Mathematics II
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Lecture - 17
Diagonalization Part - 2

Welcome viewers, today I will be talking on diagonalization. This lecture is in continuation to my earlier lecture on Diagonalization. There, I had discussed diagonalization of real matrices. Today, I will discuss diagonalization of complex matrices. This lecture constitutes two parts. The unitarily diagonalizable and application to the concepts applied to solution of system of ordinary differential equations.

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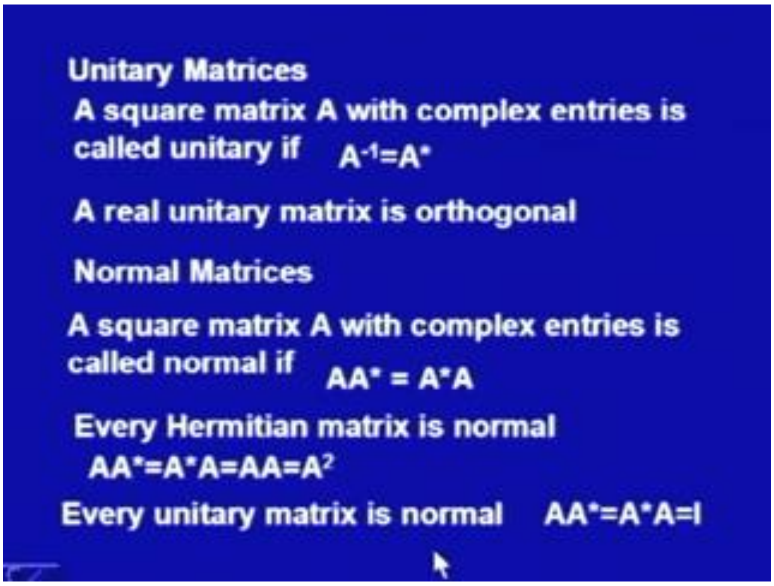

I will first start with unitary matrices. So, unitary matrix is defined as, a matrix with complex entries, satisfying the condition that A inverse is equal to A star. Here, A star is transposed conjugate. And that means, we first take the transpose and then we take the conjugate. Or we first take the conjugate and then take the transpose. And this, the matrix is the same as A inverse. So, first thing is that, A has to be invertible only then we can say that A inverse is equal to A star.

And if the matrix happens to be a real matrix, then A star is nothing but A transpose because, conjugate of a real matrix is the matrix itself. So, A star is nothing but A transpose. And in that case we have a unitary matrix as the same as orthogonal matrix. In other words we say that a real unitary matrix is orthogonal matrix, because there we will have A inverse is equal to A transpose. Similarly, the we define normal matrices; a square matrix A with complex entries is called normal, if A A star is equal to A star A.

So, this product A with A star that is transpose conjugate. And A star multiplied by A, they should be equal. The difference here is, that if A A star is equal to A star A is equal to identity. Then we can write down A inverse is equal to A star. But, say the product may not be identity, but still these two products are equal. And then we say the matrix A is a normal matrix.

So, every Hermitian matrix is a normal matrix. Because A A star is equal to A star A if it is a normal matrix. But, we know that A star is equal to A for a Hermitian matrix, what is Hermitian matrix? Hermitian matrix is a matrix with complex entry. So, that A star is equal to A. So, when A star equal to A then this product as well as this product equal to A square, which satisfies the condition of normal matrices. And that is why we say that every Hermitian matrix is a normal matrix. And every unitary matrix is normal, it is just an extension of what we have discussed just now. You can check that A A star is equal to A star A is equal I. So, this is I, but both of them are equal, this is what is required for a normal matrix. And that is why we say that every unitary matrix is a normal matrix.

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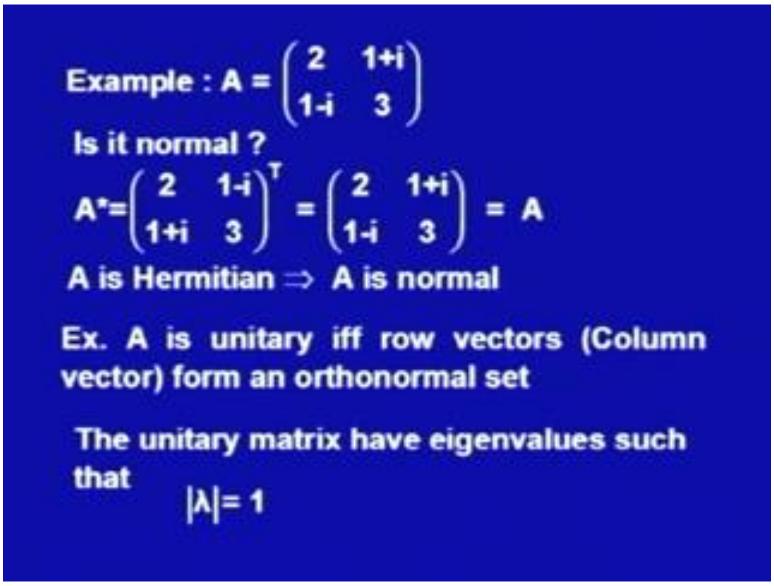

So, let us elaborate this with this example A, where I have a matrix of 2 by 2 order. Here, some of the elements are complex, we have to check whether it is normal or not. So, for this purpose, I have to first constitute A star. So, first I will take the conjugate and then transpose. The conjugate of this matrix means, a particular element here, will be transposed here. So, 1 plus i it is conjugate is 1 minus i, conjugate means this imaginary part will be negated, and then 1 minus i will become 1 plus i.

So, this is A star and when we take the transpose row becomes columns and 2 1 minus i first row will become first column. And second row will become second column, so that is a transpose. And if you compare this matrix with the given matrix, we can notice that this is nothing but A. So, A star is equal to A, so it is a this matrix A star is equal to A. So, it is a Hermitian matrix and we have all ready proved that Hermitian matrix is a normal matrix. And in this sense the given matrix A is normal matrix.

So, now I will state some simple results. And I leave them to the viewers they can prove them. First simple result is, if A is unitary then row vectors form an ortho normal set. And if row vectors form an ortho normal set, then the matrix is unitary. We write this as A is unitary if and only if row vectors form an ortho normal set. This result is not only true for row vectors, this is applied to column vectors also. And the results read like this A is unitary, if and only if column vectors form an ortho normal set, this is first result to be proved by the viewers. The second is the unitary matrix, have Eigen values, such that the magnitude of an Eigen value is 1. So, all the Eigen values in unitary matrix have magnitude 1. So, this is another result which is to be proved by the viewers.

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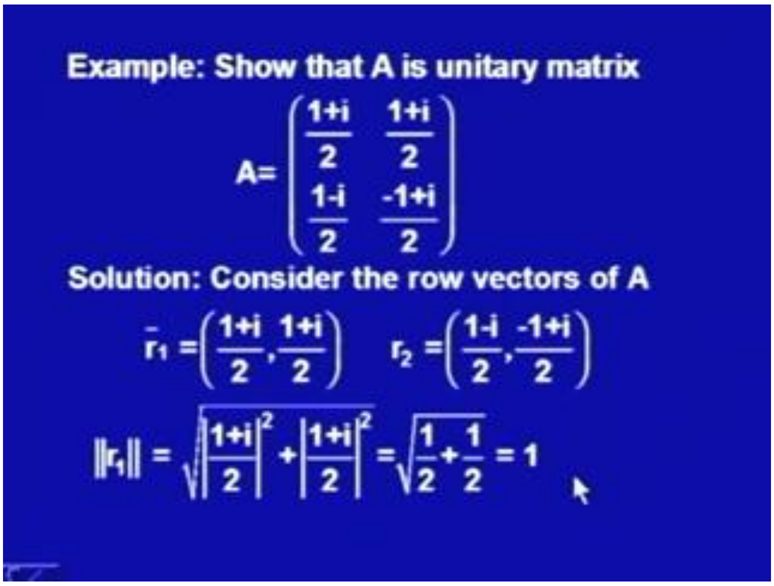

Now, in this example a matrix is given you have to show that, it is a unitary matrix. So, 2 by 2 matrix it has complex entries. And what we do is, we consider the row vectors of A they are 1 plus i by 2 and 1 plus i by 2. And the second row vector is 1 minus i by 2 and minus 1 plus i by 2. So, these are two row vectors. We calculate their magnitude, so norm of r 1 bar is equal to square of first component plus square of second component and mod of 1 plus i square is equal to real part square plus imaginary part square divided by 2. So, it is 1 plus 1 2 divided by square of this 4, so this number is equal to 1 by 2. Similarly, this is also 1 by 2, so we have 1, so norm of row vector is 1.

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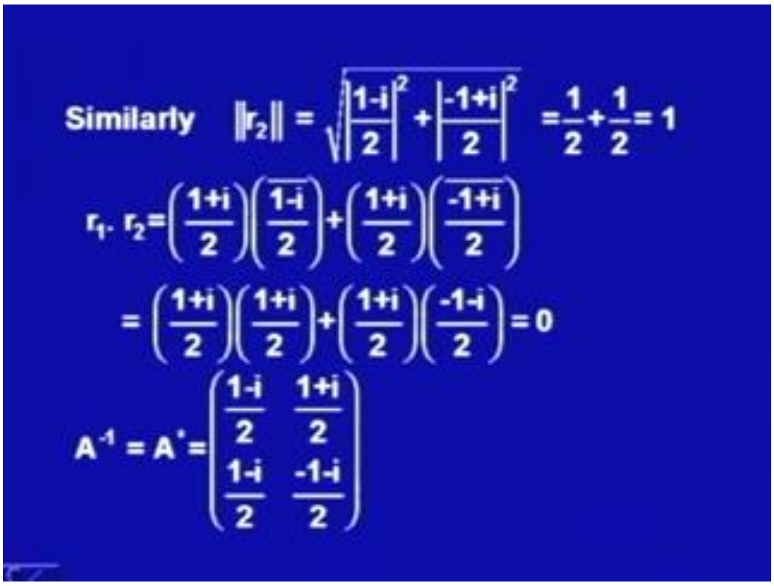

Then, on the same lines one can find the norm of second vector and it is 1 minus i by 2 magnitude of this plus minus 1 plus i by 2. This is the second component and it is square. And when you calculate this it is 1 by 2 and this is also 1 by 2. So, sum of this is equal to 1. Now, r 1 dot r 2, that is inner product of r 1 and r 2 it is 1 plus i by 2 multiplied by 1 minus i by 2 and it is conjugate plus 1 plus i by 2 into minus 1 plus i by 2 it is conjugate.

