Mathematics-II
Prof. Sunita Gakkhar
Department of Mathematics
Indian Institute of Technology, Roorkee

Module - 2
Lecture - 18
Solution of System of Linear Equation

Welcome viewers, in this lecture we are going to discuss Solution of System of Linear Equations. We will use all those concepts, which we have developed during my lectures on matrices. To start with we will first discuss solution of system of homogeneous linear equations.

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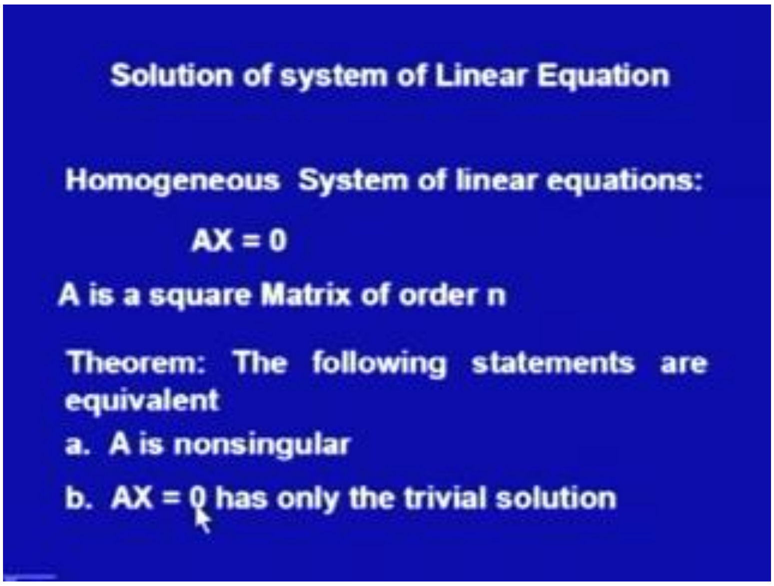

The homogeneous linear equation can be expressed in a matrix form as AX is equal to 0, here A is a square matrix of order n and X is a vector of unknowns. So, we n equations in n unknowns. Now, we call this equation homogeneous, because the right hand side is 0 and all the equation are having a homogeneous terms that is all terms are involving an unknown variable X in degrees 1. And there is no terms, which is constant and that on the right hand side we have 0 only, so a is a square matrix of order n.

To solve the system, we first develop a theorem according to which, the following two statements are equivalent. First statement says that if A is non-singular and second says that AX is equal to 0 has only the trivial solution by this I mean to say that if A is non-singular, then AX is equal to 0 has only the trivial solution or if AX is equal to 0 has only the trivial solution, then A is non-singular.

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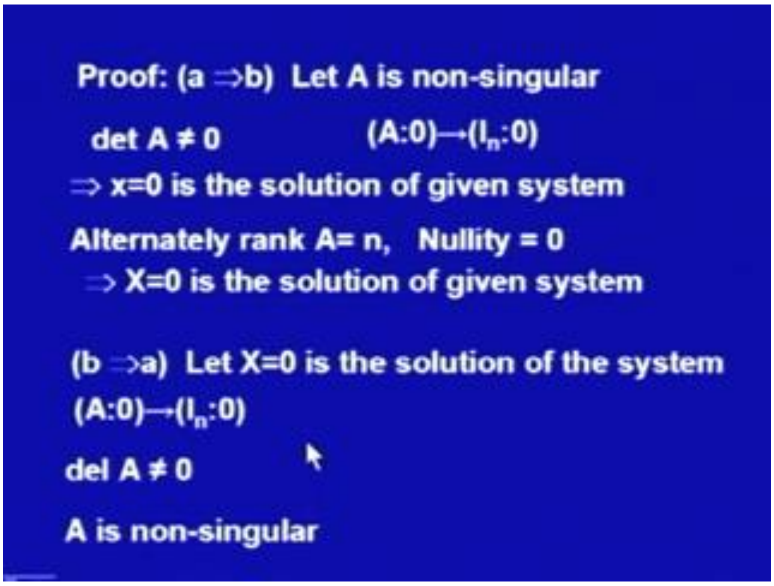

So, let us first prove this result, to start with we proof the forward side that is A is non-singular by this statement I mean to say the determinant of A is not 0 this is by the definition of a non-singular matrix A. And if determine A is not 0, then the augmented matrix formed by the column 0 that is A colon 0 will be transform to the matrix I n colon 0 in which, we will have identity matrix on the right hand side. That means, we can apply a elementary operations on the augmented matrix and this augmented matrix can be convert to a matrix like this.

And, this simply means that we will have a identity matrix as a coefficient matrix and the right hand side is 0 and by this I mean to say that if I am having n is equal to 3 and I am having 3 unknowns x y and z. Then, from this z is equal to 0, y is equal to 0 and x is equal to 0. That means, 0 0 0 is solution of this matrix, this is I have explained for n is equal to 3. But, this is true for n values for any n. So, this simply means that x equal to 0 if the solution of given system.

So, I have prove this part this is one way of proving this another way may be that rank A is equal to n if determinant A is not 0. Then we know rank A is equal to n and according nullity will be 0. And if nullity is equal to 0, then the system AX is equal to 0 has the trivial solution X is equal to 0. So, we have prove that X is equal to 0 is solution of given system when A is non-singular, so this is first part.

According to the second part b implies a that is if X is equal to 0 is a solution of the system then we will provide that the matrix A is non-singular. To prove this we say that the augmented matrix A 0 can transfer to I n 0, because X is equal to 0 is a only solution of the system and this means determinant A is not 0, and this by definition A is non-singular. So, we have prove the theorem for a system of homogeneous equations. So, if the term system is homogeneous will have only the trivial solution provided determinant A is non zero.

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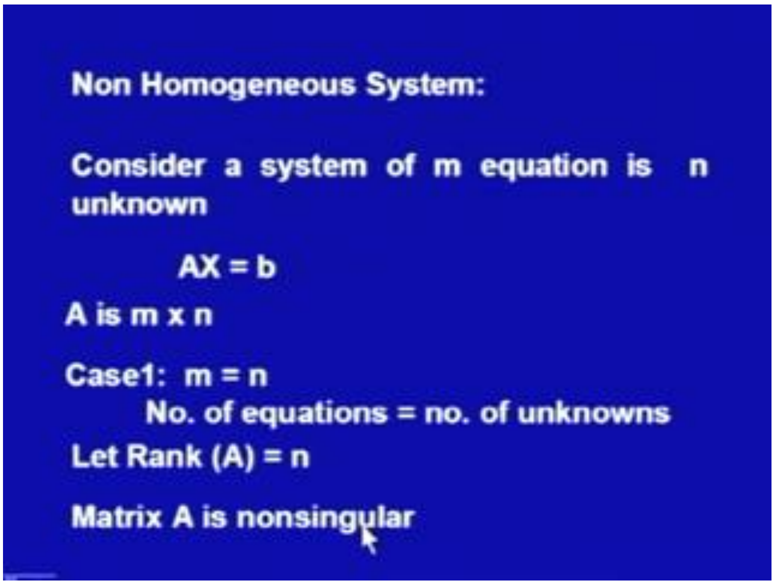

Now, we consider non homogeneous system, so let us consider a system of n m. Let us consider a system of m equation in n unknowns that is AX is equal to b. Then the matrix A is of order m by n, where m are the number of equations and n is the number of unknowns.

