Mathematics-II
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Module - 2
Lecture - 19
Functions of Complex Variables Part-I

Welcome viewers, today we are discussing Functions of Complex Variables. In this lecture, I will be covering function of a complex variable.

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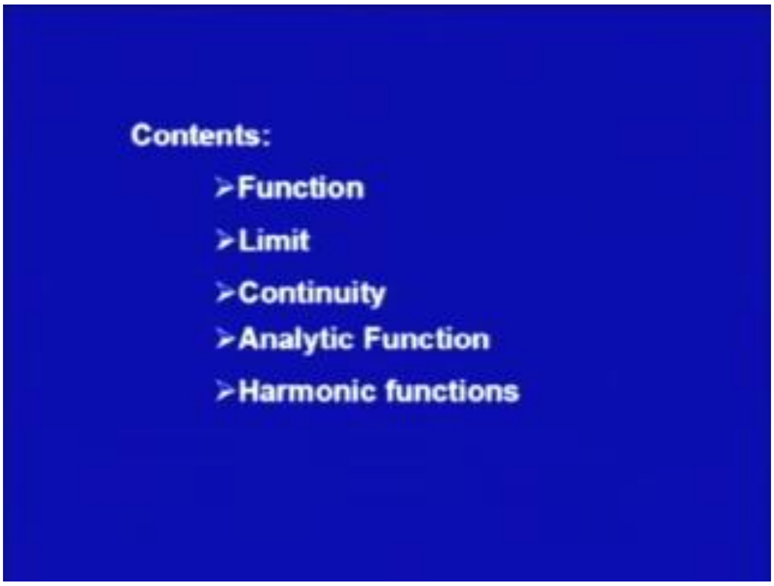

Then, I will introduce the concept of limit, I will continue with continuity of a complex variable. Then, I will introduce analytic function. And finally, we discuss harmonic functions.

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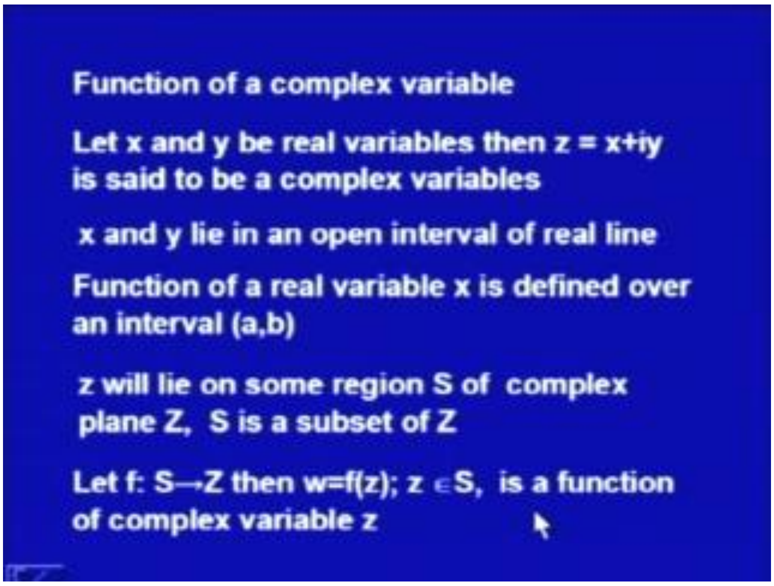

First, function of a complex variable, let x and y be real variables. Then, we denote z as x plus i y. And we say that, z is a complex variable. We know that, x and y, they are real variables. They lie in an open interval of real line. Then, function of a real variable x is defined over an interval a b. However, z will lie on some region S of complex plane and S is a subset of z.

Now, we are in a position to define the function f of a complex variable. Such that, it takes values on the set S and will map to this complex plane z. Then, w is a function of z, where z belongs to the set S. In such a situation, we say that, f is a function of complex variable z.

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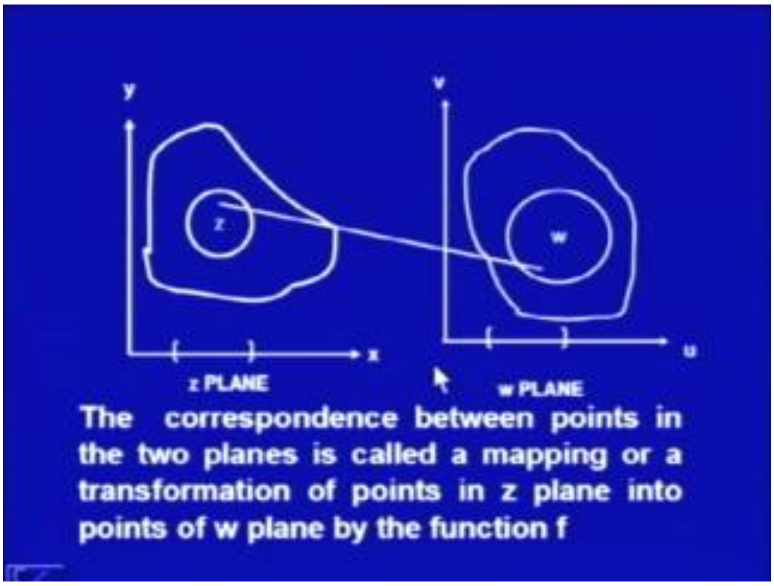

Let me illustrate on this diagram, we have real variable x. It lies in interval a b and this is the part of the real line. This is another real variable y and the variable y, will lie on this line and the z will be lying on this plane. This plane is called z plane. By a point on z plane, we will map to a point on w plane, such that w is equal to f z. Now, the values z can take in this domain, we call that domain as s.

So, s will map to z and we say w is a function of z. This plane is x and y while this plane is called w plane. Here, z is x plus i y, when it map to w, when w is written as u plus i v. So, this is real line u and this is real line v. The correspondence between points in the two planes is called a mapping or a transformation of points in z plane into points of w plane by this function f.

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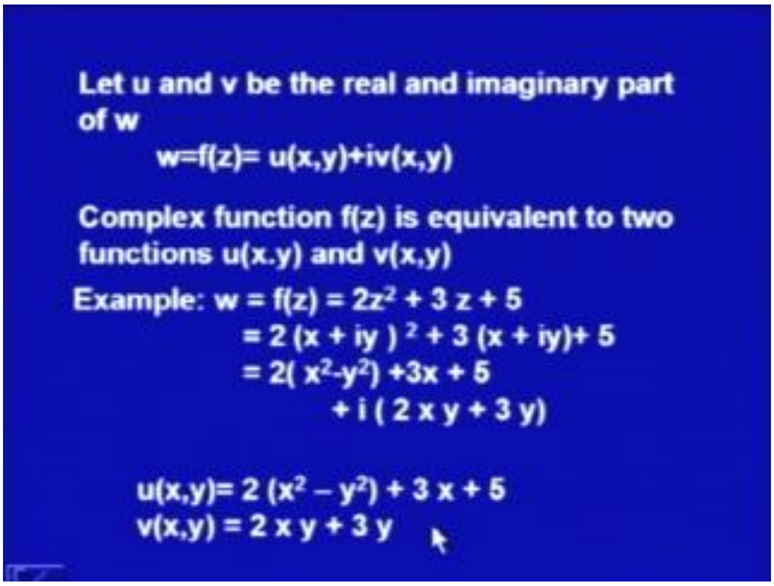

So, that is how, we introduce concept of functions. Let u and v, be real and imaginary part of w. Then, we write w is equal to f z is equal to u plus i v, normally, u is a function of x and y and v is also a function of x and y. So, w also has a real part u x y and imaginary part v is also a function of x and y. So, in this sense complex function f z is equivalent to two functions u of x and y and v of x and y. For example, we write w is the function of f z, which is defined as 2 z square plus 3 z plus 5.

So, if I write z is equal to x plus i y, then it is 2 multiplied by x plus i y square plus 3 x plus i y plus 5. Simplifying this, it is 2 x square minus y square the real part coming from this expression. And then plus 3 x plus 5, and then i times 2 x y coming from this plus 3 y is the imaginary part. So, this way, we write the function w as consisting of real part as u x y is equal to 2 times x square minus y square plus 3 x plus 5. And the imaginary part v x y is 2 x y plus 3 y.

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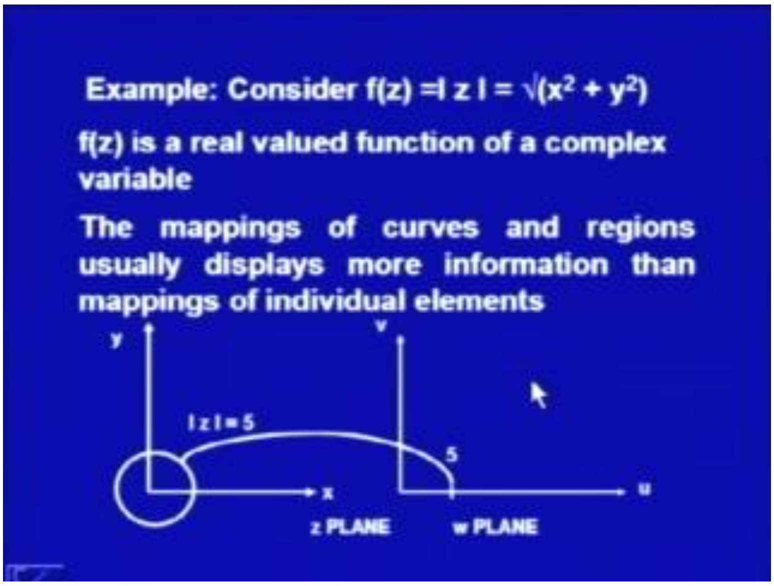

This is another example. Here, we write the function f z as modulus of z. And we write it as under root of x square plus y square x. And y being the real and imaginary part of z, f z is a real valued function of a complex variable. Because, does this function does not have an imaginary part. So, f z maps to real value, that is why; we say this function, f z is the real valued function of a complex variable z.

Normally, the mappings of curves and regions, usually displays more information, then mappings of individual elements. So, under this mapping, we write what a particular point will map to this particular point. A particular point in z plane will map to a point in w plane. But, it is more meaningful, if I say this circle mod z is equal to 5 will map to this point in the w plane. So, this boundary will map to w plane. So, normally mappings of curves and region, they display more information rather than mapping of individual elements.