And when you perform this multiplication, this comes out to be 1 plus i by 2 and conjugate of this is 1 plus i by 2 plus this remains as such, but it is conjugate will be minus 1 minus i by 2. So, this is one can take this minus common and it is 1 plus i by 2 and 1 plus i by 2 with negative sign. So, this cancels out with this and what we have is that inner product of r 1 and r 2 that is r 1 dot r 2 is 0. So, what we have is, we have two rows r 1 and r 2 there dot product is 0 and there magnitude is equal to 1.

And that means, the matrix which is given to us has row vectors which are ortho normal. They are normal because of this they are ortho normal because, there magnitudes are 1. So, in this way we say that, A star is equal to A inverse it is a unitary matrix. So, this we are not actually evaluating A inverse. And then saying A inverse is equal to A star, we are saying that this matrix has row vectors, which are ortho normal. And that is how we prove A inverse is equal to A star.

Actually, we are applying the result which we have stated in my earlier slides. This means that the matrix A is unitary matrix. Now, before we proceed further we will verify this. Now, if A inverse is equal to A star then the product of the two matrices should be identity.

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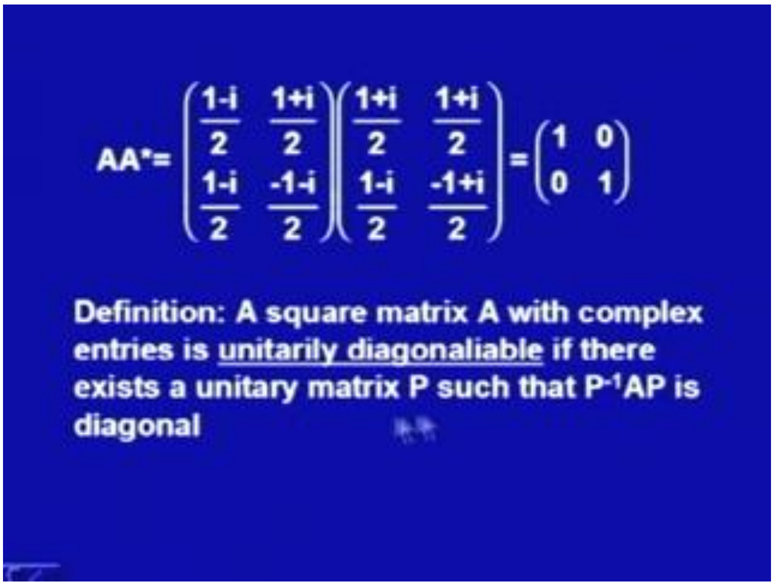

So, we verify this and we calculate this A into A star is equal to this is what I have the matrix A, and this is A star. And when we multiply it, this multiplied by this, this multiplied by this, this is 1, this row multiplied by this column is 0, this row multiplied by this column is 0 and this multiplied by this is 1. So, A A star is equal to identity matrix. Now, we define unitarily diagonalizable matrices, we say that a square matrix A with complex entries is unitarily diagonalizable, if there exists a unitary matrix P such that P inverse A P is diagonal.

So, basic difference is the matrix is diagonalizable. We should have a matrix P, which is invertible. Such that, P inverse A P is diagonal, but the difference here is the matrix P is not only invertible, but it is unitary matrix. So, if we can find such a matrix P then we say the matrix A is unitarily diagonalizable.

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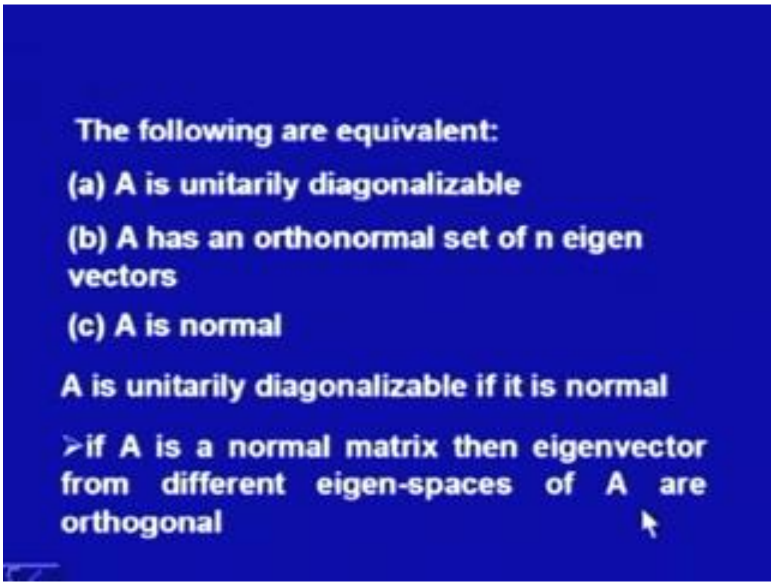

Now, to prove this, we say that the following are equivalent. First A is unitarily diagonalizable. And the second is A has an orthonormal set of n Eigen vectors and third statement is A is normal. That means, A is unitarily diagonalizable if it is normal and if A is normal matrix. Then Eigen vectors from different Eigen spaces of A are orthogonal not only orthogonal, but they are ortho normal.

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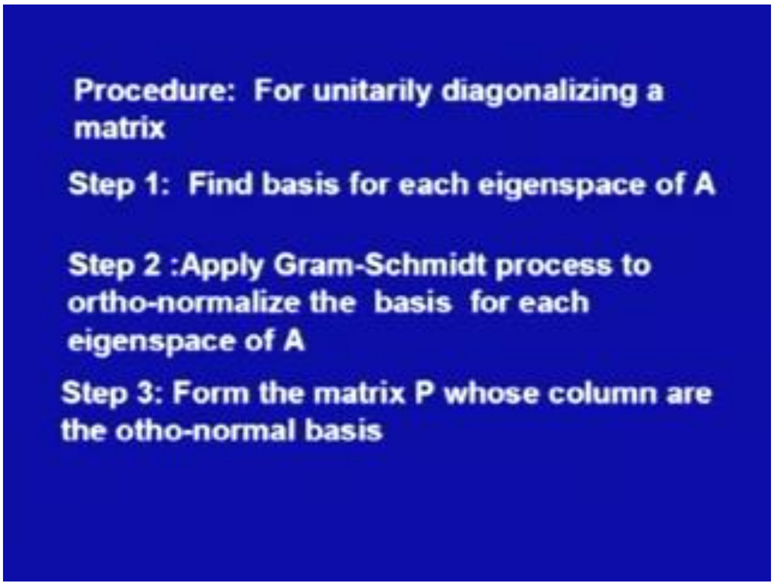

This particular result forms a basis for a procedure for unitarily diagonalizing a given matrix. So, let us discuss a procedure for unitarily diagonalizing a matrix. The step 1 is find the basis for each Eigen spaces of A. In the step 2, we apply Gram Schmidt process to ortho-normalize the basis for each Eigen space of A. And finally, in the step 3 we form the matrix P, whose columns are the ortho normal basis. Now, in the next example we illustrate this procedure for unitarily diagonalizing a given matrix A.

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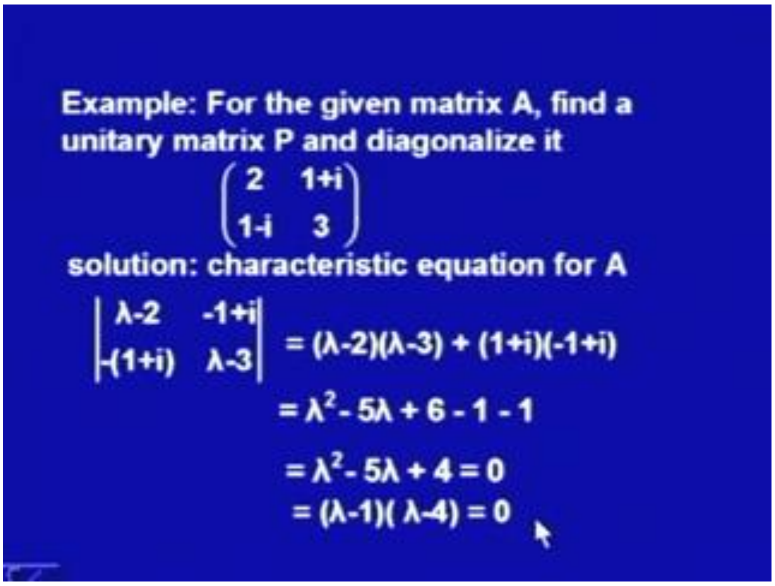

So, in this example a matrix A is given as a 2 by 2 matrix. We are considering 2 by 2 matrices because, they are simple. So, you have to find a unitary matrix P, which will diagonalize this given matrix A. To find matrix P, we have to first obtain the Eigen values and then corresponding Eigen vectors. For this, we first write down the characteristic equation for the given matrix A.