Now, we consider the different cases case 1 is when m is equal to n that is the number of equation is equal to number of unknowns in that case the matrix A is a square matrix. So, let us consider situation when rank of A is equal to n, so if rank of A is equal to n; that means, matrix A is non-singular.

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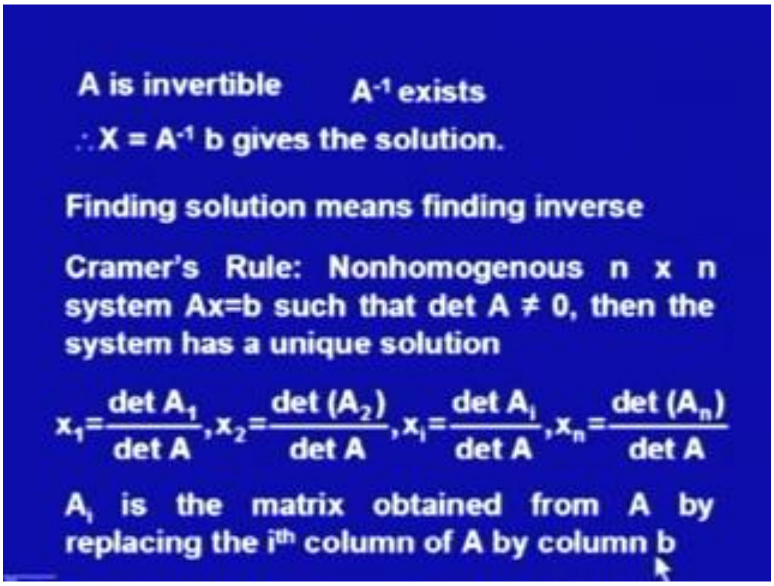

And if the matrix A is non-singular, then the matrix is to be invertible. That means, we can very easily find out A inverse and A inverse exist then we can premultiply our given equation AX is equal to b by A inverse simplifies to X is equal to A inverse b. So, this expression gives us the solutions for the non-homogeneous system AX is equal to b.

Now, this simply means that finding the solution means finding inverse of the given matrix A. So, if the solution is to be obtained, then we have to first obtained A inverse. And then A inverse multiplied by b is the solution of the given system. This actually the basis for a method developed as this method is called Cramer's rule. And according to this a non homogeneous n by n system AX is equal to b given that determinant A is not 0, then the system has a unique solution. So, this method is applicable when determinant A is not 0.

The unique solution of the system is given is given by X 1 equal to determination A 1 divided by determinant A X 2 equal to determinant A 2 divided by determinant A. The i th component of the solution vector Xi is equal to determinant Ai divided by determinant A. And the last component X n in the solution vectors is determinant A and divided by determinant A. What are A 1 A 2 A i matrices these are the matrices, which are obtained from A by replacing the i th column of A by column b.

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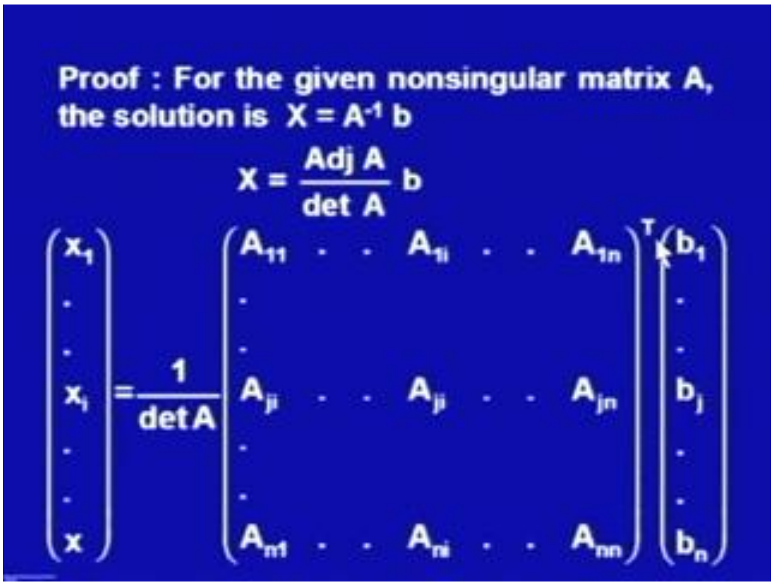

So, let me first prove this result, so for the given non-singular matrix A, the solution is X is equal to A inverse b this we have just now establish. And accordingly the solution vector X is adjoint of A divided by determinate A b this is the definition I have taken for the inverse A inverse. So, X is equal to adjoint A divided by determinant A into b.

Now, we can write down this a equation in this expanded form on the left hand side I have the column vector X 1 Xi Xn is equal to 1 over determinant A. Then I am writing adjoint A this is my adjoint of A and then the column vectors b 1 bj and bn. So, this is the matrix adjoint of A, where A 1 1 A 1 i etcetera. These are the cofactors of the corresponding elements in the matrix A and then we take the transpose.

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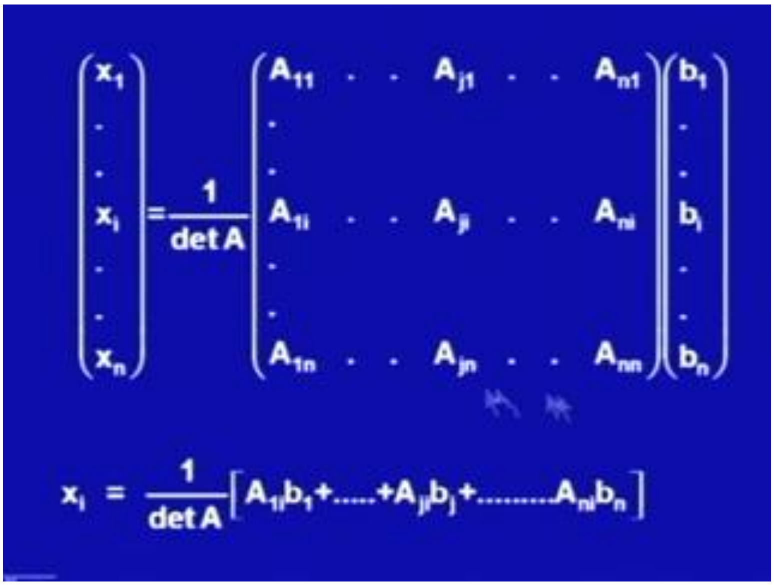

So, let us simplify this if I write down the transpose, then the columns will become rows and rows become columns. So, this is what I got the transpose matrix A 11 A 1 i A 1 n in the columns. Now, previously they were in rows and A j 1 A j i A j n they are, now in columns they previously they were in rows.

So, if I perform this multiplication then x i if I consider the i th component here is equal other one upon determinant A this is constant. So, I am writing it here, then I will see, what is the i the component from this product. So, you to get the i th components of this product this is A 1 i multiplied by b 1, then will have A 2 i multiply by b 2.

So, this i th row multiplied by this column that will give me the i th component here. So, xi is equal to this element, so this element is obtained by multiplying this row by this column. So, it is A 1 i b 1 plus A j i bj this multiplied by the j th elements here, and A n i multiplied by b n, so Xi is equal to this. To prove the results i have to show that that this is nothing but determinant Ai.

