Mathematics-II
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Module - 1
Lecture - 3
Cauchy's Integral Theorem

Welcome to the lecture series on complex analysis, for undergraduate students. Today's lecture is on Cauchy's integral theorem. We have learnt about the contour integrations, that is the integration of complex function in complex domain. Till now, what we have done the contour integrations; we had found out that for some functions, the integral depends on the contour while for some functions, integral does not depend on the contour. In the case when it is not depending on the path, we can use the indefinite integral of the function and then in then we can use the limits for that the from z naught to z one, what is our true points and we can calculate the integral independent of any path. Now today we would let us find out the reason why for some functions the integral depends on path and for some function it does not depend.

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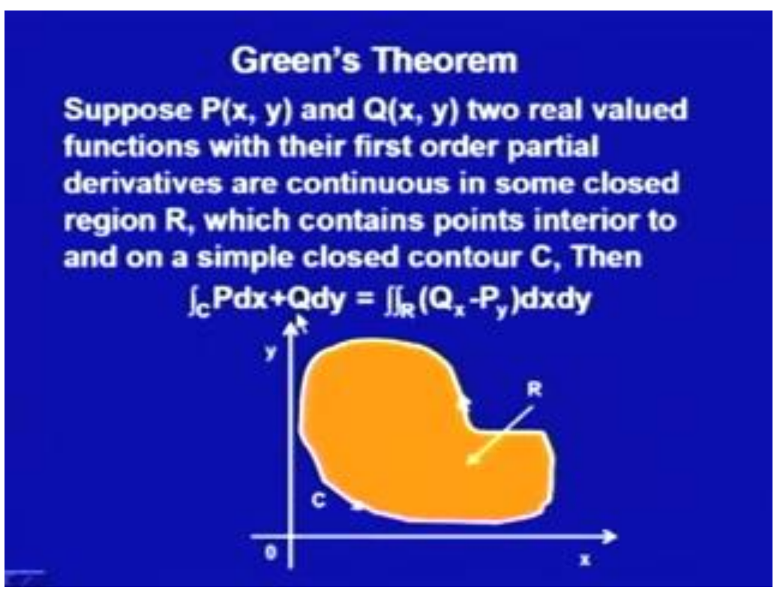

So let us first see Suppose that Pxy and Qxy are two real valued functions which are continuous with the first order partial derivatives are also continuous in some closed region R which contains points interior to and on a simple closed contour C. Then according to the Green's theorem in advanced calculus, we do know that about line integral, that integral along the contour C of Pdx plus Qdy is same as the integral over the reason of Qx minus Py with respect to dx dy. Lets see that is what it is saying. This is the region xy - so this is xy domain, that is a Cartesian plane or you could say is z plane. Here is some region R which is enclosed by a contour C; this is the contour C. We have taken this contour in positive orientation, that is in anticlockwise. So that every point which is interior to this C is left the to this one. It's a We are saying is that the integral along this contour of two functions P and Q - P with respect to x and Q with respect to y - is same as that double integral in this complete region. of Qx is the partial derivative of Q with respect to x while Py is the partial derivative of P with the respect to y; with respect to both the variable x and y, that is this reason double integral. Now let us use this result in our complex functions.

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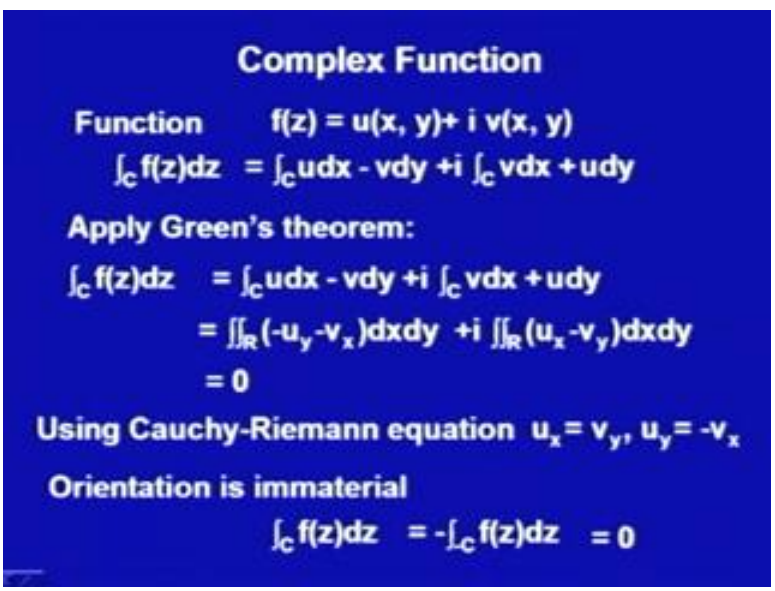

So we have started with the complex function fz which we could say is that uxy plus ivxy that we could write. We had seen in the contour integration, that is integral fz dz in the last lectures we done it can be given as integral C udx minus vdy plus i times integral over C vdx plus udy. When we have to presented our function fz as uxy plus ivxy and we said z as x plus iy and the contour C can represented on that line - on that contour C the z we are giving in the parametric form- then we have done in the last lecture this formula. Now in the formula we could apply the Green's theorem. Why? We are assuming is that function f is analytic. f is analytic - that means f is continuous. If f is continuous, then certainly u and v would also be continuous. And its derivative if ah is continuous analytic, that means u and v; if f dash is continuous then u dash and v dash with respect to partial derivatives with respect to x and y they would also be continuous. So now we would be apply the Green's theorem in this result.

What it says is If it you remember, the Green's theorem, says is the line integral along the contour u dx plus - Q pdx plus Q dy, we have done is Qx minus Py; that is, its partial derivative with respect to x and its partial derivative with respect to y and this would have with the minus signs. So now we could write it out this one - this would be minus uy minus vx dx dy. Similarly this could be given as its partial derivative with respect to y and its partial derivative with respect to x; so ux minus vy dx dy. So what we have got the integral in the region R, where this contour C is enclosing this region R ; that integral of minus ux minus uy minus vx and of uv ux minus vy. This integrant minus uy minus vx and this integrant ux minus vy - both are 0. Why? Because f is continuous analytic. So Cauchy-Riemann equation says ux is vy and uy is equal to minus vx. So ux equal vy and uy is minus vx; this gives both integrants to be zero and it is a closed region R. So the integrant, when both integrants are zero this would give zero.