So, the characteristic equation for this is the determinant lambda minus 2 minus 1 plus i minus 1 plus i and lambda minus 3. One can simplify it, it is lambda minus 2 into lambda minus 3 plus 1 plus i into minus 1 plus i. From this multiplication and it is lambda square minus 5 lambda plus 6 this part and from here, we will get minus 1 minus 1. And that gives us lambda square minus 5 lambda plus 4 equal to 0. And when you factorize it, it is lambda minus 1 into lambda minus 4 equal to 0.

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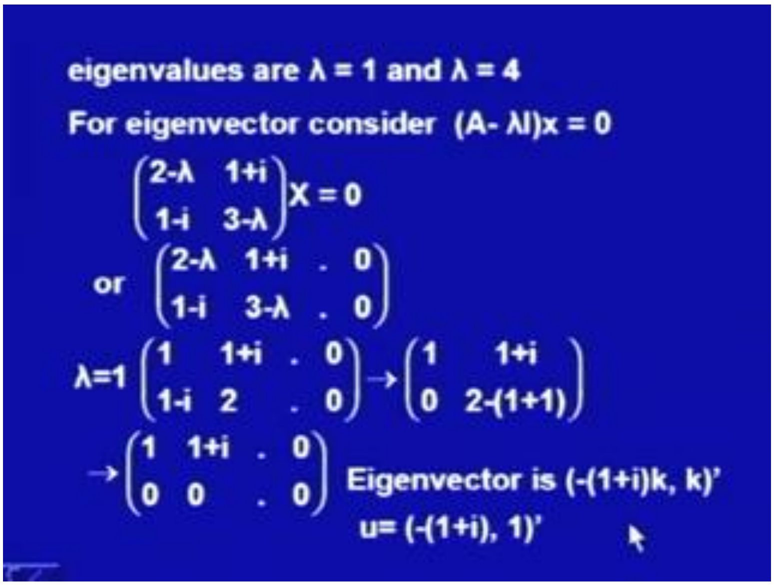

And that gives us the Eigen values as lambda is equal to 1 and lambda is equal to 4. To get the Eigen vectors corresponding to these Eigen values. We have to solve this equation A minus lambda I multiplied by x is equal to 0. So, let us write down A minus lambda I X equal to 0. So, it is 2 minus lambda 1 plus i 1 minus i 3 minus lambda X equal to 0. And for lambda is equal to 1, we have to solve this equation, we will get Eigen vectors corresponding to lambda is equal to 1. And when we solve this equation for lambda is equal to 4, we will get Eigen vector corresponding to lambda is equal to 4.

So, first for lambda is equal to 1 this equation become 1 1 plus i 0 1 minus i 2 0 this is the augmented matrix, corresponding to this system. So, this system can be transformed I am writing only this part, the coefficient matrix part here. I have applied elementary transformation and this will only the coefficient matrix is taken here because, this will not be affected. So, we will transform this matrix to this matrix and from here one can notice that this augmented matrix will finally, become this matrix.

And this matrix has one Eigen, one row completely 0. And that means, we can have Eigen vector of given matrix as minus 1 plus i k into k, where k is an arbitrary value. This is an Eigen vector corresponding to lambda is equal to 1. And when we write k is equal to 1, then let us say the Eigen vector u becomes minus 1 plus i and 1. This indicates that it is a column vector, so it is a transpose here. So, u is a column vector, which is actually an Eigen vector corresponding to lambda is equal to 1.

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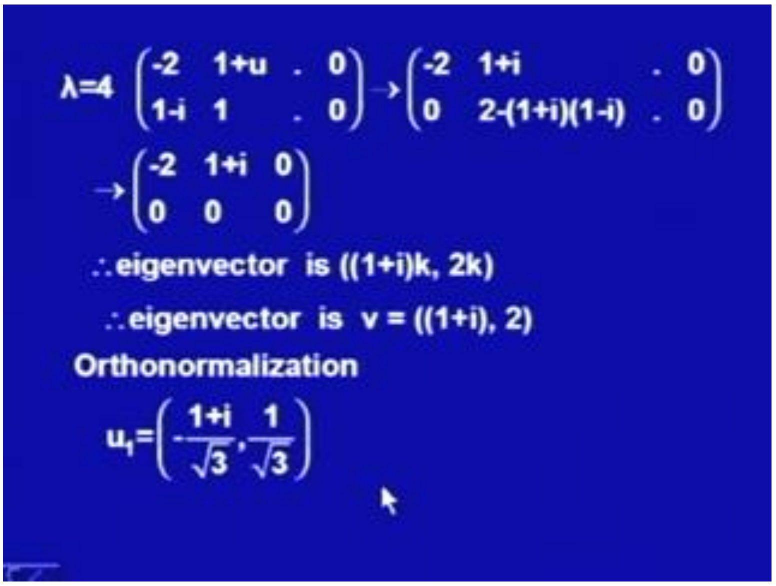

On the same lines we can get Eigen values, we can find Eigen vector corresponding to lambda is equal to 4. So, we consider this, augmented matrix apply elementary transformations, number of elementary transformations are required. And finally, this matrix will reduce to this matrix with one row completely reduced to 0.

And from here one can easily write down, the Eigen vector as 1 plus i times k into 2 k because, this will satisfy this equation. And that means, the Eigen vector is 1 plus i comma 2, when we write k is equal to 1. So, we have another Eigen vector. So, in this sense u 1 is equal to minus 1 plus i by root 3 comma 1 by root 3, this is what we have obtained after normalizing the given vector u, u. And this is a row vector because, I am not writing transpose here, so it is a u vector. So, u 1 is equal to minus 1 plus i by root 3 comma 1 by root 3 is a orthonormal vector it is magnitude is 1.

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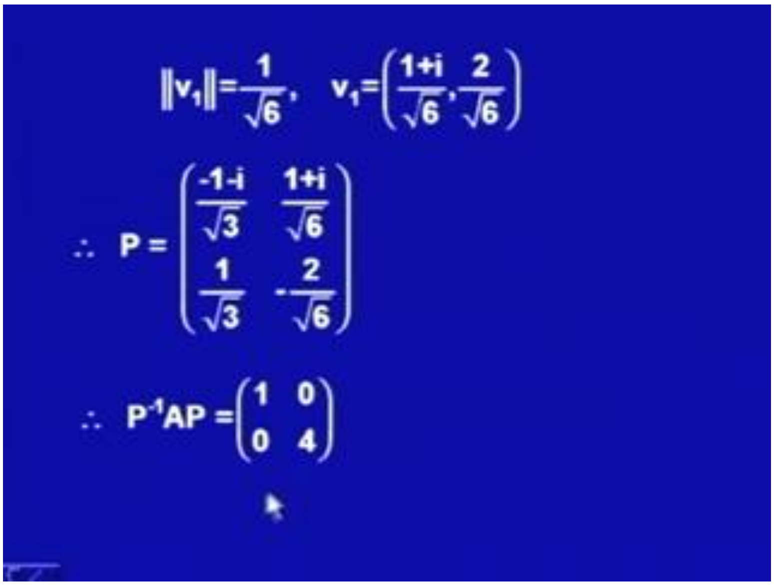

And v 1 is 1 plus i by under root 6 and 2 over under root 6, because the magnitude of v 1 is 1 by root 6. So, v 1 is a row vector with this as the first component and this is the second component and it is again a row vector. Now, u 1 and v 1 form a set of ortho normal vectors, there dot product is 0 and each of them is of magnitude one. And that means, we have a matrix P which is having the two vectors as column vectors, this is first column vector and this is second column vector.

And these column vectors are having magnitude 1 and there dot product is 0, so they are orthonormal. And this characteristic makes the matrix P as a unitary matrix. And then P inverse A P is equal to a diagonal matrix. So, we here we have in this particular step we have found a matrix P which is a unitary matrix for the given matrix. So, we say that P inverse A P is a diagonal matrix. Basically, be we are applying the result, which we have all ready stated.

The proof is not been given here and the proof of that theorem can again be left to the reader. So, P inverse A P is a diagonal matrix and the diagonal elements here are the Eigen values of the matrix A they are 1 and 4. So, in this way we say that A is unitarily diagonalizable. Now, here we discuss applications to Eigen value problem to the system of differential equations.

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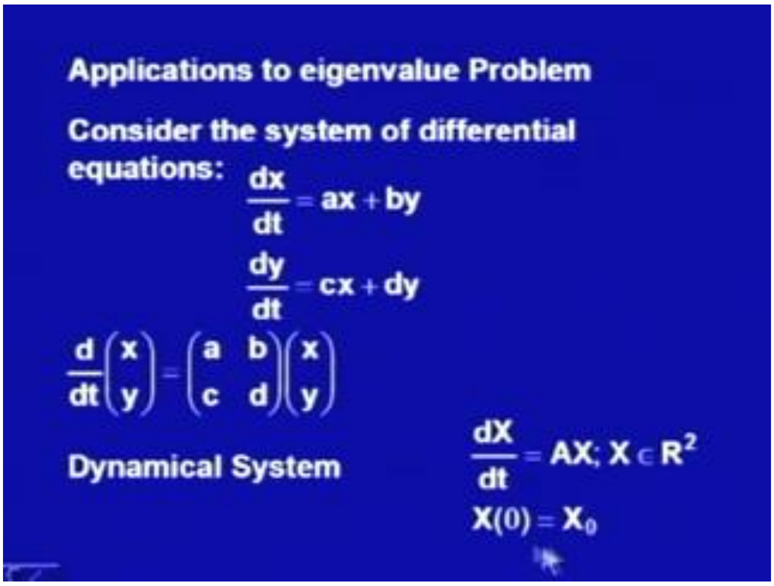

Consider system of differential equations in two variables x and y, x and y are two dependent variables depending upon an independent variable t. And we write the first equation as the first order differential equation relating the derivative d x by d t with x and y. So, we say d x by d t is equal to a x plus b y a and b being constants. And the second equation also relates derivative with respect to y with the variables x and y, so d y by d t is equal to c x plus d y.