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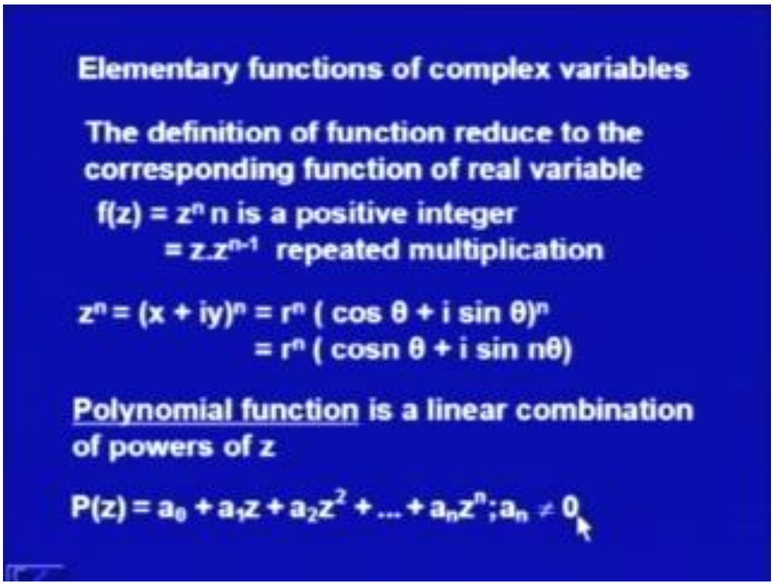

Now, will discuss elementary functions of complex variables, we have functions of real variables. We extend those functions to complex variables in such a manner, that when we consider the variable to a real variable. They will become function of a real variable. So, it simply in extension from real domain to complex domain. So, let us say, f z is a function of z raise to power n, n is a positive integer. We define this function as z times, z raise to power n minus 1.

The idea is, it is repeated multiplication and when x z is a real variable. Then, f of x becomes x raise to power n with this z square means z multiplied 2 times z raise to power n means z is multiplied n times. Now, z raise to power n can be written as x plus i y raise to power n. And x plus i y in polar form can be written as r cos theta plus i sin theta in this sense. This number x plus i y raise to power n is r raise to power n multiplied by cos theta plus i sin theta raise to power n.

And further this can be simplified to r raise to power n cos n theta plus i sin n theta. So, z n can be expressed as real part r n cos n theta plus i times imaginary part r n sin n theta. Where the real part u is function of r n theta and imaginary part is also function of r n theta. Once we have defined z raise to power n, we are in a position to define polynomial function. So, polynomial function is a linear combination of powers of z.

And we write it as a naught plus a 1 z plus a 2 z square plus a n z n, where a n is not equal to 0. So, it is a linear combination of powers of z, means we have z raise to power

0, z raise to power 1 z square z n. And then we have taken their linear combination that is each term is multiplied by a constant, and then added. Here, it is notice that n is not equal to 0, by this, I mean to say that this is a polynomial in degree n. If a n is equal to 0, then this polynomial will be of lower degree

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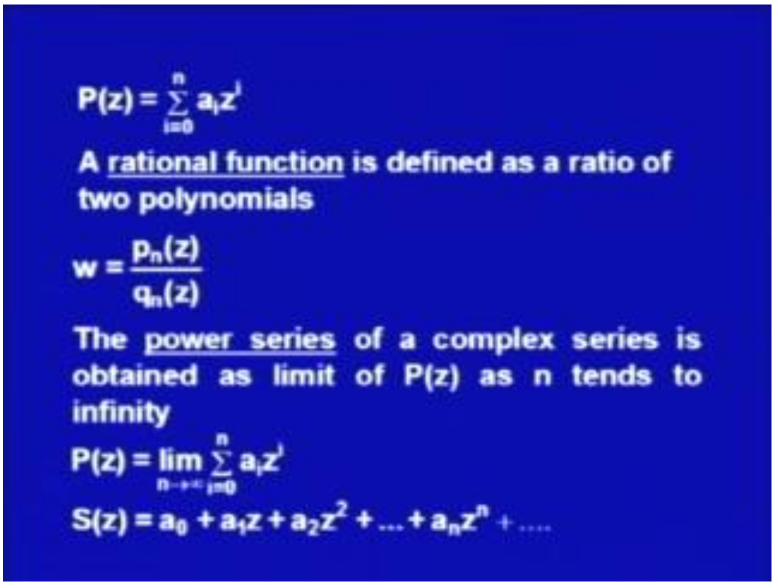

In short we write the polynomial p z as summation i is equal to 0 to n a i z i. And this i will take values from 0 to n. From here, we extend two a rational function, a rational function is defined as a ration of two polynomials. So, if I have two polynomials p and z and q m z, then w is p n z divided by q m z. Normally, these n and m are different. But, they may be same.

Here, you may note you here the important thing is that q and z is not 0. The power series of a complex series is obtain as limit of P z as n tends to infinity. So, we write this expression as P z is equal to limit of a i z i n tending to infinity. So, this becomes a power series. And we write, it as S z is equals a naught plus a 1 z plus a 2 z square plus a n z n and so on. So, this is a power series, we call it a power series, because each and every terms is express as some power of z.

Now, next is limit of a function of a complex variable, before we proceed to limit of a function of complex variable. I like to review the concept of limit in case of real valued functions real variables.

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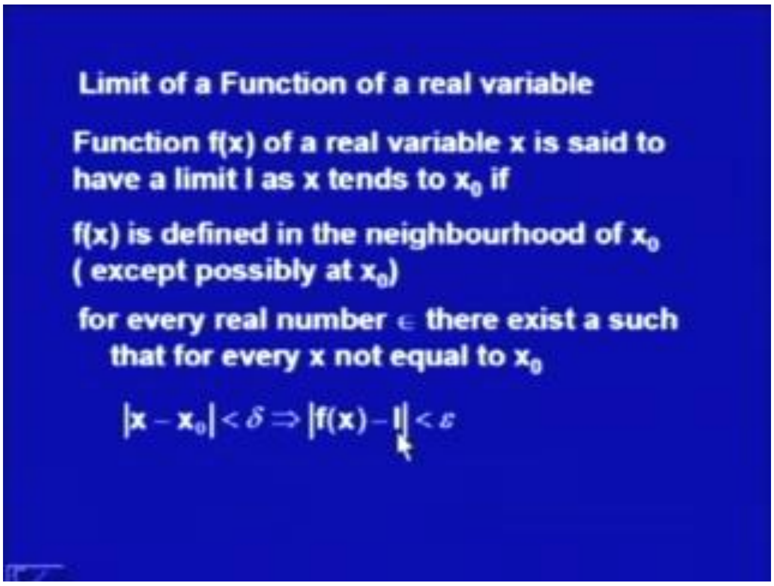

So, limit of a function of a real variable, let f x is a function of a real variable x. Then, we say that, l is the limit of this function f x as x tends to x naught. If f x is defined in the neighborhood of x naught except possibly at x naught. And then for every real number Epsilon there exist a delta. Such that, for every x not equal to x naught, we have x minus x naught less than delta implies f x minus l is less than Epsilon. By this I mean to say that, whenever, we are close to x. Then, we are close to l also.

So, if we are close to x naught and the distance between x and x naught is delta, then there will be some Epsilon. And will be close to f x, the function f x will be close to l. So, that is the meaning of limit of a function of a real variable, closeness to x means closeness to l.

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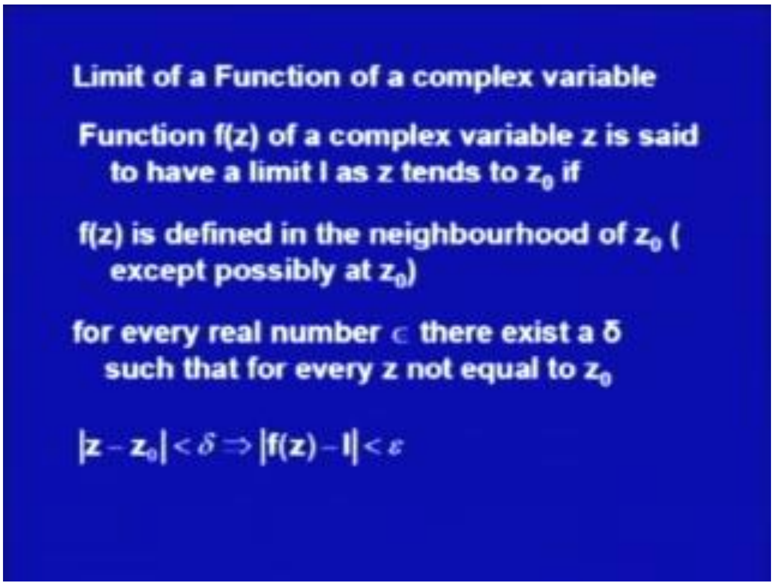

Now, with this we will go to limit of a function of a complex variable. So, let us say f z is a complex variable function z is the complex variable. Then, this function f z has a limit l as z tends to z naught, if it satisfies the two conditions. The first is f z is the defined in the neighborhood of z naught accept possibly at z naught. Further, for every real number Epsilon, which has to be positive there exists a positive delta.

Such that, for every z not equal to z naught, we have modulus of z minus z naught is less than delta implies modulus of f z minus l is less than Epsilon. This z minus z naught modulus less than delta represents a unit is circles centered at z naught. And of radius delta by f z minus l modulus less than Epsilon represents circle centered at l and of radius Epsilon. So, that means, whenever we have points in the in this circle is that plane. Then, will have points in the circle in w plane.

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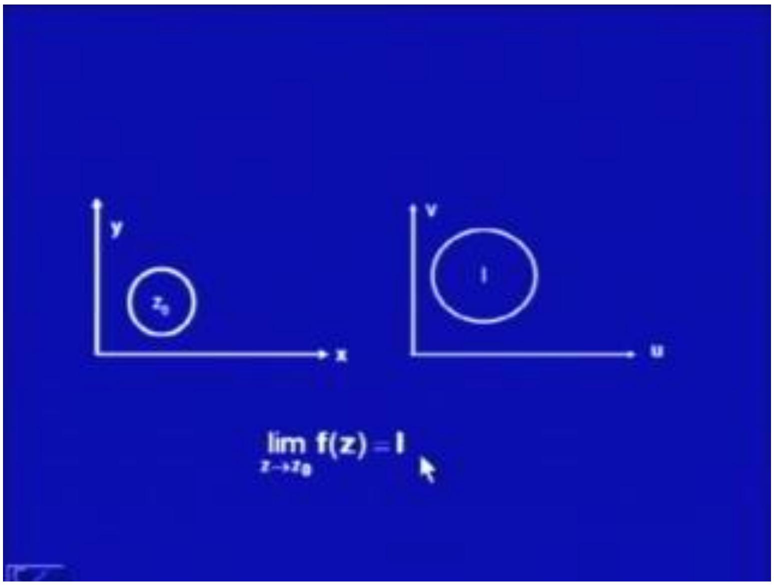

Let me illustrate this pictorially. So, let us consider z plane and w plane. In z plane, I have a point z naught and there is a mapping, which maps f of z naught to l. So, we consider all the points in the delta neighborhood of z naught. So, whenever we take a point in this neighborhood. So, this is the circle of radius delta centered at z naught. This is a circle of radius Epsilon and centered at l. So, whenever we have a point in this neighborhood of z naught. Then, correspondingly we have the point f of z in this circle.