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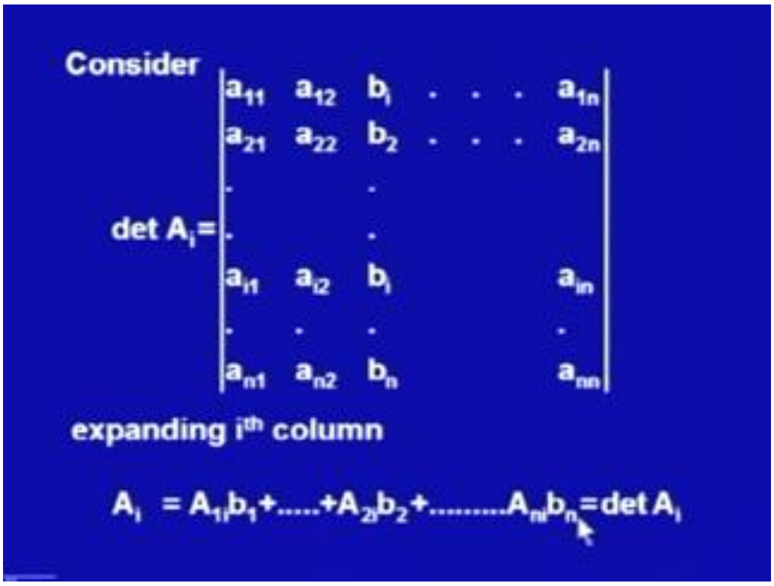

So, let me first simplify the determinant Ai for this purpose let us consider determinant Ai as this determinant. Now, what is this determinant is the determinant of the matrix A with i th column is replaced by the column b, which was their on the right hand side of equation AX is equal to b. So, this is the column which i have replace at the i th position, so this is determinant Ai.

To simplify this, if I expand this determinant along this column, then I know Ai determinant Ai is equal to A 1 i multiplied by bi plus A 2 i multiplied by b 2 plus Aii multiply by bi plus A n i multiplied by bn. So, this i know from the properties of determinant, so this is nothing, but determinant Ai.

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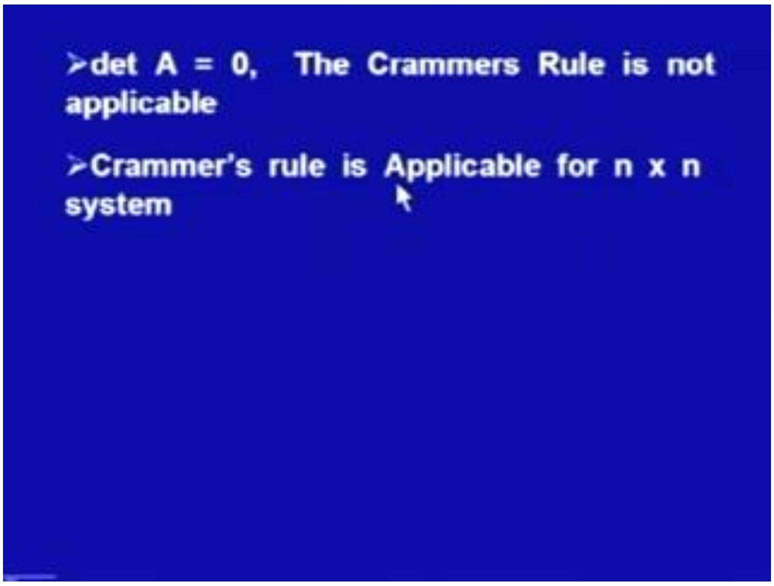

So, once I am convince that this is this expression is determinant Ai, I can easily prove the result. Now, if determinant A is equal to 0, then since the formula, which I have derived has determinant A in the denominator the Cramer's rule is not applicable, and further Cramer's rule is applicable for n by n system, because only when determinant A is meaningful. So, Cramer's rule has two limitations that is it has to be applicable for n by n system and second determinant A should be non zero.

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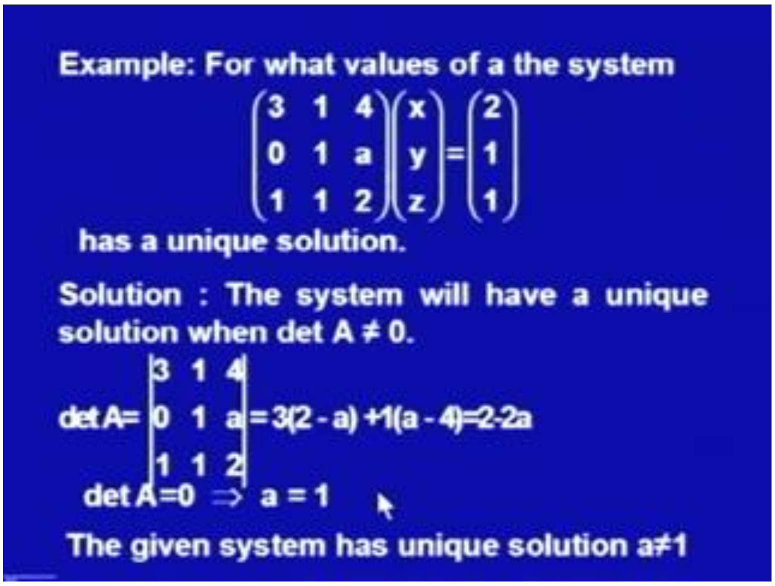

So, let us illustrate the Cramer's rule for this given system I have been given a 3 by 3 system and it is non homogeneous system, because right hand side is non zero vector. So, we have to first find that this system has a unique solution, so this involves an are unknown constant A, so the question is for, what values of a the system has a unique solution.

Now, we known that the system will have a unique solution provide a determinant A is not 0, so determinant A is to be evaluated, so to calculate determinant A, I write 3 1 4 in the first row 0 1 a in the second row and 1 1 2 in the third row of the determinant. So, if I simplify it is 3 times 2 minus a plus 1 into a minus 4 and that gives me two this is 6 minus 4 is 2 and minus 3 a plus a it is minus 2 a, so determinant A is 2 minus 2 a and this will be 0 when A is equal to 1. So, this system will not have a unique solution when a is equal to 1 for all other values the system will have a unique solution. So, the given system has unique solution, when A is not equal to 1, so this is the conclusion on the basis of this.

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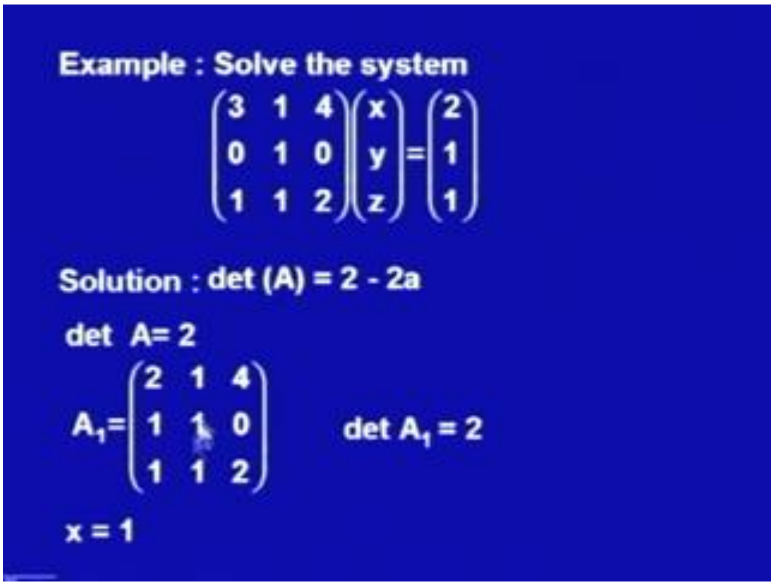

So, we solve the system when A is equal to 0, so I have taken A is equal to 0 in my earlier example, so let us solve this system and we solve this system using Cramer's rule. So, we first calculate determinant A as we have seen I have complete determinant A as 2 minus 2 a and a is equal to 0 in this example.

