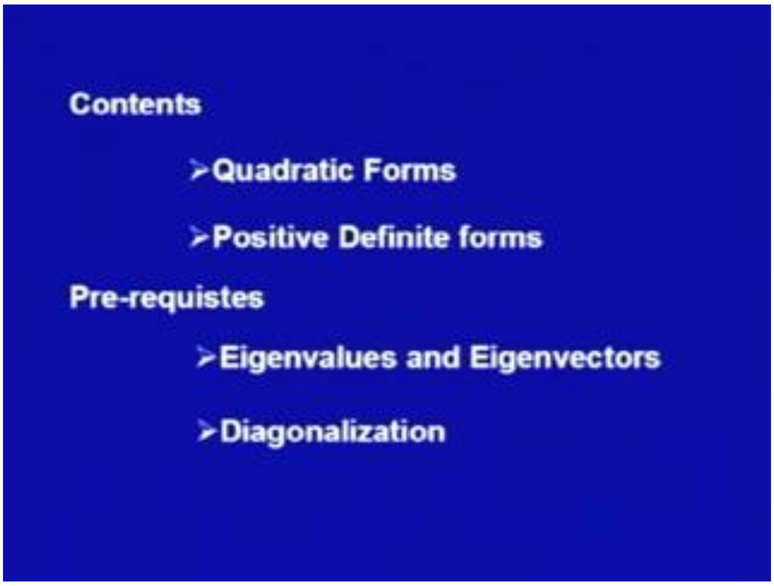
Mathematics - II
Prof. Sunita Gakkhar
Department of Mathematics
Indian Institute of Technology, Roorkee

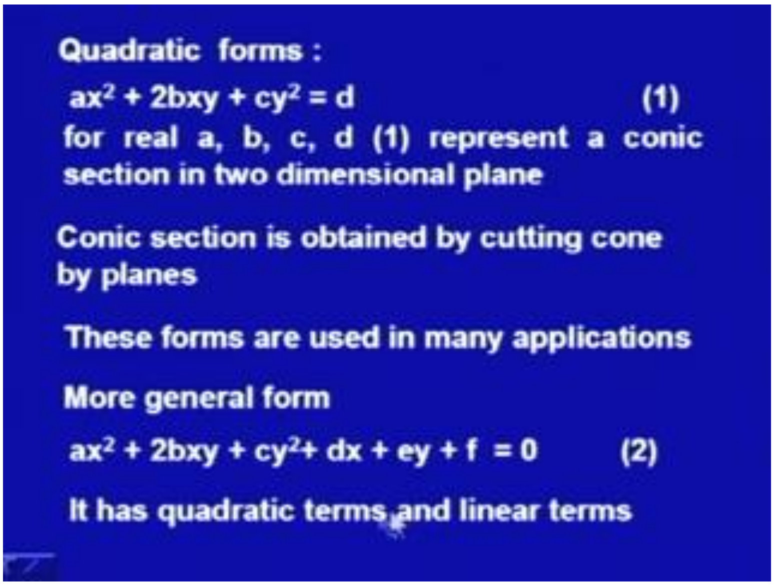
Module - 2
Lecture - 15
Quadratic Forms

(Refer Slide Time: 00:35)



Welcome viewers, today we will be discussing Quadratic Forms. This lecture includes quadratic forms and positive definite quadratic forms. The pre-requisite to this lecture is eigenvalues and eigenvectors, together with diagonalization. I suggest that before going through this lecture, the user must go through the eigenvalues and eigenvectors together with diagonalization. We start with quadratic forms.

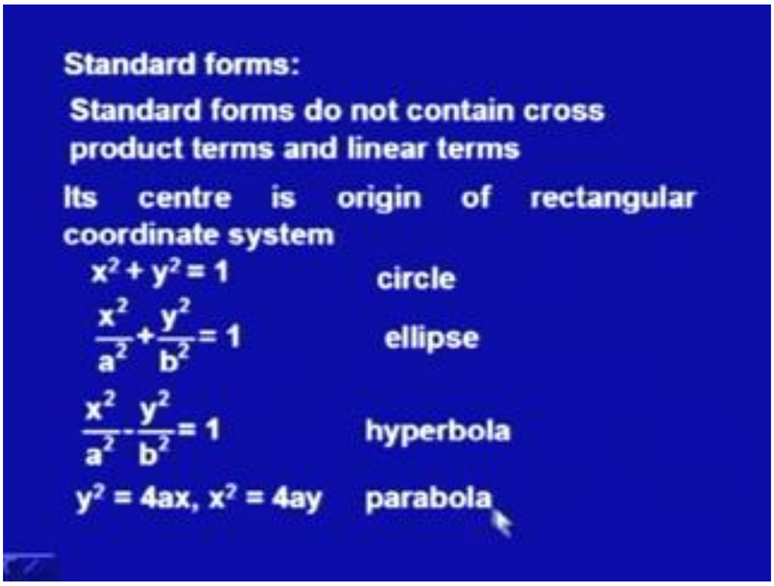
(Refer Slide Time: 01:01)



We all are familiar with this quadratic form a x square plus 2 b x y plus c y square is equal to d. We call it a quadratic form, because this has x square term, x y term and y square terms in this expression. And all these terms are quadratic. So, we say this form is a quadratic form; provided a, b, c and d are real numbers. And for these real numbers this expression represents a conic section, in two dimensional plane.