So, points here will map to points here. So, points in the delta neighborhood of z naught will map to points in the Epsilon neighborhood of the limit l. That is the meaning of the limit of a function f of z and this we write it as limit of f z as z tending to z naught as l.

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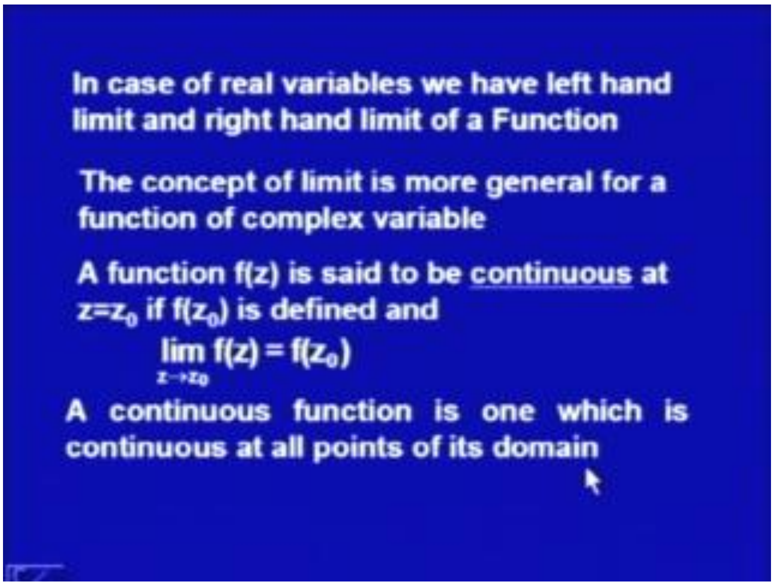

In case of real variable, we have left hand limit and right hand limit of a function. However, in case of complex variables, the concept of limit is more general and we will not have only left hand limit. And right hand limit, we have two limit is, in case of real variables. Because, we can approach to the point from left side or from the right side, because, where, the function is defined on an interval, but in case of complex variables.

We can have infinitely many paths to approach to the given point z naught. So, in this sense all the limit is form all the path should be equal. And we say this concept is more general in case of complex variables. A function f z is said to be continuous at the point z is equal to z naught if f z naught is defined and limit of f z as z tending to z naught f of z naught. Now, this definition is also an extension of the definition of continuity of continuous function of real variables. And then a continuous function is one, which is continuous at all points of it is domain.

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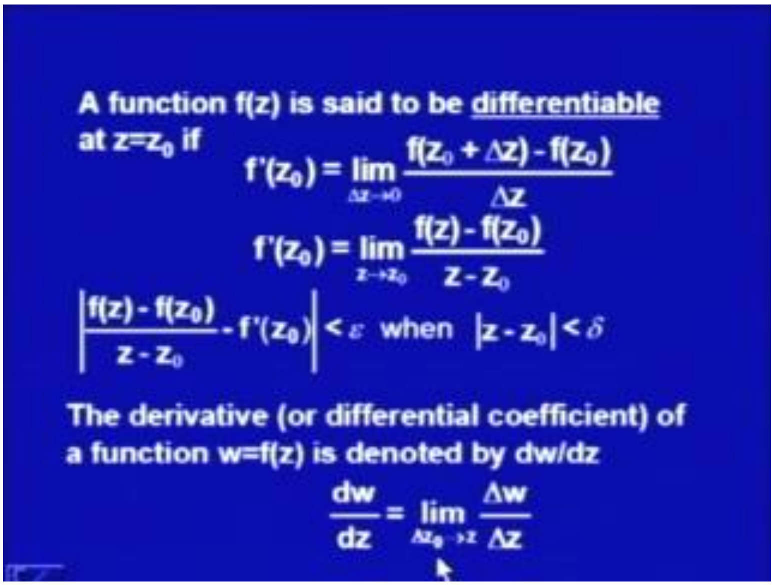

After defining, limit and continuity of a complex variable function. We are now in a position to define differentiability of a complex variables function of f of z. A function f z is said to be differential at z is equal to z naught. If limit of the expression f z naught plus delta z minus f of z naught divided by delta z as delta z tending to 0 exists. And then we say this limit is is f dash of z naught or derivative of f at z naught.

If I denotes z naught plus delta z as z, then this expression can be written as f of z minus f of z naught divided by z minus z naught. The limit of this expression as z tending to z naught is defining as f dash of z naught. Since, we are defining limit here. So, if we apply the concept of limit. Then, f z minus f z naught divided by z minus z naught minus f dash z naught. It is modulus is less than Epsilon, whenever z minus z naught modulus is less than delta.

The derivative or we call it as differential coefficient of a function w of f z is denoted by d w by d z. This is nothing but, d w by d z. And, we write d w by d z is limit of delta z naught ending to z as delta w by delta z. We say this is the change in f divided by change in z. So, this delta w divided by delta z, when delta z tending to 0 d. It is called d w by d z.

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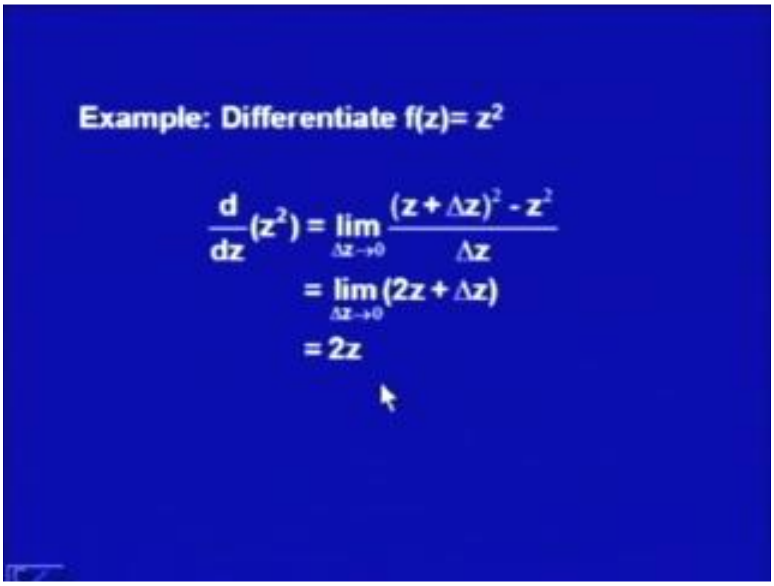

Let us illustrate this with an example, where f z is equal to z square. So, if we have differentiate this then d by d z of z square is equal to z plus delta z whole square minus z square divided by delta z. So, this is the ration we form, and then we take the limit as delta z tending to 0. So, when, we simply the numerator, this comes out to be z square and z square cancels out. And what we have is, 2 z into delta z, which will cancel with this plus delta z square, which out of this delta z square. This delta z will cancel out. And will have only delta. So, limit of 2 z plus delta z as delta z tending to 0 means, this is nothing but 2 z or we say the derivative of f z is 2 z.

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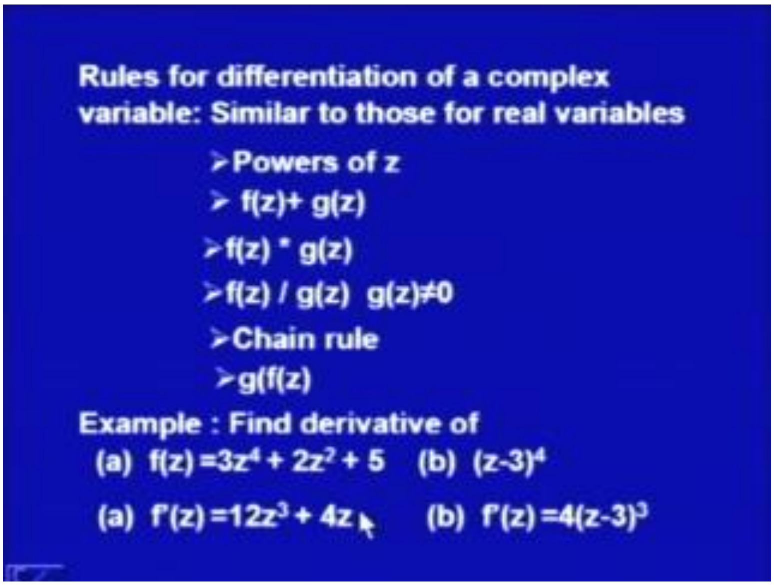

Now, we form the rules for differentiation of a complex variable. These rules are very similar to what we have for real variables. And there proof is also very similar. So, 1 by 1 powers of z. That is z raise to power n derivative of z raise to power n is nothing but n times z raise to power n minus 1. Then, we can have derivative of sum of two functions f z plus g z. Then, derivative of this sum is sum of derivatives.

Similarly, we can write a formula for product of two functions. Then, when we have ratio f z divided by g z, we can differentiate them provided g z is not 0. Then, we have chain rule. And it is a composite function; similar formulae can be obtained for these cases, when we have a function of a complex variable. And, they can be used in this example.

So, if we have to find the derivative of 3 z raise to power 4 plus 2 z square plus 5. We can consider it to be sum of three functions 1 2 and 3. And the derivative will be derivative of this plus derivative of this plus derivative of this. We know, how to find derivative of z raise to power 4, and we know how to find derivative of z square and so on. So, the derivate of this can be easily formed.

Similarly, in this expression it is z minus 3 raise to power 4. So, write down the derivative, it is a composite function. So, we can find out the derivative of z minus 3 raise to power 4. So, if we solve this the derivative of this function f of f dash of z is 12 times z cube plus 4 times z. And derivative of 5 is derivative of constant is 0. And in the second case f dash z is 4 times z minus 3 cube into derivative of z minus 3, which is one, derivative of this function is simply 4 times z minus 3 cube.

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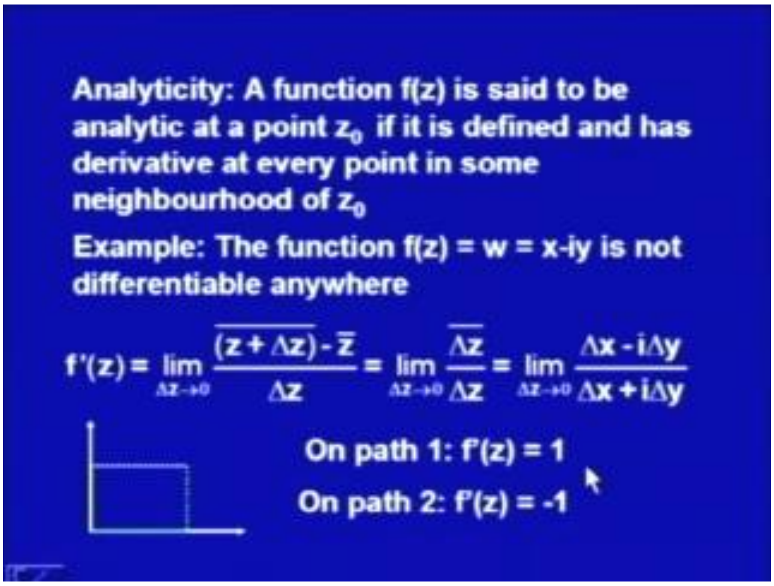

Now, we introduce the concept of analyticity. A function f z is said to be analytic at a point z naught, if it is define and has derivative at every point in some neighborhood of z naught. So, basically when we say differentiability a function is differentiable at a point, but when we talk about analyticity it has to be differentiable not only at a point z naught. But, it has to be differentiable in some neighborhood of z naught.