So what we have got actually that now if here I have taken the orientation as the positive one - that is counter clockwise - but actually this orientation doesn't matter because we have got this integral to be zero. So if the orientation has been reverse, then we do know that integral cfz dz can be given as is same minus of minus cf z tz. minus C means that is orientation has been changed to the reverse to this one. So this is minus of this one; because this is zero, this will also be zero. That says, is orientation is not mattering what we have got now. Let us say state the result which we have got from here.

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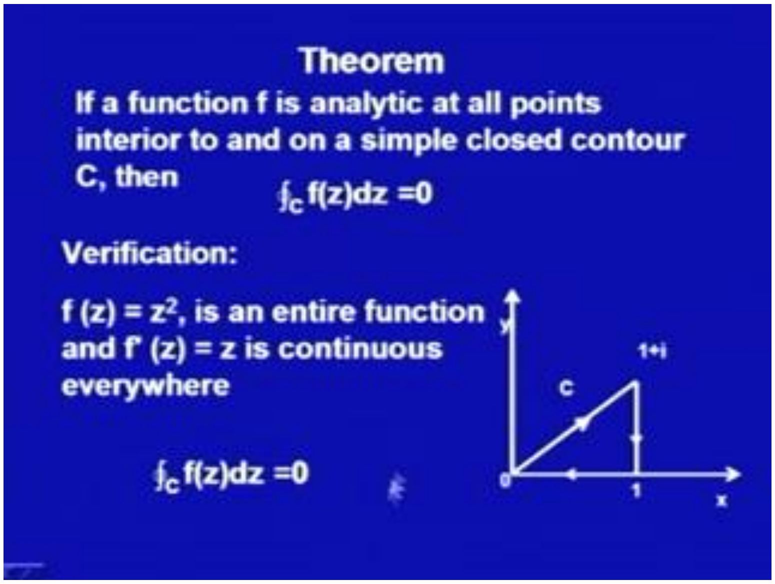

We have got the result: if a function f is analytic at all points interior to and on a simple closed contour C, then integral C fz dz is 0. That is for any analytic function, we do get that for the closed contours simple closed contour, integral of fz dz would be 0. So because to verified the proof - is a such we have actually find it out, that is that is the proof you can go ahead with this fz is analytic; we write it fz ux plus ivy and write this contour integral using that formula and is just in the last slide, what we have done the at we could call the proof of this theorem.

Actually this theorem has been given by Cauchy and ah the proof was also been given; there we had use the continuity of the f dash. Next this Gosset has done it; that is, without using the continuity of f dash he has given more general result which said is that only f is analytic and here also shown that f dash is also analytic. Actually any derivative of f would be analytic in this region; that we will do later on. So here we just first verify this result for our examples. You do remember a contour integral we have done - one function z square. z square - we do know is an entire function, that is, it is analytic everywhere. So and f dash z is two z, which is continuous everywhere. So all these conditions are being satisfied; that says if I do take any closed contour -simple closed contour, I must get integral to be zero.

Now if you you remember, in the last lectures we have done one example where we have taken the integral of z square from the point zero to one plus i. If you have to remember, we have done this example using different contours. One - we have done is using this line, straight line from zero to one plus I; another, we have used this using the 0 to 1 and then 1 to 1 plus i and we have got that in each case our integral has come up same. So now you see I am making this a closed contour. This C is now my closed contour from zero to one plus I, then to one and then to this one.

Now since the integral along this line as well as this plus this line they were same, so if I am changing this direction for these two integrals - whatever integral we have got over here that would be negative of that one; and that gives that is in this direction, it should be 0. And that we had verified it actually we had find out in the last example that it is coming out to be same. So the sum of both these integral would be zero and this theorem also says if i take any close contour so.

This is an example because we have done in the last lecture, so I have taken. Actually you can take any closed contour; you can take any circle any ellipse or any close contour, this will always give you the integral as zero because of this theorem - since z square is an entire function. Now let us move to the general results which we are calling as the Cauchy's theorem. How do we find it out that when we could say it would happen? So let us move to some more results. One definition - we are defining connecting domains.

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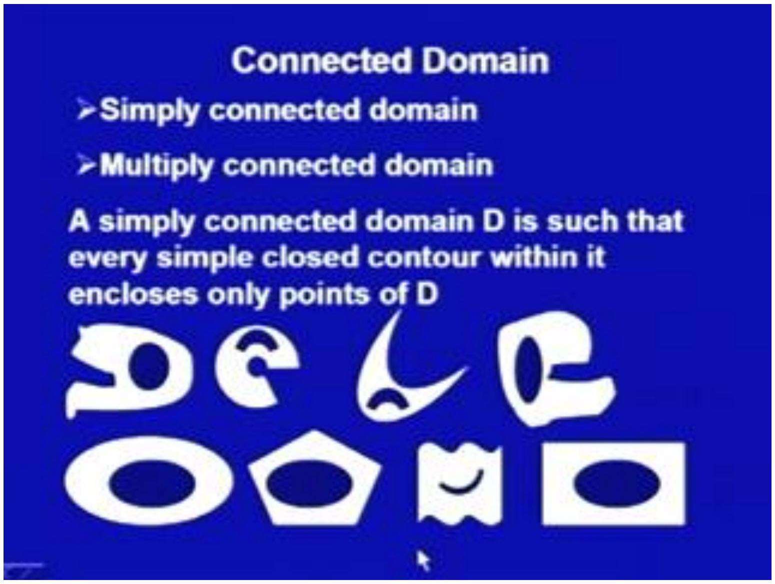

Connected sets we mean is that is the sets which cannot be partitioned further with limits not being common to all. So we are not very much interested in connected set definition as such; if you are not remembering or if you have not done that over there, here we will define it simply in two manners. One we, would call simply connected domain and another we would call multiply connecting domain. What is simply connecting domain?
A simply connecting domain D is such that every simple closed contour within it encloses only points of D. What it is saying is, if I take any simple closed contour, its interior points are only points of D. Let us see some examples here. This is a domain D. If I take here any closed contour - simply closed contour; suppose this closed contour I have taken.