So, here we have two equations two differential equations which are of first order they are coupled. Because, first equation involves x as well as y and the second equation in y also involves the variable x. So, this is coupled with x and this is coupled with y, so we have a system of coupled first order differential equations. Now, this system can be written in this form, in this matrix form here d by d t of this matrix x y. So, we have a column vector on the left hand side and we have taken it is derivative.

So, d by left hand side is d by d t of x y is equal to a b c d. This is the coefficient matrix on the right hand side multiplied by the column vector x y. And this is 2 by 2 matrix this is a column vector, this is the column vector. And we write the dynamical system, we call this as dynamical system and we write it as d x by d t is equal to A X. I denote this matrix by A this column as X, so this is nothing but d x by d t and the whole system is d x by d t is equal to A X.

So, X is a column vector in R 2 because, dimension of X is 2, so X belongs to R 2. Now, together with this differential equation it is not a single equation, it is actually a system of equations. And this system of two, differential equation can be written in a compact form in this matrix differential equation and this has to be solved under specific initial conditions. So, we specify the initial conditions as x at t is equal to 0 is equal to X naught. Actually this condition simply means that x at t is equal to 0 is X naught and y at t is equal to 0 is Y naught.

So, that initial condition is also written in a compact form and this system is called a dynamically system. We call it dynamically system because, it is changing with time. So, we are considering how the dynamics of the vector x is changing with time. So, that is the meaning of a dynamical system.

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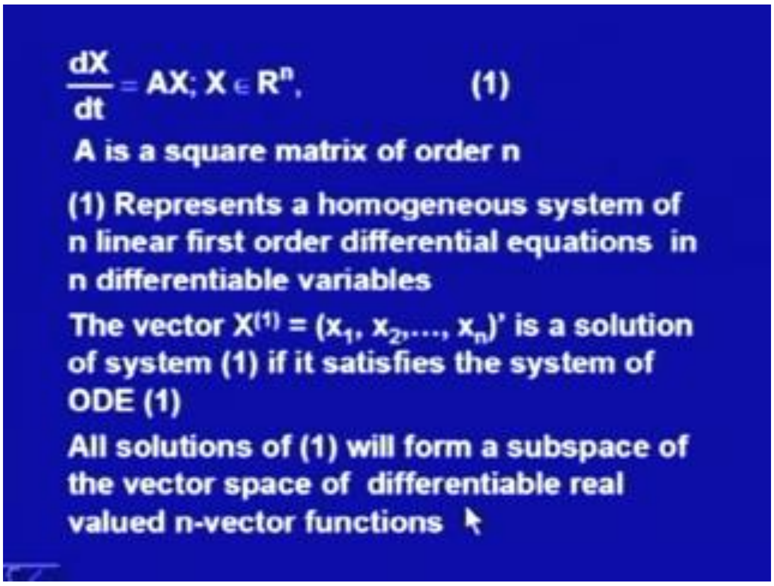

Now, we generalize this concept and we write down d x by d t is equal to A X, where X belongs to R n. So, earlier I have written a matrix equation in two variables, but X need not be belonging to R 2 it maybe a column vector consisting of n components. In that case we say X belongs to R n and this equation d x by d t is equal to A X, means that A is a square matrix of order n. Now, 1 represents a homogeneous system of n linear first order differential equations in n differentiable variables.

Because, X is a column vector, so this left hand side is actually n terms. And the right hand side we have this product and X is a column vector. So, actually this system represents n differential equations because, each equation is having d by d by d t involved in it. And since we are considering derivative, so we should have the functions as differentiable functions, this all these variables should be differentiable.

We call this as a homogeneous system, because all the terms on the right hand side they are involving X or Y there is no term, which is free of the variable X and Y. So, there is no constant matrix added here. And in this sense we say this represents of homogeneous system of n differential equations. Each equation is linear in it is own right and that is why it is a linear system of differential equations. The vector X 1, which is having n components it is a column vector, we say it is a solution of the given system 1. If it satisfies the system of Ordinary Differential Equations or in short we say it ODE.

So, any function which satisfies this differential equations is called the solution of this system. So, X we denote this solution vector as X 1, all solutions of 1 will form a subspace of the vector space of differentiable real valued n vector functions. If you look at this, this is the solution what is the x 1, x 1 has to be a function of t it has to be a differentiable function, x 2 also has to be a differentiable function. Because, all this is they are to be substituted here. If any of them is not differentiable then you cannot say that it is d x by d t, when you cannot form d x by d t there is no question of satisfying this differential equation.

So, first thing is that this has to be differentiable function and when you substitute it here this should satisfy this differential equation. So, this each solution vector is a vector in n dimension. So, if we consider all vectors which is having n components each of them is a differential function, then it will form a vector space. But, if we consider all the solution space, that is the space constituting this solutions of this differential equation. Then we say it will form a subspace of this vector space of n differentiable vector functions.

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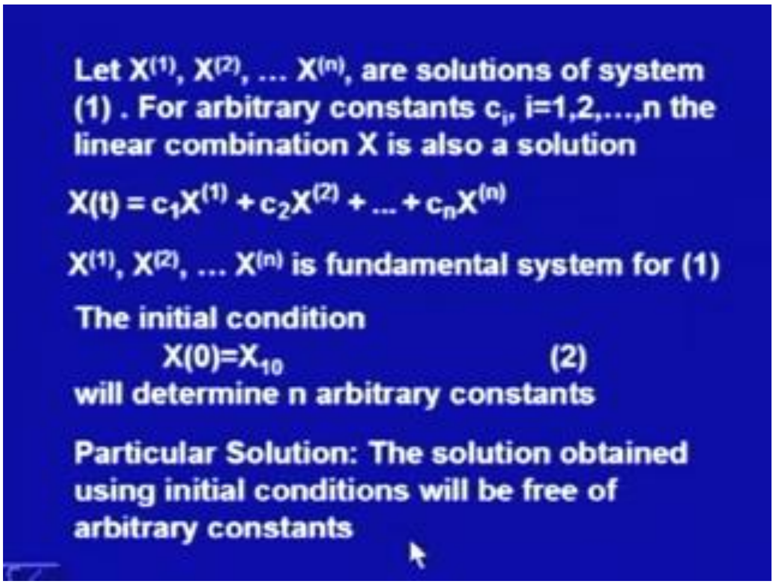

The next if X 1, X 2, X n are solutions of a system 1, then for arbitrary constants the linear combination X is also a solution. So, X t is written as c 1 X 1 plus c 2 X 2 plus c n X n. If we can write down the solution as X t consisting of this, then we say that X 1, X 2, X n is a fundamental system for one. The initial condition X 0 is equal to X 1 0. If we denote, the initial condition in this manner. Then when we substitute this initial condition here, then all these constants c 1, c 2, c n it can be obtained.

So, in this state at this state these are arbitrary constants, so X 1, X 2, X n they are n solutions there linear combination is also solution. But, if you use this initial condition then all these c 1, c 2, c n will be determined and what we have is a particular solution. So, a particular solution is obtained using initial conditions, which will be free of arbitrary constants. So, these n conditions used with this will eliminate these arbitrary constants. And what we have is a solution free of arbitrary constants and that such a solution is called a particular solution.

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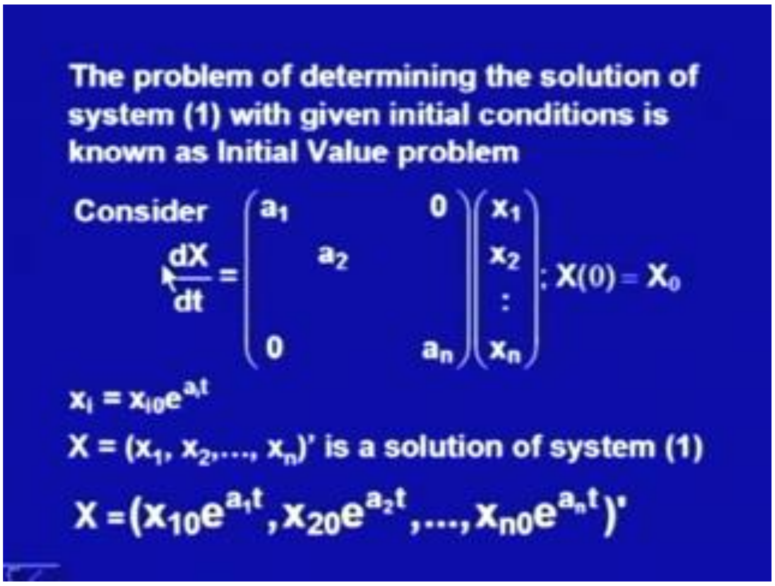

The problem of determining the solution of system 1 with given initial condition is known as initial value problem. So, let us consider d x by d t is equal to a 1 a 2 a n and n by n diagonal matrix multiplied by the column vector x 1 x 2 x n which is nothing but X together with the initial condition X 0 is equal to X naught. So this, constitute an Eigen value problem, but it is a special case; in this sense that, the matrix A happens to be a diagonal matrix.

That means, only diagonal entries are non zero rest of the entries are zeroes, I am considering this because, this is easy to solve. So, let us solve this system, here you can reduce this system as d X 1 by d t is equal to a 1 x 1. So, this equation is no more coupled with rest of the equations d X d t is equal to a 2 times x 2 it is again decoupled with other equations. So, if the equations are decoupled specially, when we have a diagonal matrix here. Then each equation can be solved independent of other equations.