Example, the function f z which is equal to x minus i y is not differentiable anywhere. Let us check this f dash z is obtain as limit of this ration, this is delta z delta of f. So, function f is z bar x minus i y is z bar. So, z plus delta z bar minus z bar divided by delta z and then we take the limit as delta z tending to 0. So, this is nothing but delta z bar divided by delta z and this is equal to limit of delta x plus delta x minus i times delta y.

Because, we are taking conjugate, so it is delta x minus i times delta y divided by delta z, which is delta x plus i times delta y. So to take this limit, what we do is, we are here this is my point 0 and this is the point delta z. Now, I can approach to this point in number of ways to get this limit, what I will do is, I will first take this path. On this path, first, we have delta y drop to 0 and then delta x is tending to 0.

So, if I take delta y is equal to 0 in this expression. Then, this and this will cancel out. And, they there will be 0 and what is using, delta x divided by delta x which is 1. Whenever, delta x tending to 0, this expression gives me limit as 1. So, on path 1 limit of this expression that is f dash z is equal to 1. While, if I consider the second path on which delta x is drop to 0 and then delta y is tending to 0.

So, on path two when this happens; that means, delta x this they becomes 0 and delta y tending to 0. So, this and this are equal. So, this ratio becomes minus 1 and when, we take the limit delta z tending to 0, these remains minus 1. So, we see that along this path limit is comes out to be 1 by along this path limit comes out to be minus 1. But, from the definition of limit both the limit should be the same in whatever way, we have be approach to the point z. So, in this sense, we say limit does not exist. So, f dash z does not exist and this is true for all values of z. And that is why we say this function is not differentiable anywhere.

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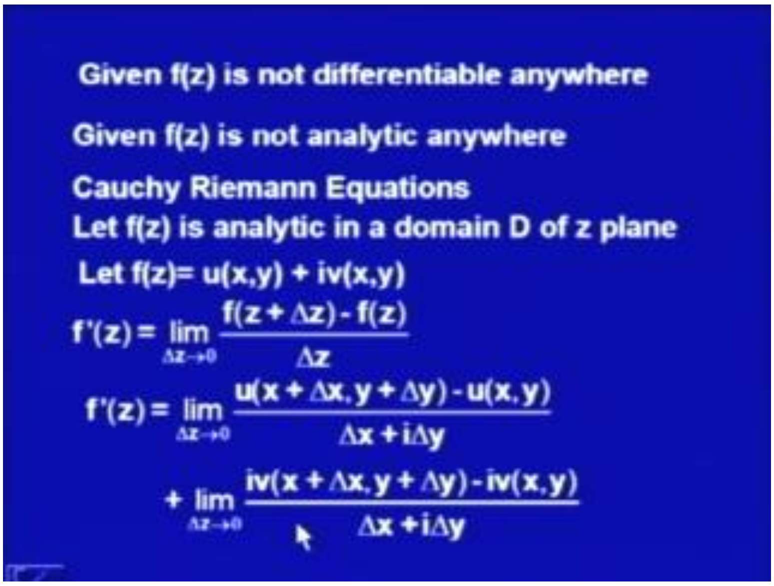

Now, this given f z is not differentiable anywhere. So, there is no question that it will be analytic. So, we say f z is not analytic anywhere. Now, we discuss Cauchy Riemann equations. And these equations, they are required for checking the analyticity of a given function. So, we first state that Cauchy Riemann equations and then will prove it. So, let f z is analytic in a given domain D of z plane. And we write the given function f of z as u x y plus iv x y is; that means, u is the real part and v is the imaginary part.

Then, f dash z is equal to f z plus delta z minus f z divided by delta z. The limit of this ration is f dash z by definition of differentiability. So, we write down f dash z as limit delta z tending to 0. And f z plus delta z is written as u x plus delta x y plus delta y plus i v x plus delta x y plus delta y. So, this is the function of this is written as u plus i v. So, this is the u part and this is the v part.

Similarly, f z is u plus i v. So, this is the u part and this is the v part and here x is the function of x and y and we are giving increment. So, x plus delta x and y plus delta y, but here we have only x and y. So, I have written this into parts first the real part and the second is the imaginary part of f.

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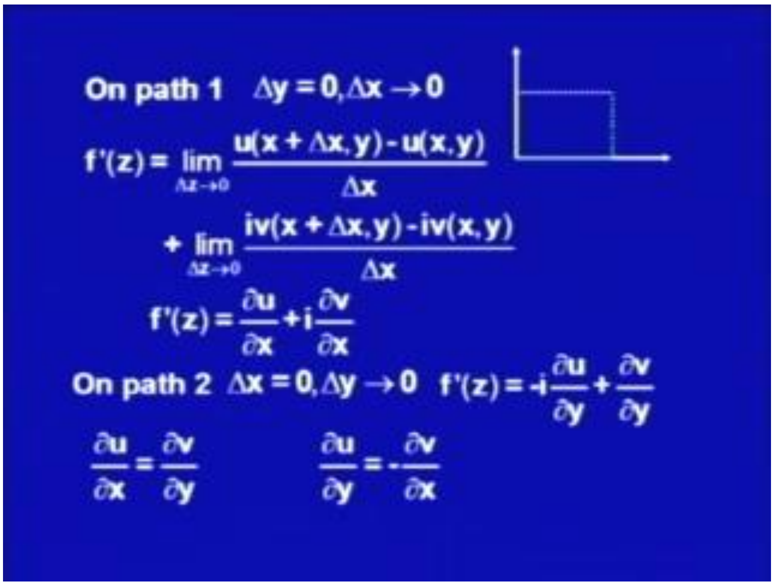

Now, we to take the limit I again consider two paths. First the path 1, that is this green path on which delta y is 0 and delta x is tending to 0. So, along this path f dash z is written as u x plus delta x comma y minus u x y divided by delta x and delta z tending to 0 and delta z tending to 0 means delta x is tending to 0. Then, limit delta z tending to 0 for I v x plus delta x comma y delta y is said to 0 minus i x y divided by delta x.

And then we know from the definition of see u is a real variable function of two variables x and y. And this is nothing but the definition of partial derivative of u with the respect to x. And this is partial derivative of v with respect to x. So using that definition, we can write f of f dash of z is equal to delta u by delta x plus i times delta v by delta x along this path 1.

Similarly, on path 2 delta x is tend set to 0 and delta y tending to 0. So, if you proceed on same lines then f dash z comes out to be minus i times delta u by delta y plus delta v by delta y. So here, I have obtained the limit of two expressions, one along this path another along this path and we have seen along one path it comes out to be this. On the other path, it comes out to be this, if limit has to exist then both the limit is should be same.

So, if we compare these two expressions then for equivalently of them. We should have delta u by delta x the real part of this must be equal to the real part of this. It is delta v by delta y and the comparing the imaginary part here. And the imaginary part here, it is delta u by delta y is equal to minus delta v by delta x. Now, these conditions are called Cauchy Riemann conditions.

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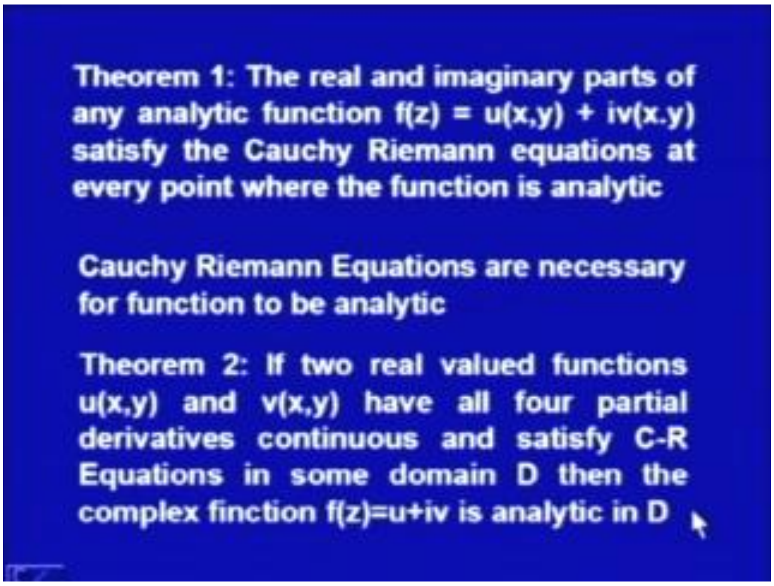

With this we are now in a position to state the theorem. It says that the real and imaginary part of an, any analytic function f z is equal to u x y plus i v x y, satisfy the Cauchy Riemann equations at every point where the function is analytic. Now, these Cauchy Riemann equations are necessary function are these Cauchy Riemann equations are necessary for function to be analytic.

Now, the second theorem states that these conditions can be made sufficient. According to this theorem, if two real valued functions u x y and v x y have all four partial derivatives continuous and satisfy C-R equations is some domain D. Then, the complex function f z, which is equal to u plus iv then it is analytic in D. So, we require only the continuity of four partial derivatives together with C-R equations.

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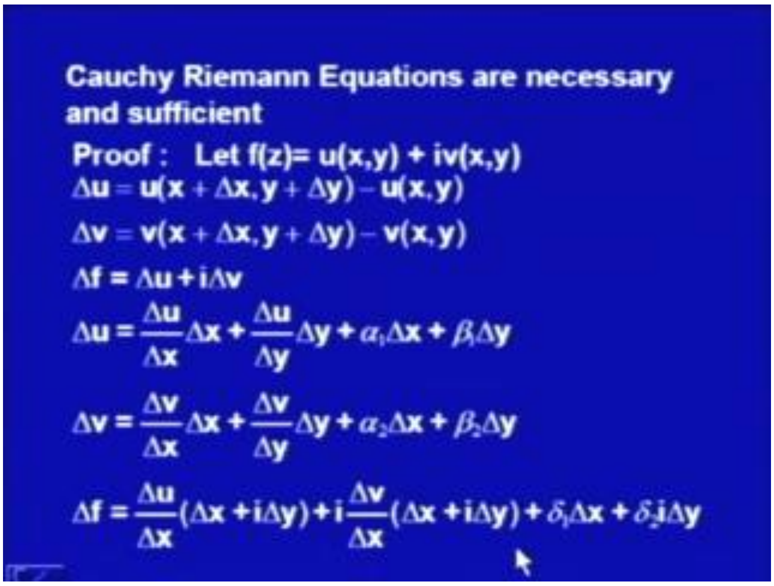

And then they imply the analyticity. So, in this sense C-R conditions are necessary and sufficient. X plus i delta y plus beta 1 delta x plus delta 2 times i delta y, so this is nothing but delta z.

