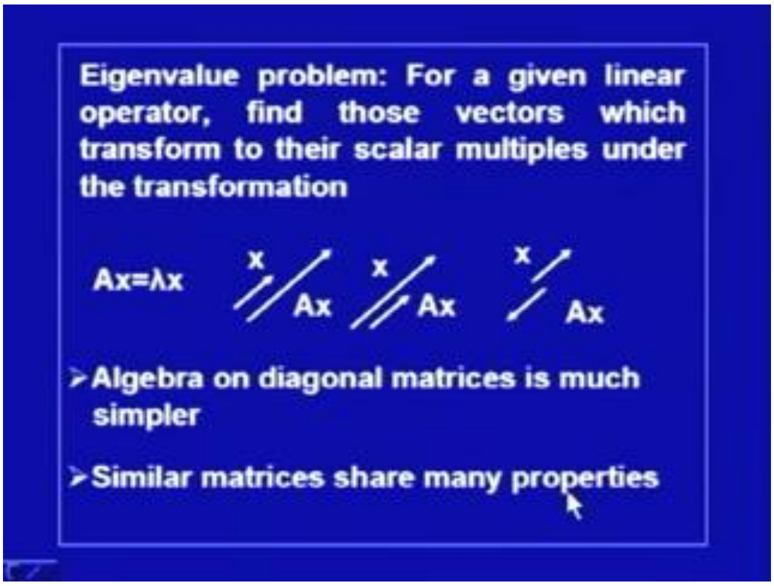
Mathematics-II
Prof. Sunita Gakkhar
Department of Mathematics
Indian Institute of Technology, Roorkee

Module - 2
Lecture - 16
Diagonalization Parts-1

Welcome viewers. Today's topic is that Diagonalization of square matrices. To start this topic, I will first like to review, what we have done in the Eigen value problem.

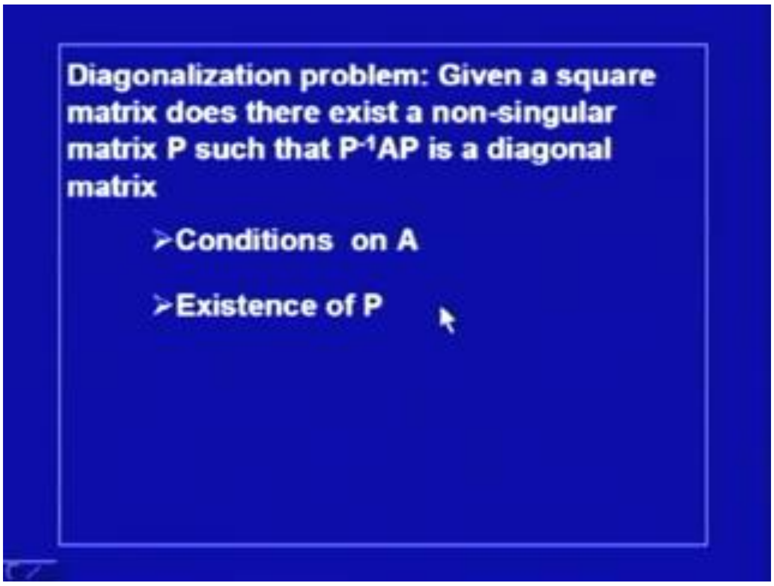
(Refer Slide Time: 00:39)



The Eigen value will be, that for the given linear operator. Find those vectors, which transform to their scalar multiples under the transformation, A x is equal to lambda x; where A is square matrix associated with the linear operator. X is called an Eigen vector and lambda is the associated Eigen value. By this I mean, that if a vector x is given. That under this transformation, it will become A x.

In such a way, that the vector, we transformation vector x is parallel to x. But, it is scale has changed. If the scaling factor lambda is bigger than 1, then it is magnified. And if scaling factor is lambda is less than 1, then x will be contracted. And then lambda is negative the direction will be changed. We have a performed various manipulations on Eigen value problem. And we have seen that, the algebra on diagonal matrices is much simpler. And similar matrices share many properties. This is also been established in one of my earlier lectures.

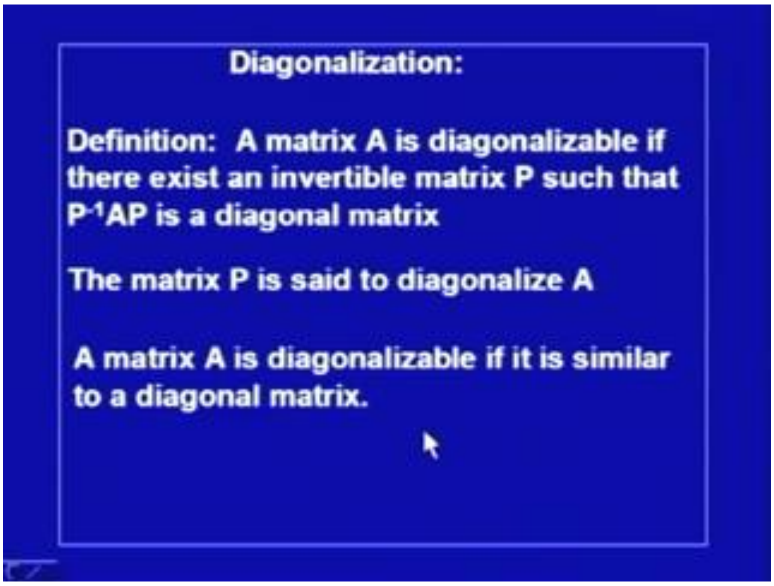
(Refer Slide Time: 02:01)



Given a square matrix, does there exist a nonsingular matrix P, such that P inverse A P is a diagonal matrix. But, this I mean to say, that if I have a square matrix A, can I find a matrix P, so that it is similar to of diagonal matrix. And the matrix is obtained as P inverse A P. We have seen that P inverse A P is equal to is similar to A. And this will be a diagonal matrix. And that we can easily perform variety of algebraic operations on the diagonal matrices.

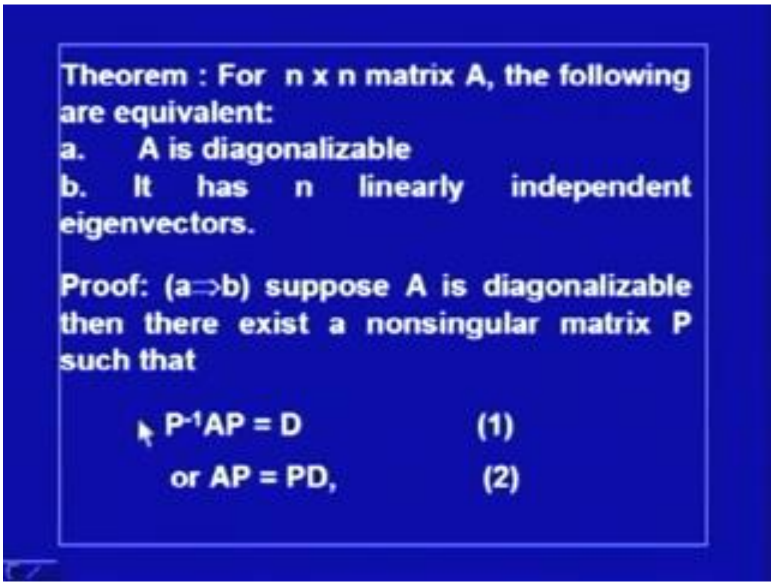
So, if we can perform operations on the diagonal matrices. We can perform operations on the square matrix A. So that is the purpose of diagonalization of a square matrix. So, we want to find out, under what conditions the matrix A can be diagonalized. That is under, what condition it is possible to find P. And if it is possible to find P, then what is P. So, first we have to establish the existence of P. And then we have to find P. So that, these are the issues, which are related with diagonalization problem.

(Refer Slide Time: 03:18)



So, I will first give the definition. A matrix A is diagonalizable, if there exists an invertible matrix P, such that P inverse A P is a diagonal matrix. And in such a case, we say that the matrix P is said to diagonalize A. A matrix A is diagonalizable, if it is similar to a diagonal matrix. We have already seen that, if there exists a matrix like P inverse A P. Then, A and P inverse A P, they are similar. So, a matrix A is diagonalizable, simply means that, it is similar to a diagonal matrix.

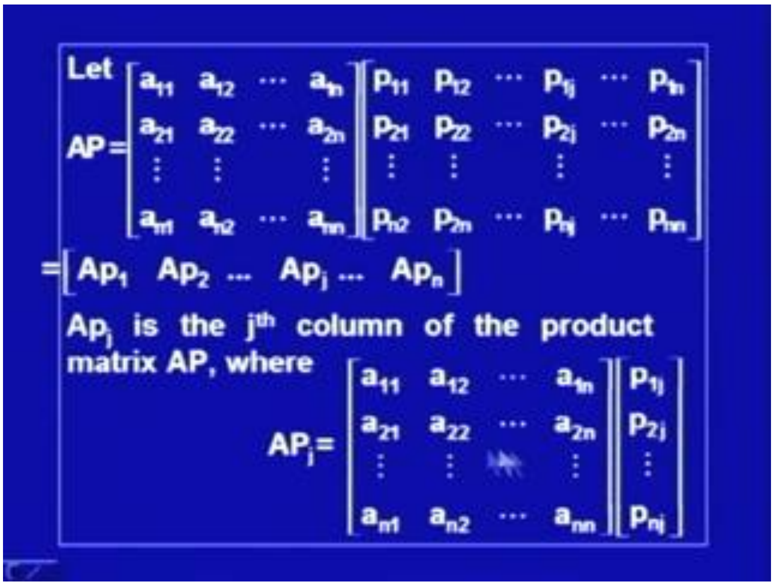
(Refer Slide Time: 04:04)



Now, for this theorem will be helpful. So, let me first state the theorem. For a square matrix A, the following are equivalent. That A is diagonalizable and b is that, it has n linearly independent Eigen vectors. Now, to prove this theorem, I will first prove that if A is diagonalizable. Then, it has n linear independent Eigen vectors. But, since this theorem suggests that, these two statements are equivalent. So, I have also to prove that, it has n linearly independent Eigen vector means, that A is diagonalizable.