Now you see the interior points are only the interior points of this domain D. Now suppose if I shift this contour to little bit this side, then it will be become outside this one; so this will not be inside the domain D. If i take any contour in this domain D, it will always take the interior points - would be only the interior points of D. Now you see another example. Here is a domain; if I take any closed contour in over here - you can just check with yourself also - this is also a simply connected domain. Similarly you see this is also a simply connect domain, because if I take any contour over here which is inside this domain D, that will contained only the points of D. Certainly disk is not a simply connected one. See this is also an example of connected simply connected domain. Here is also example. So any closed bounded circle or ellipse or this would be your simply connected domains; this is also a simply connected domain, this is also a simply connected domain, this is also a simply connected domain. now if a domain What is multiply connected domain?

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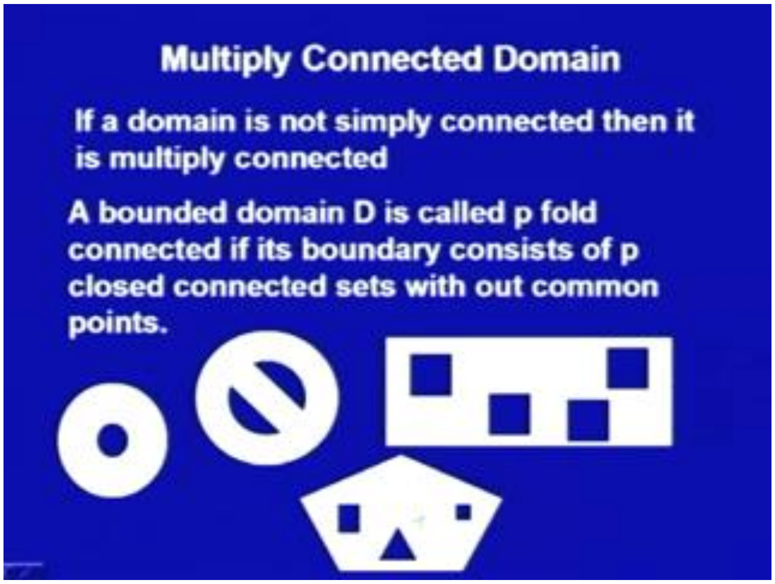

So multiply connected domain The domain which is not simply connected is called multiply connected. We had seen that is in the last examples, all the bounded domains we have taken about for the simply connected domain. So if you have talked about all the bounded domains, in that case what could the multiply connected domain be? We could actually rewrite this definition. If Bounded domain D is called p fold connected. Similarly we called simply connected as one connected and multiply connected as p connected also, if its boundary consist of p closed connected sets without common points.

In the last example I said is that disk is not a simply connected. You see What is the disk? This disk is having the points; we are having this complete disk as the domain. The points which are interior to this inside inner circle they are not in the domain D. So now, if I take a closed contour over here, say let us say this circle, then the interior points are the points which are not in D. So of course, it is not a simply connected domain. But we can call it doubly connected or two connected. Why? Because you see in this domain we do have two boundaries; one is this outer boundary and another is this inner boundary; the points on this boundaries are these points and points on this boundaries are this. So the set of this points on the two boundaries they are disjoint. So we do have this domain has two disjoint boundaries, that is domain the boundary consists of two sets which are disjoint; so it is doubly connected.

Similarly you see this is a triply connected domain. Why? We do have here one boundary - this is outer boundary; another is this inner boundary and then third one is this inner boundary. Certainly this is not a simply connect because if I do take a contour just like this kind of circle, so this contour is completely inside the domain, but its interior does not contain all the points which are interior points of the domain. So this is a triply connected because the boundary consist of three boundaries and each boundary is disjoint to each other; so we do have this triply connected.

Similarly this - is you see is that is we are having the one, two, three, four and five boundaries. So we will call it five connected. This is one outer boundary, two, three and four; so we will have this four connected. What We have got one more simple conclusion from here; if it is a bounded domain, we will call it p connected if it has p minus one holes. So doubly connected - one hole; triply connected - two holes; four connected - three holes; five connected - four holes. So if it is has having p minus one holes, then we will call it p connected; so this is the definition of multiply connected domain. Now in the reference of this simply connected domain, let us see we have defined the Cauchy theorem which says is that for a simple closed contour, the integral will always integral of analytic function will be 0. So now rewrite this uh theorem in simply connected domain; we will call it Cauchy integral theorem.

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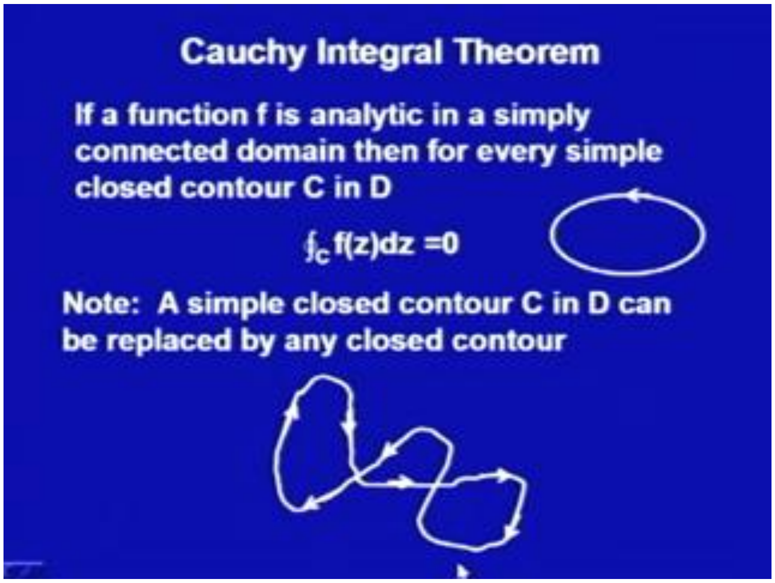

If a function f is analytic in a simply connected domain, then for every simple closed contour C in D integral along that closed contour C of fz is zero. Now we have not taken any other condition, that is whether f dash is ah continuous or anything, we are just taking that it is it is lying in the simply connected domain. Now so this is simple contour. Now if it is not this thing This theorem can be extended to any closed contour rather than simply closed contour. Simply closed contour means is that is it is not intersecting itself at any point. If I do have any contour that is it is not A simply closed contour C in D can be replaced by any closed contour.