So, that gives me determinant A is equal to 2, but this Cramer's rules means that I have to evaluate determinant of A 1, where A 1 is this matrix and if we evaluate this determinant then this determinant comes out to be 2, that can be very easily obtained it is if we expand about this particular row, then determinant A 1 comes out to be 2. And, determinant A is equal to 2 and determinant A 1 equal to 2, so according to the Cramer's rule determinant A 1 divided by determinant A gives me the solution, so x is equal to 1, 2 divided by 2 gives me X is equal to 1.

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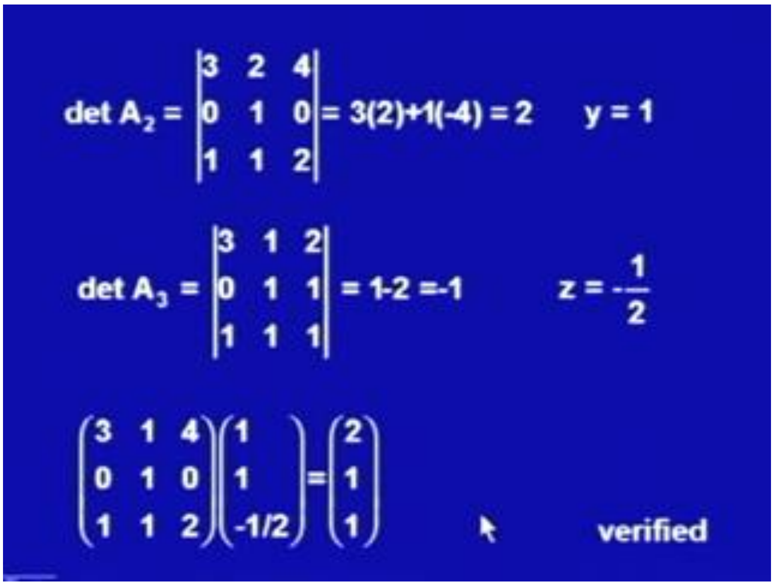

For the second component I first calculate determinant A 2, which is obtained by writing the matrix A and replacing the second column of the matrix by the column vector b, so column vector b is 2 1 1, so I am replacing the second column by 2 1 1. So, if I evaluate this a determinant about this particular row, then it is 3 into 2 minus 4 that gives me 2 so 3 into 2 is 6 minus 4 that gives me 2 and; that means, y is equal to 1, because y is equal to determinant A 2 divided by determinant A, which I have already complete as 2, so it is 2 divided by 2 that comes out to be 1.

Similarly, determinant A 3 is computed by replacing the third column of the matrix by the column vector 2 1 1. And, now when I expand this determinant about say this row then this comes out to be 1 minus 1 is 0 and for this it is 1 minus 2, so the result comes out to be minus 1 and then from here, I can compute z is equal to minus 1 divided by two. So, z is equal to minus half, so my solution is x is equal to 1 y is equal to 1 and z is equal to minus half.

Let me check whether it is satisfy the given system or not, so to verify I write down the matrix A and write down solution 1 1 and minus half, if I compute this it is 3 into 1 plus 1 and minus 2. So, the result is 2, then this multiplied by this is simply 1 and this row multiplied by this column gives me 1. So, 2 1 1 is the given right hand side and this verifies that whatever solution I have obtained here that is 1 1 minus half is the solution of given system.

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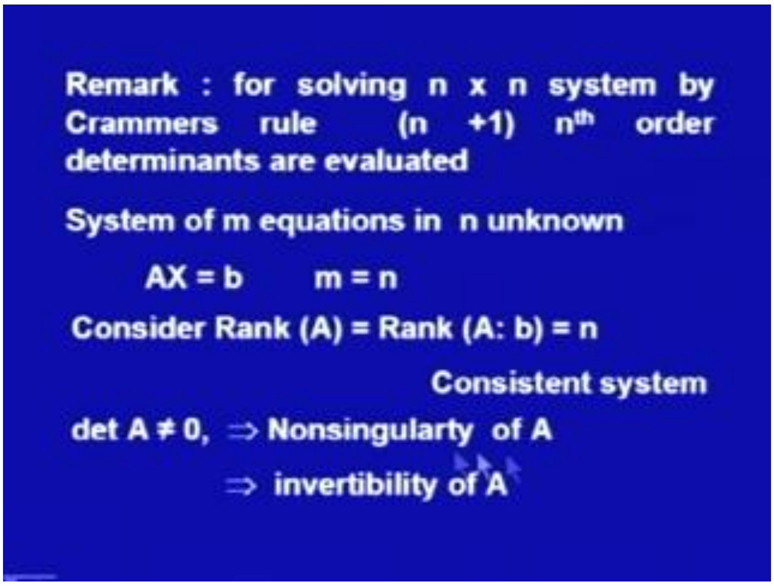

So, the that is how we apply Cramer's rule and one may notice that for solving 3 by 3 system we have evaluated 4 third order determinants, but if we have an n by n system, then will have evaluate n plus 1 n th order determinants. Now, this means lot of computational effort is required for solving Cramer's rule further there are some limitations on the Cramer's rules that is the matrix has to be non-singular if determinant A is equal to 0 this cannot be applied further the matrix has to be a square matrix, so these are some of the limitations of Cramer's rule.

Now, we will discuss a general method for solving m equation in n unknowns, so let us consider a system of m equations in n unknown. So, we have AX is equal to b and for a particular case m is equal to n, so if take m is equal to n, then rank of A is equal to rank of A colon b the augmented matrix is equal to n.

So, if this condition is satisfied, then in fact if this condition is satisfy then we say the solution this the system of equations are consistent. So, first thing is there has to be a consistency only then we can talk about the solution of the system, further if this is equal to n; that means, the matrix A is non-singular and the system is consistent. So, determinant A is not 0, so this rank is equal to n means determinant A is not 0, so non-singularity of A is there and this implies invertibility of A, and that implies the uniqueness of solution.