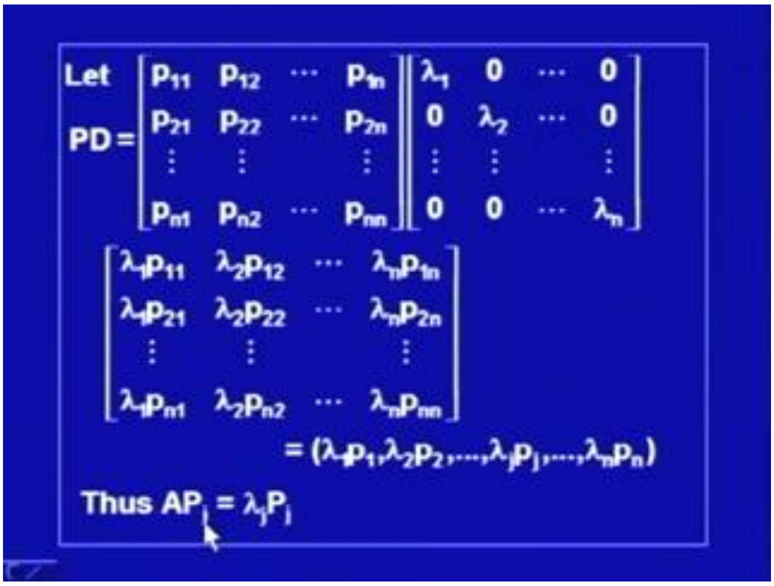
So, first I will prove the forward part, that is suppose A is diagonalizable. And then there exist some non-singular matrix P, such that P inverse A P is equal to D. Now, P has to be invertible, because we need P inverse. So, A has to be diagonalizable means, P inverse A P is equal to a diagonal matrix D. And, we can write down this equation in an alternative form, A P is equal to P D. And this can be obtained by pre-multiplying equation 1 by P.

(Refer Slide Time: 05:27)



Now, let us calculate A P, so if A is this matrix n by n matrix and P is this square matrix. Then, let us try to find out the product matrix. Now, I write down this product matrix as this row vector, in which A P 1 is the product of A matrix with the column P 1. So, this I call as P 1, so first element in this row is A p 1. This matrix multiplied by this column. The second element will be these matrices multiplied by this column. And this is the j th entry, in this product and A P n is the last entry. Then, A P j is a j th column of the product matrix A P; where A P j this is I have already explained.

(Refer Slide Time: 06:25)



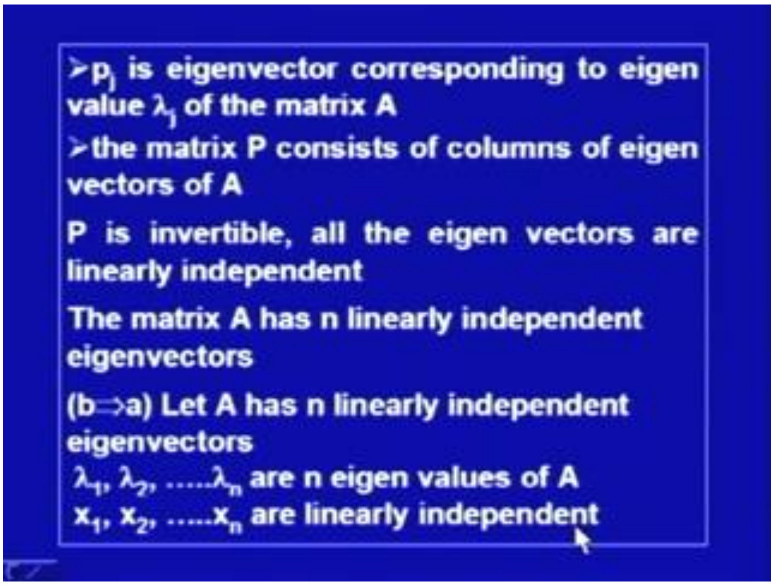
Now, considering the right hand side P D is equal to the matrix P, multiplied by the diagonal matrix. Now, this matrix P is already been explained. And this diagonal matrix will have Eigen values in the diagonal. So, in the matrix is diagonalizable means, that we have a matrix A is equal to P inverse P D; where D is a diagonal matrix consisting of Eigen values. So, if we have Eigen values lambda 1, lambda 2, lambda n for the given matrix A. Then, D will be given by this diagonal matrix.

And then I can perform the multiplication, it is P 11 and lambda 1, first element. Then, we will have P 12, lambda 2, this is a second element. And then we will have lambda n P 1 n, this is will be obtained, when this is multiplied by this column. Similarly, P 21 multiplied by this is this element and P 2. This row multiplied by the second column, will be this element and so on. So, this is the product of these two matrices.

Then, we can rewrite this matrix as a row matrix, where we will have lambda 1. P 1 is the first element, lambda 2, P 2 is a second element and so on. What is P 1, P is actually a column vector. This, so P 1 is nothing but first column vector. P 2 nothing but the second column vector and P n is the nth column vector of this matrix. So, we have simplified P D in this form.

And now, if we compare, what we had for the earlier expression. Then, we say that A P j, which we have compute in the last slide is equal to lambda j P j. So, this is lambda j P j; and what we have obtained earlier is A P j.

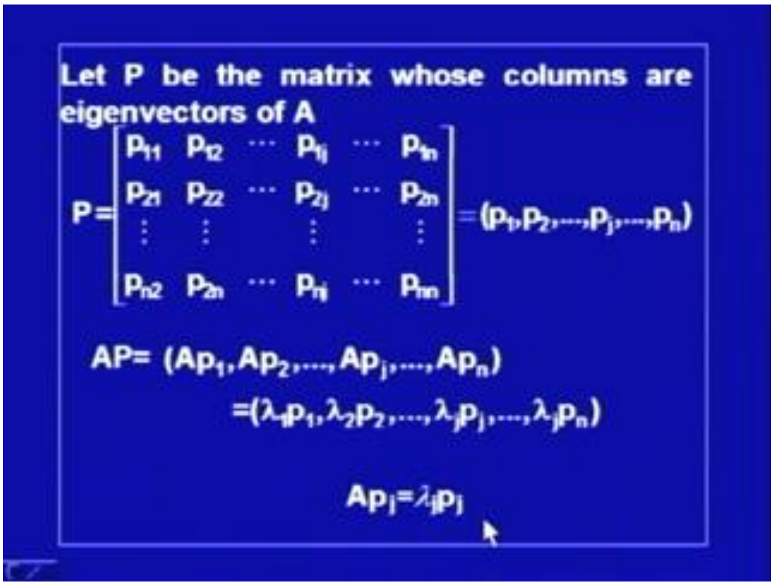
(Refer Slide Time: 08:35)



Then, we can say that, P j is the Eigen vector corresponding to Eigen value lambda j of the matrix A. And the matrix P consist of column. So, Eigenvector of A, that is what we have seen. And since, P is invertible all the Eigen vectors are linearly independent, this result which we have established earlier. So, the matrix A has n linearly independent Eigen vectors. And if they are linearly independent, then we have proved to the result.

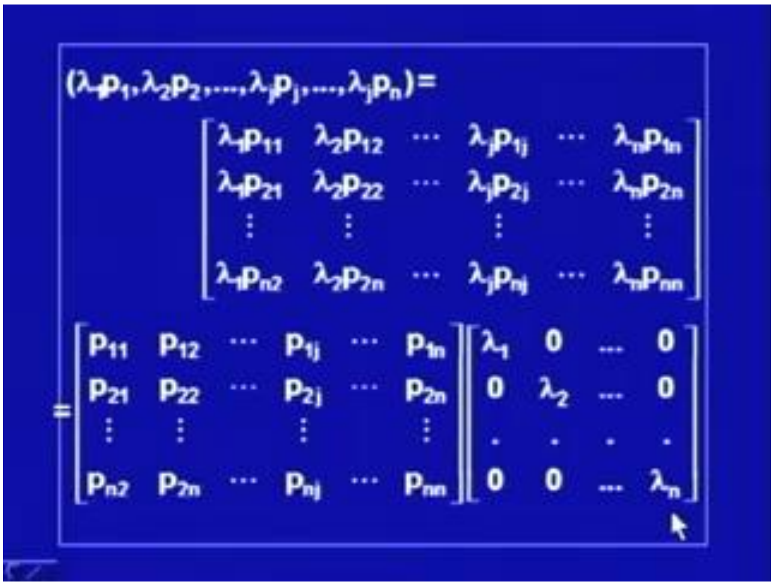
So, if we have a diagonalizable, if A is diagonalizable. And then it will have n linearly independent vectors. Now, if it has the inverse part is that. If it has n linearly independent vectors, then it is diagonalizable. So, let us prove the reverse part of the theorem. So, we have, let A has n linearly independent Eigen vectors. Then this means, if lambda 1, lambda 2, lambda n are n Eigen vectors of Eigen values of A. Then, x 1, x 2, x n are linearly independent, that is being given to us.

(Refer Slide Time: 09:50)



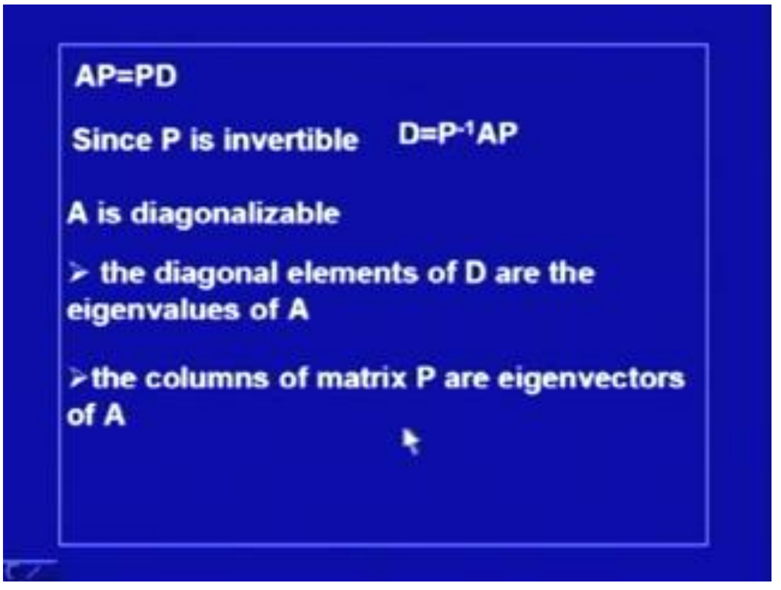
And let P be the matrix, which columns are Eigen vectors of A, so this the matrix. Let us form the matrix P with Eigen vectors. And they are linearly independent, so this is the first column P 1, then the second column P 2 and so on. Now, A P will be A P 1, A P 2, A P j, A P n, this we have already explain. And now A P 1 is equal to lambda 1, P 1, A P 2 is equal to lambda 2, P 2 and so on. So, we can say that A P j is equal to lambda j P j.

