex 2 is e of x 1 plus x 2 that is the corresponding formula for real variable. So, this formula is also applicable for complex variables.

This is what we have, then cos square Z plus sin square Z is also 1. One can check from here, I am writing one upon 4 e i Z plus e minus i Z whole square for cos square Z. And this is written for sin square Z. We can expand these two and it comes out to be e to i Z plus 2 times e minus 2 i Z by expanding this. And from here, I will be getting e 2 i Z minus 2 plus e minus 2 i Z.

And we can see that these terms will get cancelled and will have 2 plus 2 that is 4 and this 4 and 4 will get cancel and will have 1. So, similarly we can develop all those trigonometric formulae's, which we have studied for a real variables, they can be they can be proved for complex variables.

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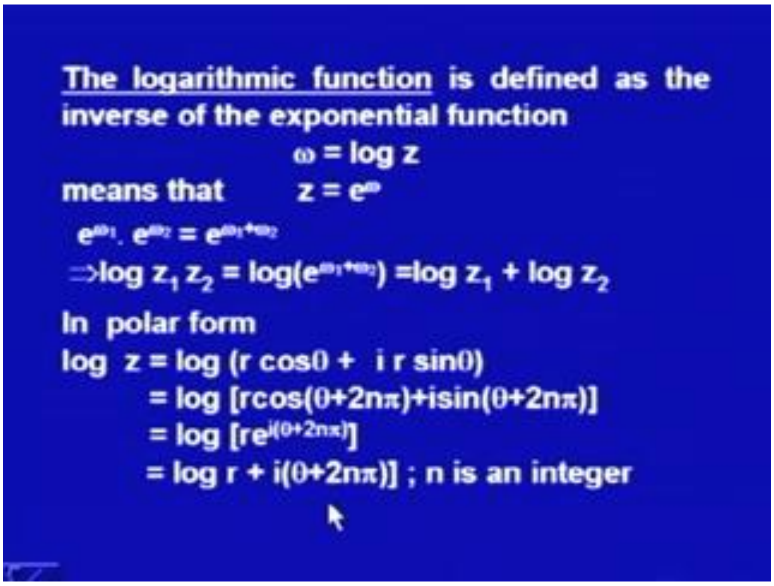

Then, we come to logarithmic function. Logarithmic function is defined as the inverse of the exponential function. So, we write w is equal to log Z. Then, Z is equal to e w. So that is, how we write that logarithmic function is inverses of exponential function. So from here, if we multiply e w 1 and e w 2, then it will be equal to e w 1 plus w 2. You can check this is nothing but log z 1. If we take the log of this, then it is log of Z 1 into Z 2 is equal to log of e w 1 plus w 2. We can simplify it and comes out to be log Z 1 plus log Z 2.

We can express this function in polar form. So, log of Z is equal to log of Z. I am writing as r cos theta plus i r sin theta. So, log of z is log of r cos theta plus i r sin theta. And since cosine theta and sin theta are periodic functions. So, I write log of r cos theta plus 2 n pi 2 i sin theta plus 2 n pi. And this can be written as log r e i theta plus 2 n pi or I can rewrite it as log r plus i times theta plus 2 n pi. And n is an integer in this case. So, this gives me the log of Z in polar form.

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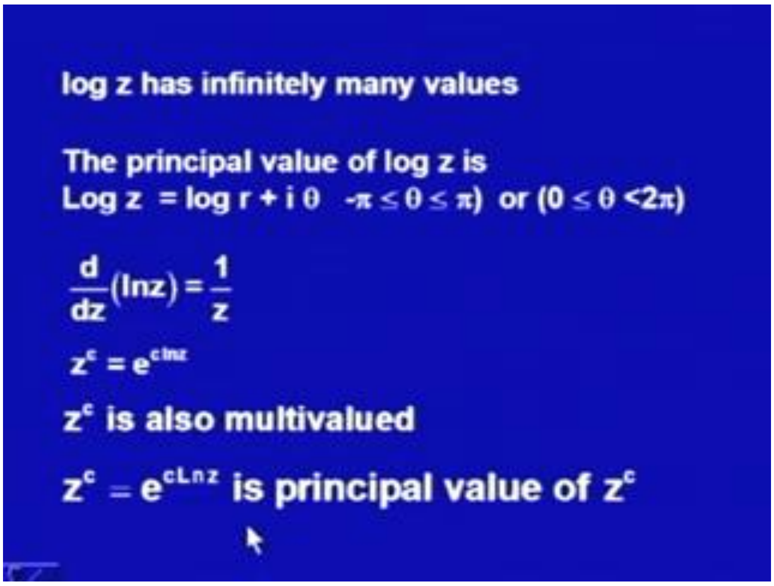

One can notice from the earlier formula that log Z has infinitely many values. And the principal value of log Z is written as log of Z. Notice the difference, this is equal to log r plus i theta, where theta varies from minus pi 2 pi. Or sometimes we take from 0 to 2 pi. One can also calculate the derivative d by d Z of l n Z is equal to 1 upon Z. And then we can define the function Z raised to power c as e times c l n Z. So, Z raised to power c is defined as this.

And, since this function is multi valued, so Z c is also multi valued, so we can define Z c in terms of log function. So, Z square Z cube etcetera, they are defined as repeated multiplications. But, if c is a different number not necessarily 2 3 or real or a natural number. Then, we can define z raised to power c in this particular manner. Since, I said that Z c is a multi valued function, so it will have a principal value. And for principal value, we write Z c is equal to e raised to power c l n Z. So, this is the principal value of Z raised to power c.

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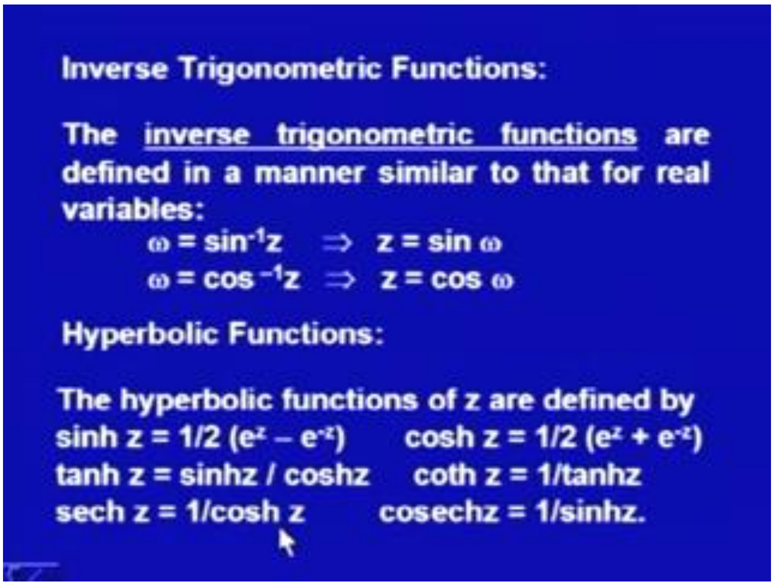

We after defining trigonometric functions, we can define inverse trigonometric functions. And they are defining in the same manner as that for real variables. That is w is equal to sin inverse Z means Z is equal to sin w. And w is equal to cos inverse Z implies Z is equal to cos w.

We can we are here, defining hyperbolic functions. They are defined as sin hyperbolic Z is equal to half of e Z minus e raised to power minus Z and cos hyperbolic Z is defined as half of e Z plus e minus Z. And once we have defined hyperbolic sin and cosine functions, we can define other functions also. Tangent hyperbolic Z is sin hyperbolic Z divided by cos hyperbolic Z. Cot hyperbolic Z is 1 upon tangent hyperbolic Z. Sec hyperbolic Z is 1 upon cos hyperbolic Z. And cosec hyperbolic Z is 1 upon sin hyperbolic Z.

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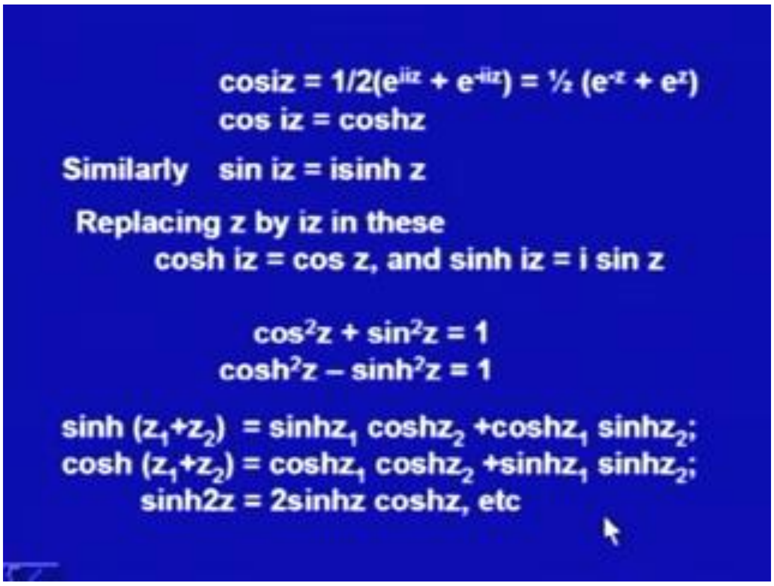

On the basis of this, we can prove the same formulae sin square Z plus sin square Z is equal to cos square plus sin square Z is equal to 1.And, one can prove on the same line that cos hyperbolic square Z minus sin hyperbolic square Z is equal to 1. And this is, what we have for real variables also. So, in place of x in place of Z, suppose we have x then this is a formula which we have already developed.