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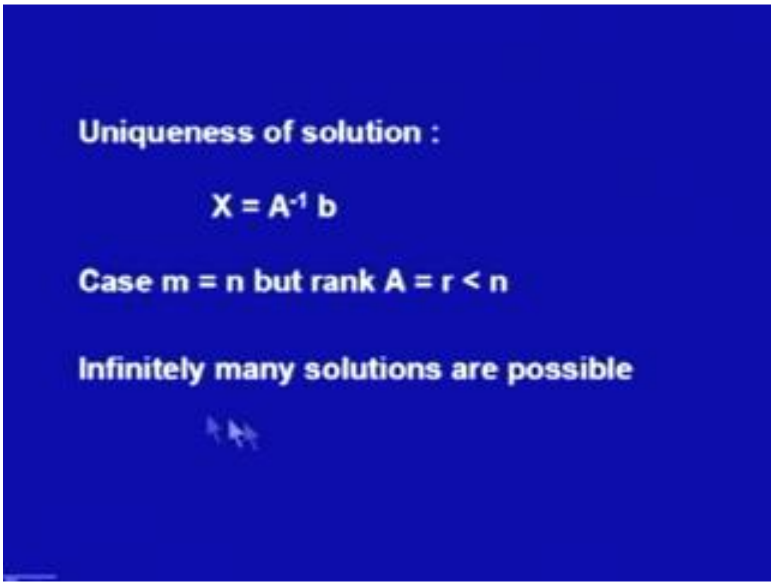

So, this means X is equal to A inverse b is the unique solution of the system AX is equal to b. This is the case when m is equal to n, but suppose we have a case when m is equal to n, but rank of A is not n it is r, which is less than n then we may not be able to get the unique solution of the system.

So, for in this case if rank A is equal to r, which is less than n, then in fact we will be having infinitely many solutions for the given systems. So, now we have to case is m is equal to n, then we may have unique solution when the matrix A is invertible, but if the matrix is not invertible and rank of A is less than n, then there will be infinitely money solutions possible.

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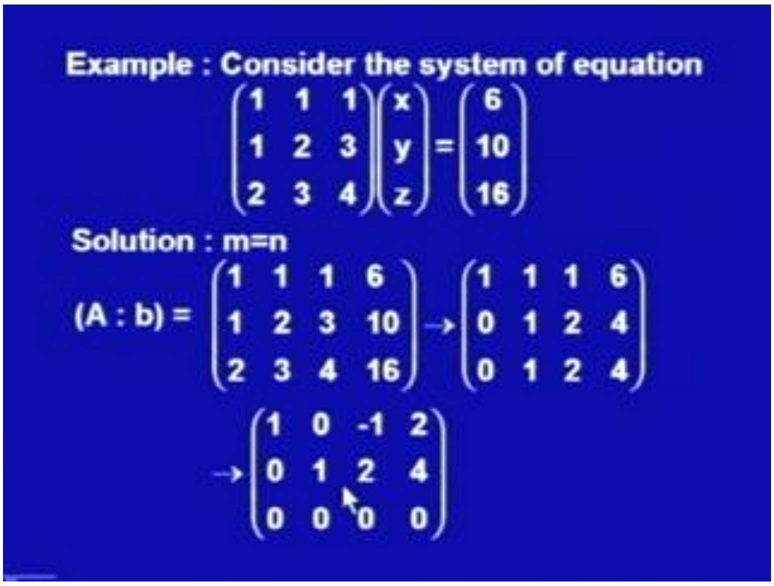

For example, consider a system of equation the three by three system is a square matrix m is equal to n and we have to see whether we have finitely many solutions or a unique solution for the system. So, m is equal to n we can notice here and then we consider the augmented matrix that is matrix A the coefficient matrix is written here, followed by the column vector 6 10 16 we apply elementary transformation elementary operations on this augmented matrix.

So, that this and this become 0 and this matrix will be transform to this matrix, which may be further transformed see in this in this case this row is equal to this row, so if you subtract this row from this row, then will have a 0 row in this matrix. So, we can notice that rank of A cannot be three the size of the system, so rank may be 2, in fact it is two because we have A 2 by 2 matrixes, which is having determinant non zero, so in this matrix has rank 2.

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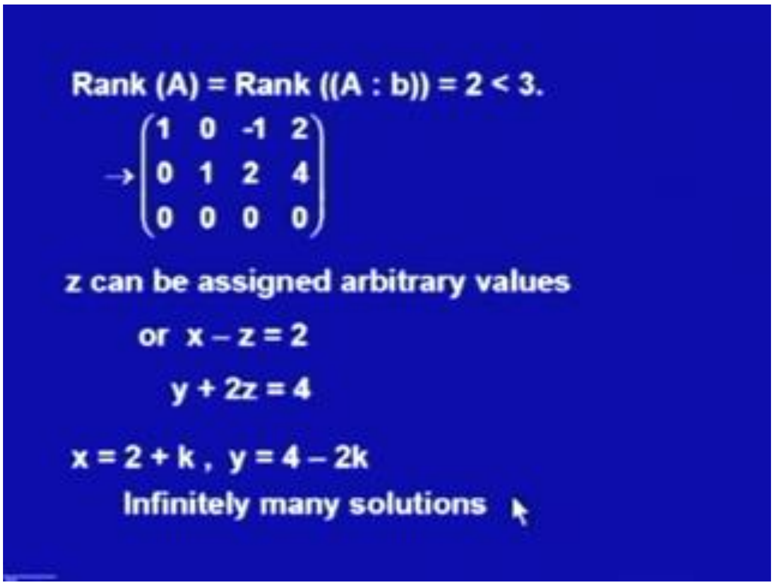

So, according to the result here we will having infinitely many solutions why, because rank of A is equal to rank of the augmented matrix and this is 2, which is less than 3, then we have this augmented matrix. Then, from here this equation is not giving any information only these two equations are two be solved in three variables from here one can notices that z can be a assigned any arbitrary value.

Then, x minus z is equal to 2 from the first equation and from the second equation y plus 2 z is equal to 4. So, now we have only two equations in 3 variables we can assign any arbitrary value to z, let us say k when x is equal to 2 plus and y is equal to 4 minus 2 k and these two equations are satisfied and; that means, you can given arbitrary values to k and we will be having different solutions.

So, as we change values of values of k x and y will vary and; that means, we will be having infinitely many solutions for the given system. And this what this is happen, because the determinant of a is 0 and rank of A is 2, which is less than 3.

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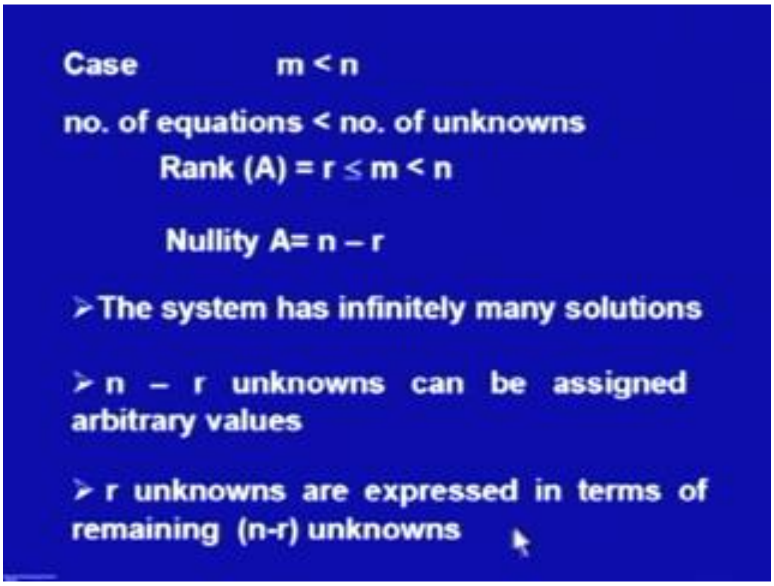

So, we have infinitely many solutions in this case, now I consider a case when m is less than n, so previously we were consider case when m is equal n we have a square system. But, suppose numbers of unknowns are less than a number of equations are less than the less than the number of unknowns, then what will happen.