So, we first solve the first equation in the system that is d X 1 by d t is equal to a 1 into x 1, this is the first order differential equation and it is very easy to solve. And it is solution will be x 1 is equal to x 1 0 e times e or exponential a t, here what is x 1 0 is the arbitrary constant and we can calculate this using this initial conditions. Similarly, we can calculate the second solution d X 2 by d t is equal to a 2 into x 2 and so on. So, in general the solution of this system is written as x i is equal to x i 0 e a i t I will take values from 1 to n, i is equal to 1 will have solution of first equation, i is equal to 2 will have solution of second equation and so on, so this is the general solution.

Once, we have obtained this then X is equal to x 1 comma x 2 comma x n and then the column vector is a solution of system 1, this system 1. So, this represents a solution of this equation. And if you substitute these values, the way we have computed here, this is the solution vector of this differential equation.

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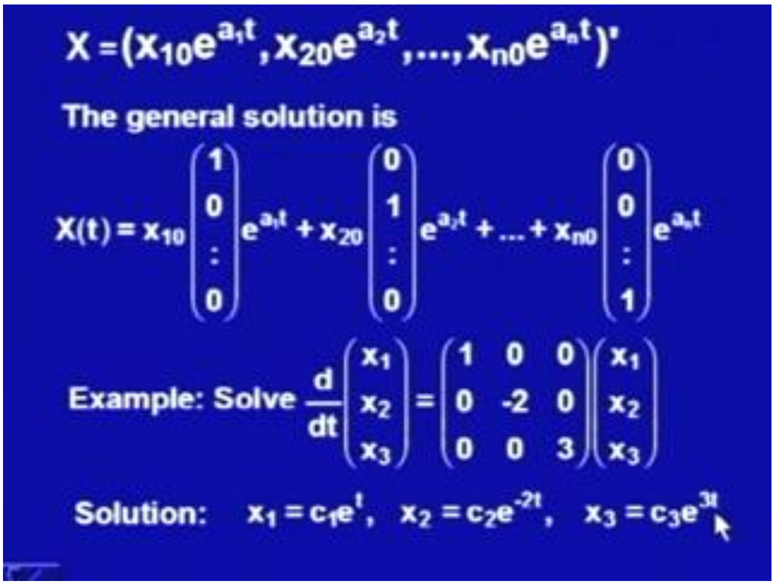

And with this the general solution is written in this form, where I have written this first term as x 1 0 1 0 0 is a column vector e a 1 t. And the second corresponding to this I have written x 2 0 a column vector multiplied by e a 2 t and so on. So, if you sum it up it will be first component will be x 1 0 e a 1 t, only this contribution will be coming. For the second time contribution will be coming from this time only and for the n'th term it is contribution is coming from this term.

So, one can write this general solution in this form, so X t is a linear combination of these solutions. We illustrate this with this example, so if I have given a 3 by 3 system d x 1 by d t d by d t of x 2 d t and d by d t of x 3 is equal to this diagonal matrix multiplied by the column vector x 1 x 2 x 3. So, let us solve this system, this has diagonals 1 minus 2 and 3 and that makes x 1 as c 1 e t x 2 as c 2 e minus 2 t and x 3 as c 3 e 3 t.

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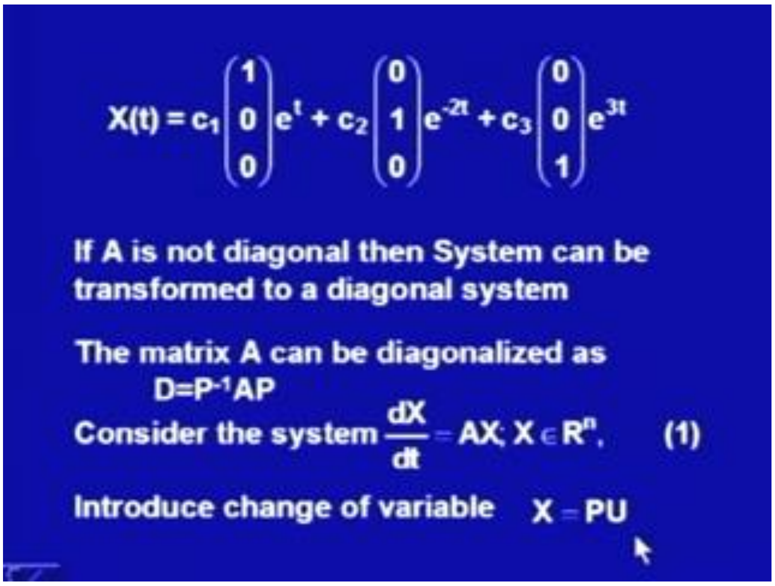

And that means, X t is equal to c 1 times the column vector 1 0 0 e t plus c 2 times the column vector 0 1 0 e minus 2 t plus c 3 as 0 0 1 e 3 t. So, we have obtained the general solution, now if the matrix A is given in the diagonal form. Then things are simple, but if A is not diagonal then system can be transformed to a diagonal system. And then we can solve it. And here comes the importance of diagonalization what we have discussed in these lectures.

So, if the matrix A can be diagonalized as this D is equal to P inverse A P, then we can solve the dynamical system d X by d t is equal to A X; X belonging to R n on the same lines as I have illustrated just now. So, if this is the system given to us, then we can change the variable X to P U, so U is a new variable and we write down X as P U.

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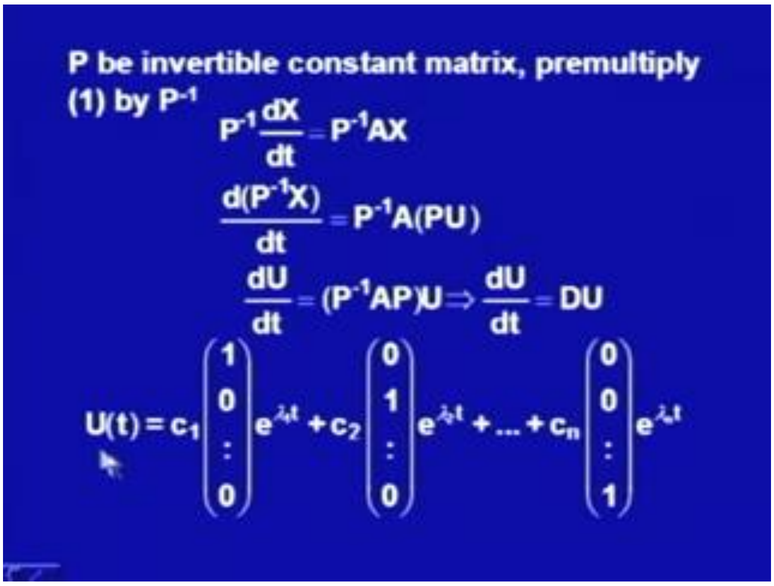

With this transformation P be invertible constant matrix, then we can pre multiply the given equation by P inverse. So, before we apply this transformation we have to make sure that P is invertible and it is a constant matrix. So, if you do that then P inverse d X by d t is equal to P inverse A X. Since, P inverse is a constant matrix, so we can write down this left hand side as d by d t of p inverse X and that is why I have assumed that it is invertible and constant matrix.

So, left hand side is this and right hand side is P inverse A and A this X I am replacing by P U and this simply means d U by d t is equal to p inverse A P into U. So, that gives me d U by d t is equal to D U. So, with this transformation this matrix D is diagonal and I have obtained a system of equations, in which the matrix D is diagonal. And then we can very easily solve this system of equations and we know from our earlier discussions that U t can be expressed as c 1 an arbitrary constant multiplied by this column vector multiplied by e into lambda 1 t, what is lambda 1, lambda 1 is the diagonal first diagonal element in the matrix D.

Then, we are add the second term here c 2 arbitrary constant multiplied by this column vector, where the second term is 1 and rest of the terms are 0. And this will be multiplied by e lambda 2 t lambda 2 being the second term in the diagonal matrix D and so on. The last term will be c n the n'th arbitrary constant, n'th component in this vector will be 1 rest of them are 0 and multiplied by exponential of lambda n t. So, we have obtained solution U t, but where our equation was in X. So, we have to transform this solution to it is original coordinate system.

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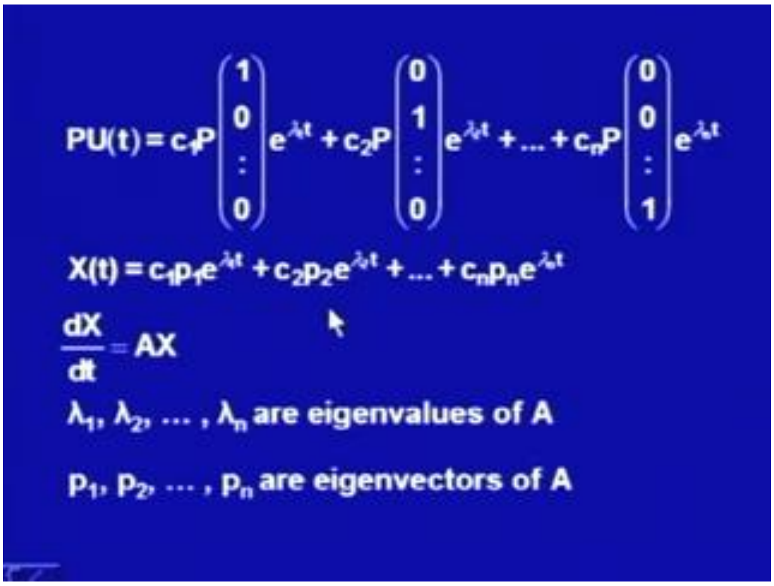

So, that means, we apply P on both the sides, so P of U t is equal to c 1 P of this vector e lambda 1 t plus c 2 P of the vector multiplied by e lambda 2 t and so on. And that makes P U t as X of t, where X is a function of t is equal to c 1 and I write down P into 1 0 0. If you perform this multiplication this simply means the first row first column of the matrix P. So, it is p 1 e lambda 1 t plus c 2 and this multiplication is P times 0 1 0 0 is nothing but P 2, so it is e lambda 2 t and so on. And the final term will be the last value P n, so it is P multiplied by this is nothing but P n, so we write down X t as c 1 p 1 e lambda 1 t plus c 2 P 2 e lambda 2 t and so on.