Similarly, one can write down cos i Z is equal to half of e i Z plus e minus i Z. And this is nothing but half of e minus Z plus e Z. And that gives me, cos of i Z is equal to cos hyperbolic Z. Similarly, sin i Z is i times sin hyperbolic Z that is, how we relate the trigonometric functions and hyperbolic functions. So, cos hyperbolic Z is cos Z and sin hyperbolic Z is sin hyperbolic i Z is equal to i times sin Z.

We can further prove the formulae which we have done for real variables. For hyperbolic functions also sin hyperbolic Z of Z 1 plus Z 2 is equal to sin hyperbolic Z 1, cos hyperbolic Z 1 into sin hyperbolic Z 2. So, all these are the formulae which we have already discussed in our school days. But, that they were for real variables. Now, we have for complex variables. So, all these formulae can easily be established.

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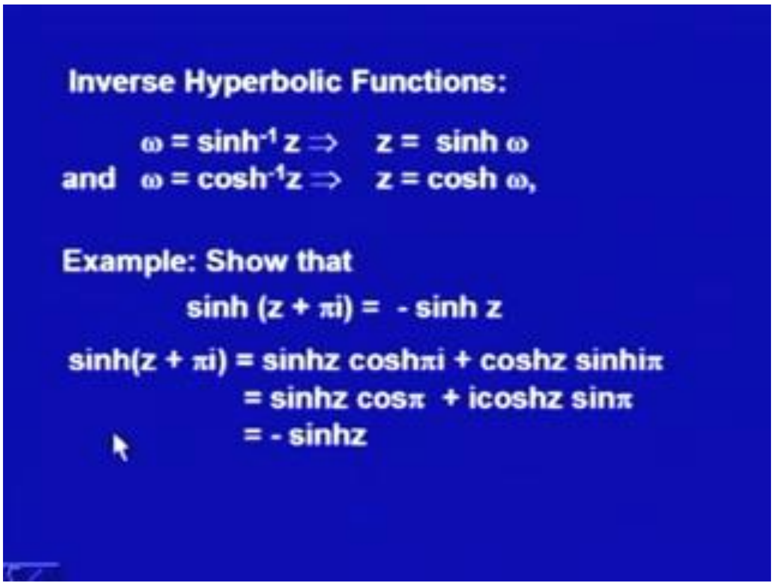

After that, we discus inverse hyperbolic functions, w is equal to sin hyperbolic inverse Z implies Z equal to sin hyperbolic w. And w is equal to cos, cos hyperbolic inverse Z implies that Z is equal to cos hyperbolic w. And from these definitions one can easily prove that sin hyperbolic Z plus pi i is equal to minus sin hyperbolic Z. This is, how we prove this result.

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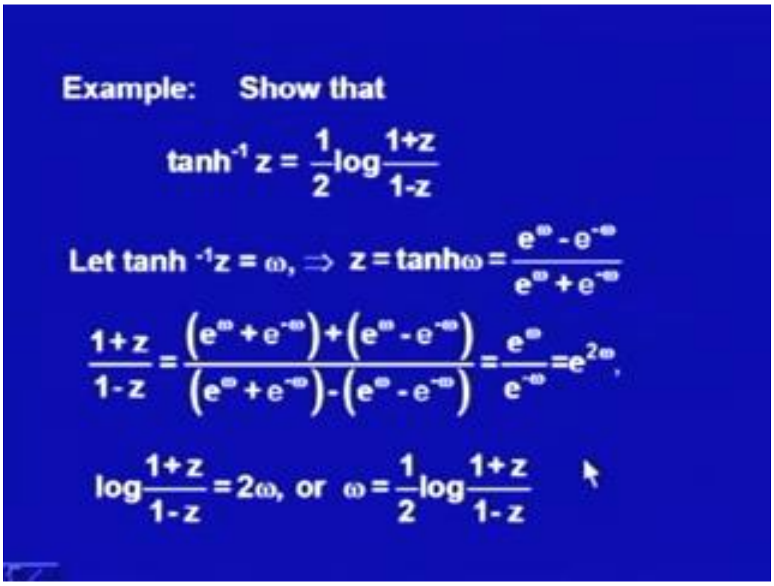

Now, in this example, I show that tan hyperbolic inverse Z is equal to half of log 1 plus 1 Z over 1 minus Z. To prove this I consider tan hyperbolic inverse Z is equal to w. And from here, we can see Z is equal to tan hyperbolic w. And tan hyperbolic is sin hyperbolic w divided by cos hyperbolic w. So, this is the formula for I have for sin hyperbolic this is for cosine hyperbolic.

Then, we simplify this expression as one plus Z over 1 minus Z is equal to e w minus e minus w plus e w minus e minus 2. And similarly, this denominator is 1 minus Z and simplifying it this gives me e 2 w taking logs on both sides. We get log 1 plus Z over Z minus Z is equal to 2 w. And simplifying it, w is equal to half of log 1 Z over 1 minus Z. So this is what we have to prove. So that is, how we apply various operations on tangent hyperbolic functions and log functions.

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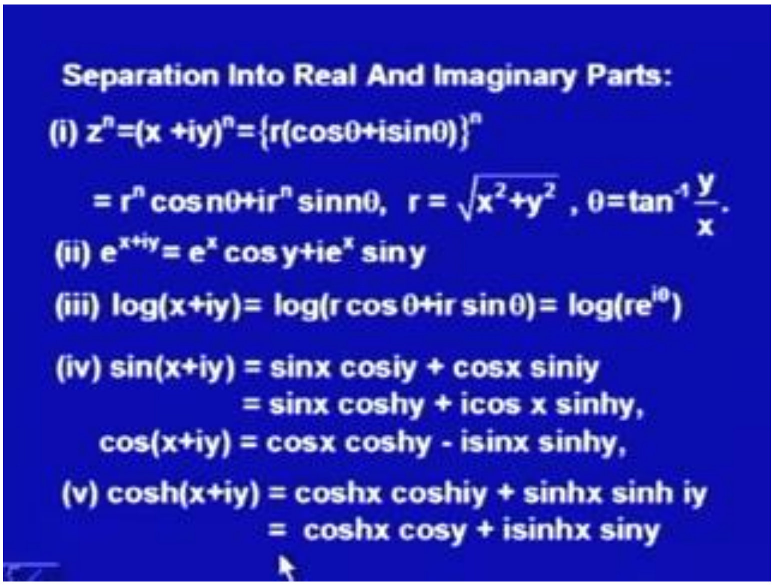

Now many times, we need to separate the real and imaginary parts of a function. So, I will give some examples to illustrate this. So let us say, we have been given Z raised to power n and we have to express it in its real and imaginary part. So to do this, let me right down this as x plus i by raised to power n. Or I write it as r cos theta plus i sin theta raised power n.

And, expanding it is r raised to power n cos n theta plus i r n sin n theta, where r is equal to under root of x square plus y square and theta is equal to 10 inverse y by x. So, this function has real part as r raise to power n cos n theta and imaginary part is rn sin n theta. Similarly, e x plus iy can be written in real and imaginary parts as ex cos y plus i times ex sin y. So, real part is x cos y and imaginary part is ex sin y.

Log x plus i y is log of r cos theta plus i r sin theta. And this can be written as log r into e i e raise to power i theta. Taking log, we can say r is the real part and theta is the imaginary part of log x plus i y. Then, sin x plus i y can be written as sin x cos i y plus cos x sin i y. And from here, sin x is equal to cos hyperbolic y and sin i y, I write it as i times sin hyperbolic y. So that gives me imaginary as cos x sin hyperbolic y.

Similarly, cos x plus i y is expressed as cos x cos hyperbolic y is the real part and sin x sin hyperbolic y as the imaginary part of cos x plus i y. Cos hyperbolic x plus i y is equal to cos hyperbolic x cos hyperbolic i y, plus sin hyperbolic x into sin hyperbolic i y. Again making use of the definitions, which we have already given for hyperbolic functions I write cos hyperbolic iy as cos y. So, real part of this function is cos hyperbolic x cos y. And from here, we get the imaginary part as i sin hyperbolic x into sin y. So, these are how we can separate out the real and imaginary parts of some elementary functions.