(Refer Slide Time: 10:31)



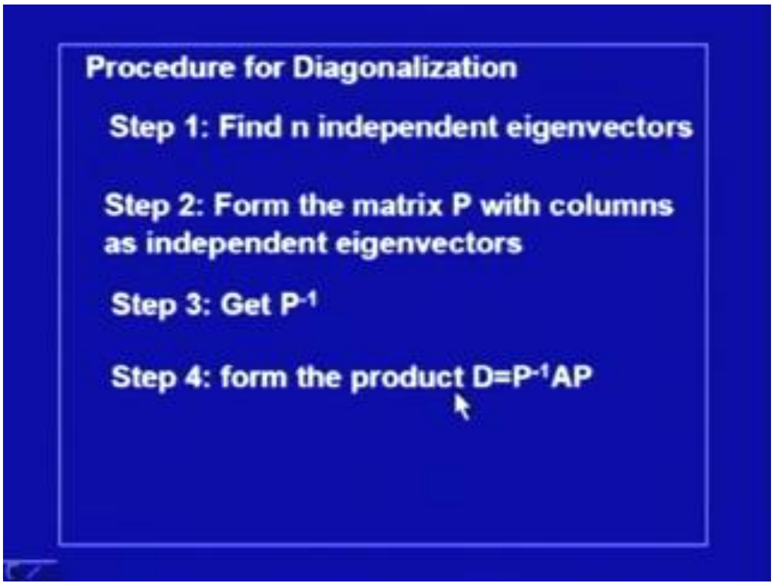
And from here, we can say lambda 1, P 1 lambda 2, P 2 lambda j, lambda P j. And lambda j P n is equal to this matrix, this we have already seen. So, it is nothing but P 11, P 21, P n j, the first column P 1to P 2 to P 2. And the second column and this is multiplied by this matrix. So, actually we are steps, what we have done in the forward part. So, this product is equal to P D.

(Refer Slide Time: 11:08)



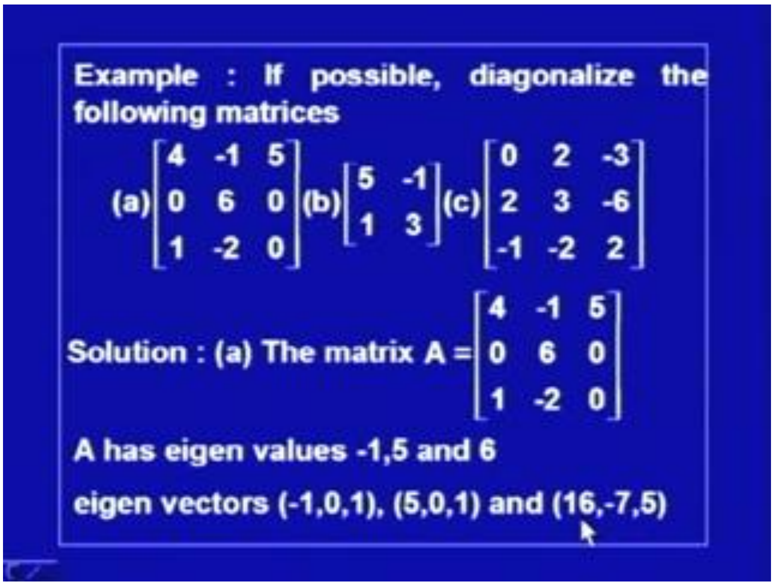
So, we have A P is equal to P D. And from the here, since P is invertible. We can write down D is equal to P inverse A P. So, we have started with n independent Eigen vectors. And we could prove that the matrix A is similar to D. And that proves that, A is a diagonalizable. Now, the diagonal elements of D are the Eigen values of A, this we have already observed. And the columns of matrix P are Eigen vectors of A. So, we know to how to form the matrix P and this matrix P is invertible, because it forms and columns of n independent vectors.

(Refer Slide Time: 11:53)



So, this forms a basis for finding diagonalizing a given matrix. So, let us discuss the procedure for diagonalization. So, the first step is that, you have to find n independent Eigen vectors for a given matrix A. And then in the second step, we form the matrix P with columns as independent Eigen vectors. Then, third step is calculating P inverse. And forth step is form the product P inverse A P. And this will be the diagonal matrix D.

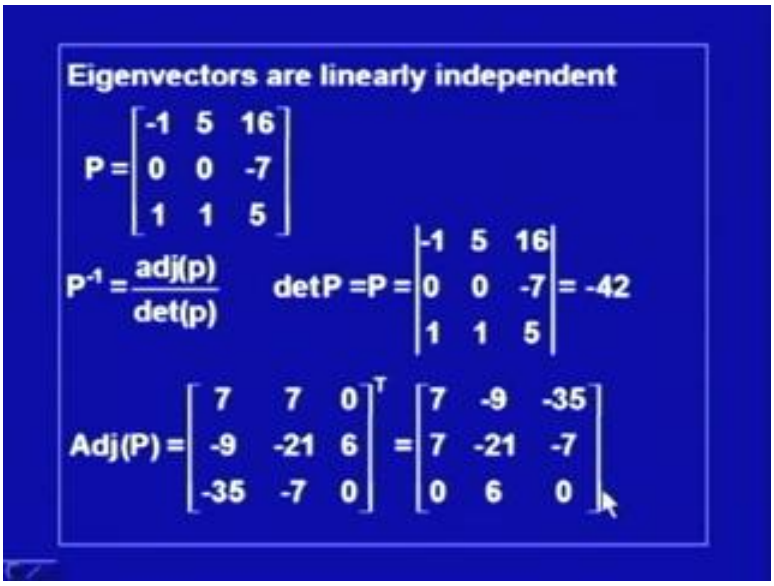
(Refer Slide Time: 12:36)



So, we apply this procedure to diagonalize, the matrices the square matrices. So, I will start with this example, first we consider a 3 by 3 matrix, 4 minus 1, 5, 0, 6, 0, 1 minus 2 and 0. So, let us consider this square matrix. One can find out that, the Eigen values of this matrix is minus 1, 5 and 6. And actually, this problem was solved earlier, where we have obtained Eigen values of this matrix. And we have obtained the Eigen vectors of this matrix.

Either, you go through the lecture. Or you can take it as an exercise to compute the Eigen values and corresponding Eigen vectors. For this example, minus 1, 5 and 6 are the Eigen values. And the corresponding Eigen vectors are minus 1, 0 and 1 for lambda is equal to minus 1. For lambda is equal to 5 Eigen vectors are computed as 5, 0 and 1. And for 6 Eigen vectors are computed as 16 minus 7 and 5.

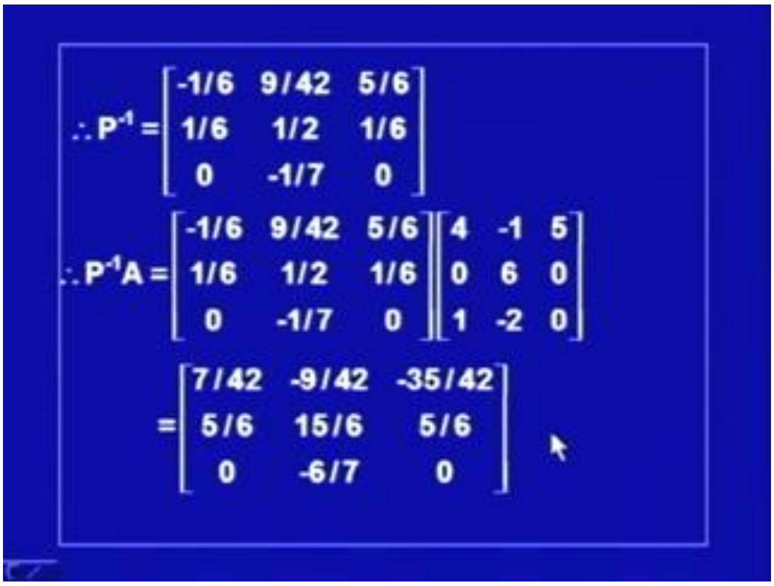
(Refer Slide Time: 13:46)



With these Eigen vectors, which are linearly independent. One can form the matrix P. So, this is the first Eigen vector, forms a first column. The second Eigenvector form the second column. Third Eigen vector forms the third column. So, that is my matrix. Now, since this matrix is invertible consisting of independent, linearly independent Eigen vectors. So, we have to find out and it is inverse adjoint P divided by determinant P.

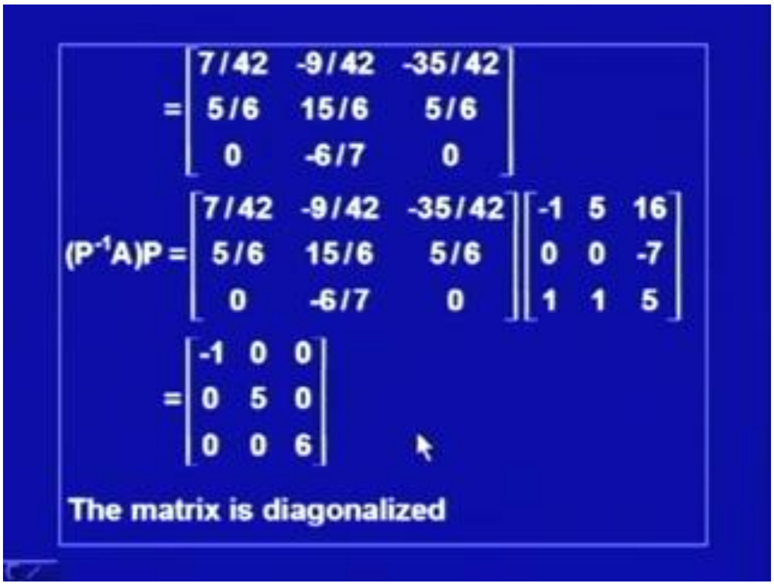
So, let me first calculate determinant P and this comes to be this determinant, which is computed as minus 42 and we calculate the adjoint P, and this matrix of sub determinants, we take the transpose and this can be easily verified that this is adjoint of P and once we get this adjoint. We can write down P inverse as this matrix.

(Refer Slide Time: 14:40)



So, once we have P inverse, we can calculate P inverse A. So, this is my P inverse and this is a given matrix A, perform this multiplication. And this comes out to be this matrix, one can check different terms here.