We call it a conic section, because it is obtained by cutting cones by planes. These forms are used in many applications. And that is why, we are discussing them in this lecture. This is a particular form, the more general form maybe a x square plus 2 b x y plus c y square plus linear terms d x plus e y plus f equal to 0. This more general form than this. It has all the terms, which are quadratic. But, this may have linear terms also.

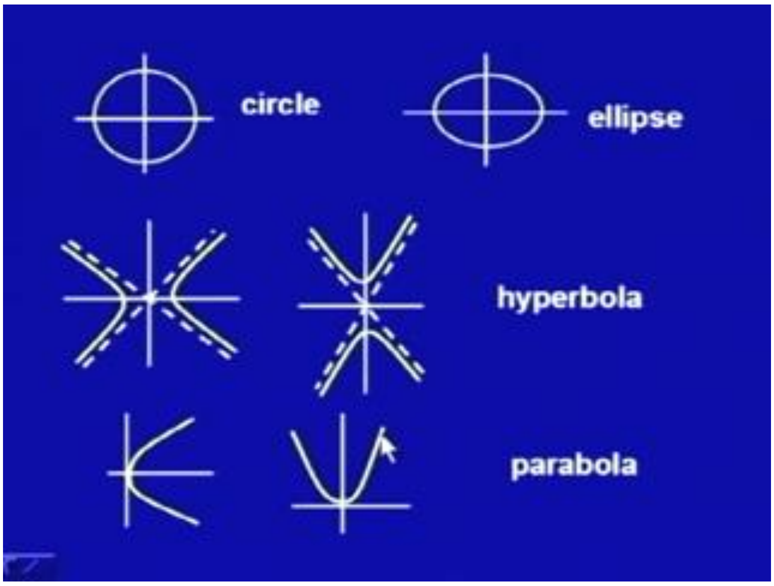
(Refer Slide Time: 02:17)



Now, general form is very difficult to recognize. So, we consider forms as a standard form, in which we do not have any cross product terms and linear terms. Like x y, x square or y square and terms like d x plus e y, etcetera. So, the term an expression, which is free of these linear terms. As well as product terms is called a standard form. It is centre is origin of rectangular coordinate system, that is why this is preferred.

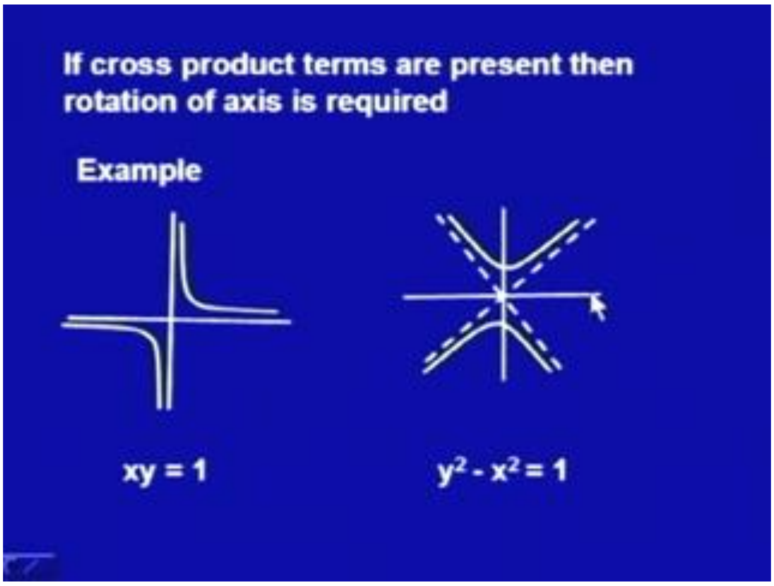
Like we have x square plus y square is equal to 1, we know it is a circle. It does not have any cross terms, does not have any linear terms. So, it is a standard form of a circle, having centre at origin and radius 1. Similarly, we have x square by a square plus y square by b square is equal to 1 represents an ellipse. Again this is free of cross terms and linear terms. Another form maybe x square over a square minus y square over b square is equal to 1, it represents hyperbola. And we may have a form y square is equal to 4 a x. So, x square is equal to 4 a y, it represents parabola.

(Refer Slide Time: 03:52)



This is the circle we all are aware of this. Now, this represents and ellipse. Here these are equal, but in case of ellipse the major axis and minor axis, these components are different. Now, this represents a hyperbola, the yellow lines, the yellow curves. These are the axis and in fact, these are the asymptotes of the hyperbola. In case of parabola, this is a parabola or parabola maybe like this.