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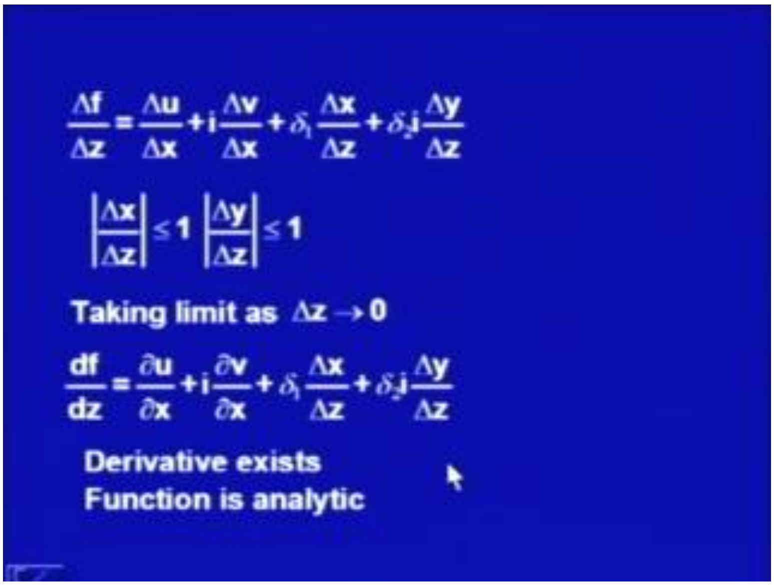

So, this delta f by delta z is nothing but delta u by delta x plus i times delta v by delta x plus delta 1 times delta x by by delta z plus delta 2 i delta y by delta z. And since, delta x by delta z, what is delta z, delta z is delta x square plus delta y square delta z square is delta x square plus delta y square. So, delta x is always smaller than delta z. And that is why; delta x by delta z modulus is less than equal to 1.

Similarly, delta y by delta z is also less than equal to 1 and in this taking the limit as delta z tending to 0. We will have this becoming d f by d z is equal to this is delta u by delta x plus this becomes delta v by delta x and plus delta 1 delta x by delta z plus delta 2 i delta y by delta z. And, they tend to 0 and that is why, we say derivative exist and when derivative exist function is analytic. This is coming because the derivatives are these derivatives are continuous. And, that proves the second theorem.

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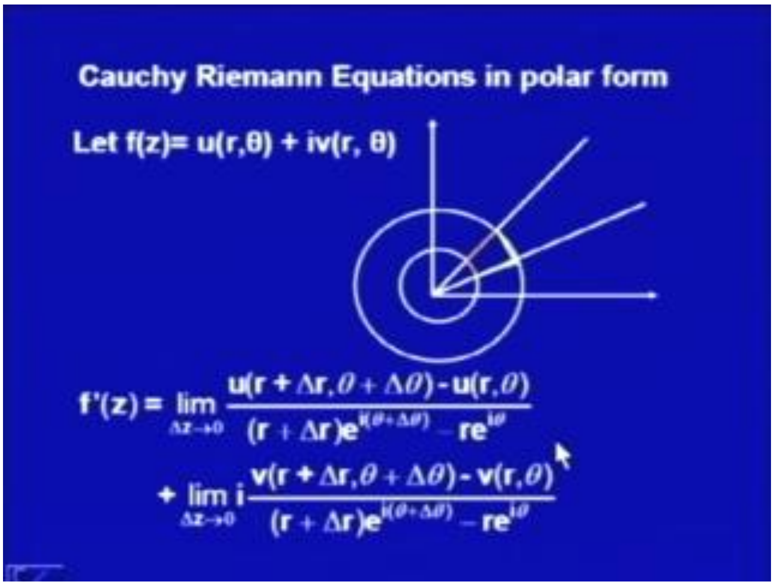

Now, the Cauchy Riemann equations which we have developed they are in Cartesian form. Then, we write z is equal to x plus i y, when we write down the function z, when we write down the variable z as re i theta, then Cauchy Riemann equation they are in polar form.

So, to develop Cauchy Riemann equation in polar form, what we do is, we consider z plane and this is out point z and this is the increment here. So, if you have to take limit from this point to this point, then any number of paths can be taken to approach to this point. But, if I consider two different paths, one is along this polar, one along this circular path and then on radial line that is path 1 and the other path is you move along the radial line first and then on this circle.

So, you can approach from this point to this point in two different ways, so if I consider these two different paths. I get the value of the limit and when the two limit is are equated what I get is, C-R equations in polar form.

So, let us consider f z is equal to u r theta plus i v r theta and then I write f dash z is equal to u r plus delta r theta plus delta theta minus u r theta and this is delta r is written as r plus delta r e i theta plus delta theta minus r e i theta this is nothing but delta z. Similarly, I write down the second component and then I take limit along two different paths.

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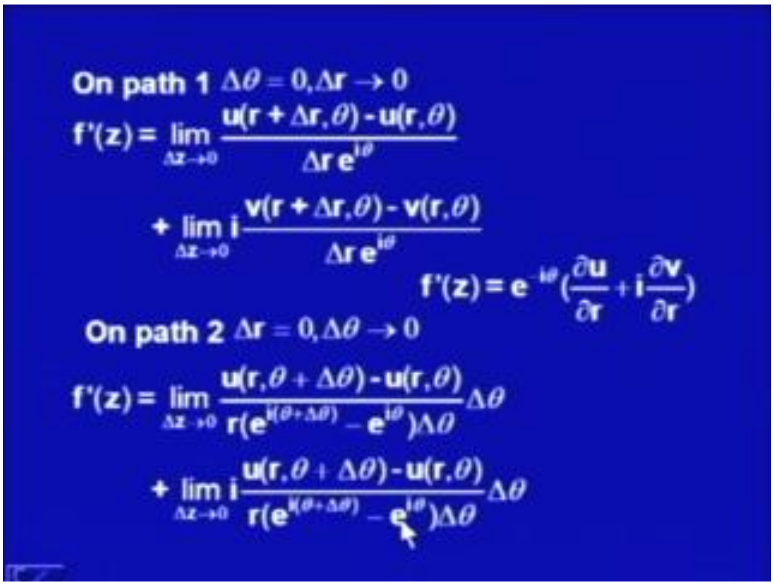

On path 1 delta theta is 0 and delta r tending to 0, we got this expression and from here we can easily see that f dash z come out to be e i theta delta u by delta r plus i times delta v by delta r. Similarly, on the path 2 delta r is equal to 0 and delta theta tending to 0 and we simplify the expression in this form and from here.

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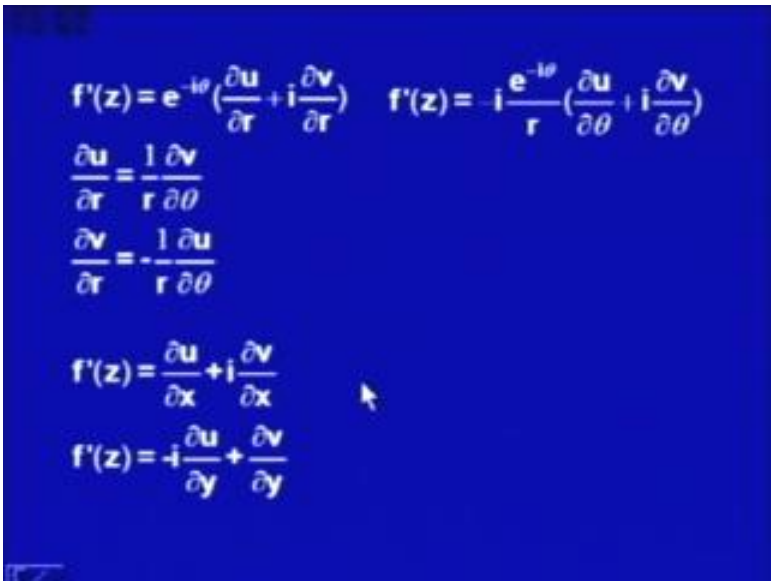

We will get f dash z as this, so f z is f dash z on path 1 is this f dash z on path 2 is this. And from here, we can have delta u by delta r is equal to 1 over r delta v by delta theta and from the imaginary parts will get delta v by delta r is equal to minus 1 upon r delta u by delta theta. These are polar forms of C-R equations. These are the Cartesian form of C-R equations, C-R equations we say this is the short form Cauchy Riemann equations or Cauchy Riemann conditions.

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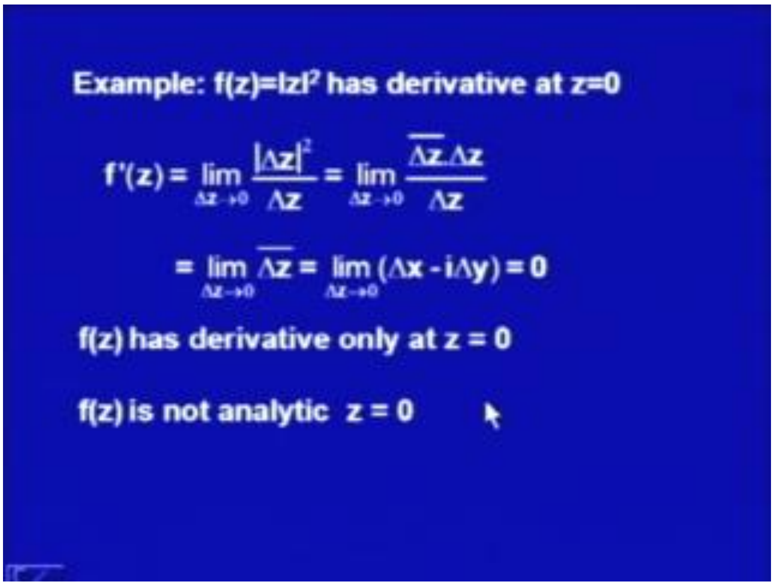

So, if a function is given f z is equal to mod z square and we have to see, what is it is, derivative at a given point z is equal to 0. Then, we can make use of the definition of the derivative and we can find it is derivative, so let us compute the derivative using the basic definition. So, f dash z is limit of delta z tending to 0 modulus of delta z square over delta and modulus of delta square i can write it like this and this delta z and this delta z they will tend to they will be cancelled out and what we have is, that f dash z is equal to 0.