So suppose this is closed contour which is intersecting itself at finite number of times. Even then, this theorem would hold or this result of this theorem, that is integral along this closed contour C of fz, would be zeroif this contour is lying inside a simply connected domain. Why we could do is, if it is lying inside the simply connected domain, say for example if I am talking about this contour; this closed contour, I can break into three parts - one is this closed contour, another is this closed contour and then another is this closed contour. For each closed contour - this is simple course contour - for each simple closed contour, this Cauchy theorem will hold true. That says is If i add up all these three, I would get the final integral as zero and then we would say that any closed contour, it would be zero. So here, the Cauchy theorem is actually about the simple closed contour. But we can extend this result to any closed contour in this manner.

Let us see that is what it is actually referring to. We are actually basically interested in finding it out in what sense we could say that for a function the interior will depend upon the path, and for what functions the integral does not depend on the path. So now, we are talking about analytic functions. We are finding out if it is on any simple closed contour, the interior along that contour will always be zero for analytic function f.

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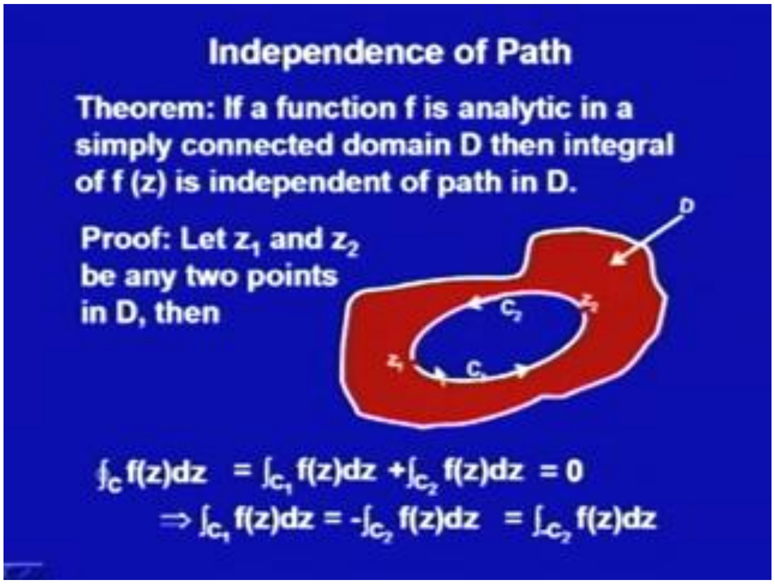

This says is that our independence of path. So i am writing that result in form of theorem. If a function f is analytic in a simply connected domain D, then integral of fz is independence independent of path in D. Let us say z 1 and z 2 are any two points in a simply connected domain D. So let us see this is my simply connected domain D; z 1 and z 2 these are any two points.

Now I want to say is that integral of f is analytic in the whole domain. I want to say that integral along this ah integral of f from the point z one to z two would be same, whether I reach in this manner or i reach from this manner or I go with any other path; it should be independent of path. See what we are saying is I would use this Cauchy's integral theorem. This is a simply connected domain; I had made one simple closed contour passing through these two points - z 1 and z 2. I have taken it positively oriented. Then according this Cauchy's integral theorem, I do have that integral fz dz - is now this closed integral, this closed contour - the integral should be 0.

Now this closed contour - I am dividing into two parts. One is from z one to z two - this C one. And another is from z two to z one as C two. So could write it as C one fz dz plus C two fz dz; that we do know by a simple definition of the our integrals, we we could write it out that is the path we can write as the summation of these paths; so this is equal to 0 according to the theorem.

Now what it says is from here integral C one fz dz should be minus of integral C two fz dz. Now if I take the since reversal property, then minus of integral along C 2 that could be given as integral along minus C 2. So what we do get : integral from on the path C 1 is same as integral on the path minus of the integral on the path C 2 which is same as integral on the path minus C 2. That says is, whether I am using this path or I am using this path, this would be same.

Now here we have taken this closed contour - actually we can make infinite many closed contours passing through these two points and for each path, I would get it like this. Now here, I have taken the simple closed contour, so my paths are not intersecting each other. Now if suppose the path from z one to z 2 I do talk about the two paths such that they are intersecting each other.

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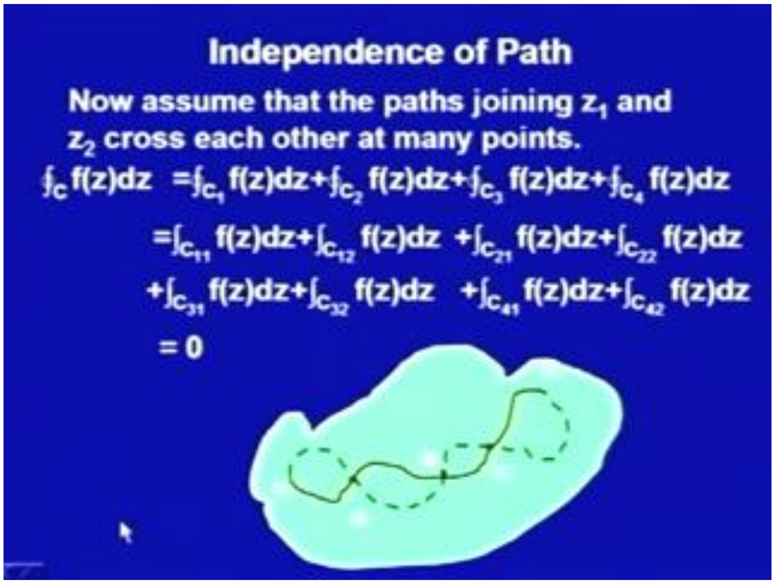

So see what we could say is Suppose these are the two points z 1 and z 2, and we do have that one path is this red path and another path is this green dashed path. These paths are intersecting each other at a number of points. Now Here for example, I have taken that they are intersecting at three points. We want to say even if this is happening, is still the integral of an analytic function from z 1 to z 2 will remain independent of path. So we are taking it that it is intersecting at three points - a, b and C; so crosses each other at many points.