So, what I am writing is that the solution vector X t is this linear combination in which P 1 is the Eigen vector corresponding to Eigen value lambda 1 of the matrix D, which is the diagonal matrix similar to the given matrix A. So, solution can be easily obtained from the matrix D. So, basically what we are doing is we are diagonalizing the matrix A and then we are finding the solution in this particular manner.

So, if d X by d t is equal to A X and lambda 1, lambda 2, lambda n are Eigen values of A and P 1, P 2 and P n are Eigen vectors of A. Then the solution of this differential equation is expressed as this in terms of Eigen values and Eigen vectors.

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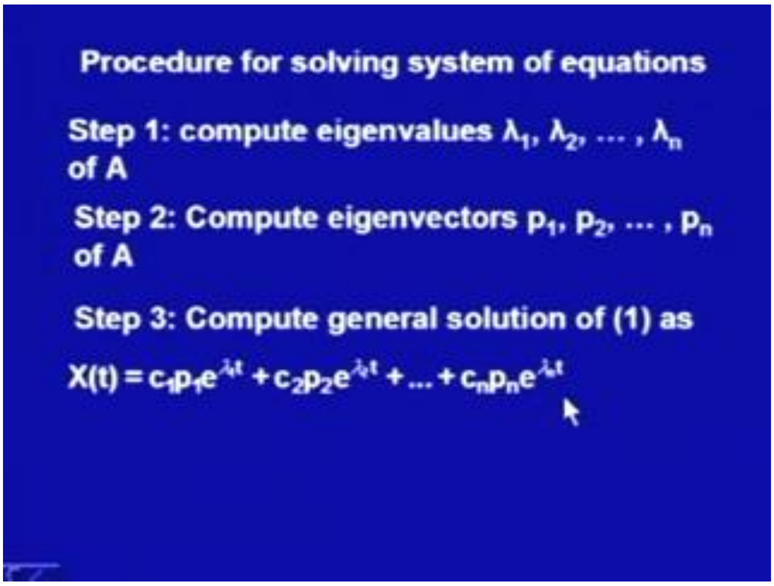

This gives us a procedure for solving system of equations according to this the step 1 constitutes, the computation of Eigen values lambda 1, lambda 2, lambda n of the given matrix A. In the step 2, we compute the eigenvectors P 1, P 2, P n of A corresponding to the Eigen values lambda 1, lambda 2, lambda n and the step 3 is. We compute the general solution of 1 as X t is equal to c 1 p 1 e lambda 1 t plus c 2 p 2 e lambda 2 t and lastly c n p n e lambda n t.

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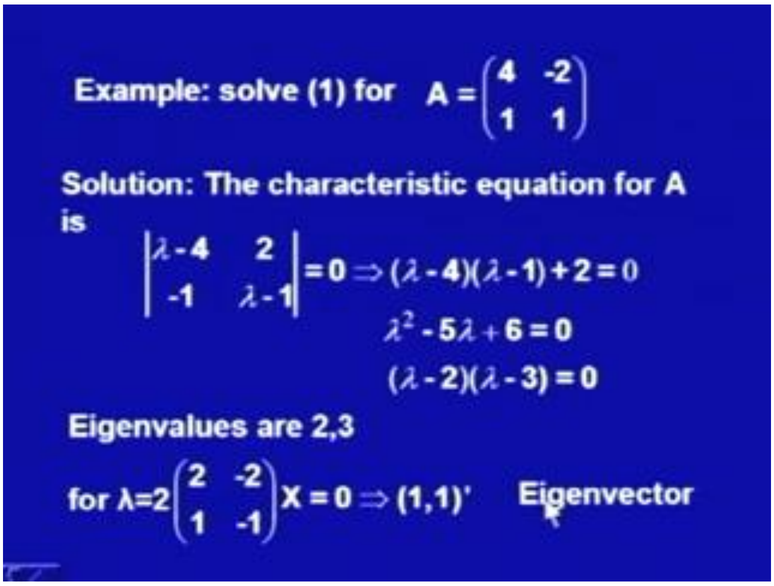

We illustrate this procedure in this example, so a 2 by 2 matrix being given to us, first according to first step, we have to first form the Eigen values for this the characteristic equation for A is the determinant of this 2 by 2 matrix, it is simplified to lambda minus 4 multiplied by lambda minus 1 plus 2 equal to 0. We, will further simplify it is lambda square minus 5 lambda plus 6 equal to 0 and factorizing it is lambda minus 2 into lambda minus 3 equal to 0 and that gives us Eigen values as 2 and 3. And from here for lambda is equal to 2 we calculate the Eigen vector by solving this equation 2 minus 2 1 minus 1 X equal to 0 and one can easily see that one row can be transformed to 0 and the Eigen vector will be 1 comma 1 transpose.

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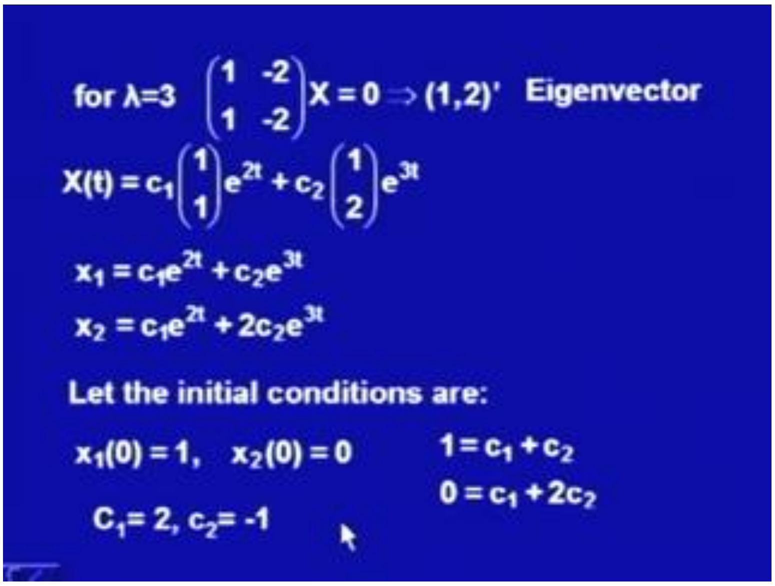

For lambda is equal to 3, we have to solve this system 1 minus 2 1 minus 2 X equal to 0 again these 2 rows are identical and this gives us the Eigen vector as 1 comma 2. And once, we have obtained Eigen values and corresponding Eigen vectors, then X t is equal to c 1 multiplied by the first vector e 2 t and then plus c 2 the second vector 1 2 into e raise to power 3 t. So, this is the solution of given system of differential equations, in this example I have two distinct roots of the characteristic equation and correspondingly I have got two independent Eigen vectors.

But, there may be a situation where we have repeated roots, matrix cannot be diagonalizable, then things will be more involved I am not discussing all these things at the moment. So, once we got the solution, then in the component form we can write down x 1 as c 1 e 2 t plus c 2 e 3 t that is the first component and the second component of this vector x 2 is written as c 1 e 2 t plus 2 times c 2 e 3 t, so we can get this as a solution of given differential equation.

Let, the initial conditions be specified as x 1 0 is equal to 1 x 2 0 is equal to 0, see in this equation we have two arbitrary constants involved in it. So, we in fact, we will have infinitely many solutions of the given system of equations because, these coefficients can take infinitely many values. So, to fix the ideas we can specify the initial conditions and for the given initial conditions they are two in number, so we can substitute them here two equations, two unknowns and one can solve for c 1 and c 2 and what we get is the particular solution.

So, if you substitute this here t is equal to 0 x 1 is 1, so left hand side is 1 and right hand side t is equal to 0, it is c 1 this is 1 for t is equal to 0 this factor is also 1, so c 1 plus c 2 from the right hand side. And similarly, substituting this in the second equation we will have 0 on the left hand side and on the right hand side will have c plus 2 c 2, these two equations can be solved easily and the solution will be c 1 is equal to 2 and c 2 is equal to minus 1.

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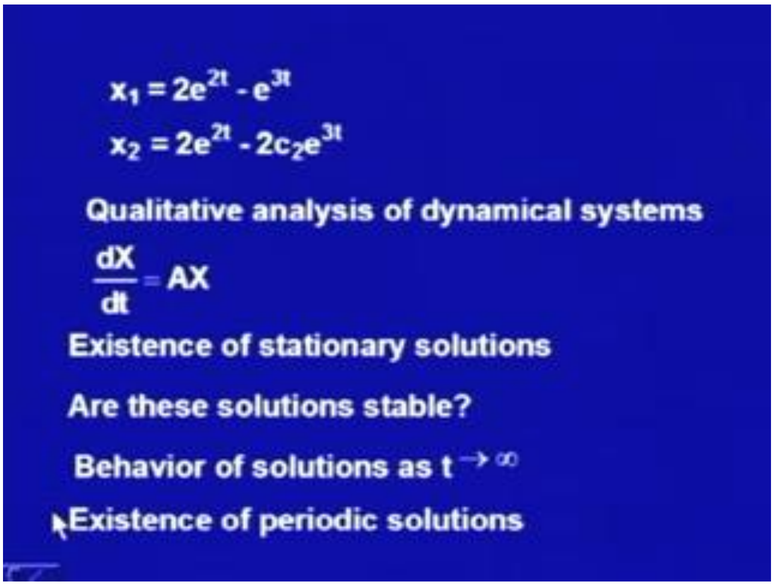

When, we substitute this here we get the a solution x 1 as 2 e 2 t minus e 3 t and x 2 as 2 e 2 t minus 2 c 2 e 3 t. So, in this example we have been given a system of differential equations together with initial conditions or we say we have been given an Eigen value problem. And this Eigen value problem is solved using the concept of Eigen values and Eigen vectors, so this concept is useful in solving system of first order linear homogeneous differential equations.