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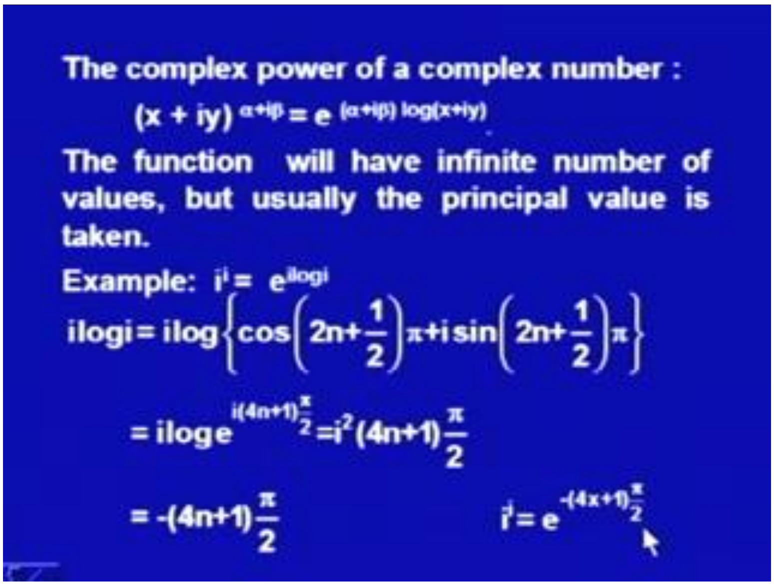

Then, complex power of a complex number, so let us say I have a number x plus i y. And here, the power is alpha plus i beta. So, this can be expressed as e raised to power alpha i beta into log of x plus i y. This, I have already discussed that you can express this in terms of log function. The function will have infinitely many values, because this value will be multi valued. And the principal value is taken most of the cases.

Let me, illustrate this with the simple example i raised to power i, how we simplify i raised to power i as e exponential of i times log i. Actually, i raised to power i is e raised to power i raised to power i and i raised to power i. I am writing as i log i. So, simply for taking log i log i is equal to i log of this is nothing but cos 2 n plus half pi plus i sin 2 n plus half pi.

I am writing the general value here and simplifying it. This e i times log e 4 n plus 1 pi by 2. And this is simplified as i square 4 n plus 1 into pi by 2 which can be further simplified to minus 4 n plus 1 multiplied by pi by 2. From here we write down i raised to power i is equal to exponential of minus 4 n plus 1 multiplied by pi by 2.

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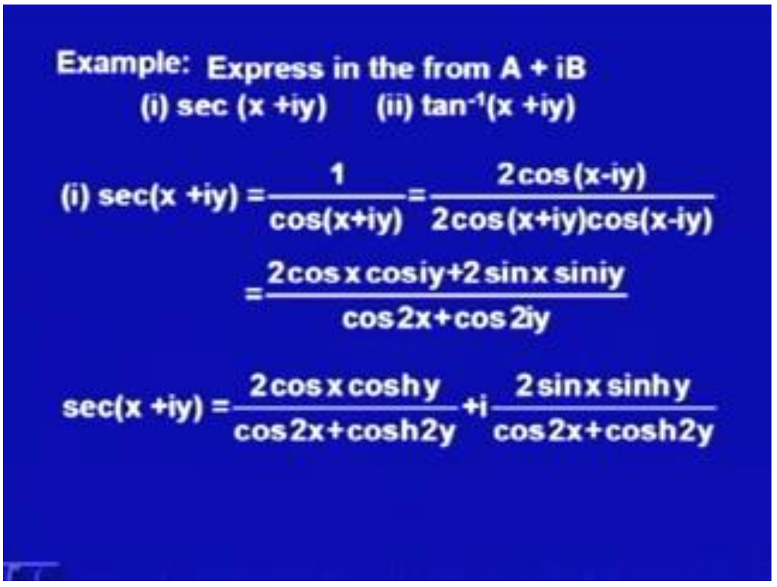

The next example, the function sec x plus i y is expressed as A plus i B; that means, we want to write down this function in its real and imaginary parts. So, first sec x plus i y is equal to 1 upon cos x plus i y. And then I multiply numerator and denominator by 2 cos x minus i y. And this on simplification gives 2 cos x cos i y plus 2 sin x sin i y. And this we apply CD formula here, it is cos 2 x plus cos 2 iy.

And, this means sec x plus i y is equal to 2 cos x cos hyperbolic y here, divided by cos 2 x plus cos hyperbolic 2 y plus i times i is coming from this. And 2 sin x sin hyperbolic y divided by cos 2 x plus cos hyperbolic 2 y. So, this is my A and this is B. So, sec plus i y is expressed as real part plus i times the imaginary parts.

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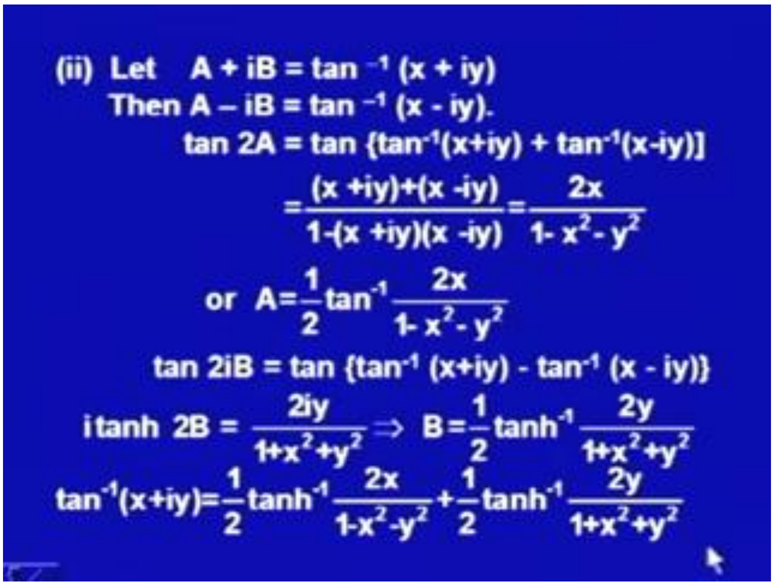

For the second part, tan inverse x plus i y, we take A plus i B is tan inverse x plus i y. And we write A minus i B is tan inverse x minus i y. Then adding these two, we will get tan 2 A is equal to tan, tan inverse x plus i y plus tan inverse x minus i y. So, what I have done is? I have added and then taken the tan on both the sides in after that this is expanded.

So, it is x plus i y plus x minus i y divided by 1 minus x pus i y into x minus i y adding. We will have 2 x in the numerator and this will be simplify to 1 minus x square minus y square. So from here, we can say 2 A is equal to tan inverse 2 x over 1 minus s square minus y square. Or A is equal to half of tan inverse 2 x over 1 minus x square minus y square, so that gives me that A part.

And, to calculate B part I will subtract the two. So, it will be tangent 2 i B is equal to tan of tan inverse x plus i y minus tan inverse x minus i y. Then, I apply the formula. And this gives me 2 i y divided by 1 plus x square plus y square. So then we simplify it I will get this and on the left hand side, I will have i times tangent hyperbolic 2 B plus tangent 2 i B is i times tangent hyperbolic 2 B.

And from this, I can say B is equal to 1 upon 2 tangent hyperbolic inverse 2 y divided by 1 plus x square plus y square. These I will cancel get cancel with this. And then further simplification will give me with co imaginary part B as this. So, I have obtain A and I have obtain B to tan inverse x plus i y is expressed as A plus i B. So this, what I have written here this is A plus i times B.

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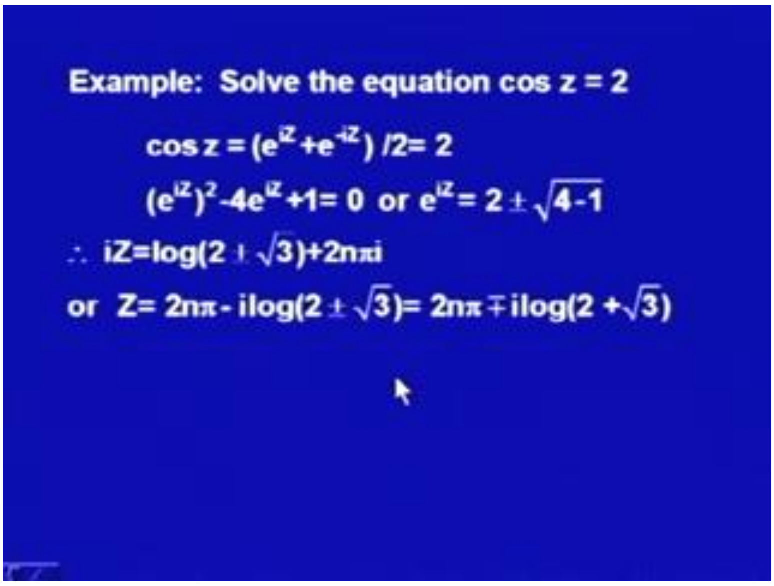

In the next example, we solve cos Z is equal to 2 Z is a complex variable. So cos Z, we write it as e i Z plus e minus I Z by 2 is equal 2. And then we can write e minus i Z. We can write it as 1 upon e i Z. And simplify this equation, it will give me e i Z square minus 4 times e i Z. This is the terms which are coming from the right hand side plus 1 equal to 0.