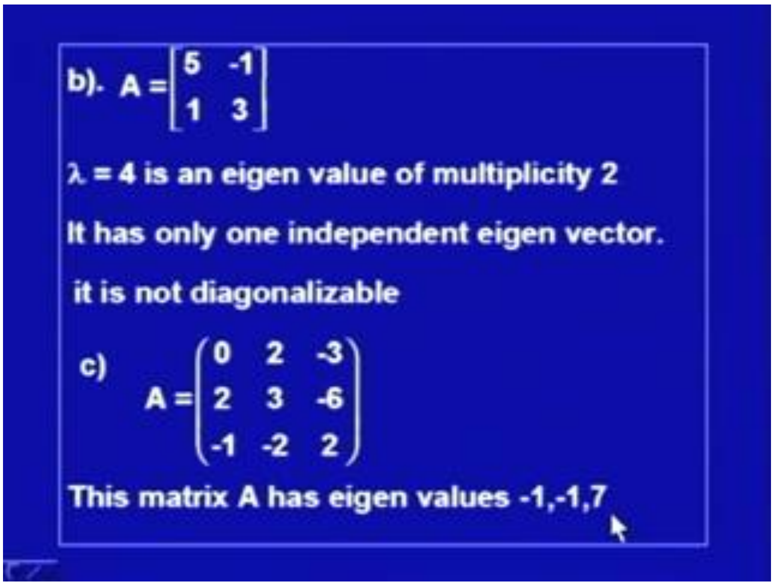
(Refer Slide Time: 15:07)



After that, given this P inverse A, one can write down P inverse A P. That means, multiply this matrix by the matrix. So, this is a matrix P, which I had obtained. So, this multiplied by this, when you perform this multiplication. This comes out to be this matrix and one can notice that, this minus 1, 5 and 6. These are the diagonal elements, which are actually the Eigen values of this given matrix A.

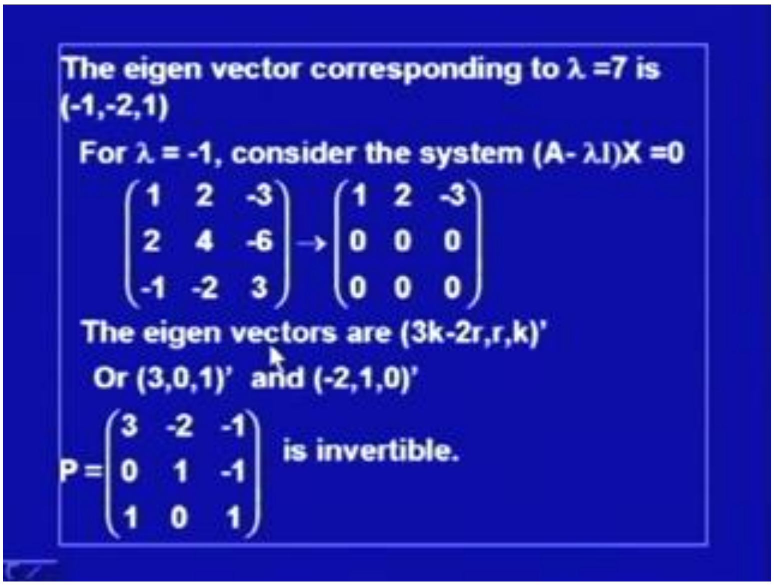
One can check, if you first perform this row multiplied by this. We will get minus 1 and when this is multiplied by this. One can check that this comes out to be 0 and so on. So, P inverse A P is a diagonal matrix; and we say the matrix is diagonalized, but if you consider the second problem, consisting of a 2 by 2 matrix.

(Refer Slide Time: 16:04)



Then, we had already observed, that lambda is equal to 4 is an Eigen value of multiplicity 2. This I have done in one of my earlier lectures. All you can take it again as an exercise. And check that lambda is equal to 4 is an Eigen value of multiplicity 2. But, we have seen that, it has only one independent Eigenvector. And this means that, it is not diagonalizable. We cannot find a matrix P, which is not invertible, so it is not diagonalizable. Now, this is another example a 3 by 3 matrix. This matrix is Eigen value minus 1, minus 1 and 7, one can take this is an exercise to check.

(Refer Slide Time: 17:05)



Then, lambda is equal to 7 is the Eigen vector corresponding lambda is equal to 7 is minus 1 minus 2, 1. Again this may be an exercise for you. For lambda is equal to minus 1. We consider system A minus lambda I X is equal to 0. And the system is 1, 2 minus 3, 2, 4 minus 6, minus 1 minus 2, 3. That is the matrix A minus lambda I, we apply linear transformation on these matrix. And, we find that this row and this row. They are proportional not only this row and this row they are proportional.

So, there is only one independent row. And the two rows can be transform to 0 rows by using appropriate elementary transformations. And these suggest that this matrix has only nullity T and rank 1. That means, we can find two independent Eigen vectors corresponding to lambda is equal to minus 1. So, this problem is different and what we had in the earlier problem.

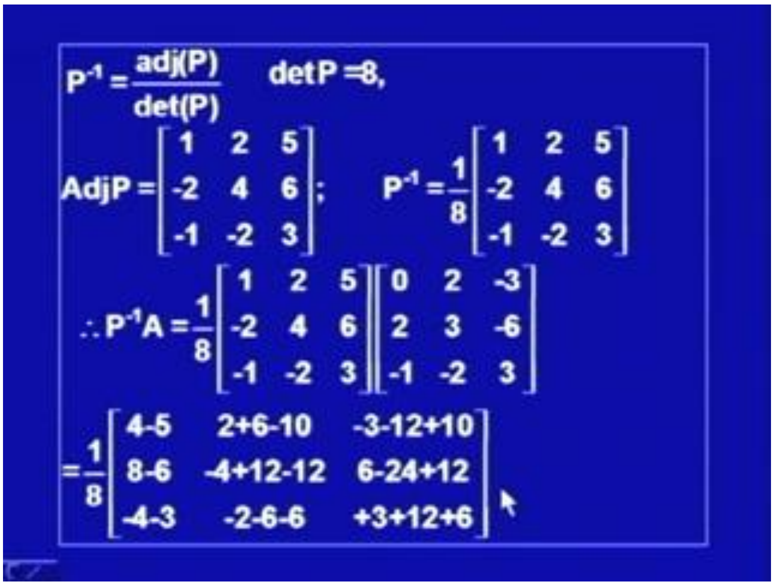
And the earlier problem, we had a root of multiplicity 2. But, we could find only one independent Eigenvector, but in this case the multiplicity is 2. And we can find two independent Eigen vectors. Now, these two independent Eigen vectors can be obtained as 3 k minus 2 r, r and k. How we can assign x by and z arbitrary values of r and k? And then if you substitute this as y r, y is equal to r and z is equal to k in this equation. Then, 3 k minus 2 r comma r comma k, we will satisfy this multiplied by x equal to 0.

So, the Eigen value corresponding lambda is equal to minus 1, will be given by this. And now, we can assign independent values to r and k. So, if I write down r is equal to 0.

Then, Eigen value will be 3, 0, 1 and when I independently assign 0 to the arbitrary constant k. Then, minus 2, 1, 0 will be an Eigen vector. And since, I can assign, these two values independent of each other. So, we can say that, these two are independent Eigen vectors.

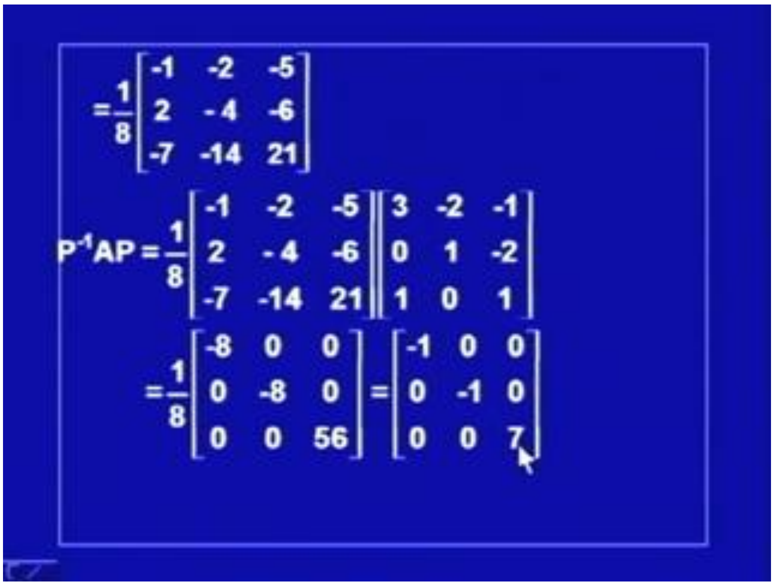
So, corresponding to lambda is equal to minus 1. We could obtain two independent Eigen vectors. So, we have three Eigen values lambda is equal to 7, lambda is equal to minus 1. And lambda is equal to minus 1. And we could get three independent Eigen vectors. So, the matrix P will now be 3, 0, 1 cos. This minus 2, 1, 0 corresponding to lambda is equal to minus 1. And the third one is minus 1 minus 2, 1. So, this matrix P is invertible.

(Refer Slide Time: 20:22)



Now, we can obtain is determinant as 8. And this value can be used here, to get the adjoint P. And from where can get P inverse. So, adjoint P is calculated as this and P inverse comes out to be 1 by 8 of this matrix. We have obtain P, we have obtain P inverse, Let us check whether, they give rise to Eigen. They give rise to a diagonal matrix. So, for this purpose, we compute P inverse A. So, this is my P inverse, which I had computed here and this is given A. So, I compute P inverse A as the product of these two matrices, which can be computed here. And the details are given here.

(Refer Slide Time: 21:19)

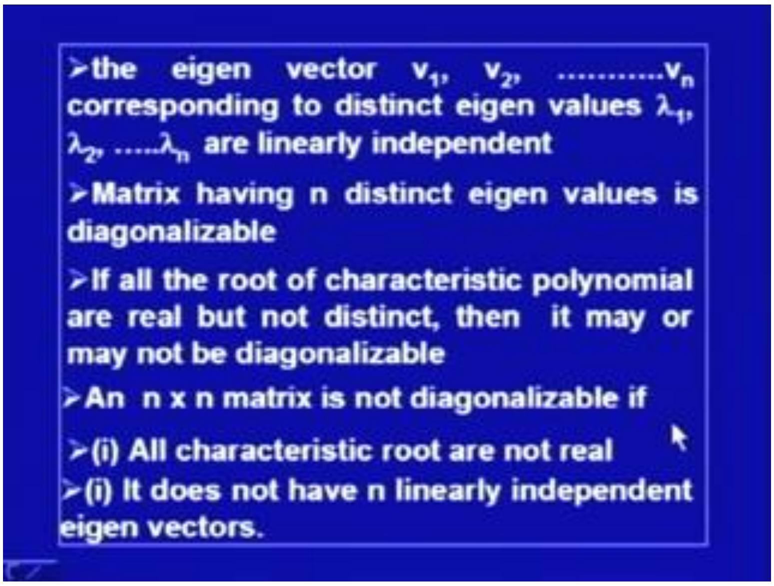


And finally, P inverse A comes out to be this 3 by 3 matrix. After computing P inverse A, we can multiply it, by P to get P inverse A P. And this calculation can be done here. And if you multiply this row by this column, we will get minus 8. But, when you multiply this row by this, it is 0. This multiplied by this is again 0. So, this way, we can perform, you can calculate individual elements of this matrix. And, this comes to be this and you can simplify as minus 1 minus 1, 7 in the diagonal. And this is a diagonal matrix.