Now, this can also be used in performing qualitative analysis of dynamical system by qualitative analysis of a dynamical system I mean to say, that a given system d X by d t is equal to A X, we may like to answer questions like, existence of stationary solutions. That means, does this system have stationary solutions or does they are exist stationary solutions for the system or we may like to know are these solutions stable. So, if this stationary solution exist are they are stable or what is the behavior of solutions as t tends to infinity and if there any possibility that this system has a periodic solutions.

So, when we discuss qualitative behavior of dynamical systems we are actually trying to answer these questions. And these questions can very easily be answered with the help of Eigen values and Eigen vectors related to this differential system

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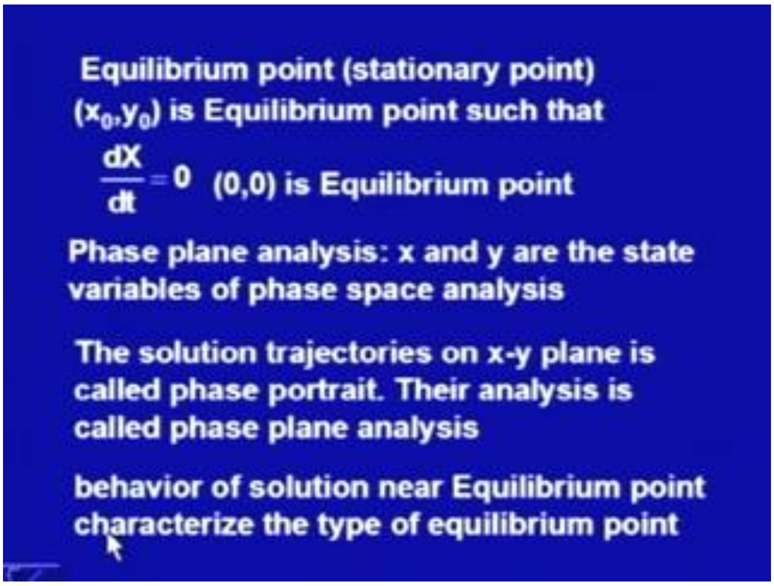

So, let us illustrate this for this we require concepts like equilibrium point sometimes we call it stationary point. We, define a stationary point as x naught, y naught for the given system such that d X by d t is equal to 0 by this I mean that 0, 0 is equilibrium point because, this system means d X by d t is equal to A X is equal to 0. Now, A X is a homogeneous system and this homogeneous system will always have 0, 0 as a solution, so 0, 0 is equilibrium point, there may be other equilibrium point possible for this dynamical system.

For the this equilibrium point or more equilibrium points we may like to perform the phase plane analysis by phase plane analysis I mean to say, that if x and y are state variables of phase space analysis. Then, what are the solution trajectories on x, y plane and these trajectories when they are drawn on x, y plane they constitute phase portrait and this analysis is called phase plane analysis. So, analyzing these trajectories on the phase plane comes under phase plane analysis, it is a two dimension, then it is phase plane analysis, if the system has more than two variables, then what we have is a phase space analysis.

So, we try to see how these trajectories look like on x y plane and then the behavior of solution near equilibrium point characterize the type of equilibrium point. So, what are the equilibrium points, what are their characteristics we illustrate with the help of Eigen values and Eigen vectors.

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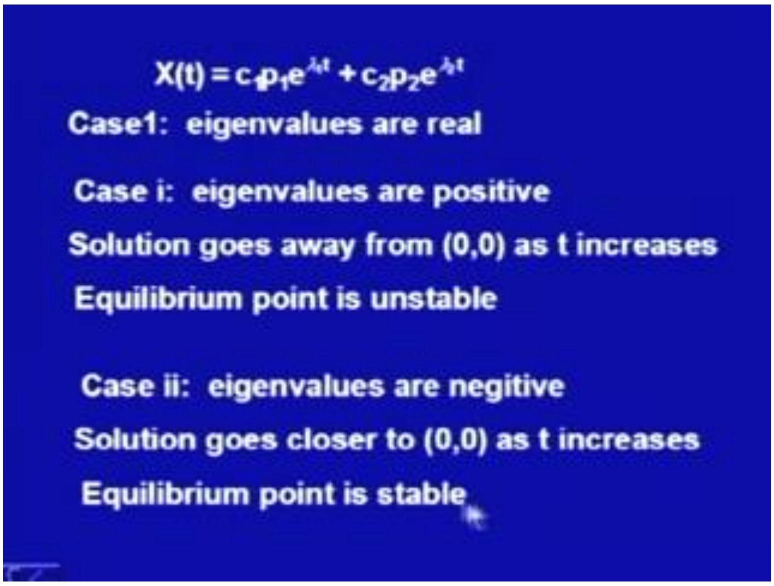

Let us see that, we have the solution of a system as X t is equal to c 1 p 1 e lambda 1 t plus c 2 p 2 e lambda 2 t for a 2 dimensional system. Now, since the solution depends upon Eigen values lambda 1 and lambda 2, so the behavior of the solution depends upon the Eigen values lambda 1 and lambda 2. So, we consider different cases, the case 1 is when the Eigen values are real and in this case I subdivide into case 1 as Eigen values are positive, Eigen values maybe negative, Eigen values may be one positive, one negative.

So, I will first discuss the case when Eigen values are positive, so if when the Eigen values are positive and distinct, then solution goes away from 0, 0 as t increases, this can be observed from this solution, when lambda 1 and lambda 2 are positive, this and this component will increase as time increases and we will be going away and away from 0, 0 and such in such a situation we say the equilibrium point 0, 0 is unstable.

While in the second case when the Eigen values are negative, then these negative values means that this term as well as this term will become smaller and smaller as t increases. That means, we will be going closer and closer to the equilibrium point 0, 0 and this means that the equilibrium point is stable.

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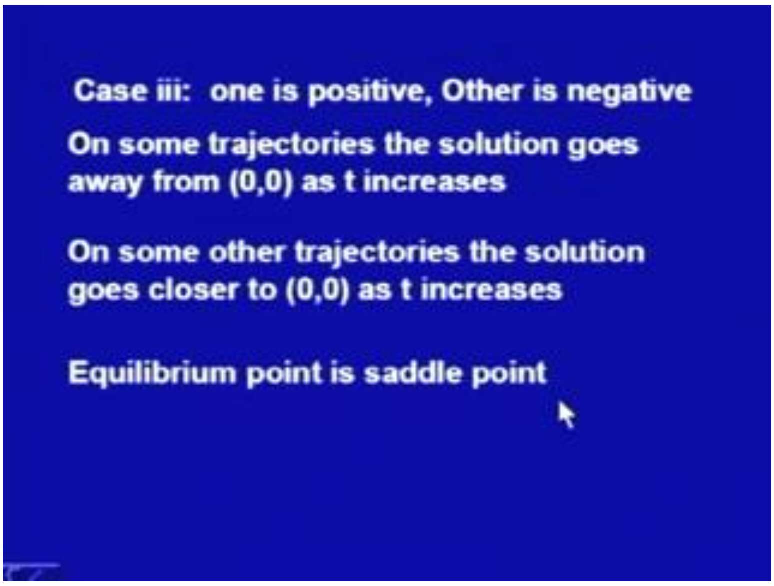

In case 3, when 1 Eigen value is positive other is negative, we say that they are some trajectories, where the solution goes away from 0, 0 as t increases because, corresponding e lambda 1 t increases. But, on some other trajectories the solution goes closer to 0, 0 as t increases because, lambda 2 is negative means e lambda 2 t tends to 0 as t tends to infinity for negative lambda 2.

So, there are some trajectories which will go close to 0, but there are some other trajectories where solution goes away from 0, 0 in such a situation we say that the equilibrium point is saddle point.

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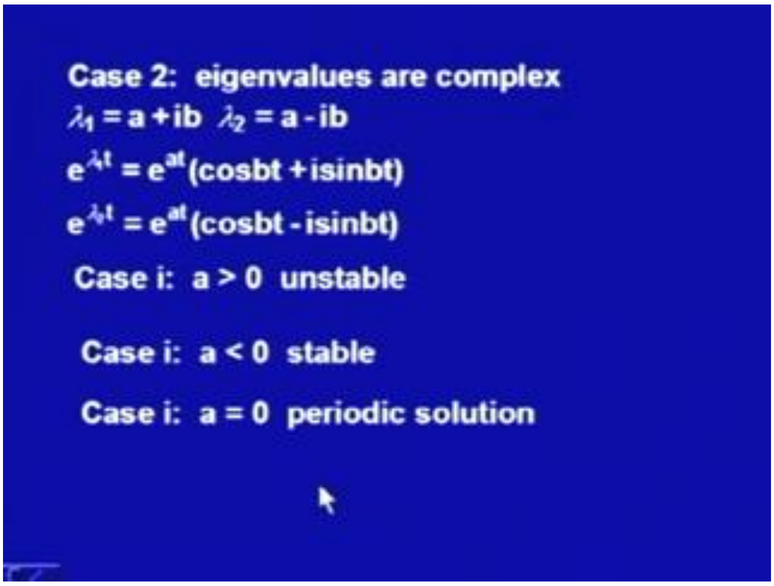

In the next case, we consider the Eigen values are complex, when the Eigen values are complex, we can write an Eigen value lambda 1 as a plus i b. And since, the complex Eigen roots of a characteristic equation always occur in pair, so the second lambda 2 will be a minus i b it is the complex conjugate of lambda 1. And if we have this lambda 1 and lambda 2 then e lambda 1 t is written as e raise to power a t plus i b t and i b t is written as cos b t plus i sin b t.