Now what we will do is As I said, as that is the Cauchy's theorem holds to even if I do take the path - not simple contours but the contours with with any contour. So here, I am taking this as any contour; rather we will just use this Cauchy theorem for different segments. So let us first say - this from z 1 to a; this is a closed contour. And for this, the Cauchy theorem holds true. That is, along this path of this closed contour this integral will be - integral of analytic function f - could be 0. Similarly along this path from a to b- this contour, closed contour; then from b to c - this closed contour; and then from c to z - two this closed contour.

Now so I am writing this complete contour - that is complete with red and green, from containing both z one and z two - this complete contour this I can break into four contours. This first contour from z one to a - we are calling C 1; the second contour from z to b - in this again, I am taking all the positively oriented, that is the inside points - interior points - are on the left of this one; and then this is from b to c, this contour when we are talking about this closed contour C 3 and this is close contour C 4now you see.

So, all these four are the simple closed contours in a simple connected domain. So integral along this of any analytic function along these closed contours would be 0, that is says this final integral would be 0. Now I am writing this first integral, that is first this simple closed contour. First, I will take this red path from z 1 to a. This is C 1 1 and then this green path from a 2 back to z 1 - this direction - this is C 1 2. Similarly for C 2, I will take this red path from a to b in this direction and then I am taking in this direction.

So you see, here I am actually changing my direction according to my convenience; that is, in this one i am taking this positive direction, in this i am taking the negative direction. Why? Because I want to keep the direction - same direction - for one path; that is why I am making it. And we do know that this Cauchy theorem is holding true whether we have taking is the direction as immaterial, when the function is analytic, on this simple close contour. So here, I am taking this first this direction first and then C 2 - 2 is this one. Similarly for b to C, I am taking the again the positive direction; that is from C 3 1 is this red one and C 3 2 is this green one in this direction. Then C 4 1 is your this red one in the negative direction - now you see - and then C four two is green path from this direction.
Since of complete sum has to be 0, so what it says is ah from here what i would get it actually that C 1 1 - now I am writing it in two different manners. Now I would break up the red path and the green path. I would keep red path on the left side and the green path on the right side. Red path - you see all the second indices as 1 - that is what is our red path; and all the second indices 2 - that is our green path. So C 1 1 - this one - plus C 2 1 - two you to C is that is how do have taken C two one we have taken this path connecting from a to b; so C 1 1, C 2 1, C 3 1, C 4 1 - the path from z 1 to z 2 , this path. This would be same as.

Now if I take C 1 2. C 1 2 is this path; so minus of C 1 2 means the path from z 1 to a. Then C 2 2 C two two is the same path; C two two was we have taken is that the second one we have taken in this this orientation. So C 2 2 was actually from C from b to a; so now minus C 2 2 would be again from a to b. Then we do get from this side - and minus of C 3 2, then minus C 4 2, this one. So what we are getting is actually we can rewrite it by your manner and see it clearly we would be getting that the integral along this path would be same as integral along this path. So whatever be the path, if the function is analytic in this whole domain D which is simply connected, then for any analytic function integral is independent of path. If it is independent of path, then in last lectures we had seen is if indefinite integral does exists, are that is also we have called is antiderivative - we could write the integral very easily; that is, we don't have to see from which path and we do not have to the find out the function along this path or we don't have to write the parametric equation of the path. We just have to know the points z 1 and z 2, and the function and its antiderivative, and that what we can write the integral value. What does this Cauchy theorem also makes that antiderivative exist You see the Cauchy theorem also says that we can use the Cauchy theorem to say that antiderivative exists.

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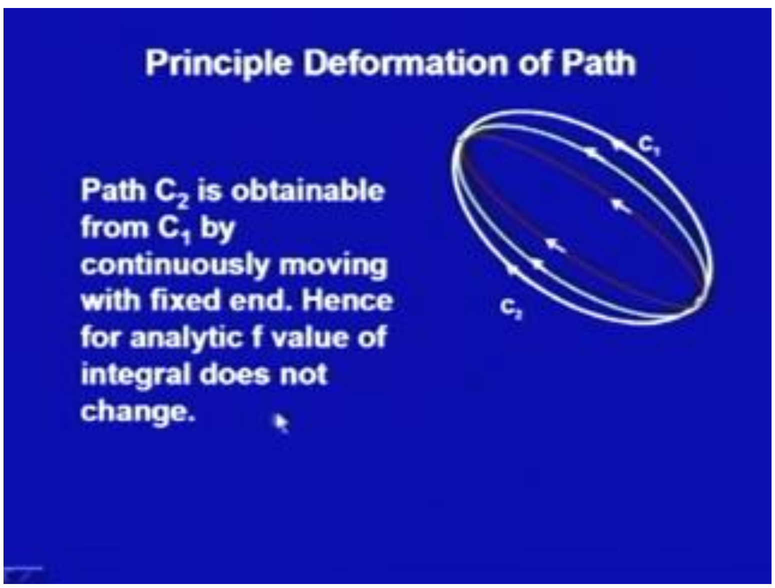

You see, one more thing in between we are talking about - principle of deformation of path. As we are saying it is independent of path. So if the function is analytic and every path here - whatever the paths i have talked about from C one to C; two these are two points we are talking about and keeping these fixed ends, if we just take any path, we can we are you see that is we are changing our paths in this one; this is one example you could says you can make it any other manner. Integral of any analytic function, when the function is analytic on these paths and it's interior or rather you could say only on the path, we do find out are these integral will remain same This is called actually the principle of deformation of path. That is, we can deform the path from C 1 to C 2. So just try I am writing this thing: path C 2 is obtainable from C 1 by continuously moving with fixed ends. Hence for analytic f value of integral does not change. This is called the principle of deformation of path. So now, come to this existence of indefinite integral.