And, this is a quadratic in e i Z. So, e i Z is equal to 2 plus minus under root 4 minus 1. And that means, i Z is equal to log 2 plus under root 3 plus 2 n pi i is the general expression. And Z is equal to 2 n pi minus i log 2 plus minus root 3. Or we can say it is 2 n pi minus plus log 2 plus root 3. So, this is the general value for Z, which will satisfy this equation.

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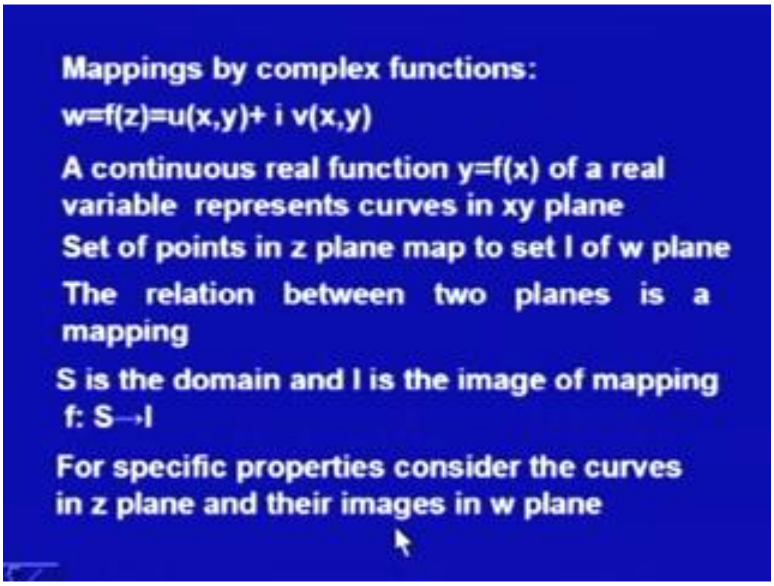

Now, we will discuss mapping by complex functions, so let us consider a complex function of a complex variable Z. So, w is a mapping from Z plane to w plane, so we will write w is equal to f of Z and we expressed this function as u x y the real part and imaginary part is v x y.

Now, when we define this function, then what we are doing is, we are taking a point in Z plane. And that will map to a point in the w plane. But, when we take continuous real functions y is equal to f x of a real variable. Then we know that it represents a curve in x y plane. So, we want to see how curves in Z plane will map to curves in the w plane. So, set up points in Z plane map to set I of w plane. So let us, consider the set S of points in Z plane they will map to an image I of w plane.

So, if the setup points lie on a curve, then how the mapping will be done or what is the relationship between two sets in to different planes. So, let us say S is the domain and I the image of mapping. And we have a function f from S to I. For specific properties, we consider curves in Z plane and their images in w plane, so this is what we are going to do next of the examples.

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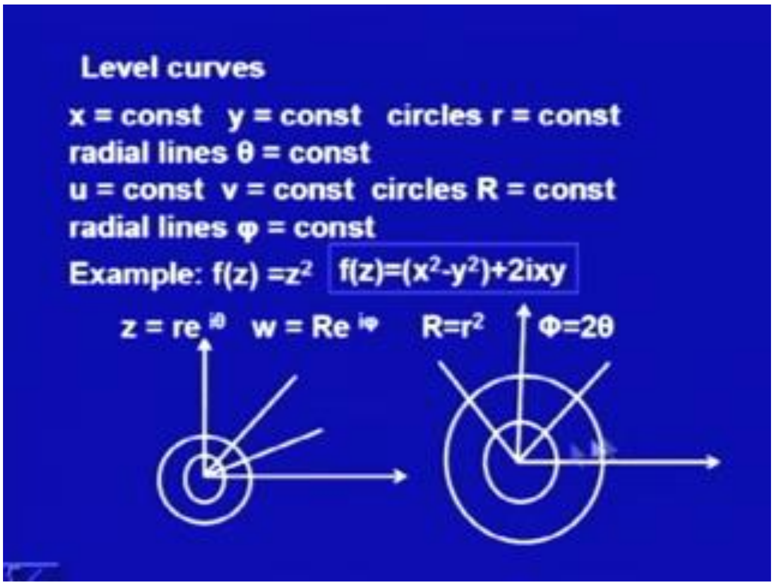

Now to describe this, we discuss level curves. So, the level curves are x is equal to constant. These are horizontal lines, how horizontal lines will map to w plane under the given transformation. Or how vertical lines y is equal to constant will map to w plane. Circles r is equal to constant in Z plane, how they map to w plane and radial lines passing through origin, how they map to w plane. So, these are some of points of our interest, so these curves are called level curves in x plane.

Similarly, we can talk about level curves in w plane. That is, what is the curve, for which u is equal to constant, is an image, v is equal to constant, is an image. Or what are the curves in Z plane? That will map to circles in w plane or circles are r is equal to constant. Or what are the curves in Z plane? They that will map to radial line phi are equal to constant in w plane.

So for this, let us take an example we have f Z is equal to Z square. And this means f x is equal to x square, minus y square plus 2 times i x y. So, this is the real part and this is the imaginary part. And if you express this function in polar form, then Z is equal to r e i theta. Or we can say w is equal to R e i phi. So, this is a point Z by in w plane. It will be Z square, so I am writing f Z as w.

Let us call it R e i phi. So, if I compare then one can notice that R is nothing but R square. Because, Z square is r square e to i theta. So comparing the two, I will have R is equal to r square and pi is equal to 2 theta by this. I mean to say that this is Z plane and this is a radial line theta is equal to constant. This is another radial line theta is equal to constant this is a circle Z is equal to a, a constant. This is another circle center at origin, but with the different constant.

So let us see, how these four different curves will map. One w plane under this transformation, so let us see this is the situation. When we consider this line, theta is equal to theta 1. Then, this suggests that this line will map to a radial line with angle equal to 2 theta. So, this radial line will move to this position if this angle is theta. Then, this angle will be 2 theta.

Similarly, this line will be mapping to this line, where the slope of this line is 2 theta 1. Then, slope of this line will be 2 times theta 1. So, the slopes are double. So, this is the mapping of this whether slope is double of this and this line will map to this line. Now, when we come to this circle when this circle R, but this is its image its radius will be r square. So, this radius is different than this if it is r is not equal to 1.

Then, this will be the image of the interior circle. And then this circle will map to this circle. So that is, how circles are mapping to circles and radial lines are mapping to radial lines circle are mapping to circles. But, the radiuses are changing. And similarly, radial lines are mapping to radial line. But, there slopes are changing. So that is, how under this transformation, this level curves circles and radial lines will map to circles and radial lines in w plane.

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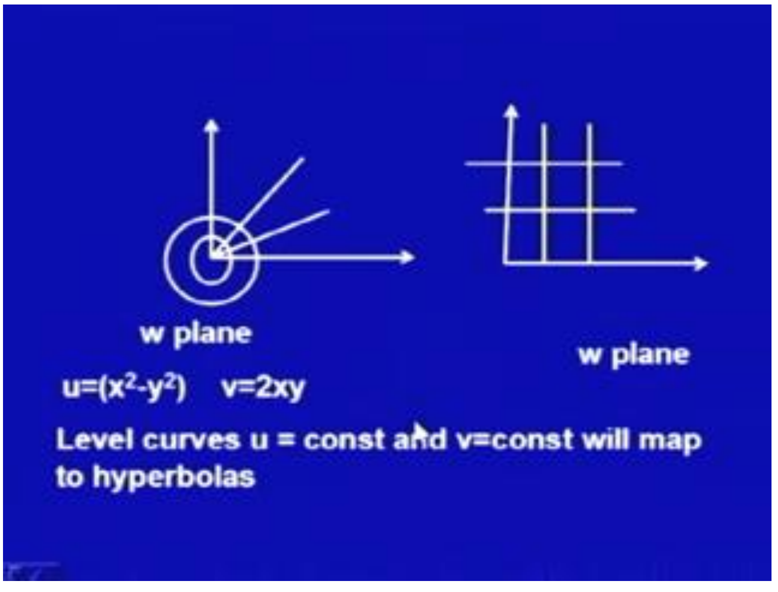

Now, many we have interested to know, how this is u is equal to constant lines. And this is v is equal to constant lines, what are the curves, which will map to these lines. So, if we see our function x square minus y square is equal to u and 2 x y is equal to v. Then, this line will be actually having a hyperbola on x y plane. The hyperbola is x square is x square minus y square is equal to constant.