Similarly, corresponding to lambda 2 we can write down e lambda 2 t as e a t multiplied by cos b t minus i sin b t. Now, in this case solution will depend what is the sign of a, if a is positive then this value will remain bounded because, they are sin, cosine function, so they there values cannot increase beyond 1. So, 1 or minus 1, so this value this remains bounded, so as t tends to infinity this may go to infinity or may tend to 0 depending upon whether a is positive or negative.

So, in the case when the a is positive, then we will always go away from 0 because of this positive exponent and in that case we say the solution is unstable. However, in the case 2 when a is negative, then this term will always go closer and closer to 0 and this remains bounded. So, the total term will be going closer to 0 and we say that the solution is unstable, in the third case when a is equal to 0, then we will have a periodic solution what we can do is that when a is equal to 0, then first part will be cos b t plus i sin b t the second part will be cos b t minus i sin b t and the solution will be linear combination of these two. And this linear combination can be solved and one can find out the trajectory, which is periodic, so when a is equal to 0 we will have a periodic solution.

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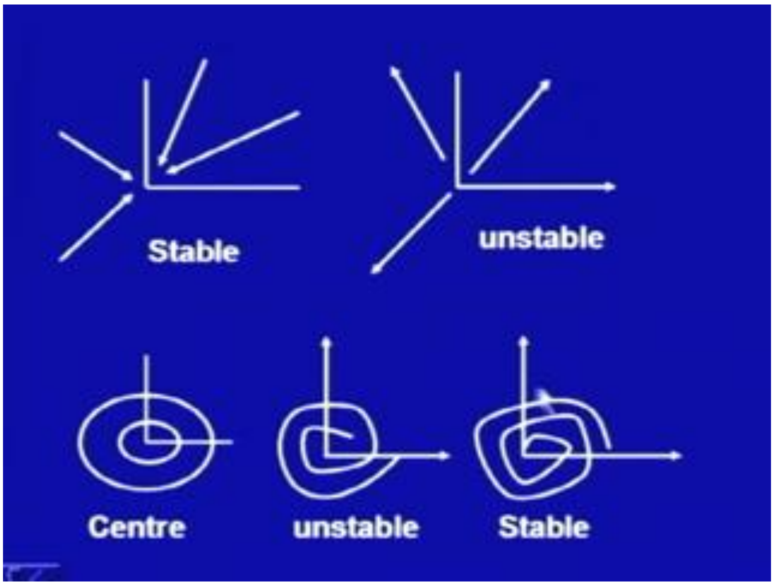

And what we are doing here is, that this is the phase space in fact, the phase plane because, this two dimension this is x axis and let us see this is y axis. And when both the lambdas are positive, they the solutions will 10 to 0 from all the sides whatever be the value of lambda 1 will always tend to 0, if they are negative. So, we say the solution is stable, but when both the lambdas are negative, then we will go away and away from the solution and we say solution is unstable.

So, this is the behavior of the solution whether we are going towards the equilibrium point or we are going away from equilibrium point. We are going closer to the equilibrium point we say the solution is stable, when we are going away from it is unstable So, these are cases when lambda 1 and lambda 2 are real and negative, this is the case when lambda 1, lambda 2 are real and positive

In all these cases I am considering that the two values of lambda 1 and lambda 2 are distinct. But, if it is complex value and the real part is 0, then we will always have a close trajectory and close trajectory means the solution is periodic, we start from this point and will always come back to that point and this describes a periodic nature of the solution. And whether we will be moving on this curve or we will be moving on this curve that depends upon the initial distribution or initial condition specified with the dynamical system.

This is the case, when the solution is unstable and this is the case when we have complex Eigen values. And if the real part is positive, then if we start from this equilibrium this position, then we will go away and away from the 0 equilibrium point and we say the solution is unstable. So, when lambda 1 and lambda 2 are complex and real part is positive, then this is the direction in which we will be moving on the phase space and we say the solution is unstable.

But, if we are having a the real part of the complex root as negative, then if we start from this initial condition, then we will move towards the 0, 0 equilibrium point in this manner and we say the solution is stable. So, from wherever we start we ultimately tend to this 0, 0 equilibrium point and such a point is called stable equilibrium point, so that is how we describe the behavior of the solution on the basis of Eigen values and Eigen vectors of the given dynamical system.

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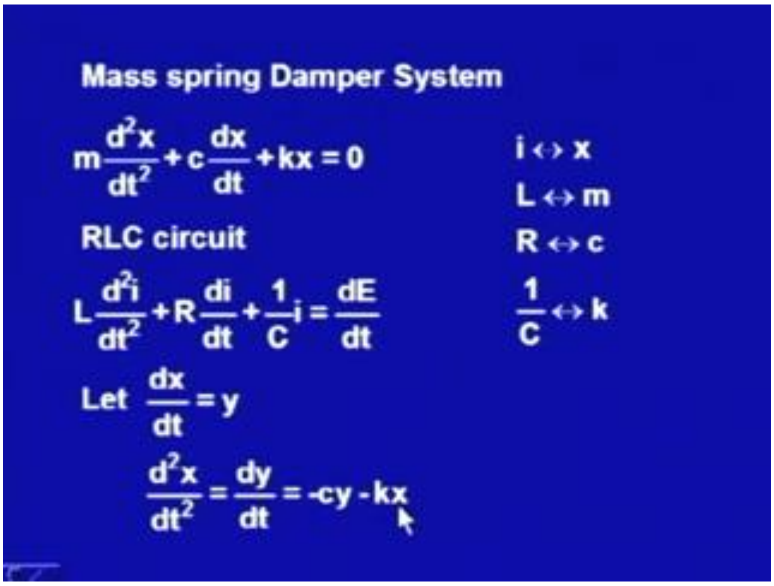

So, we illustrate this for some simple physical problems for this purpose I will start with mass spring damper system. So, we have a simple system consisting of a mass m, which is attached to a spring with spring constant k and a damper is present in the system. In which the damping is there, which is proportional to the velocity d x by d t and c is a damping coefficient. So, we will have m d 2 x d t square plus c d x by d t plus k x is equal to 0, so this second order differential equation represents mass spring damper system.

A similar type of second order equation can be obtained in RLC circuits, where L is the Inductance, R is the Resistance and C is the Capacitance and E is the External EMF applied in the circuit. So, we write down the differential equation as L d 2 i d t square plus R d i by d t plus 1 over C i is equal to d E by d t i being the current in the circuit. Now, if you compare these two equations then one can easily see that these two systems are similar, when i is related to x, x is the displacement in this and i is current in this.

So, the two systems are identical i is same as x, the L inductance is identified with m in this system, R the resistance in this RLC circuit is identified by c in this mass spring damper system. And one over C, C being the capacitance and this is identified as k in mass spring damper system, in this I mean to say that whatever be the effect of k in mass spring damper system, the same effect is observed by 1 over C in RLC circuit.

Whatever effect c is having here same effect R is having in the RLC circuit, to solve this system we first introduce d x by d t is equal to y and this substitution will reduce this second order equation into two first order differential equations. So, we write down d x by d t is equal to y here and d 2 x by d t square will come out to be minus c y minus k x.

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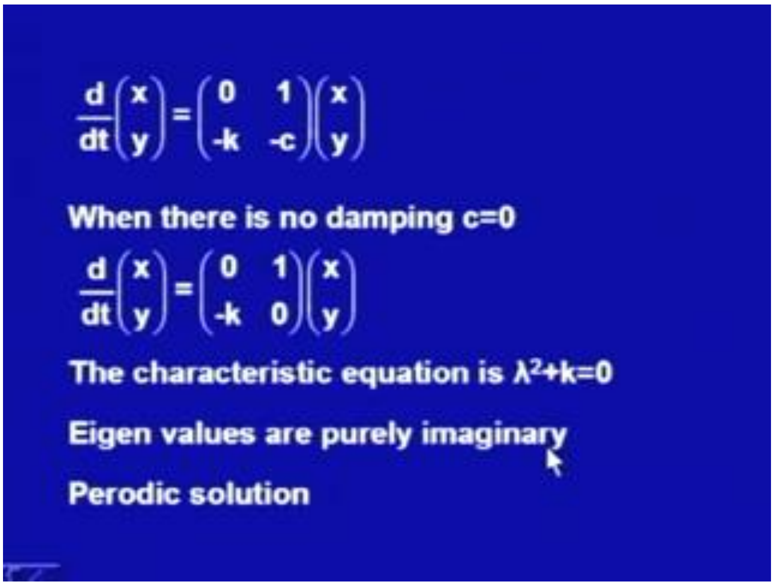

And that makes our system as d by d t x y is equal to 0 1 minus k minus c multiplied by x y. So, a second order system can be reduced to two first order differential equations. When there is no damping present; that means, c is equal to 0, then we have d by d t of x y is equal to 0 1 minus k 0 x and y. If you have to solve this system we say the characteristic equation is lambda square plus k is equal to 0 and from here one can find out that Eigen values are purely imaginary.

And the on the basis of our discussion we know that this solution is periodic solution as the values are purely imaginary. So, we can solve and we can get the solution of differential equation with the help of Eigen values and Eigen vectors. So, this is the end of this lecture. And in this lecture we have started with some diagonalization of complex matrices. Then we have discussed how we can solve the ordinary differential equations, how we can solve system of ordinary differential equations. What is the role of Eigen values and Eigen vectors in predicting the behavior of solution of such systems.

Thank you.