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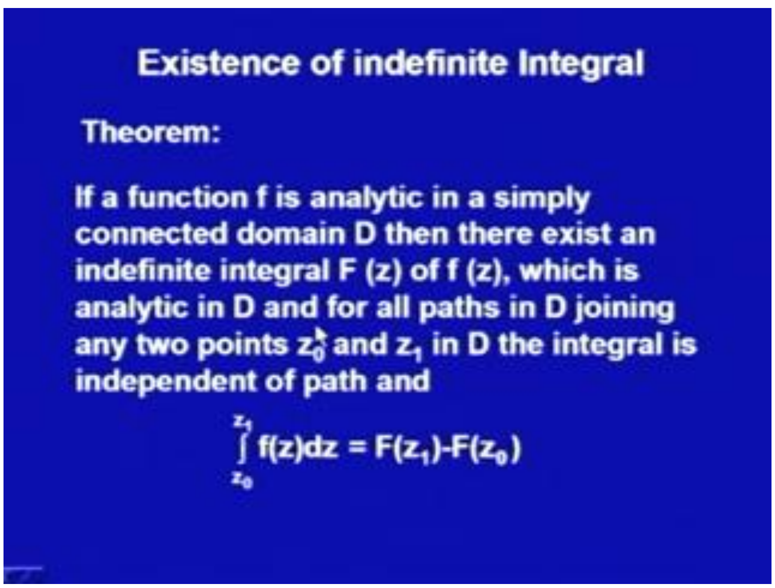

If a function f is analytic in a simply connected domain D, then there exists an indefinite integral capital Fz of small fz, which is analytic in D, and for all paths in D joining any two points z naught and z 1 in D, the integral is independent of path and can be given as integral from z naught to z one - now you see, I am not writing it that is integral along the path C , I am just writing from z naught to z 1. of fz dz is as capital F of z 1 minus capital F of z 2. You see, we had proved the Cauchy theorem when we had assumed the derivative of f was continuous. As I told you, that f dash the continuity of f dash was not required - that the proof was given by Gosset; but I have not done that here, because it is little bit more involved.

So you see, now we are moving towards that is when this fz is only analytic, we are not talking about any condition on the f dash being continuous; but what we have added up that is we have added up the condition that it is simply connected domain. So here what we are saying is Of course, we do know along any closed contour, the integral is 0; from there, we had obtained that it is independent of path; so these points we have got that is because f is analytic in the simply connected domain; then any path - if z naught to z 1, I do take a closed contour - we do get is that the integral along to that would be zero; that gives us our independence of path. Now the thing here what we are saying is that indefinite integral capital fz of fz is existing. This capital Fz - sometimes you also we call antiderivative. So let us see the proof of this theorem. How are we going to do?

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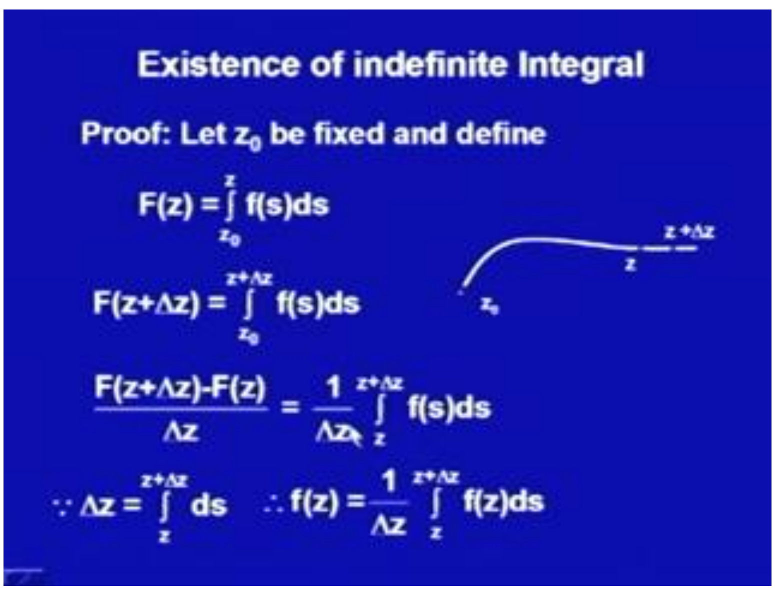

Let us have these two points, that is fix point z naught and i am having any path in the domain D from z naught its moving one to z. So be this one, and i define now capital Fz this is being define as integral from z naught to z fs ds along this path; this path is in the domain D. Now we want to show that this function actually exists. This function will exist because this is path in simply connected domain D. Its integral along any path from z naught to z - either i take this path or this path or that path - it will always be same. That says is, since the value for each one would be same, so existence you could say from here we are just saying is that does this does exists.

So I have defined that this exists. Now thing which we want to say is that is this is actually antiderivative of f; that is if I do not take these limits, I should say capital Fz is integral of its small fz dz. For that, what we have what we will do is will take this function fz and will show that the derivative of this function capital Fz is small fz.That is For that we will use the first definition of derivative. So for that what will we take? We will take one point, z plus delta z, in the neighborhood of z and we will extend this path - straight line we could say - till this one. Now this, I have taken in a small neighborhood of in a neighborhood of z. So it is again inside our domain and function is analytic because function is analytic in whole domain - so we are saying is the function is analytic in that path or on that path, extended path also.