Similarly, this line will be the image of hyperbola in Z plane. And that plane will be that hyperbola will be x y is equal to constant. So, these are two different w planes here. We have the circles and here. We have the radial line here, we have the level curves. So, this u is equal to x square minus y square. And this is v is equal to 2 x y. So this, we have already discuss.

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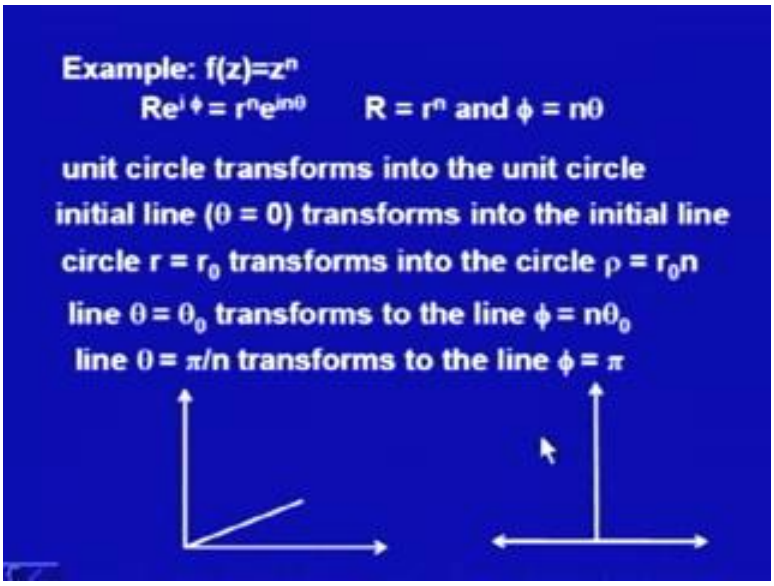

Then, we come to another function f Z is equal to Z raised to power n. Now in this case, again I am writing R e i phi. The w plane, this will be r n e i n theta in terms of r n theta. And from here, we can say that R is equal to r raised to power n. And phi is equal to n theta. So, this means unit circle transforms into unit circles. So, if we have a unit circle r is equal to 1. Then, capital R will be also one further initial line theta is equal to 0. Then, phi will also be 0.

So, initial line will transform to initial line under this transformation circle, r is equal to r naught transforms into the circle, rho is equal to r naught raised to power n. And line theta is equal to theta naught transforms to the initial. To the line phi is equal n theta naught; that means, the angle will be rotated. And it will be times the original angle theta naught. And line theta is equal to phi by n in particular will transform to the line phi is equal to pi.

So, if we consider this then if, this is the line with angle theta is equal to pi by n. Then, this line will map to this line will map to this line having slope theta is equal to pi. So, if we consider all such points here. Then, we can say thus the region enclosed by this line. And this initial line will be the upper half plane of w. So, this region will map to the upper half region of w plane.

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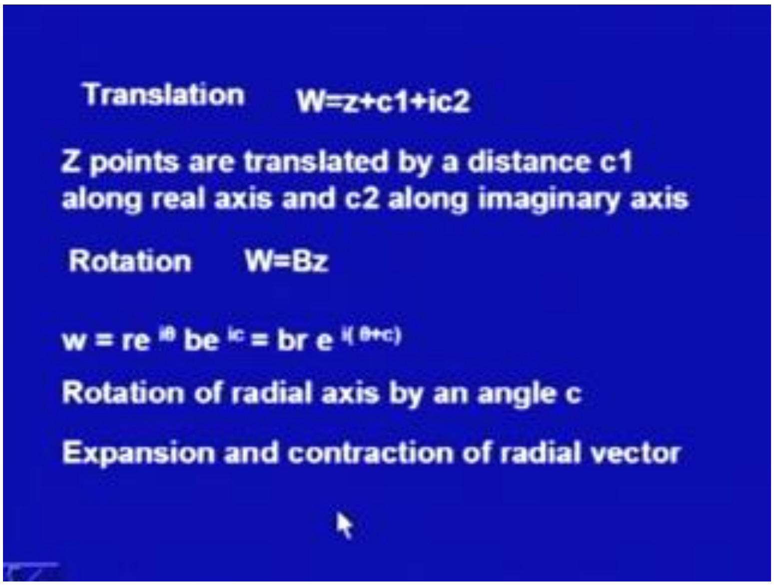

Now, we discus translation under the mapping W is equal to Z plus c 1 plus i c 2. So, if I add a constant this time of complex constant. Then, translation of points will take place. So, let us say that Z points are translated by distance c 1 along real axis and c 2 along imaginary axis. And if we consider transformation W is BZ. Then, this transformation means that W is equal to e re i theta that is Z.

And, let us say B is a complex number which be expressed as b times e i c, c is a constant b is a constant. So, if we combine the two it is the magnitude is b r and angle is theta plus c. So, this gives me r is equal to b times r and angle will be rotated by c. So, rotation of radial axis by an angle c will occur under this transformation. And the radial vector becomes b r, it will be expanded or contracted depending upon the magnitude of b, if b if greater than 1, then capital R will be greater than r. So, it is expansion and b is less than 1. Then, capital R will be contracted.

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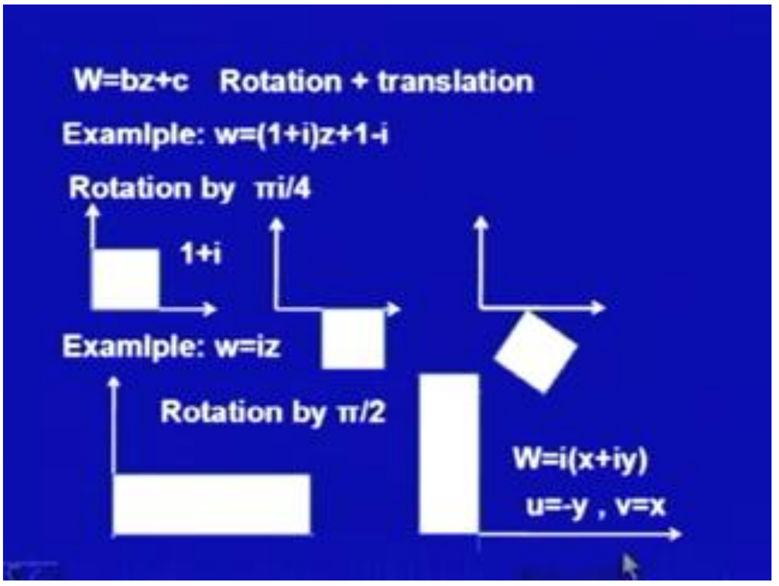

So, we have the situation. Now, we will have a combination of rotation plus translation W is equal to b Z c, so rotation plus translation. So, example is w is equal to 1 plus i Z plus 1 minus i. So, this is my b and this is my c. So under this transformation, let us consider this region. A rectangular region and let us let me say that this point is 1 plus i. Then, since I have to rotate the point will be translated by a distance 1 minus i. So under this translation, this rectangle will move to this rectangle. And because of this term it will be translated. And this will rotate an angle of pi by 4. So this will finally, be move to this position under this transformation.

In another example, w is equal to i Z which is more simplified than this. Let us, consider this infinite straight this is x axis this is y axis. And let us say this is of distance one and this is infinite straight. So under this transformation, it is this strip is rotated by an angle pi by 2. And with this w becomes i x plus i y and this gives me u is equal to minus y and v is equal to x. And that is, how this strip will move to this strip in w plane. So, different regions are mapped to different regions.

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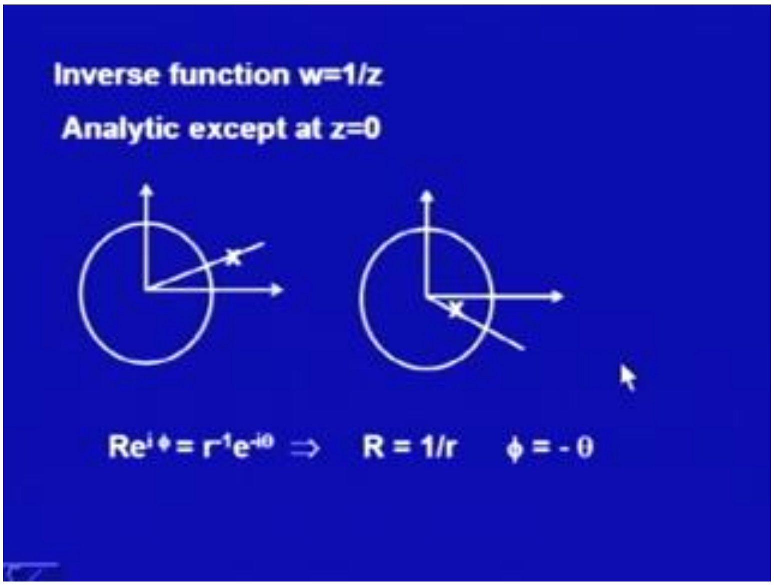

Now, we consider inverse function w is equal 1 upon Z. Now, we may notice that this function is analytic except at Z is equal to 0. And when we write down this function as R e i phi on w plane and R minus 1 e minus i theta the right hand side. Then, this is equal to this implies that R is equal to 1 upon r. And phi is equal to minus theta; that means if I have a point on Z plane as this.