Rank of A is equal to r, which will be less than or equal to m, but this n will, since it is given to be less than n, so rank r will always be less than n. So, this is the situation and, then according to the rank nullity theorem nullity of A is n minus r and that is case that the system again will have infinitely many solutions. In fact, n minus r unknowns can be assigned arbitrary values and only r unknowns will be written interms of remaining unknowns.

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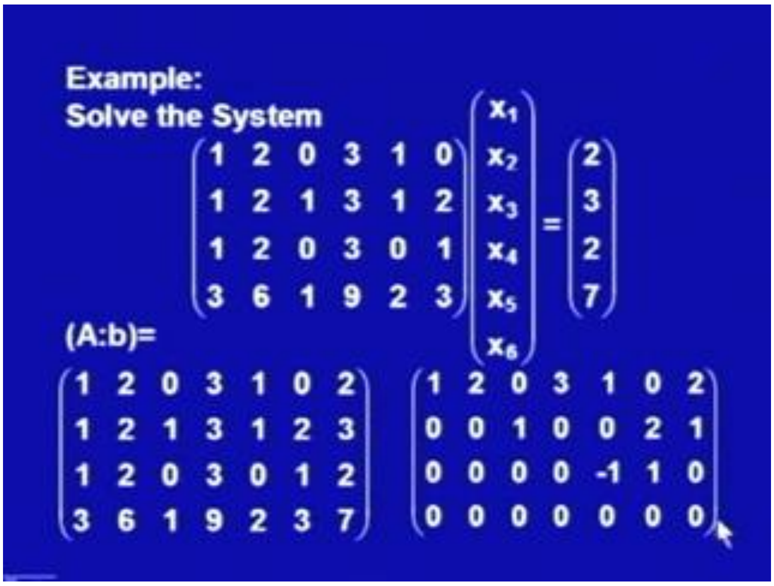

So, r unknowns are expressed in terms of remaining n minus r unknowns and this I illustrate for this system. So, let us four equations in six unknowns, so I have the coefficient matrix as this is the column vector of unknowns they are six in number and we have four equations.

So, in this system if I consider the augmented matrix A colon b, then a this is the coefficient matrix and here I have added the column vectors 2 3 2 and 7 I can apply elementary transformations and this can be reduced to this, this can be done very easily you can subtract this from this, you can subtract this from this. And then one can add the 3 and subtract, so this can easily be transform to this by using elementary transformation, now in this case one row is identically 0.

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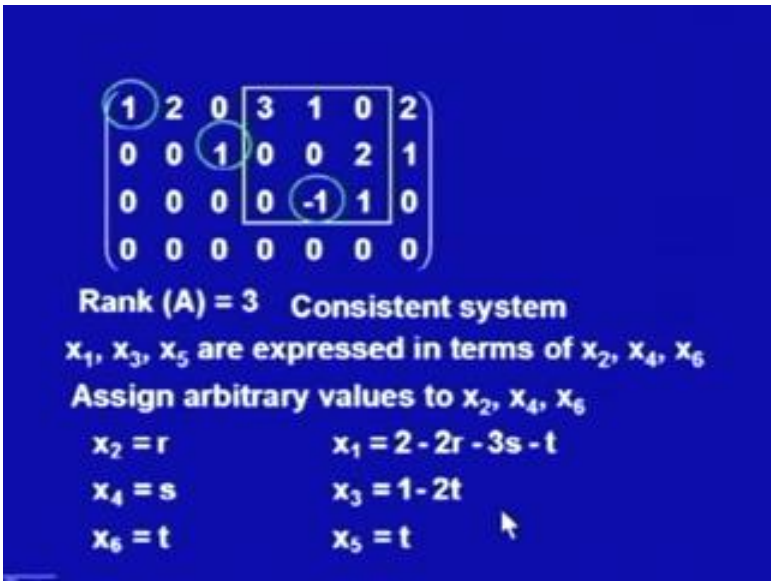

We can further simplify, so this what I have obtained I my last slide one can notice that this is a square matrix, which is having non zero determinant and; that means, the rank matrix a is three. Further, if you consider this, then again the rank comes out to be three; that means, the given system is consistent as the rank of A and the augmented matrix both are equal and they are three.

Now, if this is we transform matrix, transform augmented matrix, then one can notice that this row this row this row has the leading once in this column this is the leading one here, and the leading one is here leading term is this in the matrix and from this discussion we conclude that rank A is equal to 3 and we have a consistent system. And this is leading one corresponding to the unknown x 1 this leading one is corresponding to the unknown x 3 and this is corresponding to x 5.

So, we can say that X 1 x 3 x 5 are expressed in terms of the remaining unknowns that is x two x 4 and x 6. So, we assign arbitrary values to these unknown x two x 4 x 6 and we can write down x two is r r being arbitrary x 4 is s and x 6 is t, so we have assign arbitrary values r s n t to x two x 4 x 6 and then the remaining unknowns x 1 x 3 x 5 they are expressed as x 1 is equal to 2 minus 2 r minus 3 s minus t.

So, from the first equation I am writing x 1. In fact, I am using first equation for x 1 and this third this equation second equation is used for expressing x 3. So, x 3 is equal to 1 minus 2 t this is x 6 and 6 is equal to t that is, what I have assigned here, so from this equation I am getting this and this equation is used for writing x 5. So, on the right hand side we do not have any we on the right hand side we have 0 here, so we simply say x 5 is equal to t.

So, this third equation is obtained from this third equation and forth equation is a trivially satisfied. So, we have these solutions in terms of unknowns r s and t and as we very r s and t we will be getting different values of x 1 x 3 and x 5 and; that means, we will be having infinitely many solutions to be given system.

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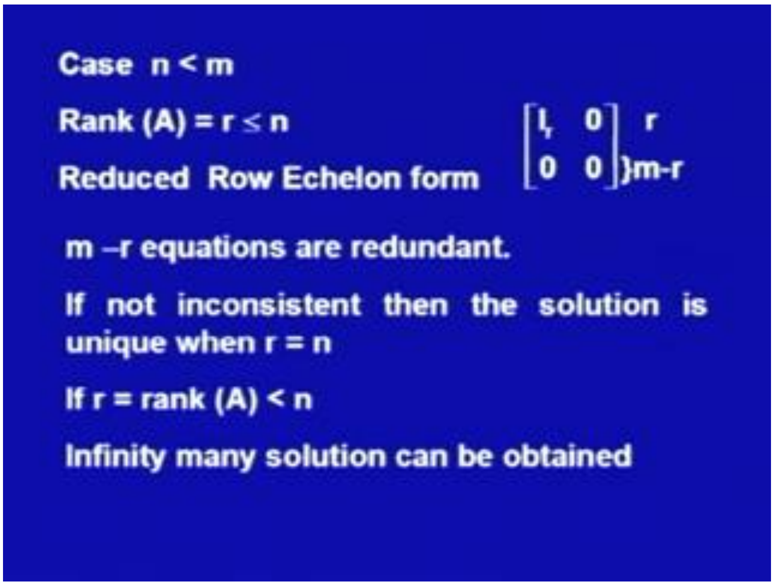

So, with this we come to the last case, where n is less than m that is the number of unknowns is less than the number of equations, clearly even in this case rank of r will be r and it will always be less than equal to n, because this rank r cannot exceed n. So, we will have rank a is equal to r.