So if it is happening, then according to this definition how we have defined this capital Fz? I could write this capital Fz plus delta z as integral from z naught to z plus delta z of fs ds. Now for using the first definition, I want a difference of Fz plus delta z with fz. What will be that difference? By this definition, it would be simply integral z naught to z plus delta z fs ds minus integral z naught to z fs ds. That is, integral from here to here minus integral from here to here : we do know from this contour integration that this would be nothing but the integral of fs - that function f on this small path from z two delta z. So that is what we are writing - is it is nothing but z to z plus delta z fs ds. And if I am dividing it by delta z - that's why, I should have this one upon delta z outside.

Now what we have to show that this limit of this is small fz. How do we do? Again we will go with the first definition. That is, I take its difference with a small fz and show that this difference can be made arbitrarily is arbitrarily small when delta z is made arbitrarily small or when delta z is this z plus delta z is approaching to z or this is small this z delta z is approaching to zero. The difference of this function with small fz can be made very very small; that can also approach to zero. So let us try for that one. let us first From z to delta z z plus delta z, the length of this path is only delta z.

So now, write this the length of path formula; we do get delta z is integral of z plus delta z ds. That is says Now, if I take f of z, that is at this point that is independent of all the points which are inside this on this simple line from z to delta z, so from here what we could write f of z I can write as one upon delta z integral z plus delta z fz ds. By this, fz is independent of any point, that is it's its not integrable over here; the this integral is this fz is constant with respect to the all the path of integrations; that can be taken out and this integral would be nothing but delta z. So delta z upon delta z is one; so this is fz, that we could write. Now, come to this Fz plus delta z minus Fz upon delta z minus Fz. You see, both I am getting is one upon delta z is common for both and in both the things, i am having the integral from z to z plus delta z. Of course, here is the function fs and here is the function fz. Again using the properties of this contour integration, we could say we could rewrite it as the same integral from z to delta z and we would write fs minus fz. So we let us see that is how we are writing it.

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$$\frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z)$$

$$= \frac{1}{\Delta z}\int_{z}^{z+\Delta z}(f(s)-f(z))ds$$

$$\left|\frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z)\right| = \frac{1}{|\Delta z|}\left|\int_{z}^{z+\Delta z}(f(s)-f(z))ds\right|$$

$$\leq \frac{1}{|\Delta z|}\varepsilon\Delta z = \varepsilon$$

$$\because |f(s)-f(z)| \leq \varepsilon$$

$$\therefore \lim_{\Delta z \to 0}\frac{F(z+\Delta z)-F(z)}{\Delta z} = f(z)$$

We are writing it as Fz plus delta z minus Fz upon delta z minus fz as one upon delta z z integral z to z plus delta z of fs minus fz ds. Now you see we want to make it arbitrarily small; so we take the modulus of this one. Modulus of this one would be modulus of this one; the modulus property we could write one upon mod of delta z and then mod of integral absolute value of the integral z to z plus delta z fs minus fz ds. Now, for finding out the absolute value of this one what we actually we have to show that this is arbitrarily small. So for this now, what will we do? We will use this ML inequality - if you remember. Now you see: fs minus fz. What is my region of integration? z to z plus delta z and we are talking about this path. So what we are talking about in this path you see, any s over here - its difference with z that will always be less than delta z; since I am taking a small neighborhood, so lets take this delta z is small enough. Since f is analytic - so this is small f is analytic - so f would be continuous in this domain; by the definition of continuity, if I am taking any a small neighborhood of the z, then the difference between the two points is a small, that is says that the fs minus fz would also be small - let us say this is epsilon. So what we are saying is that fs minus mz - mod of this - would be less than epsilon.

1 upon delta z is as such - this is less than epsilon - and then the length of path is delta z. So using this an ML inequality, I am getting this is less than epsilon. How have we done? I am just rewriting it. Since fs minus fz would be less than epsilon; whenever we are having is s that is in the small neighborhood of z because of the continuity of f. What it says is that Now this I am taking in this path- direction - this path now because I said is that is I am taking z plus delta z as in the neighborhood. So it's not necessary that I to take only this way, I can take here also, I can take here also till it is inside that domain D – that is simply connected domain D. So -whatever be side we could take it – here I am just using this condition - whatever direction I do take, this z plus delta z all the times the path i would be having is that a straight line over there.

And it would be always the independence of paths says that it will always be - the all the definition of fz plus delta z and fz - will remain same and we will always get because of the continuity of a small f, that this thing would be holding true. That is, says this would be can be made this difference can be made arbitrarily small for any delta z, that is in whole neighborhood of z. What it says is Now use the first definition of the derivative. Its says that Limit as delta z approaches to zero of fz plus delta z minus fz of upon delta z would be small fz. What does it say? It simply says by the definition of derivative that the derivative of capital Fz is a small fz or in other words we write that capital Fz is nothing but the antiderivative of small fz.

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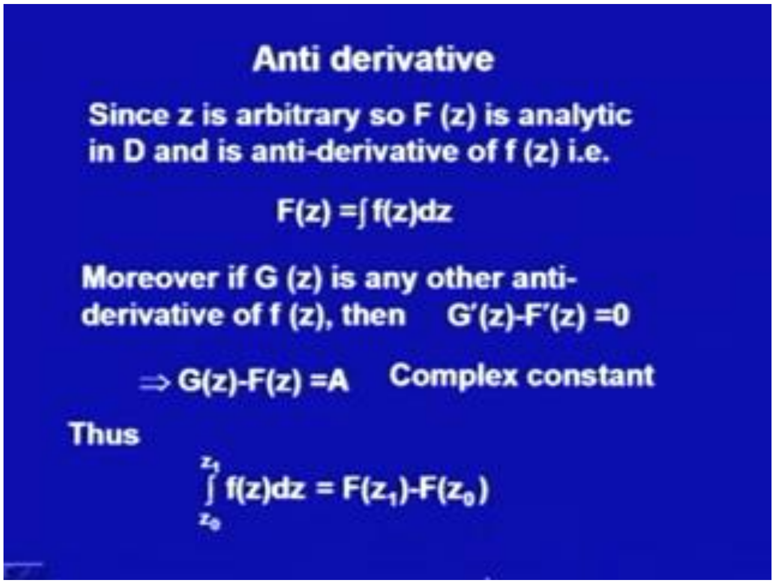

So we have got z is also arbitrary; so we have got that ah and that z also we have taken the arbitrary so now we have got this capital Fz is analytic in whole of the domain D and its antiderivative is and is the anti derivative of a small fz, that is we could say Fz is integral fz dz. Now, you are seeing is that is I am not taking z naught to z. Now we are saying is that since my z i have taken arbitrary, the z naught also I have taken arbitrary point. So now, I can use it general and I can write this antiderivative does exist. Antiderivative does exist, that is says my function capital F is analytic in that whole domain D. Now if it is happening and suppose there is another suppose there is some other antiderivative of this is small f which is capital Dz.