Then, this point will map to w plane here by how I arrive at this point see this. Let us say this is a unit circle. Then, this is a point outside the unit circle. So, one upon r will be this r is greater than 1. So, one upon r will be smaller than 1. So, this point will map to a 0 inside the unit circle. So that is, why this point is inside the unit circle. So, this is R is equal to 1 upon r.

But, what is the effect of this phi is equal to minus theta. So, this angle is theta. Then, this point will lie on the line which is mirror image of this line above this initial line. So, this line is will map to this 0. So, this point will map to a point here which indicate that is. It is inside and it will lie on the line phi is equal to minus theta. So, these way different points will map to different points in w plane.

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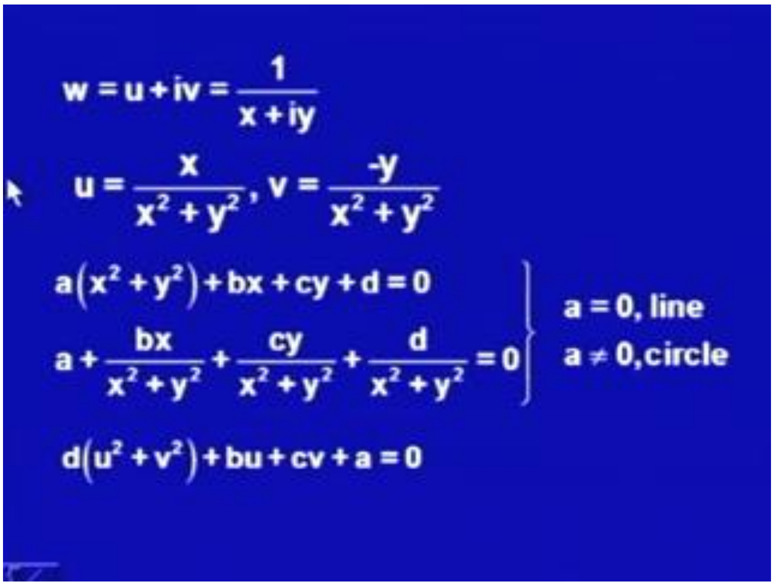

Let me rewrite this, W is equal to u plus iv as 1 upon x plus i y. And we can simplify it u is equal to x over x plus y square and v is equal to minus y over x square plus y square. This can be obtained by getting read of this plus i y in the denominator. So, we will have you is equal to this expression and v is equal to this.

And from here, one can notice that if we have a circle. General equation of circle is ax square plus y square plus b x plus cy plus d is equal to 0. It is a quadratic equation and this will represent a circle, when a is not 0. And when a is equal to 0. This part will not be there, it will represent a straight line. So, this is actually a general quadratic equation. It may represent a line or a circle. So, if we consider a line or a circle.

Then under this transformation, it will map to this and this again is a quadratic. So, it may be a circle or a straight line. So, after this transformation this will be simplify to d u square plus v square plus b u plus c v plus a equal to 0. So this again a quadratic, it will represent line or a circle. So that, the idea is the circles and lines in the Z plane will map to lines or circle. In the W plane different situations are possible.

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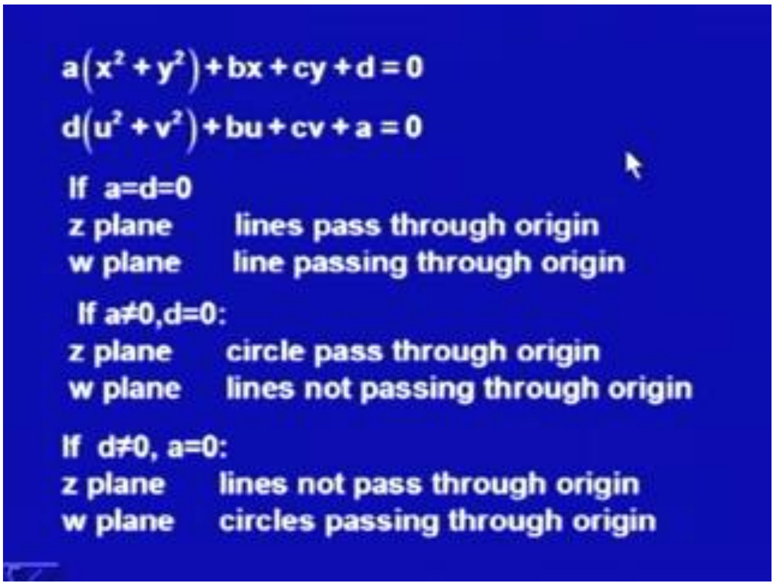

So let us see, what will happen in different cases. So, this is the x y plane. And this is the curve in Z in w plane. Then if I consider the case, when a and d are 0 a is 0 means, this part is not their d 0 means, this part is not there. So, we have b x plus cy is equal to 0, it represents a line passing through the origin. So, a line passing through the origin in Z plane will map to a line passing through the origin. Because d and a be in 0 Z plane.

In w plane also, it is b u plus c v. So, line passing through origin will map to line passing through the origin. However, if a is not 0 and d is equal to 0. Then, Z plane will represent a circle passing through origin. And w plane will represent lines not passing through the origin. So that is, how curve which circles are passing through origin in the Z plane. They will become lines passing through origin in the w plane lines not passing thorough origin in the w plane.

In the next case, when d is not 0 and a is equal to 0. Then in Z plane, they represent lines not pass through origin. And in w plane will be having circles passing through origin. So, these are different cases and accordingly circles and lines can be circle. And lines, the level curves in Z plane will map to corresponding curves in w plane.

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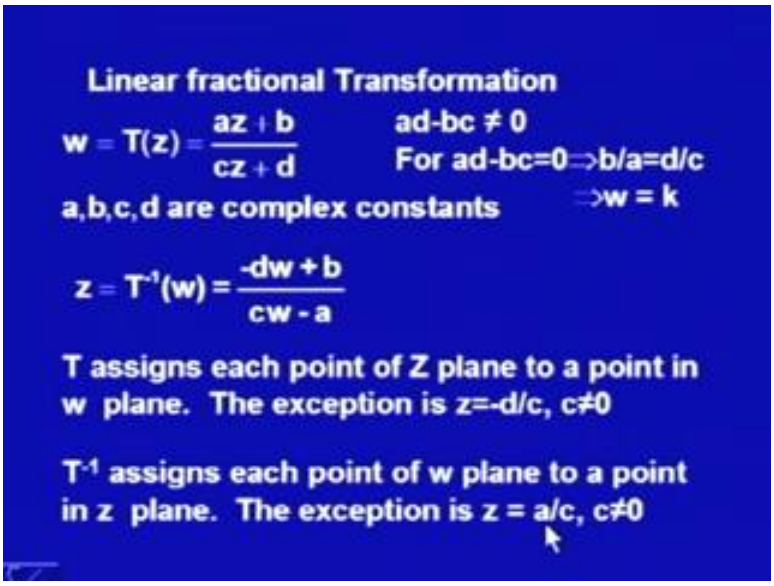

Another transformation of importance is the linear fractional transformation, we define a linear fractional transformation, w is equal to TZ as a Z plus b divided by c Z plus d. You may notice that in the numerator, we have a linear function. In the denominator also, we have a linear function. And since we are dividing them we call it, a linear fractional transformation here, a b c d are complex constants.

Now, this transformation is meaningful only when ad minus b c is not equal to 0. Because, if ad minus b c equal to 0. Then, we can write it as b by a is equal to d by c. And you can take a outside from this numerator and c outside from this denominator and if this condition is satisfied. Then, the two things will cancel out and what we have is simply w is equal to a constant. So, this mapping is of not our interest. So, we will consider the case, when ad minus b c is not 0.

Now this mapping, one can always simplified and write it as Z is equal to minus d w plus b divided by c w minus a. And that means, for every w there is a corresponding Z in Z plane; that means, the map the transformation is invertible. And invertible map invertible transformation is always 1 1 1 2. So, we can say that T assigns each point of Z plane to a point in w plane.

And similarly, T was assigns each point of w plane to a point in Z plane. The exception is a point Z is equal to minus d by c when c is not 0 this is the exception. Because, when denominator is 0, then this mapping is undefined. So, apart from this mapping is defined everywhere and you can have a corresponding point in w for a given Z. Similarly, for T inverse it assigns a point of w plane to a point Z plane and the exception is this point.