Now, if we have such a system then we can reduce the augmented matrix in this form this is reduced row Echelon form in which, is rank r, so we will be having I r here identity matrix and rest of the rows will be 0, since we are having rank s r, then n minus r rows will be 0. And, if that it the case then we can assign arbitrary values to m minus r unknowns and only r unknowns can be written interms of m minus r unknowns and that is; that means, will be again having infinitely many solutions. So, we will have in this case m minus r equations which are redundant, because they are zeros and it may happen that the system is not consistent then of course, we cannot we cannot solve the system, but if it is consistent and if r is equal to rank of a is less than n, then infinitely many solutions can be obtained.

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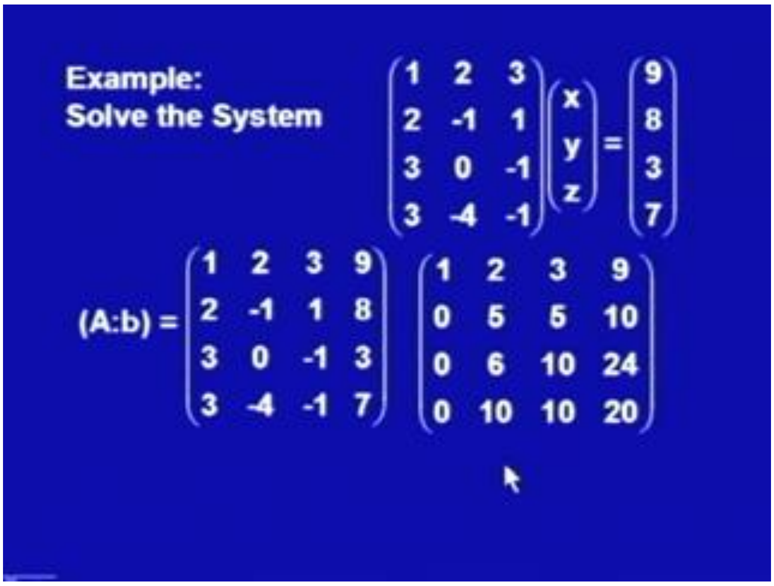

In the next example I am considering the case, when n is equal to m, so let us solve the system in 3 unknowns when 4 equations are given. So, let us have the system, which is the coefficient matrix is 4 by 3 4 rows and 3 columns. So, we consider first the augmented matrix when this column is added here, so this is the augmented matrix we apply elementary transformations to this matrix and first these entries are made zeros. So, this matrix reduce this matrix and in this matrix one may notice that 5 is common, and here 10 is common one can take 6 we can further rationalize this row and applying elementary transformation to this matrix gives.

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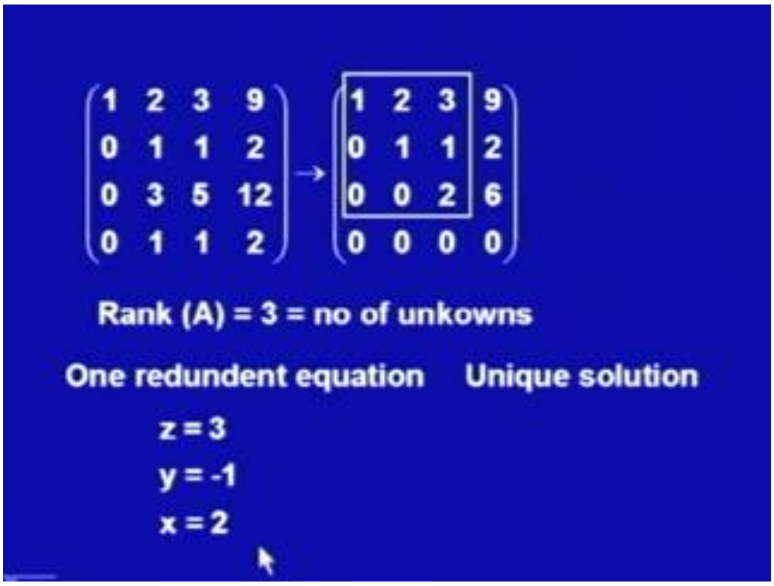

This matrix and from here one can notice that this row is identical to this row, so this row can be made to 0. And, this matrix transform to this matrix having one row as identically 0 and from here this square matrix has determinant non zero one can notice here, if you expand about this particular row then 2 into 1, so this is non zero determinant.

So, the rank of this matrix is 3 is the number of unknowns further it is a consistent system, because rank of A matrix is 3 and rank of augmented matrix is also 3. The consistent system and having one redundant equation in this case; however, one making this equation redundant we have three equations and 3 unknowns this system can be invert this determinant is non zero. So, this matrix is this matrix is invertible and then the solution of this matrix can be obtained uniquely.

And, if you solve this then third equation gives me z is equal to 3 substituting z is equal to 3 in this equation given me 2 minus 3 that is y is equal to minus 1 and substituting z is equal to 3 and y is equal to minus 1 in the first equation we call this procedure as back substitution. We start with lowest entry that the value of this substituted in this next equation and all the values will be substituted in this equation.

So, following this procedure x come out to be 9 minus 3 minus 2, so this comes out to be x is equal to 2 this is minus 2 this is 3 and 3 9. So, this 9 will get cancel with this minus 1 and 2 that is minus 2 when it goes to the other side x come out to be 2, so this system has unique solution. So, we have discuss methods for finding solution in different cases different possibilities are their when m is equal to n m less than n m greater than n.

Now, we have two situations when the system is homogeneous when the system is non homogeneous. So, what is the relationship between the system of a homogeneous system of homogeneous equations and system of non homo-homogeneous equations for this purpose.

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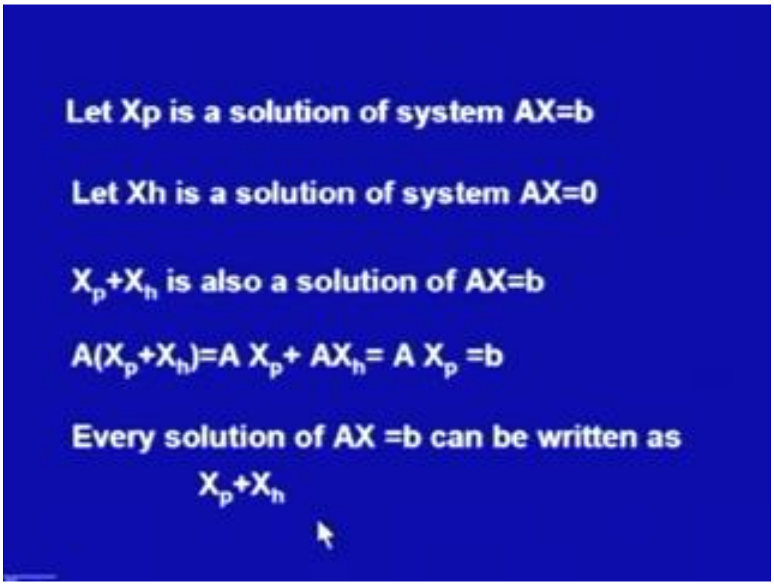

Let us say X p is a solution non-homogeneous system AX is equal to b while X h is the solution of corresponding homogeneous system AX is equal to 0. Then, one can prove easily that X p plus X h is also solution of non-homogeneous system AX is equal to b one can very easily check it, if you substitute X p plus X h in this equation, then this equation must be satisfy. So, for this purpose I write A and in place of X I write X p plus X h simple perform this multiplication it is A X p A X h, but A X h is equal to 0, so what remains is A X p is equal to b.