So, we have started with a matrix A. We have obtained the matrix P. Then, the next step, we have obtained P inverse. And when we multiply these two, we find the diagonal matrix this and mind here. That minus 1 minus 1 are repeated roots of the given matrix. And 7 is the another Eigen value of the matrix. So that this diagonal matrix constitute the Eigen values of the given matrix A.

So, in this example, although we have repeated roots, but still the matrix P can be obtained, it is invertible. And the matrix A can be diagonalized. So, we have three examples. In the first example, we have all distinct roots. In the second example, we have two repeated roots, but the matrix cannot be diagonalized. And in the third example, we have repeated roots, but still the matrix can be diagonalized.

(Refer Slide Time: 23:11)

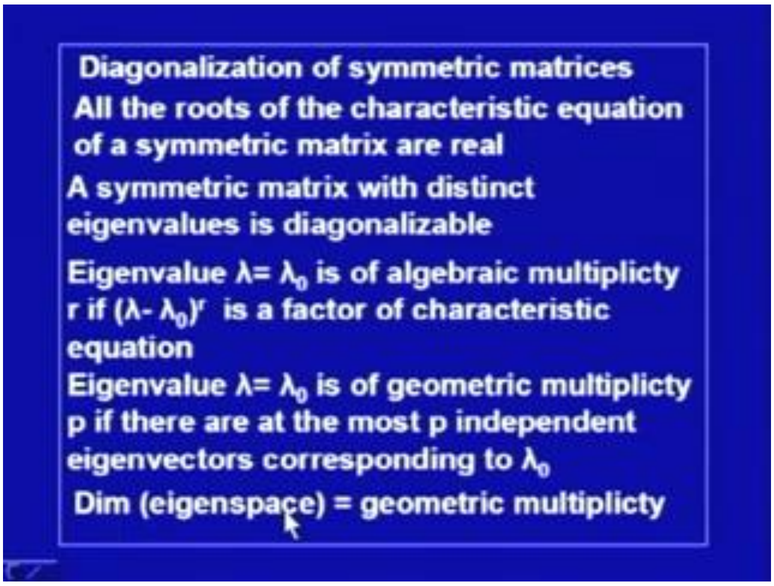


So, we can say that, the Eigen vectors v 1, v 2, v n corresponding to distinct Eigen values. Lambda 1, lambda 2, lambda and lambda n are linearly independent. This is the result which we, I have already discussed in my earlier lecture on Eigen values and Eigen vectors. So, this we have confirmed here also. And then matrix having n distinct Eigen value is diagonalizable.

So, if a matrix has distinct Eigen values. Then, we can find distinct and independent linearly independent Eigen vectors. And the matrix can be diagonalized, but if the roots of the characteristic polynomial are real, but not distinct. Then, we may be able to diagonalize it or we may not. Like in one example, early in one example, I have shown that. It cannot be diagonalized, while in another example, the matrix can be diagonalized.

Now, we say that n by n matrix is not diagonalizable. If all characteristic roots are not real, so if we have complex roots. Then of course, the matrix cannot be diagonalized. But, if they are real and distinct, it can be diagonalized. And the second is, if it does not have n linearly independent Eigen vectors. Then, again it is not diagonalizable. Now, the theorem, which we have discussed that, if useful in diagonalizing the matrix.

(Refer Slide Time: 24:48)



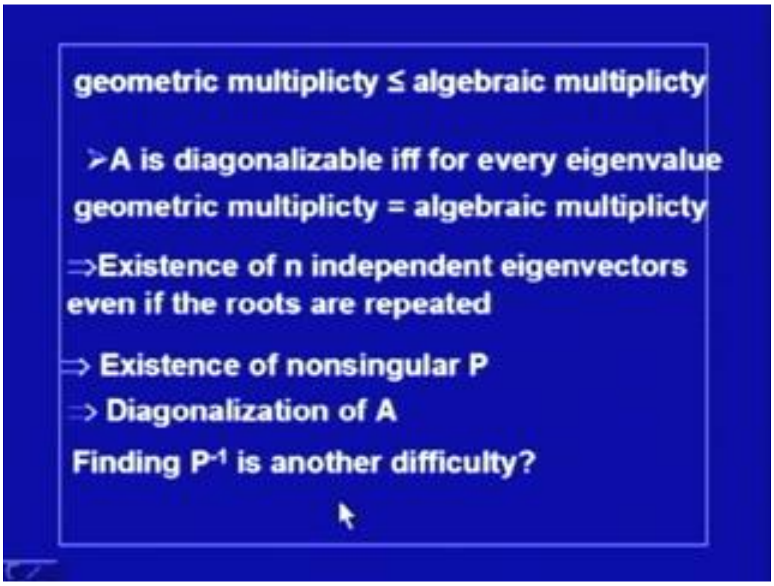
Now, we talk about diagonalization of symmetric matrices. Now, this is important, because we have already seen that all the roots of characteristic equation of symmetric matrix are real. So, this is one condition for diagonalization of a matrix. All the roots should be real, then they are distinct, then it is diagonalizable. So, if all the roots of the characteristic equation of a symmetric matrix are real.

So, matrix symmetric matrix can be diagonalized. Further a symmetric matrix with distinct Eigen value is diagonalizable. So, symmetric matrix is one category of matrices which can be diagonalized. Now, we give some definitions. If lambda is equal to lambda naught is an Eigen value, then we say it is of algebraic multiplicity r. If lambda minus lambda naught raise to power r, is a factor in the characteristic equation.

So, like in the examples, which we have taken the root is appearing twice, lambda plus 1 raise to power 2. So, we say multiplicity of the root is 2. Apart from this, we associate geometric multiplicity of the root lambda is equal to lambda naught. And we say that, P is the geometric multiplicity of the root. If there are at most p independent Eigen vectors, corresponding to lambda naught.

Like in the first example, multiplicity was 2. But, geometric multiplicity comes to be one. But, in the second example, algebraic multiplicity is 2, also that geometric multiplicity is also 2. And in that case, we have two independent Eigen vectors. And that means, the matrix can be diagonalized. Now, this is related with the dimension of Eigen space. So, we say that the geometric multiplicity is the same as dimension of the Eigen space.

(Refer Slide Time: 26:56)



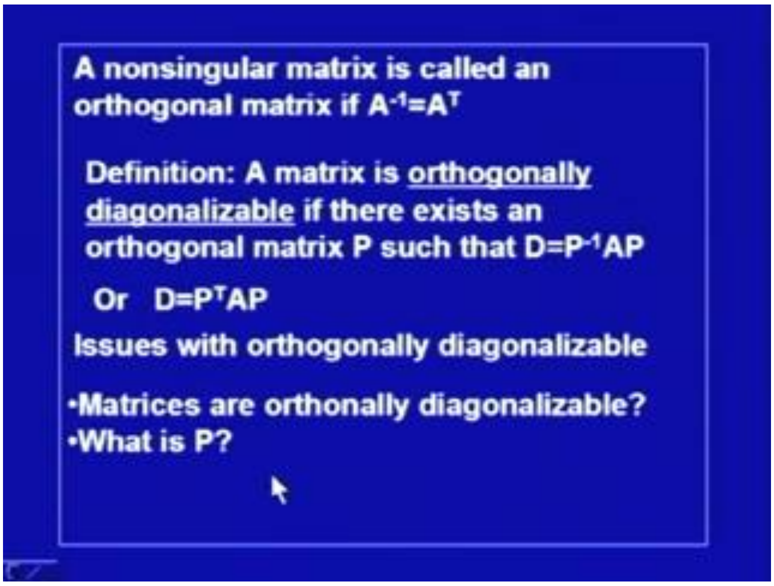
So, we can easily check, that geometric multiplicity is always less than algebraic multiplicity. The number of independent roots will always be less than the multiplicity. It can a most be equal to the algebraic multiplicity, but cannot exceed this. And in the case, if A is diagonalizable, if and only. For every Eigen value geometric multiplicity is equal to algebraic multiplicity. If this is true for each and every Eigen value, only then A is diagonalizable.

So, this is a condition on A, this suggest whether a matrix is diagonalizable or not. Because, this condition ensures, that existence of n independent Eigen vectors, even if the roots are repeated. So, we do not have to worry about whether we can get in the independent vectors or not. We simply check, what a geometric multiplicity is. And if a geometric multiplicity is each root is a same as algebraic multiplicity.

Then, existence of n, independent Eigen vectors are possible. And we can say the matrix can be diagonalized, because in that the existence of nonsingular matrix P is possible. And that leads to diagonalization of the matrix A. So, if we can find P, if you can find n independent vectors, we can find P. And once we can find P, the matrix is diagonalizable. But, still there is a difficulty.

Although, theoretically P invertible, so P inverse is possible. And once, we can find P inverse. Then, P inverse A P will be diagonalized. But many times, finding P inverse is not simple; lot of computation effort is required in finding P inverse.

(Refer Slide Time: 28:51)

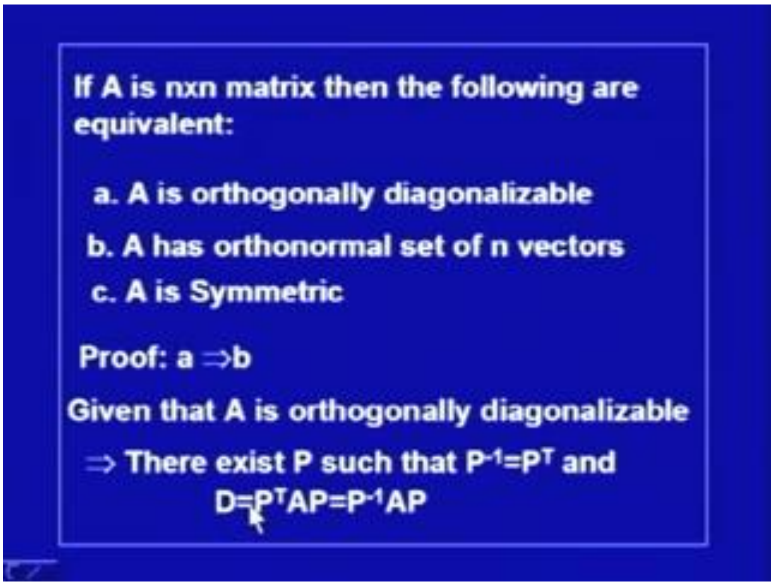


For this purpose, we introduce another concept that is called orthogonally diagonalizable. So, we say a matrix is orthogonally diagonalizable, if there exist an orthogonal matrix P. Such that, D is equal to P inverse A P. Now, it is first thing is that, the matrix A is diagonalizable. So, there has to have to some P and P inverse. So that, D becomes P inverse A P, but the matrix is orthogonally diagonalizable. Means there exist an orthogonal matrix, which is satisfying this property, that A inverse is A transpose.