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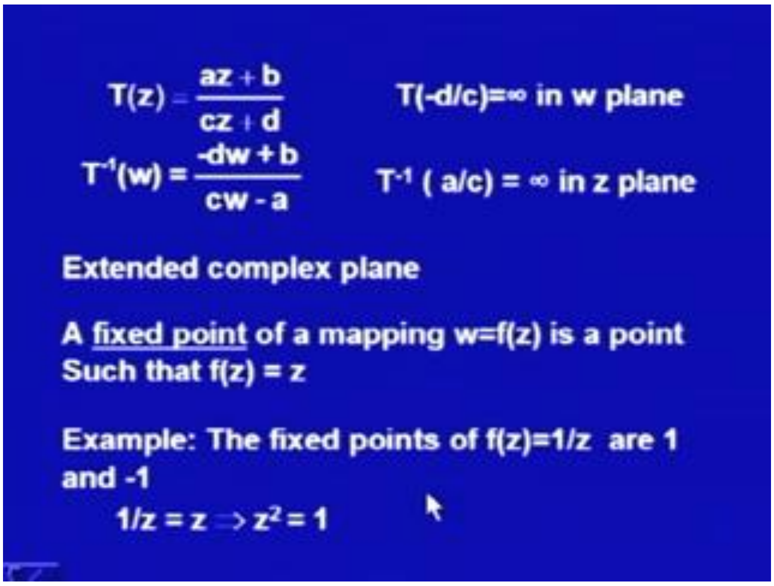

So mapping is defined for all points except these two points. In fact, we say that this mapping which is not define for minus d by Z is equal to minus d by c. If we add a point infinity in w plane then we can say that this mapping will map to infinity corresponding to point minus d by c or minus d by c. Under this transformation will map to infinity in w plane.

Similarly, we can say that a by c will map to infinity in Z plane. So, w is equal to a by c will map to infinity in Z plane. So, this way we are adding infinity in our complex plane. And the plane is then called extended complex plane. Then, we defined a fix point of a mapping a fix of a mapping, w is equal to f Z is a point such that f Z is equal to Z. Example is the fixed points of f Z is equal to 1 over Z are 1 and minus 1. And this definition suggests that one will map to 1 minus 1 under this transformation. And how we obtain this you can simply write down 1 upon Z is equal to Z. Then, it becomes Z square is equal to 1 and solving it we will get Z is equal to 1 and Z is equal to minus 1.

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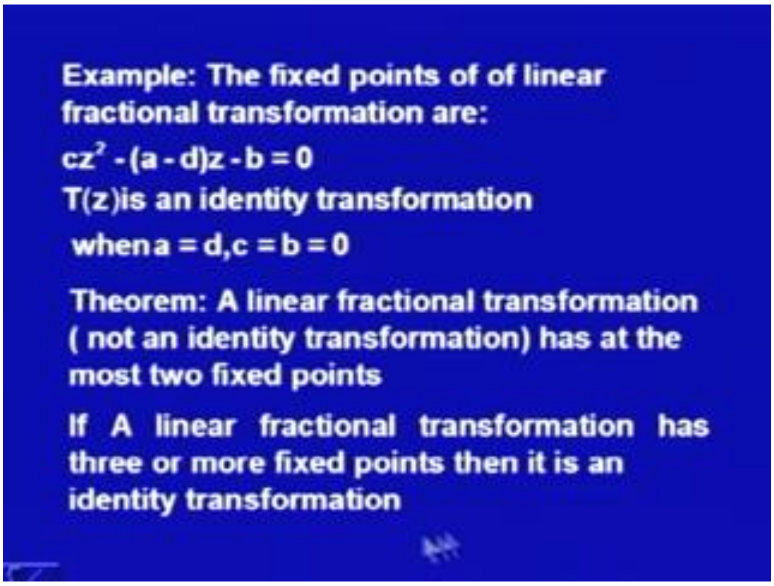

The next example, we will show that the fixed points of linear fractional transformations are given by this expression. This can be obtain by writing f Z is equal to Z. And simplifying the expression, we get c Z square minus a minus d Z minus b equal to 0. So, we say that when the transformation T Z is an identity transformation. When a is equal d this is 0 and b and c are also 0 b and c are also 0.

So, this equation is trivially satisfy; that means, every 0 will map to itself and such a transformation is an identity transformation. So under this condition, the linear fractional transformation will become an identity transformation. And on the basis of this, we establish a theorem that a linear fractional transformation which is not an identity transformation has at the most two fixed points.

One can notice that if b, these conditions not satisfy that it is not an identity transformation. Then, the fixed points are given by this quadratic this quadratic gives us only two points. So, if we are having more than those two points. Then, we say that the mapping is an identity transformation.

So, this is what the theorem is, I am not giving the prove giving you the proof of the theorem. Just a theorem and we can use it for checking whether transformation is identity transformation or not. If a linear fraction transformation has three or more fixed points then it is an identity transformation. This is the direct consequence of this.

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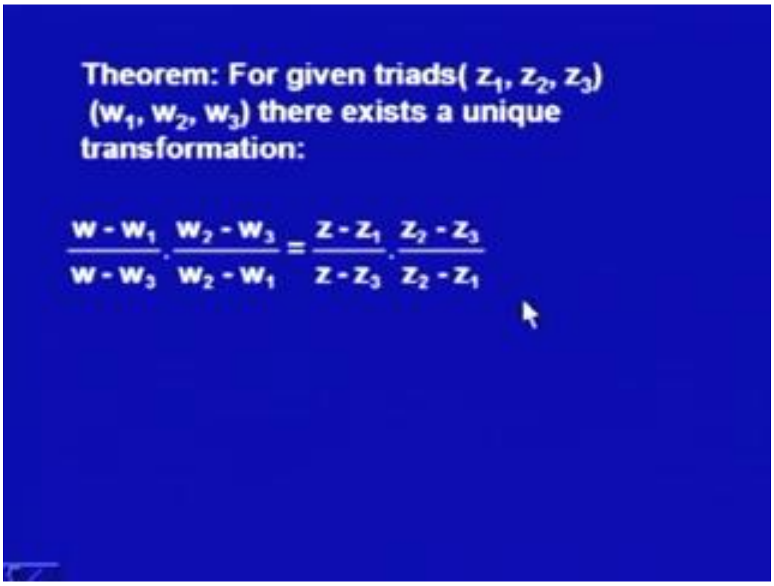

Is one more theorem, which is use to get the linear fraction transformation. And we say we can specify three different points, such that Z 1 maps to w 1 Z 2 maps to w 2 Z 3 maps to w 3 then their exist a unit transformation given by this expression. So, again the proof is not included.

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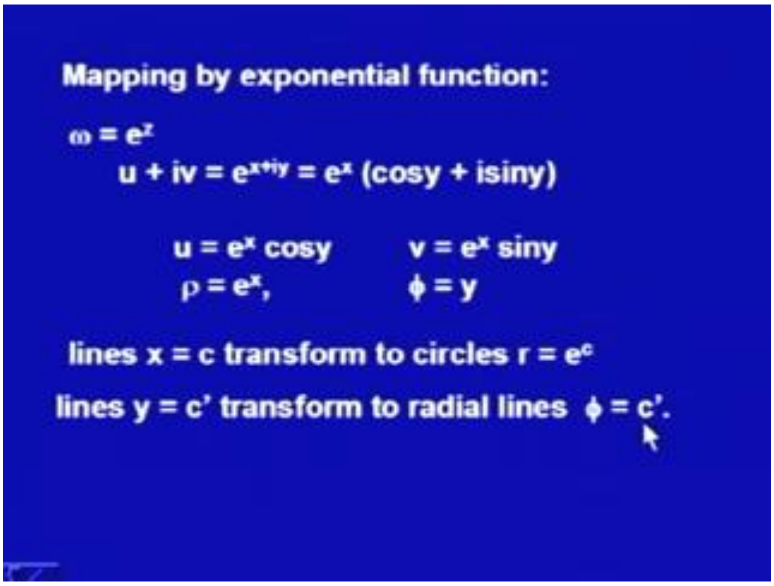

These are the mappings by simple functions, now we will discuss mapping by exponential function. So, if we have w is equal to e Z. Then, u plus iv is equal to ex plus i y is equal to ex cos y plus i sin y. By the definition of exponential function and this gives me u is equal to ex cos y and v is equal to ex sin y.

And from here, we say rho is equal to ex phi is equal to y. From here, rho is equal to ex in radial form and 5 is equal to y. And that means, lines x is equal to constant will transform to circles r is equal to e c. And lines y is equal to c dash will transform to radial lines phi is equal to c dash with this. We come to an important concept of conformal transformation.