So, and this is independent of path whatever path we choose, delta z tending to 0 this limit will be 0. So, we can say that f z has derivative only at z is equal to 0, this will happen only for z is equal to 0, but at no other point, this will happened and we say this function has derivative at z is equal to 0. But, it is not analytic at z is equal to 0.

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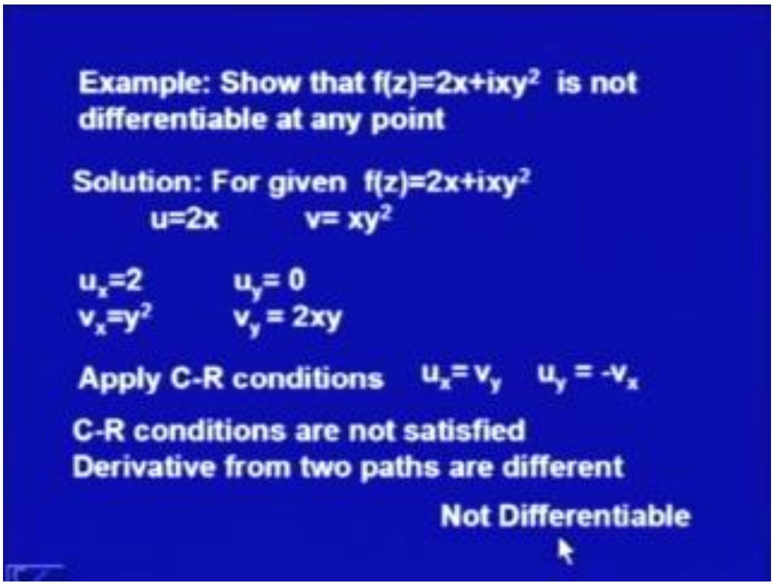

If you have to show that, this function is not differentiable at any point. We consider it is real path as u is equal to 2 x and v is equal to x y square. Then, we apply Cauchy Riemann equation and from here, we get u x is equal to u x means partial derivative of u with respective x it is 2. And, partial derivative of u with respect to y is 0 for this, partial derivative of v with respective x is y square and for this partial derivative of v with respective y is 2 x y. And, one can notice that these C-R equations will not be satisfied at any of the points, so we can say the function is not differentiable at any point in the domain.

So, C-R equations are not satisfied, when can very easily check and derivatives from two different paths are different. Because, conditions are not satisfied and that simply means that function is not differentiable.

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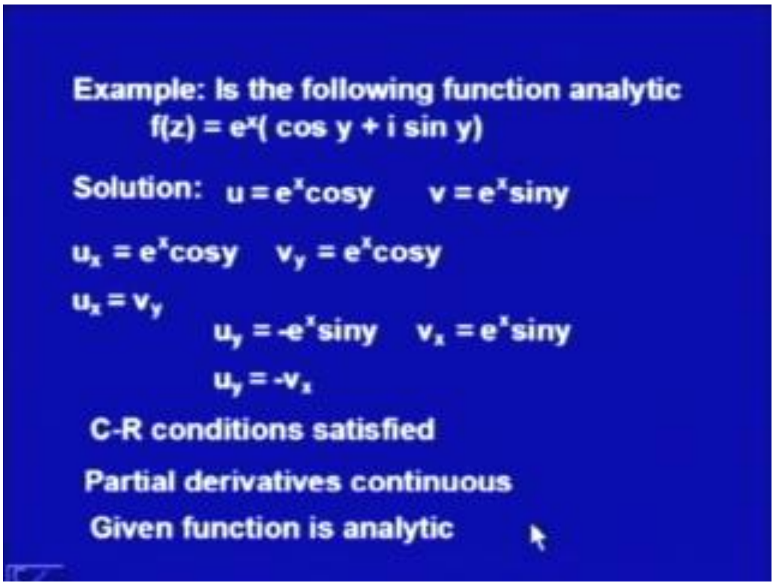

In the next example, you have to show whether function is analytic or not, so we consider f z is equal to e x cos y plus i sin y. So, accordingly u is equal to e x cos y and the imaginary part v is equal to e x sin y, we calculate the various derivatives u x is equal to e x cos y plus v y is e x cos y and we can see that u x is equal to v y.

Similarly, when you calculate u y and v x both the derivatives comes out to be satisfying this condition. So, all the C-R conditions are satisfied and it is irrespective of whatever be the value of x, these conditions are satisfied and we say that this function is analytic. Moreover these functions, these partial derivatives u x v y and u y and v x etcetera they are continuous. So, by the help of the two theorems which we have established we can say that this function is analytic. So all these things, I have already explained, so this given function is analytic.

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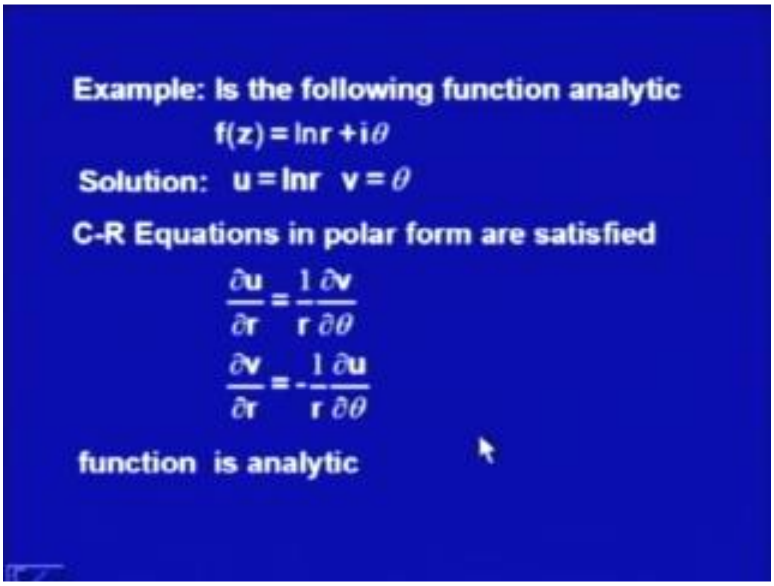

Now, this example makes use of C-R conditions in polar form, so the question is the following function analytic or not. So, f z is equal to log R plus i theta, so accordingly u is equal to l n r and v is equal to theta. We make use of polar forms of C-R equations which are given as this and one can check that these conditions are satisfied, this is straight forward. So, function is analytic.

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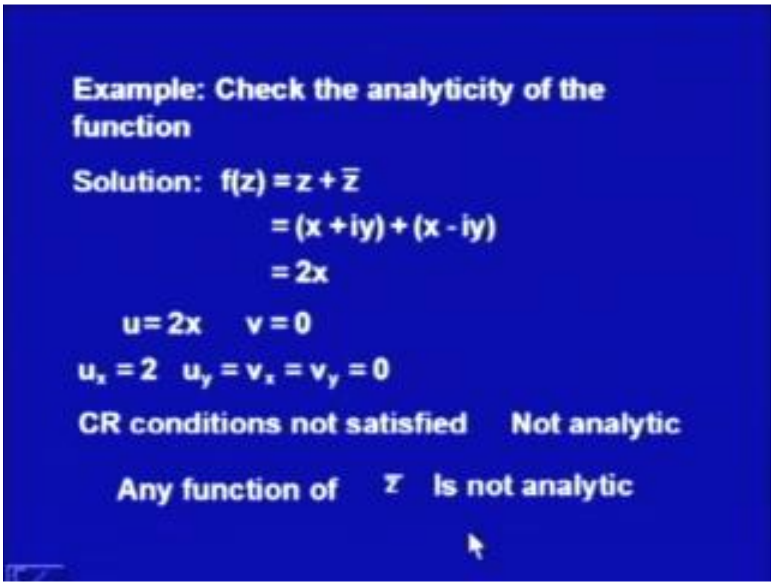

Now in this case, you have to check the analyticity of the function f z is equal to z plus z bar. We write down f z is equal to z plus z as x plus i y plus x minus i y, so this comes out to be 2 x. And accordingly, this function has u is equal to 2 x by v is equal to 0. So, when we calculate various partial derivatives u x comes out to be 2 u y is equal to v x is equal v y is equal to 0, all other derivatives are 0, so this u x is equal to 2 and v y is 0. So, this condition will never be satisfied, so C-R conditions are not satisfied and this function is not analytic.

In fact, we can say CR conditions not satisfied, function is not analytic and we say that any function of z bar is not analytic. Any function which involves z bar that function will not be analytic, like in this case the function is written as z plus z bar, so this function is not analytic.

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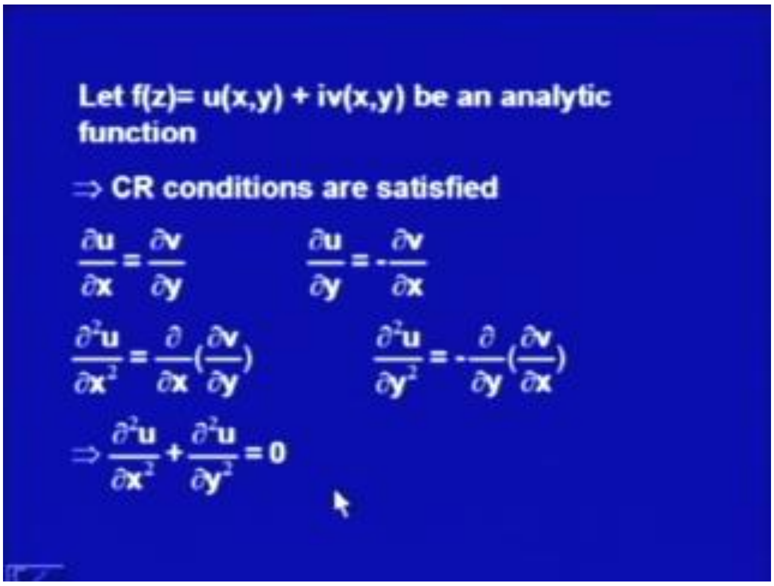

Let f z is equal to u x y plus iv x y be an analytic function, then CR conditions are satisfied; that means, delta u by delta x is equal to delta v by delta y and delta u by delta y is equal to minus delta v by delta x. And, when you differentiate this first equation with respective x partially, then left hand side is delta 2 u delta x square, while right hand side becomes delta of delta x of delta v by delta y. And similarly, we differentiate the second equation partially with respect to y, so left hand side gives me delta 2 y delta y square is equal to minus delta of delta y of delta v by delta x.

And, we have to equations if we add them together this becomes delta 2 u delta x square plus 2 u delta y square. And since, these derivatives are continuous, so we can say that delta by delta x of delta v by delta y is the same as delta of by delta y of delta v by delta x, so there will cancel out, so right hand side will be 0. Now, this is the Laplace equation in two variables x and y, so we say u satisfies the Laplace equation in two variables.