And if A inverse is equal to A transpose. We can straight away write D is equal to P transpose A P. So, we do not have to find out P inverse. In that case finding transpose is simple as compare to finding inverse. So, this is simpler than what we had earlier. So, this is the purpose of defining orthogonally diagonalizable matrices. So, we define a matrix is orthogonally diagonalizable. If we can have a matrix P, such that D is equal to P transpose A P.

Now, there are many issues with orthogonally diagonalizable matrices. The issues are, what the matrices; which can be orthogonally diagonalizable are. And what are the characteristics of such matrices. And if they are orthogonally diagonalizable, then what is P and once we can find P. Then, of course, P transpose is simple and one can easily find out D.

(Refer Slide Time: 30:40)

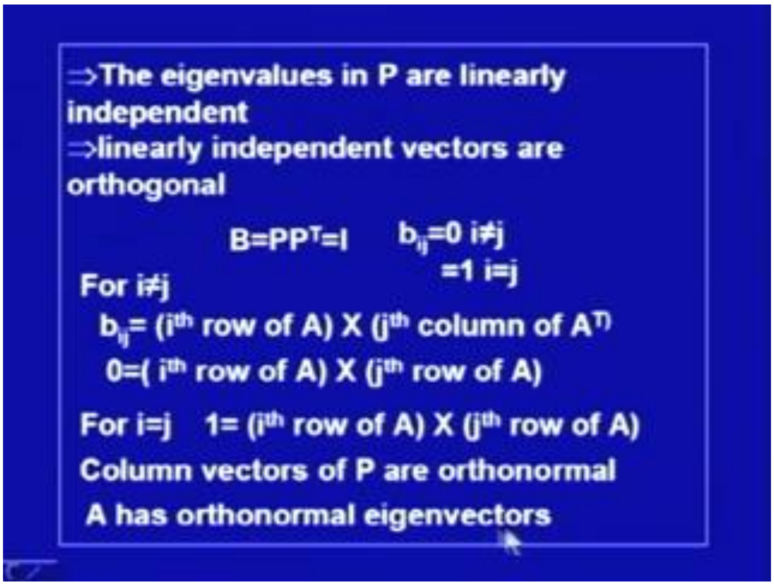


So, let us discuss this theorem according to which a square matrix A. For a given square matrix A, the following are equivalent. The first is that, A is a orthogonally diagonalizable. The second statement is that, A has orthonormal set of n vectors. And the third statement is that, A is symmetric. So, these statements are equivalent. By this, I mean to say, if A is true. Then, B is also true and if B is true, then A is also true, also B implies C and C implies B.

Not only this A implies C and C implies A; so number of things have to be proved, if this theorem is to be proved, we will do 1 by 1. So, first step is that, if A is orthogonally diagonalizable, then it will have an orthonormal set of n vectors, so that is a implies b; so given that, A is orthogonally diagonalizable. We have to prove that, it has a set of n orthonormal vectors.

So, A has orthonormal set of vectors. So, since A is orthogonally diagonalizable, so there exist P. Such that, P inverse is equal to P T, because this matrix P has to be orthogonal. So, P inverse is equal to P T and D is equal to P transpose A P, so this product is diagonal matrix. And since, P is P T and P inverse has same, so D is equal to P inverse A P, so D is similar to P inverse A P as well as it is similar to P transpose A P. So, that is which being given to us, from here one can notice that a matrix A is orthogonally diagonalizable, means that matrix A is diagonalizable, because we can find D, we can express P inverse A P as D.

(Refer Slide Time: 33:02)



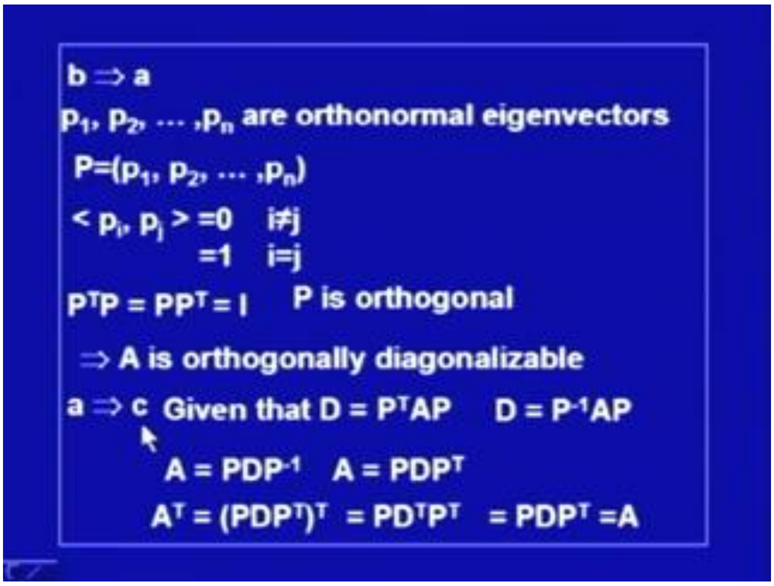
And that means, the Eigen values in P are linearly independent because a matrix P, matrix A is diagonalizable, so it is Eigen values are linearly independent. And linearly independent vectors are orthogonal we know, so this implies this and that means, the matrix B, which is set of Eigen vectors P multiplied by P T, P transpose is equal to I matrices is orthogonal.

So, this statement means that b i j is equal to 0, whenever i is not equal to j and it is 1i equal to j, this P matrix, you look at this product. Then b i j, the typical element in B is obtained as the ith row of A is multiplied by jth column of A transpose, but what is ith row of A is the ith row of A multiplied by jth row of A. It is A transpose, so this is column is replaced by row.

So, two rows are two rows A are multiplied and this is equal to 0, when i and j are distinct, because they are independent vectors. So, this is going to be 0, when i and j are distinct and when i is equal to j, then this ith vector of A is multiplied by ith, jth row of A are ith row A; that means, two vectors are multiplied. And this has to be equal to 1, according to this, because then our claim is that, B is an orthonormal matrix.

So, for I is equal to j this product is equal to 1 and that means, the column vectors of P are orthonormal, because their product is equal to 1, when i is equal to i and when they are different, then it is 1. And this means that, column vectors of P are orthonormal, so we can say that, A has orthonormal Eigen vector. So, this was to be proved, so we have started with orthogonally diagonalization of A. And we have proved that, it is the vector the matrix A has orthonormal Eigen vectors, so if the matrix is orthogonally diagonalizable. Then, it will have orthonormal Eigen vector, so this is the first part of the theorem.

(Refer Slide Time: 35:53)



The second part is that b implies is a, that is if p 1, p 2, p n are orthonormal Eigen vectors of A; that means, the matrix A is orthogonally diagonalizable. So, for this, let us consider P as set of column vectors p 1, p 2, p n and these are orthonormal Eigen vectors of the matrix A and that means, p i, p j, it is inner product is equal to 0. When i is equal to j and it is 1 i equal to j, in fact, when we multiplying ith row with jth that is nothing, but the inner product.

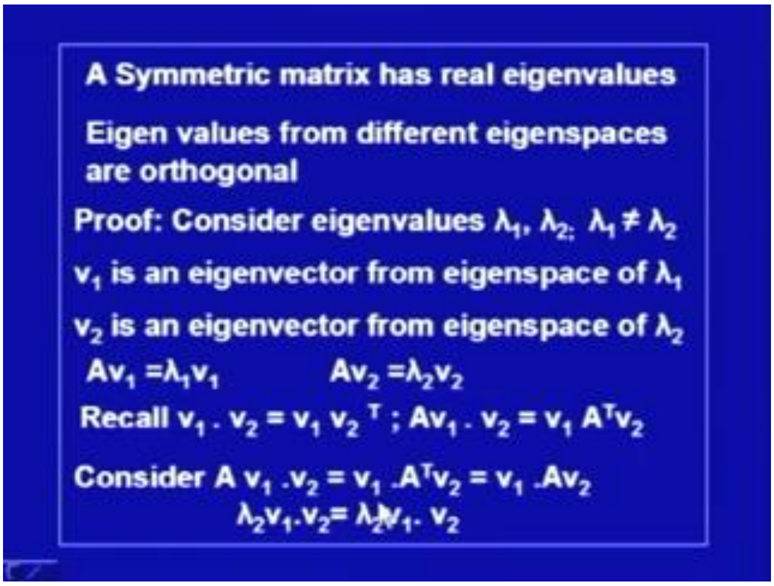
Now, if you consider the product P transpose P, then rows of P transpose of multiplied by columns of P, so each term will be of this form and that means, the term will either be 0 or 1. If it is 0, if i is not equal to j, it is 0 or it will be 1, then i is equal to j, that means P transpose P will be a matrix with one in the diagonal. And rest of the elements are 0, that matrix is nothing but an identity matrix.

So, P transpose P is equal to identity matrix, similarly one can prove that P transpose is also identity matrix and once P satisfy this property, then one can say that, P is a orthogonal matrix. And this proves, that A is orthogonally diagonalizable, then the next part of the theorem is that a implies c; that means, given D is equal to P transpose A P for a given matrix A.

It is orthogonally diagonalizable means there exist P matrix, such that P transpose A P is equal to D and since it is orthogonally diagonalizable, so P transpose is nothing but P inverse. So, D is equal to P inverse A P, so we can write down A is equal to P D, P inverse, from here and that means, A is equal to P D, P transpose. From here, we can calculate the transpose A is equal to P D, P transpose and it is transpose and further simplification will give me A transpose is equal to A, that means the matrix A is symmetric.

Now, this completes the proof of this part a implies c, so the matrix a is orthogonally diagonalizable, then the matrix is symmetric, but since we have given three statements and we have to prove the equivalence of all the three, I have to prove c implies a also. But, I leave this is an exercise for this viewer, similarly you have to show that d implies c and c implies b, this also I leave as an exercise for the viewer.