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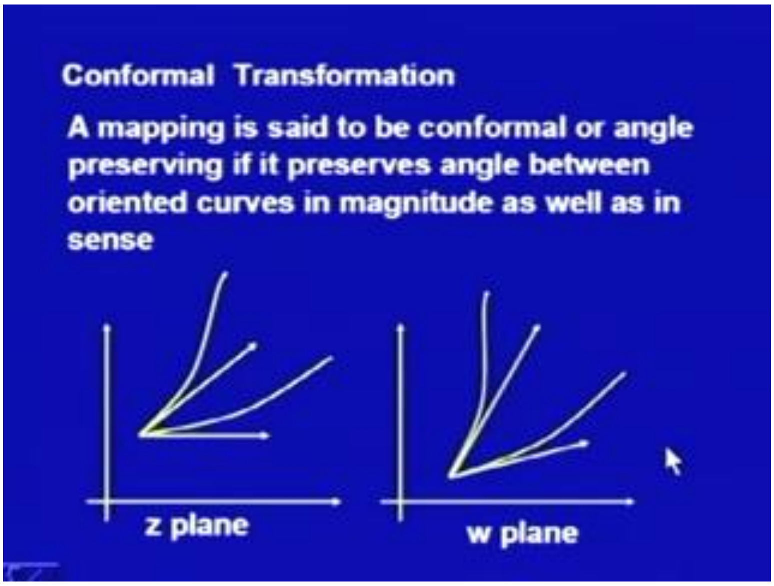

A mapping is said to be conformal or angle preserving. If it preserves angle between oriented curves in magnitude as well as in sense by this I mean to say. Let if I have two curves let us call them c 1 and c 2 in plane. And the angle between the two curves, which is the angle between the corresponding tangents. That is this tangent and this tangent.

Then, if this angle is theta 1, so when this these two curves will map to this curve and this curve in w plane then their angles are preserves. So; that means, if this angle theta this angle is also theta they are preserved in magnitude as well in sense as well as in sense means that if we measure angel from c 1 2 c 2 here we will measure angle from c 1 to c 2. So, if this angle is clockwise then this will also be clockwise. So that is the meaning of angel preserving transformation or we call it a conformal transformation.

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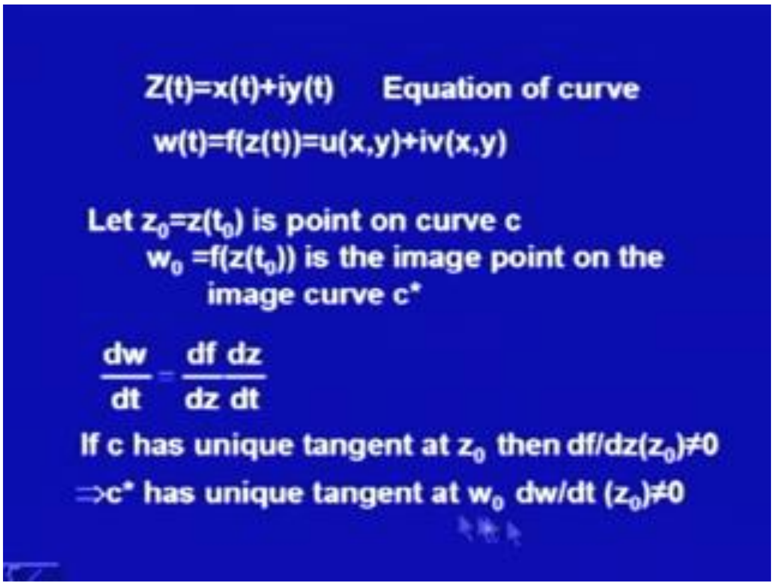

So, let us say that Z t is equal the x t plus i y t is a equation of curve. It is a equation of curve. Because as t changes, the t is the parameter of the curve as we change t will have different points. And these different points constitute the curve. And w 2 is equal to f of Z t is the image of this curve and we write it as u x y plus i v x y.

Let us say Z naught is a point on curve c. Then, the corresponding point w naught, which is f of Z t naught is the image point of the image curve c star, then d w by d t will be written as d f by d Z into d Z by d t. From here, is a composite mapping? So, d w by d t is equal to d f by d Z into d Z by d t. And if c has a unique tangent at Z naught; that means, d f by d Z at Z naught is not 0. If it is not unique tangent if this is 0.

Then, we cannot find a tangent uniquely, so this has to be non 0.So, if d f by d Z is non 0 d Z by d t is not 0, because is a very is a curve and t is varying d Z by d t is not 0. So, d w by d t is not 0; that means, if the tangent is well defined at c the tangent of its image will also be well define, so c star also has unique tangent at w naught.

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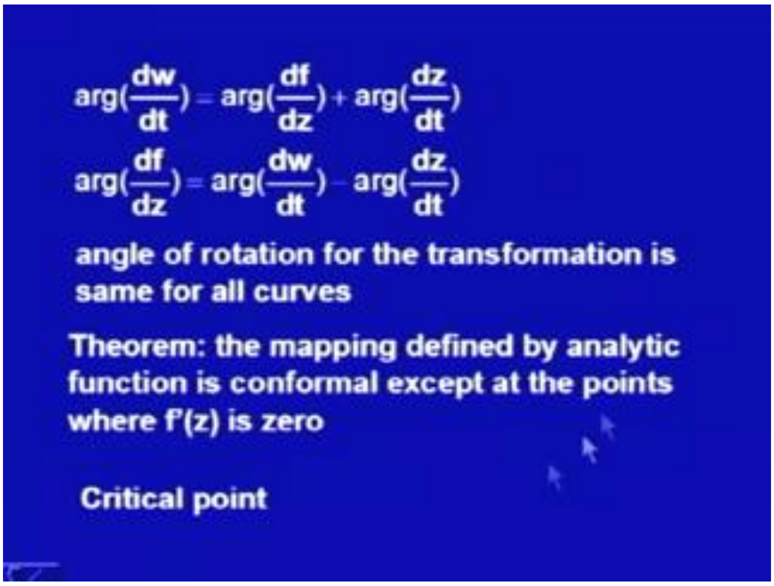

And from here, we can find out that argument of d w by d t is argument of d f by d Z plus argument of d Z by d t, which will be simplify to get argument of d f d Z is equal to argument of d w by d t minus argument of d Z by d t. And this means the angle of rotation for the transformation is same for all curves; that means, if I take different curves.

They will move differ they the angle of rotation will be the same. And that means, if we take two curves in Z plane. Then, the corresponding two curves in w plane will also move in the same angle. And so the net effect will be the no rotation. So, this gives us a theorem that the mapping defined by analytic function is conformal except at the points where f dash Z is 0.

So, f dash 0 means the tangents are not defined. So there will be a problem. But for all other zero, if the function is analytic then the mapping will be conformal. So, this is an important result. And the points were this is 0, those points are called critical points. And the angle may not be preserved. But for rest of the curves the angle will be preserved.

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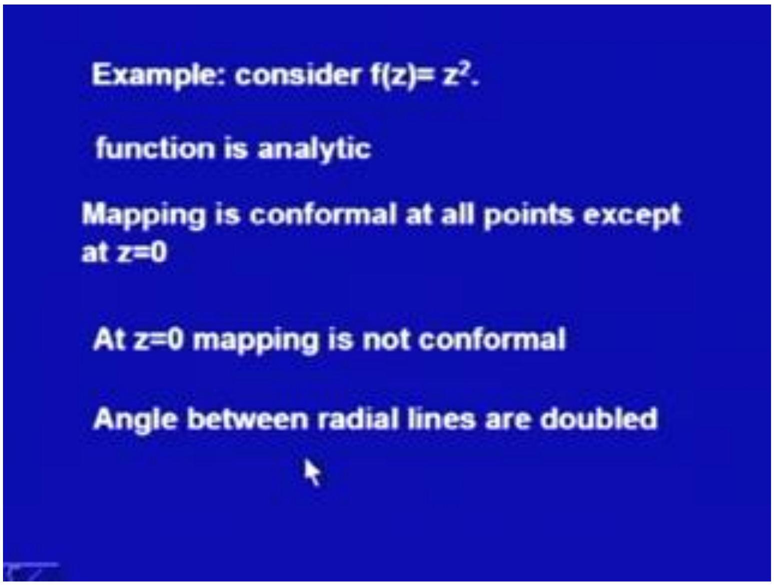

So this is an example, we consider f Z is equal to Z square. This function is analytic, which is conformal at all points except at Z is equal to 0. So, Z is equal to 0 is a critical point. One may also notice that f dash Z is 0 at Z is equal to 0. So this is a critical point and this says that at Z is equal to 0 mapping is not conformal. But, at all other points mapping is conformal angles are preserved.

And, if fact at Z is equal to 0 radial lines there, then we draw two radial lines the distance between them are doubled this, we have already seen in example. So, this function f Z is equal to Z square being analytic is conformal except at Z is equal to 0. So, with this we have covered a portion on of complex variable where we have discussed function of complex variables. We have discussed some elementary functions and mappings and finally, we have described conformal mapping that is it.

Thank you.