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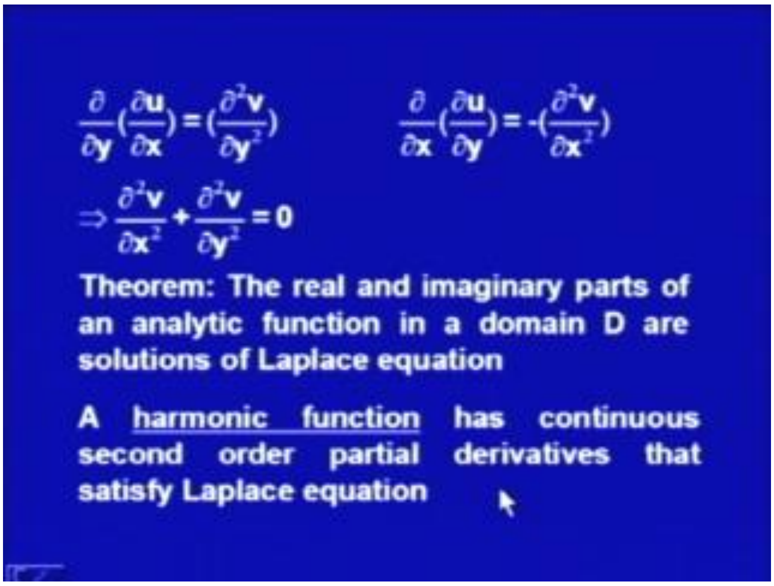

Similarly, the other partial derivative other function v also satisfies the Laplace equation. So, what we can do is, we first differentiate with respective delta by delta y, we get this expression. In the second is partially differentiated with respective x we get this, when you add the two we will have delta 2 v delta x square plus delta 2 v delta y square equal to 0.

And, this means that the real and imaginary parts of an analytic function in a domain D are solutions of Laplace equation, this is an important result. And now, the basis of this we say that harmonic functions has continuous second order partial derivatives that satisfy Laplace equation. So, we define a harmonic function which satisfies the Laplace equation and has second order continuous partial derivates; that means the analytic function there real and imaginary part of an analytic function will be harmonic functions.

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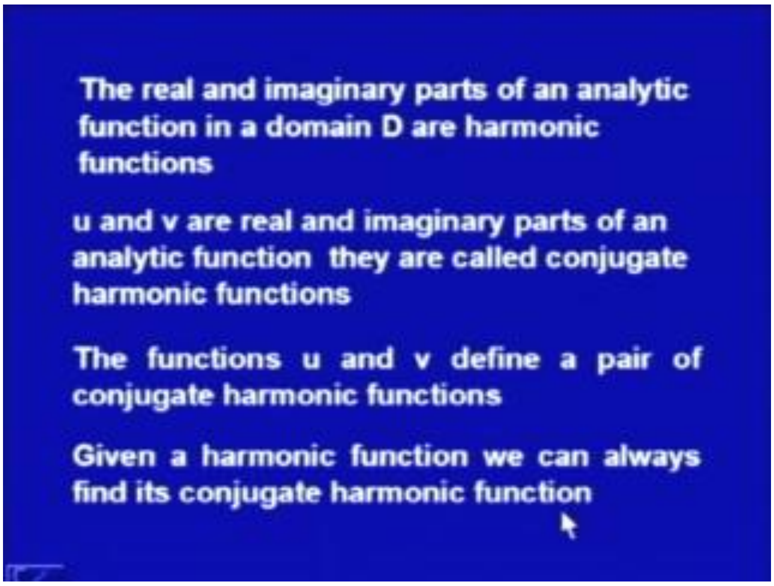

This is what I have stated; the real and imaginary parts of an analytic function in a domain D are harmonic functions. So, if u and v are real and imaginary parts of an analytic function then they are called conjugate harmonic functions. And finally, the functions u and v define a pair of conjugate harmonic function, we say u is conjugate of v and v is conjugate of u. And this says that u and v will define a pair of conjugate harmonic functions. So, given a harmonic function we can always find it is conjugate harmonic function, so if u is given then using CR equations we can find v and if v is given we can find u.

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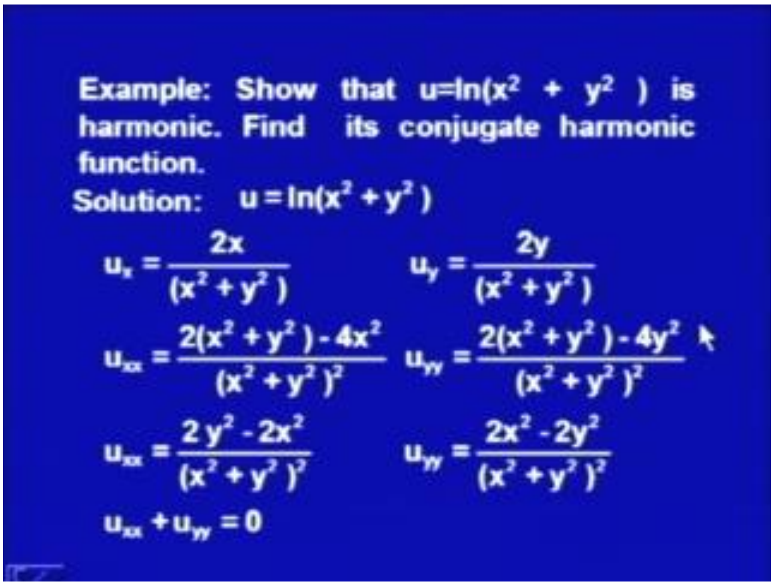

This we do in this example, here function u is given as l n x square plus y square, first we have to show that this function is harmonic and then we have to find it is conjugate harmonic function. So, to show that this function is harmonic, one has to show that it satisfies the Laplace equation for this u is u is equal l n x square plus y square, so we calculate the derivative u x which comes out to be 2 x divided by x square plus y square.

And, it is second derivative u x x if you differentiate it once again, then it is 2 times x square plus y square minus 2 x times derivative of this it is 2 x, so this is the numerator divided by denominator square x square plus y square whole square that comes out to be the second derivative of u.. Simplifying this to u x x comes out to be two y square minus 2 x square divided by x square plus y square whole square.

On the same lines one can differentiate this expression partially with respect to y that gives me u y as 2 y over x square plus y square and again differentiating partially with respective y gives me this expression for u y y. And, when you add the 2 u x x plus u y y this comes out to be 0 that simply shows that u satisfies the Laplace's equation.

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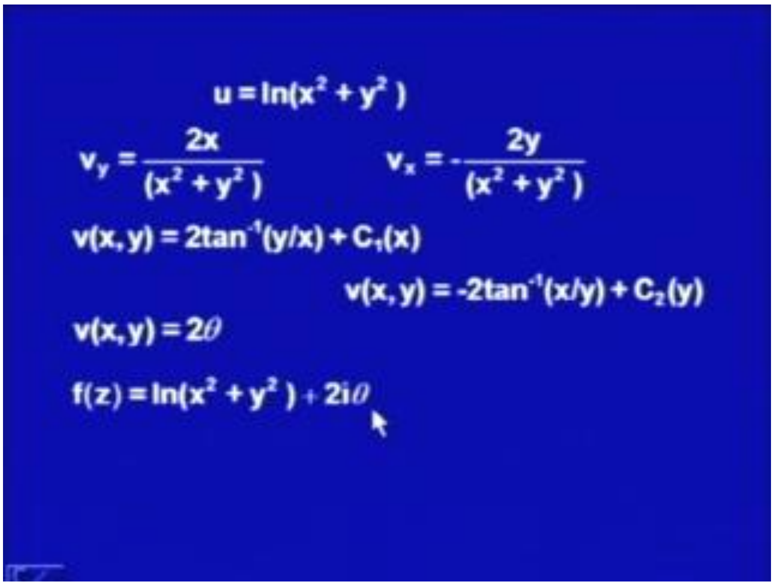

And that shows the first part that u is a harmonic function, once it is established that u is a harmonic function, we can make use of CR equations to find it is conjugate. So, u x we have already computed this will be equal to v y according to CR equation, so v y is 2 x over x square plus y square.

Similarly, v x will be minus of u y which we have computed as this, so using CR equations I have expressions for v y and v x. This can be integrated partially with respect to y, see it is a partial derivative with respect to y, so I am treating x as constant while i i am getting the derivative.

So, to get v from this expression, what I have to do is, I have to integrate this expression with respect to y keeping x fixed. So, when I integrate this, I get v x y is equal to 2 times 10 inverse y by x plus constant of integration, now normally we have to have constant of integration C 1. But in this case, I am getting a constant inside of having a constant I am writing C 1 of as a function of x, because in this integration I am treating x as a constant.

Similarly, when you integrate this partially with respect to x will have v x y is equal to minus 2 tan inverse x by y plus C 2 y. So, if we compare these two we will say that v x y is equal to theta this C 1 x and C 2 y there should not be functions of x and y, so we have v x y is equal to 2 theta So, once we have obtain v x y is equal to 2 theta I can write down the analytic function f z as l n x square plus y square the real part and this is the imaginary part. So, it is 2i theta, so my analytic function is this.

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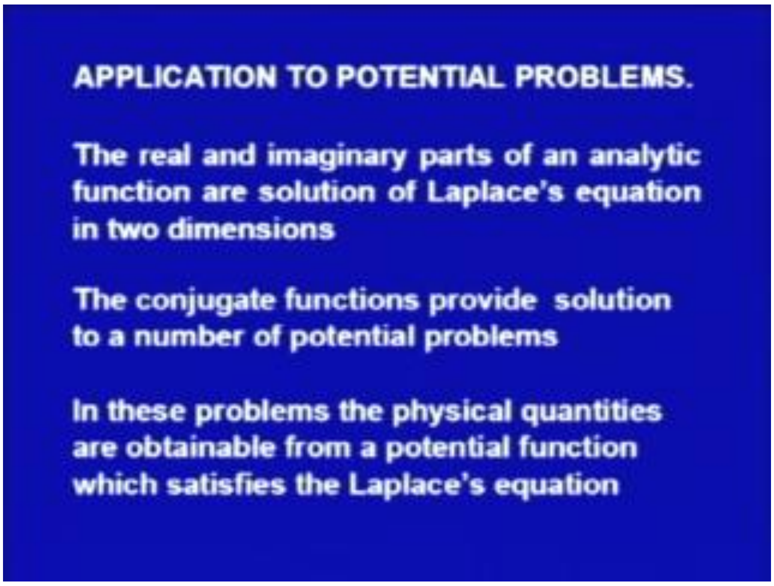

Next, we will discuss application of complex variables to potential problems. The real and imaginary parts of an analytic function are solution of Laplace equation in two dimensions.

This is what we have seen just now? Now, the conjugate functions provide solution to a number of potential problems in these problems, the physical quantities are obtainable from a potential function, which satisfies a Laplace' equation.