(Refer Slide Time: 39:22)

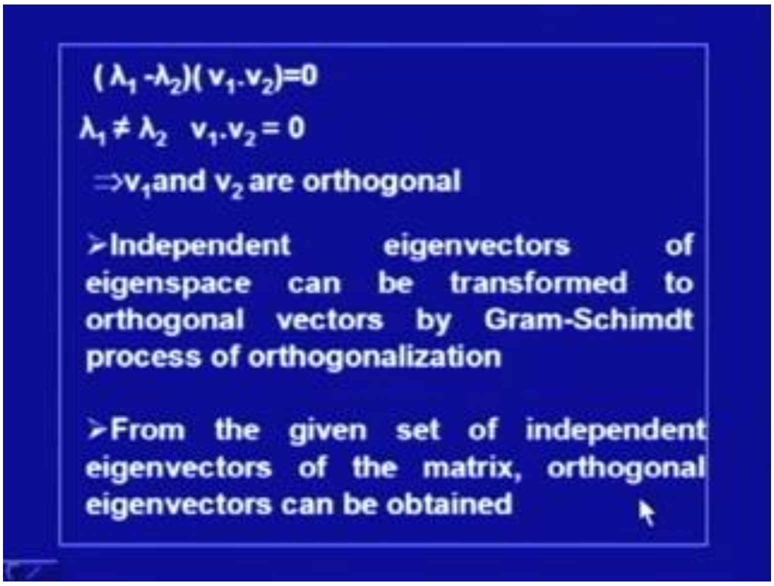


Now, and we come to this is statement, let us a symmetric matrix has a real Eigen values, this we have proved in one of my earlier lectures and Eigen values from different Eigen spaces are orthogonal. So, this is the statement, which I am going to prove now, so to prove this, let us consider that there are two Eigen values of the symmetric matrix A, which are lambda 1and lambda 2. They are not equal to that is lambda 1, this is not equal to lambda 2.

And let us assume that, v 1 is an Eigenvector corresponding to lambda 1and v 2 is an Eigen vector corresponding to lambda 2, that means A times v 1 is equal to lambda 1, v 1. From this and from here, we can say A v 2 is equal to lambda 2,v 2, now we recall some of the definitions from inner product, that v 1.v 2 is equal to v 1 multiplied by v 2 transpose.

And A times v 1.v 2 is equal to v 1 times A T. v 2, so these properties, we use to establish result. So we say, we start with A v 1.v 2 and this is equal to v 1, A T, v 2 which is from this property and since the matrix is given to be symmetric, so A transpose equal to A, so we can write down this is equal to v 1.A v 2. Now, I am using this in the first expression A v 1.v 2 is lambda 2, v 1, v 2 and from here it is coming to the lambda to v 1, v 2.

(Refer Slide Time: 41:14)



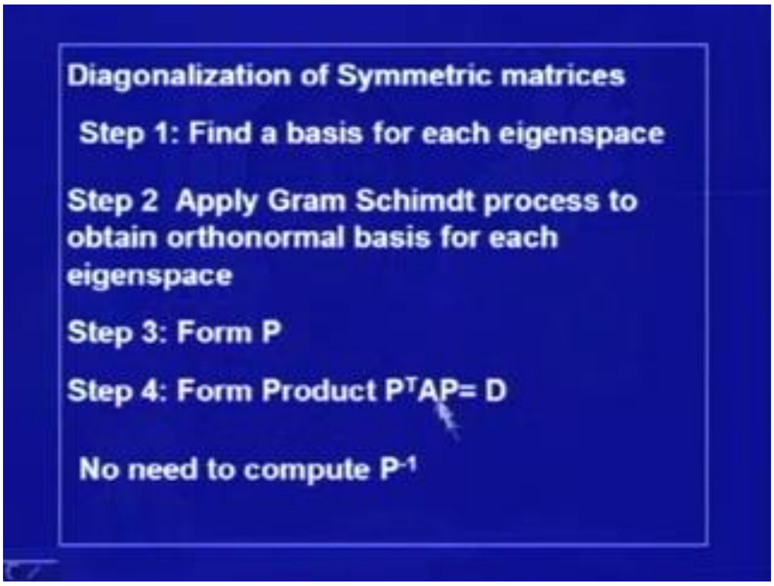
So, from here we can rewrite that equation as lambda 1minus lambda multiplied by v 1 dot v 2 is equal to 0. Now, this means, that either this is 0 or this product is 0, but since lambda 1is not equal to lambda 2, which is being given to us, so v 1.v 2 is equal to 0 and that means, v 1 and v 2 are orthogonal vectors. That proves the result, that if we have two distinct Eigen values of symmetric matrix, then the two vectors are orthogonal.

Now, independent Eigen vectors of Eigen space can be transformed to orthogonal vectors by Gram-Schmidt process of orthogonalization, now this means that we have a given matrix A, it will have some distinct Eigen values. And those Eigen value, those Eigen vectors will be orthogonal, but there are some vectors, they are some Eigen values, which I have repeated.

And suppose we can find independent Eigen vectors corresponding to those repeated Eigen values, the question is can we also orthogonalize them and guess the answer is and what we can do is, we can perform Gram-Schmidt process of orthogonalization to transform the independent Eigen vectors of the Eigen space. So, if we have repeated roots and have independent Eigen vectors, then they can also be transformed to orthogonal vectors.

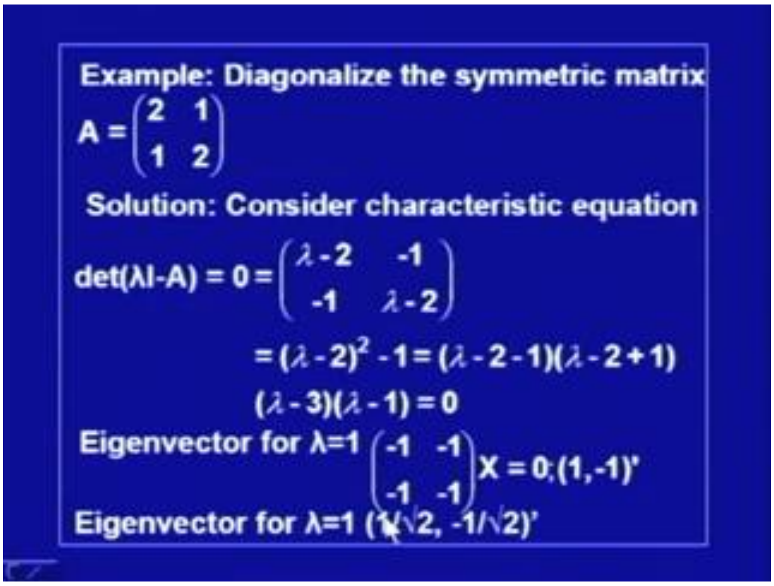
And that way, the whole a set of Eigen vectors for a given matrix A can be made orthogonal and once we can have orthogonal Eigen vectors, we can orthogonal, we can make them orthonormalize. We can perform orthonormalization by dividing each vector by it is magnitude, so from the given set of independent Eigen vectors of the matrix. We can get orthogonal Eigen vectors and then we can make them orthonormal also.

(Refer Slide Time: 43:32)



And this gives us some method for diagonalization of symmetric matrix, so the step 1is find a basis for each Eigen space apply Gram-Schmidt process to obtain orthonormal basis for each Eigen space and from that we can form the matrix P. And finally, we form the product P transpose A P and this will be nothing but the diagonal matrix D consisting of Eigen vectors, Eigen values in the diagonal. So, this procedure can be applied for diagonalization of symmetric matrices and you may notice here, that we do not have to calculate P inverse, simply P transpose is to be obtained to now make the matrix diagonalized.

(Refer Slide Time: 44:33)

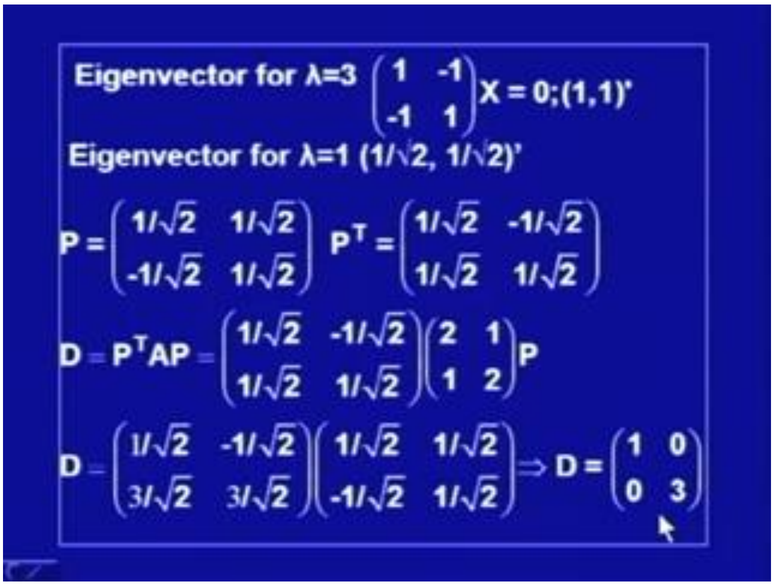


So, we do not have to compute P inverse simply P transposes is required for the process, so let us illustrate this with an example; we have a 2 by 2 matrixes for simplicity, so we have 2, 1 and 1, 2. We consider, it is characteristic equation lambda I minus A is equal to 0, it is determinant of lambda I minus A is equal to 0 is to be obtained and for this, if you calculate this determinant, then this comes to be lambda minus 3 into lambda minus 1 equal to 0.

And that means, lambda is equal to 1 and lambda is equal to 3 or these are two distinct Eigen values of this matrix A, so they are distinct Eigen values, so the vectors will be orthogonal. So, we start with lambda is equal to 1, so we simplify the matrix, the matrix equation A x is equal to lambda x for lambda is equal to 1, this will be the system of equation minus 1 minus 1 minus 1 minus 1, x.

And this simply means that, 1 and minus 1 is n Eigen value is an Eigen vector corresponding to lambda is equal to 1, because this vector will satisfy this system, so we have obtained one Eigen vector corresponding to this value. Similarly, we can obtain Eigen vector corresponding to lambda is equal to 3, however, we need not only the orthogonal vectors, we need orthonormal vectors. So, we divide this by it is magnitude, so lambda is equal to, so for lambda is equal to 1, we have this vector, it is a unit vector.

(Refer Slide Time: 46:35)

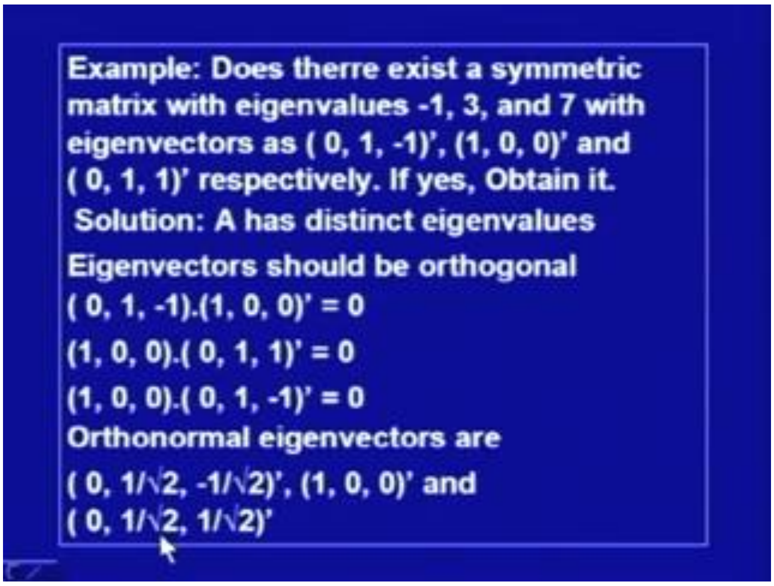


For lambda is equal to 3, this is a system and corresponding to this, we have 1,1 as it is Eigen vector, which can be divide by it is magnitude. So, corresponding to lambda is equal to one we have this Eigenvector and one can notice that, this vector which we have obtained corresponding to lambda is equal to 1 and this for lambda is equal to 3, they are orthonormal.