But, we know that X p is solution of AX is equal to b, so this equation is satisfied and; that means, X p plus X h is also solution of AX is equal to b. Now, this is very similar to, what we have in case of differential equations and we have a result that every solution of A X is equal to b cab be written as X p plus X h. So, it has two component one is a solution of homogeneous part and this is a solution of a non homogeneous part we call it up X p is called as a particular solution.

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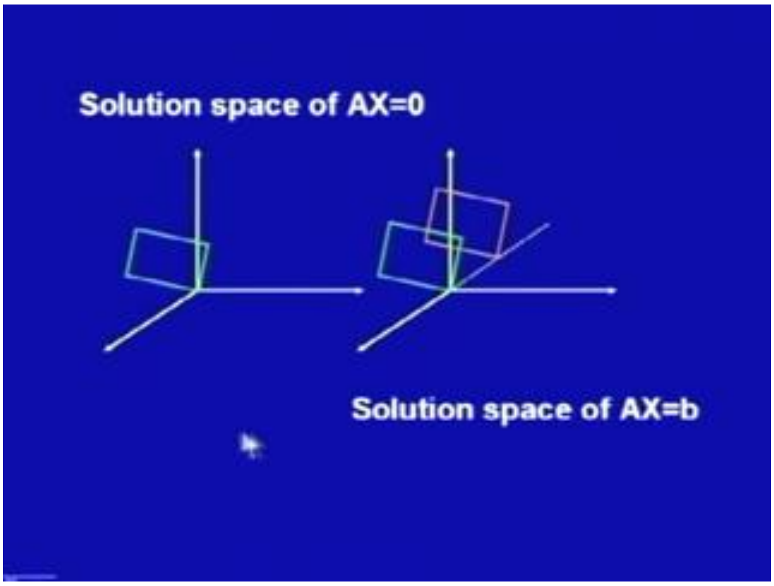

So, if we represent the solution of AX is equal to 0 on the plane see for AX is equal to 0 0 0 is also solution then we may have solution number of solutions other than x is equal to 0 if determinant A is 0, then possibility of other solutions are there and we know that they all the solutions of this equation they will form a vector space.

So, I am denoting that plane consisting of all the vectors be satisfy this equation as this plane, so this plane means, so the solution of all the solutions of AX is equal to 0. But, if I consider AX is equal to b, then this plane will be parallel to this plane and all the solutions here they are shifted this distance they will form a solution of AX equal to b that is the relevance of X p plus X h. So, the solution with these two planes are parallel planes, so this is the geometrical interpretation of X p and X h.

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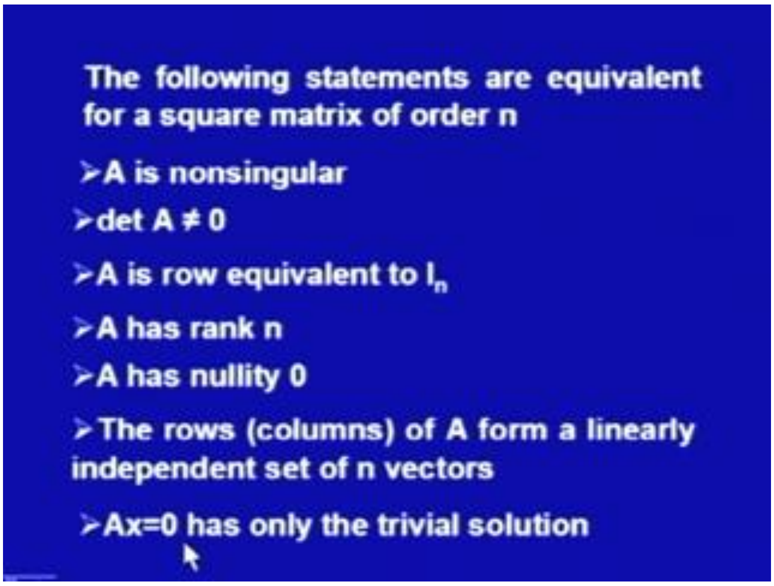

Now, we have discuss number of results related to the two systems AX is equal to 0 and X is equal to b, now we say that the following statements are equivalent for a square matrix of order n. So, the first statement is A is non-singular, second is determinant A is not 0, we are in fact, using this definition for a non-singular matrix.

Then A is row equivalent to I n whenever the determinant A is not 0, then we can always reduce A 2 I n by applying elementary row transformations, A has rank n, determinant A is not 0, so rank will be n, A has nullity 0 due to rank nullity theorem the rows columns A form a linearly independent set of n vectors this is also we have proved earlier. And finally, AX is equal to 0 has only the trivial solutions. All these statements are equivalent related to AX is equal to 0.

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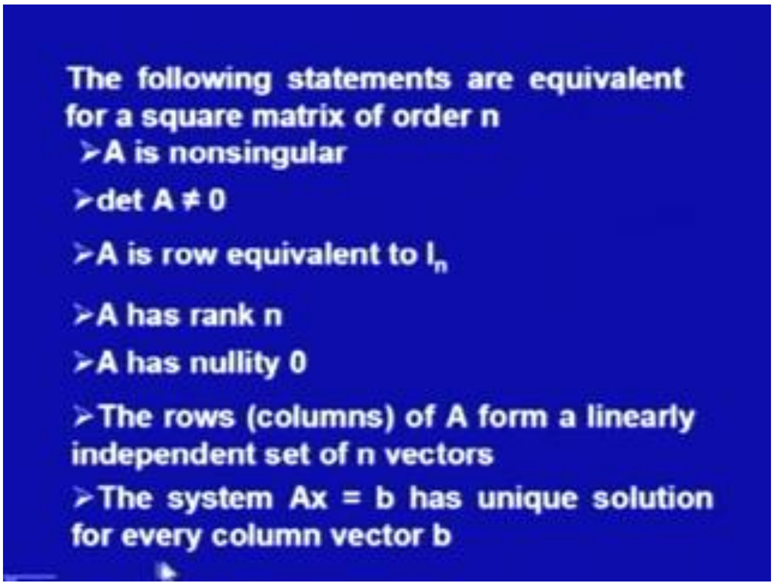

Similarly, the following statements are equivalent, when we have A is non-singular determinant A is not 0 all these things are same, which we have discussed the last point is slightly different and it says that the system AX is equal to b has unique solution for every column vector b.

So, we have x is equal to b non homogeneous system it will have a unique solution for every column vector b if matrix A is non-singular or determinant A is not 0 or A is row equivalent to I n or A has rank n or A has nullity 0 or the rows of A or columns of A form a linearly independent set of n vectors, then the system AX is equal to b has unique solution for every column vector of b.

Now, with this we have write at the end of this lecture and in this lecture I have discuss different situation for solving system of equations we may have a homogeneous system of equation or we may have non homogeneous system of equations. If it is homogeneous system, then we check the determinant and accordingly we say that whether we are having a unique solution or we can we can have a trivial solution or we have infinitely many solutions. Similar cases, will discuss for AX is equal to b, when we have 0 solutions, when we have unique solution, when we have infinitely may solutions. All these cases we have discuss for solution of system of equations.

Thank you.