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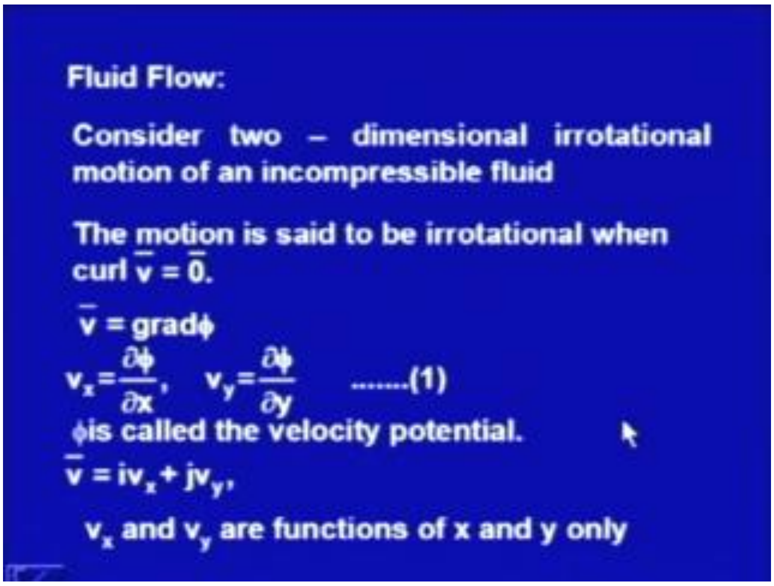

Now, there may be number of problems, but I will be considering a problem from fluid flow. So, consider two dimensional irrotational motion of an incompressible fluid. To explain this, the motion is said to be irrotational, when curl of v bar is equal to null vector. From differential calculus, we say that v bar is equal to grade of phi v bar is the velocity field here, velocity of fluid flow. Then, this suggest that v x the x component of velocity will be delta phi by delta x and v y the y component of velocity will be delta phi by delta y. Now, such a function phi is called the velocity potential

And, we write velocity v bar as x component in the i direction plus y component in the j direction. So, we write v bar as i v x plus j v y, so v x and v y are functions of x and y only, because we are considering two dimensional flow, so they are functions of x and y only no more z is involved in it.

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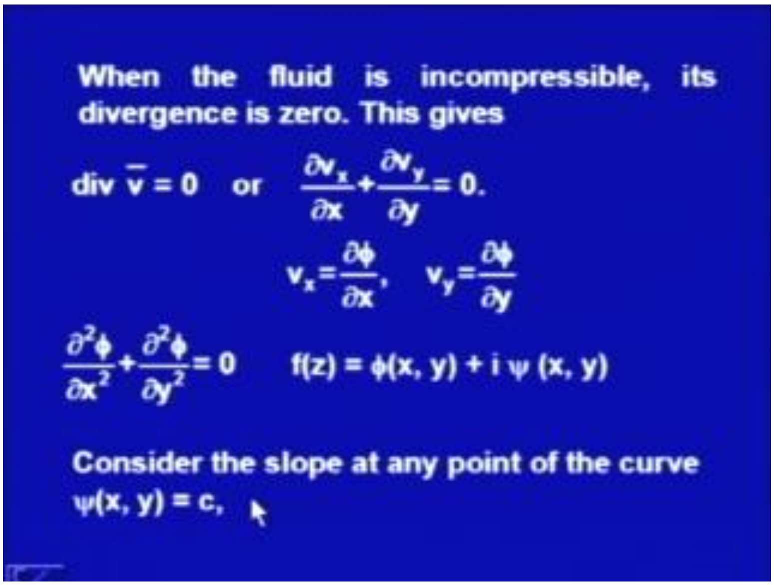

When the fluid is in compressible it is divergence is 0, so this gives divergence v bar is equal to 0 or delta v x over delta x plus delta v y over delta y is 0.And since, v x is equal to delta phi by delta x and v y is equal to delta phi by delta y, that is what we have assumed, then we can substitute these v x v y in this equation and we have delta 2 phi delta x square plus delta 2 phi delta y square is equal to 0.

In other words, we say that the velocity potential phi is an harmonic function, so if it is an harmonic function, then we can associate a function f z which is expresses phi x y plus i psi x y. So, f z is an analytic function in which the real part is the velocity potential phi, but the question is what is this function psi, we consider the slope at any point of the curve psi x y is equal to c.

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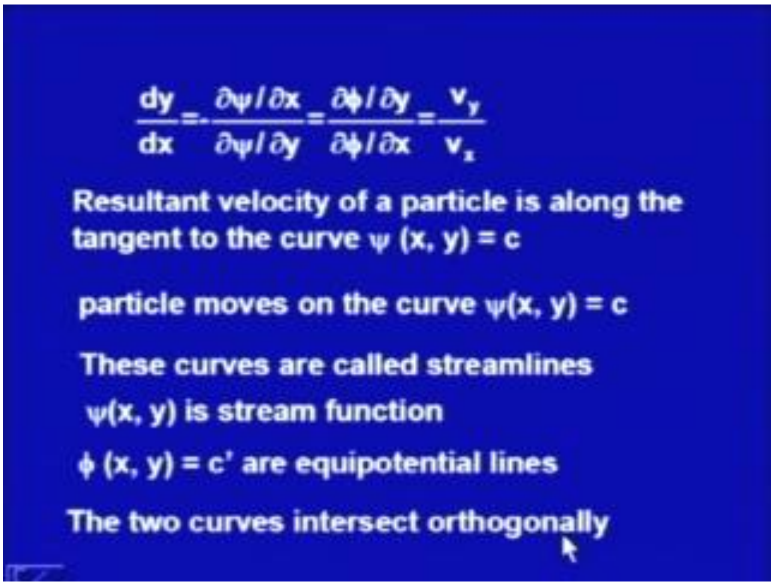

And it is computed as d y by d x is equal to minus delta psi by delta x divided by delta psi by delta y which is equal to delta phi by delta y divided by delta phi by delta x from Cauchy Riemann equations and this is equal to v y by v x. So, d y by d x is this.

So, we say that the resultant velocity of a particle is along the tangent to the curve psi x y is equal to c. So, we say that a fluid particle moves on the curve psi x y is equal to 0 and it is velocity is given by phi x y. These curves are called streamlines that is psi x y is equal to c curves or called streamlines and psi x y is called a stream function. Phi x y is equal to another constant c dash are called equipotential lines, the two curves intersect orthogonally that can be easily be checked.

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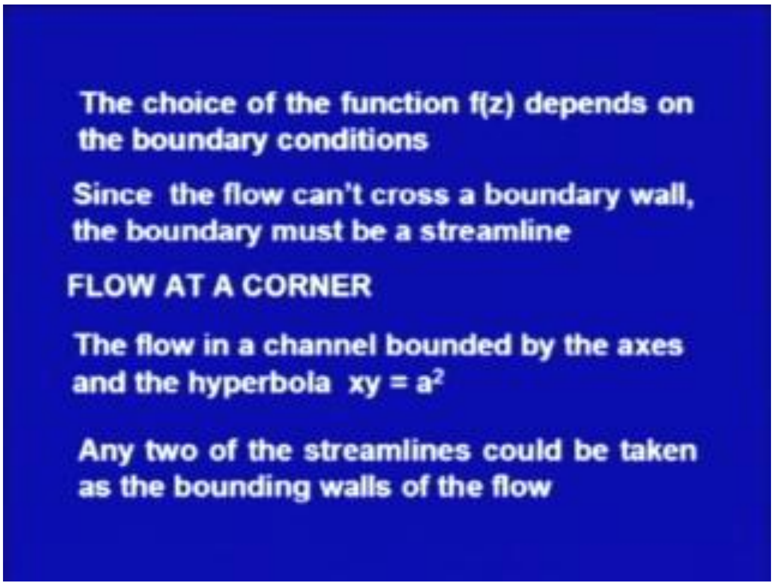

The choice of the function f z depends on the boundary conditions. Since the flow cannot cross a boundary wall, the boundary must be a streamline. And on the basis of this, we can solve fluid flow problems, so if we consider flow at a corner. Then, the flow in a channel bounded by the axis and the hyperbola x y is equal to a square. Any two of the stream lines could be taken as the boundary walls of the flow.

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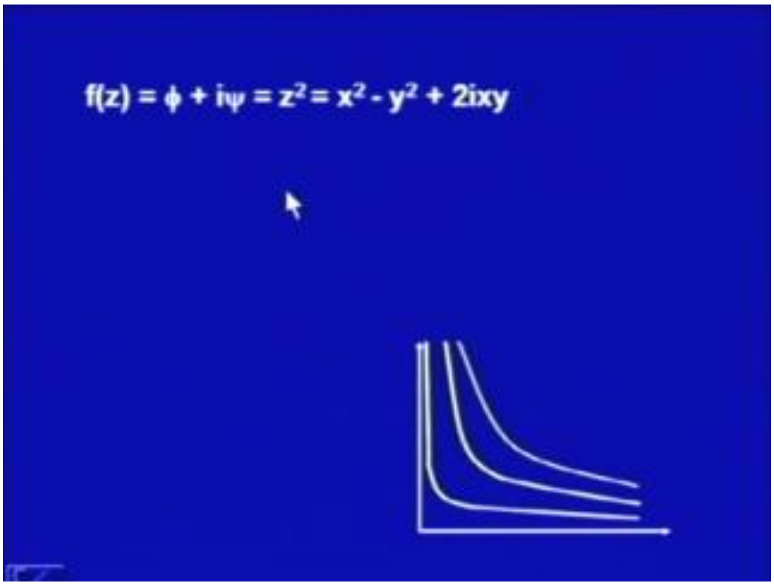

And then f z is equal to phi plus i psi is equal to x square minus y square plus twice i x y and this function is nothing but z square. So, what I have done is here, I have a corner which is given by coordinate axis and this is the boundary 2 x y it is nothing but hyperbola. So, the flow is between these boundary this boundary and this boundary.

At inside this boundary, the stream lines are moving along these curves and velocity potential will be given by x square minus y square. So, these are again these gives us equipotential lines, so if you draw hyperbola's x square minus y square is equal to a then they give us the velocity potential. So, we can see that if the fluid is flowing inside this region, then any particle of the fluid will follow one of these paths. So, that is how we apply theory of complex variables to solve real life problems.

So, with this we have completed this lecture on function of complex variables. In this lecture, I have started with the very definition of the function, then I have introduce the concept of limit and this is done on the basis, what we already know, function of our real variable, and then we have extended the ideas.

Then, we have discuss the continuity after that the differentiability and then I have discussed the motion of analyticity of a complex variable. I have obtain CR equations for checking analyticity of a function, I have given some examples to illustrate the concept of analytic function. And then I have introduce the harmonic function and then some problem and it is application to real life problems that is all for today's lecture.

Thank you.