And once we have P we can obtain P transpose by making its by taking it is transpose, so we have this P transpose, now we have to check, whether it is really being diagonalize or not. So, we form the product P transpose A P, so this is my matrix P transpose, this is the given matrix A multiplied by P. So, first I compute this product and this product comes out to be this multiplied by P.

If this is matrix P, which is given to me and if we perform this computation, we perform this product and you will have D is equal to 1, 3, 0, 0, so this matrix is diagonalized. So, I have been given a symmetric matrix, which is diagonalized with the help of P and P transpose, in fact what we have done is, we have done orthogonal diagonalization.

(Refer Slide Time: 48:17)

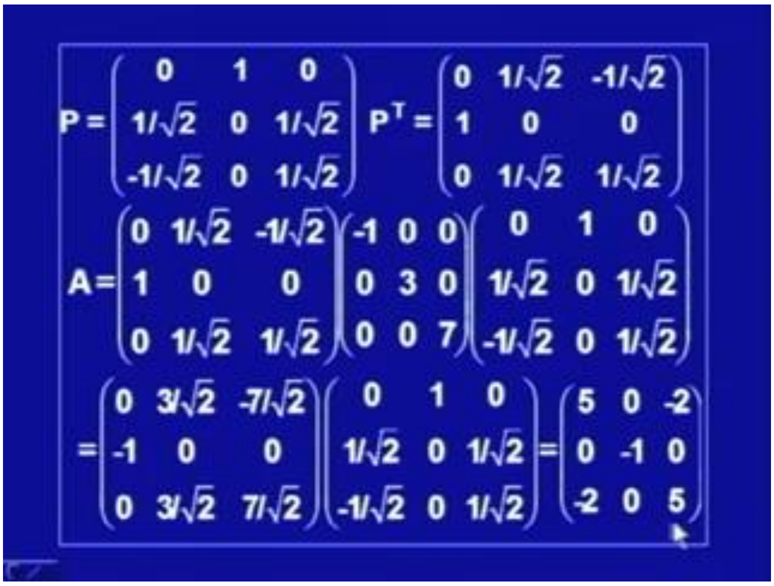


Now, in this example, the question is does there exist a symmetric matrix with Eigen values, minus 1, 3 and 7 with Eigen vectors this and this and if, then obtain the matrix A; that means, if we have Eigen values given. If, we have corresponding Eigen vectors can we find the matrix A, so this is, what we are going to do in this example, so we have been given three distinct Eigen values and see the Eigen values are distinct.

One can say, then Eigen vector should be orthogonal and by this, I mean to say, these are the three Eigen vectors given to me, if, whether they are orthogonal or not. So, let us check with this is perpendicular to this or not, so I perform this product comes out to be 0. I take the second product, this and this, this is also 0 and this and this, this is also 0, so these three vectors, these are Eigen vectors and there are orthogonal.

So; that means, there exist a matrix A, which is having these three Eigen values and corresponding Eigen vectors and this orthonormal Eigen vectors corresponding to these given set of Eigen vectors. They are 0, 1 by root 2 minus 1 by root 2, corresponding to first, corresponding to second one, it is 1, 0, 0 itself and for the third, we will have 0, 1 by root 2 and 1 by root 2, so this is a set of orthonormal Eigen vectors.

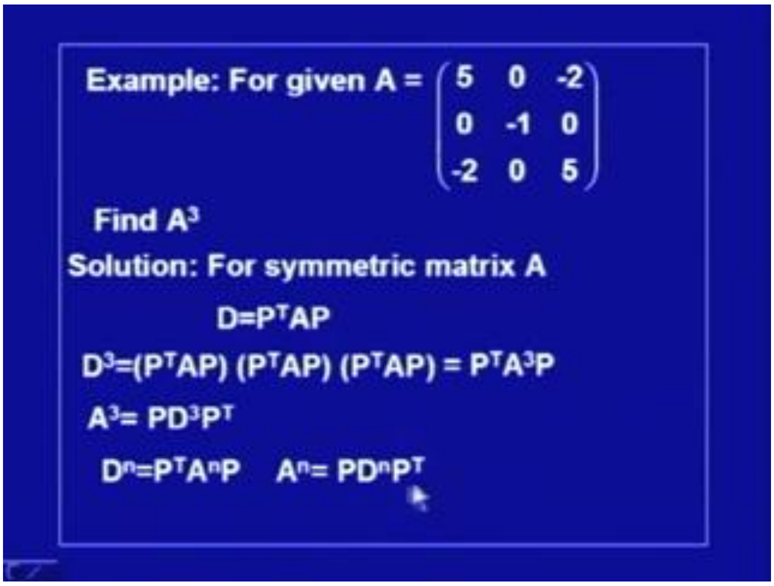
(Refer Slide Time: 49:49)



And then I can form matrix P for some matrix A, so this is first Eigen vector, second Eigen vector third Eigen vector and P transpose is this. So, the matrix A will be equal to this P transpose D and P, so A is equal to P transpose D and P, so this is being given to me. This I have computed, so from here, I can compute A, this is possible, because I could get this matrix P.

So, if you perform this product, first this product, which is coming out to be this, one can verify that, this product is actually this, then once we get this product multiplied by this; that means, this and computing this product, one can get this matrix. So, this is the matrix A, which is symmetric and we can obtain the symmetric matrix, which is having Eigen values, which are specified and Eigen vectors which are given and from there, we can find the matrix P and P transpose. So, this is the reverse problem, I have been Eigen values, I have been given Eigen vectors and I have to find the matrix, so this is possible in this manner.

(Refer Slide Time: 51:36)



In the next example, if I have the matrix A and I have to find to A cube, see in the beginning I said that with the diagonalization, a computational effort will be less, while performing algebra on matrices. So, let us see how the effort will be simplified for this matrix A, I have to compute A cube, if I have to computer A cube without doing normalize without doing diagonalization etcetera.

So, what I have to do is, I have to first perform A, I have first to multiply A into A getting A square, then A square and A will be multiplying to get A cube, but it is not only the letter of A cube. It may be A raise to the power 10, A raise to the power 30, so I have to do this way for each and every product, things will really be complex, so let me illustrate the procedure in which you can easily perform, such type of computations by using diagonalization.

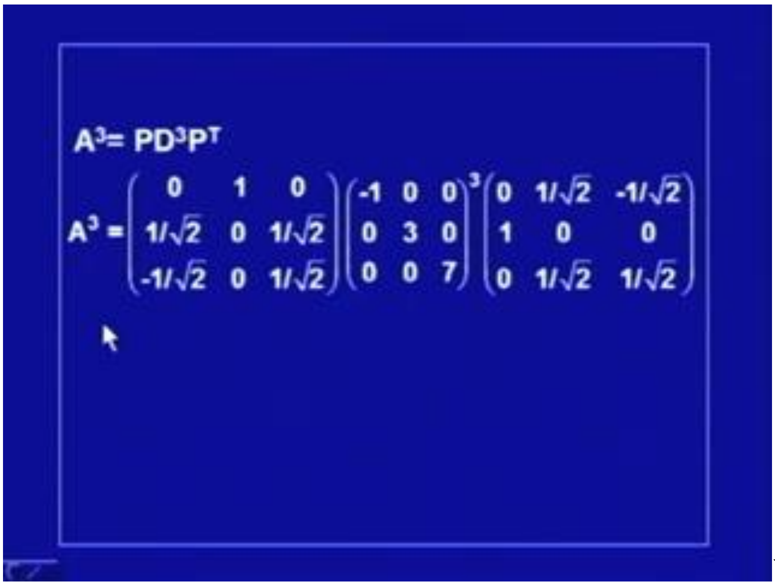
So illustrating this concept with the help of A cube, I start with a symmetric matrix A, which is this is matrix, which side given to this is applicable early for symmetric matrices, so this is a symmetric matrix. So, if I have been given symmetric matrix, then I can write down this symmetric matrix as D is equal to P transpose A P, that is D is similar to this matrix D is diagonal matrix. So, what is D cube, D cube is this multiplied 3 times and since matrix multiplication is associative.

So, one can rearrange these terms to have P, P transpose at one place, this P, P transpose at again, they can be combined and we know that P, P transpose is identity. This P, P transpose is also identity. So, what we have is P transpose A into A into A and that gives me P transpose A cube P, so for this given matrix A, I know its Eigen values.

So, I know, what is D, so diagonal matrix consisting of Eigen values and D is a diagonal matrix, computing D cube is very simple, because what we have to do is, we have to simplify cube the diagonal elements rest of the elements are 0. So, D cube is known, so from this expression, I can write down A cube as P times D cube P T, that means, if I multiply here P and post by P T, then P into P T is identity and this P into P T is also identity.

So, what remains on this side is simply A cube, so A cube is PD cube P T, so calculating A cube simply means calculate the diagonal and then make this product and you can. In fact, generalize this result and you can say D raise to the power n is P transpose A rise to the power n P. That means, you can get any power of A and this can be given as A rise to the power n is equal to P D rise to the power n P T. So, you have simply multiplied by P and P transposes, this is post multiplication, this is pre multiplication and you can get the product, so I apply this to calculate A cube.

(Refer Slide Time: 55:02)



So, A cube is P D cube P T and this D cube is this P, it is I have already computed like this and my earlier example and the Eigen values are minus 1, 3 and 7. So, one can very easily compute 3, it is minus 1 raise to the power 3, 3 raise to the power 3 and 7 rise to the power 3 and this is the post multiplication and these 3 matrices are to be multiplied to get A cube.

So, whatever be the value of n here, n may be 3, 10, 20, whatever it may be, I have to get only this product and that is how we get A cube. So, in this example, I have illustrated how we can make use of diagonalization process to get your algebra simplified.

So, today in my lecture, I have given you method of diagonalization and we have discussed orthogonal diagonalization also and with the help of orthogonal diagonalization, I have simplified the process of diagonalization in the sense, that I have do not have to now compute a inverses, I simply work with transposes and that is only for matrices, which are symmetric. So, first diagonalization of symmetric matrices is simple and then once we can get the diagonalization, things the algebra will be simpler, so this is all for today’s lecture.

Thank you.