Then what will happen? We would say its because Gz is antiderivative, that means G dash z will also be a small fz. So the difference of G dash and F dash that would be small fz minus small fz; that will always be zero for all z. If i because they are anti derivatives and this they are the derivatives of f and capital F and capital G - if I integrate it - we do know that Gz minus Fz should be a constant. Now here, because we are talking about the complex , this should be a complex constant what we say. If there does exists any other antiderivative capital G of a small f, that would be nothing but addition of complex - the difference of the these two would be nothing but a complex constant. That says is, what will happen? The output of the our theorem which said is that integral along from z naught to z 1 of fz dz that is F of z 1 minus F of z 2.

Now you see if Why am I talking about this one? If capital G is some other antiderivative, then what it will happen? It will be have it will be actually capital G of z 1 minus capital G Fz 2. So how do we say is this should be capital F of z one minus capital F of z naught. You see What is capital Gz? Capital Gz would be nothing but Fz plus A. So G of z one minus G of z two when z naught - whenever I am writing - I would again write substitute it as Fz plus A, because A is fixed constant.

So whether we are changing is z naught or z one, that doesn't make change in the A.; I will always get this one. That says is, the antiderivative is existing and the definite integral or that integral between the two points Z naught and z one in that simply connected domain of this is small fz can be given as capital F of z 1 minus capital F of z naught, where capital Fz we are defining as the antiderivative of its small fz - without a complex constant. So thus we have got Cauchy theorem says is if my f is analytic in a simply connected domain D, then its antiderivative does exist and the integral is independent of path and then we can use this simple formula. That says now explains our example which we have done in the last lecture. So let us do one example.

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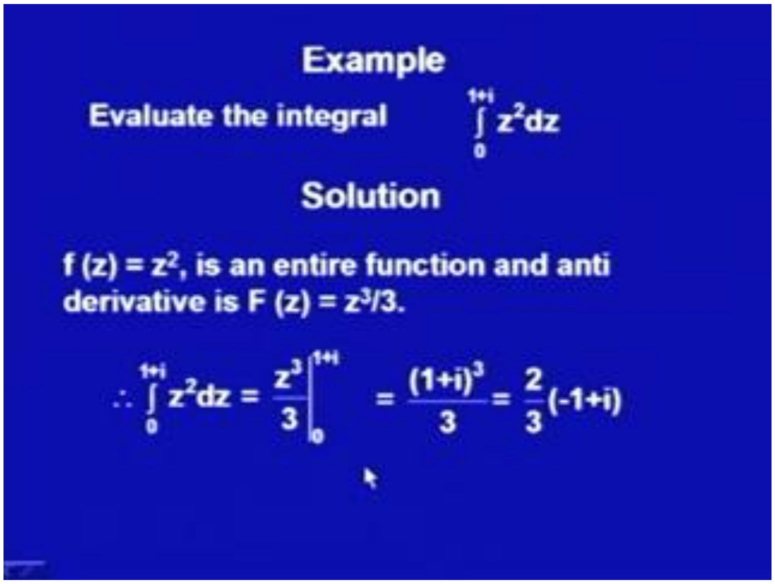

Evaluate the integral z square dz from zero to 1 plus i. You to see, that this I have we have done the example in the last lecture and we had done it along some paths. Here we do know that z square - this function - is actually entire function. So, whatever because this is an entire function, so whatever domain I am taking where this zero and one plus is lying, this fz would always be analytic. So and The antiderivative of this z square we do know is z cube by 3 -that you have done in some simple some lectures on this derivatives of this one. So if I take the derivative of z cube by three, it would be actually z square. So according to this result which we have it found out, integral from 0 to 1 plus iz square dz should be z cube by three; integral the evaluation from zero to one plus i. That says is 1 plus i cube by 3 or it is 2 by 3 minus 1 plus i.

You can compare this result which we have done in the last lecture. Similarly if I do take the function e to the power z or any integral power of z or any other analytic function any other entire function, then the derivative of anti derivative of all those functions we do know because in the differential derivatives chapters we you have done the derivatives of these functions; and since they are entire functions so whatever the two points we are taking, we can write it out. But, it is not necessary that we do talk about only entire functions. Of course, for entire functions we do not have to see any other thing, that is for entire functions, we will not see where the points of integrations have been given or whether they are laying the function is inside that one is in a simply connected domain or not; for entire functions we don't have to see those things. But its not necessary that we do apply to only the entire function. We could have analytic function in some domain. if that domain is simply connected and the points are inside that domain, is still this theorem is applicable.

So today we had learnt that when the function is analytic in any domain, then its integral is independent of path. Moreover we had learnt that its derivative is existing and its antiderivative is existing - are we called it indefinite integral - and then the integration can be found out using that antiderivative or indefinite integral evaluated at that the points, are that is the limits. And This is very simple as we have done in the definite integrals in the real analysis. So, we have come across the similar result as in the real analysis for the complex function in the complex domains also - so very nice result. That is all for this today's lecture. We will move a little bit more from here: that if function is not analytic – it is analytic everywhere, but at some points, its not analytic in the domain but it is analytic on the path - what will happen and all those things. We will discuss little more about this one; till here, we had come to that its complex analysis or integration of the complex function is similar to that of the real integrals or whatever we have done in the first course on the analysis. So, that is all for today's lecture. Thank you.