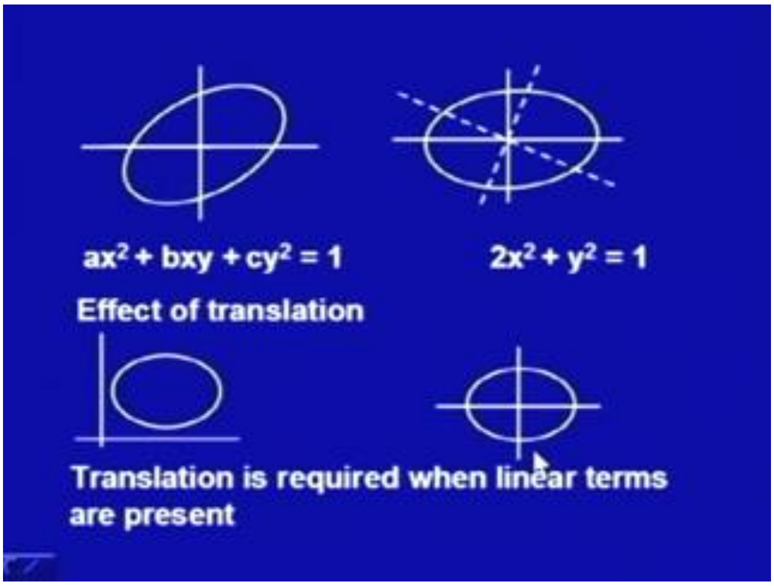
(Refer Slide Time: 04:34)



Now, if cross product terms are present. Then, we can bring again into standard form by suitably rotating the axis. So, if cross terms are present. Then, rotation of axis is required and we may get an expression which is in standard form. For example, x y is equal to 1 is this curve. So, we have rectangular axis and these are the curves representing x y is equal to 1. But, if you rotate this axis, then the curve will look like as this.

And this x y is equal to 1 will transform to y square minus x square is equal to 1, by suitable transformation. And here, we do not have any cross term present over here. But, this curve is the same as this, only thing is we have to rotate this axis. So, after rotation, this was the previous axis before rotation. And when you rotate it, it will be the new axis. So, this curve and this curve they are same the only difference is that this curve is rotated. So, after rotation cross terms can be eliminated. And one can very easily recognize this to be hyperbola.

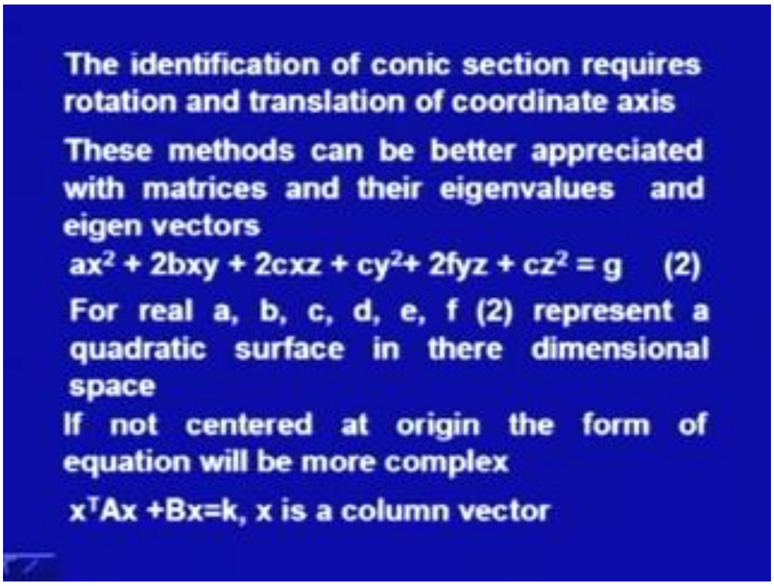
(Refer Slide Time: 05:53)



Similarly, this represents an ellipse, a x square plus b x y plus c y square is equal to 1, for suitable values of a, b and c, and then after rotation of axis this axis. Then, will have this ellipse. And the equation will be transformed to 2 x square plus y square is equal to 1. Now, this is the original axis and after rotation, this is the new axis. So, on the new axis it is 2 x square plus y square is equal to 1, which is standard form. And no cross terms are present over here.

Now, what will be the effect of translation. Translation is required, when linear terms are present to bring the system; to bring the given quadratic form into standard form. For example, if we have been given this ellipse. And the expression for this will require linear terms also. But, if you translate this axis. That means, to shift this origin to this place, then it will be a normal form. So, you are shifting this origin to the centre of the ellipse. And this will now be in a standard form. So, whenever linear terms are present one has to shift, one has to perform translation to get the standard form of the quadratic.

(Refer Slide Time: 07:27)

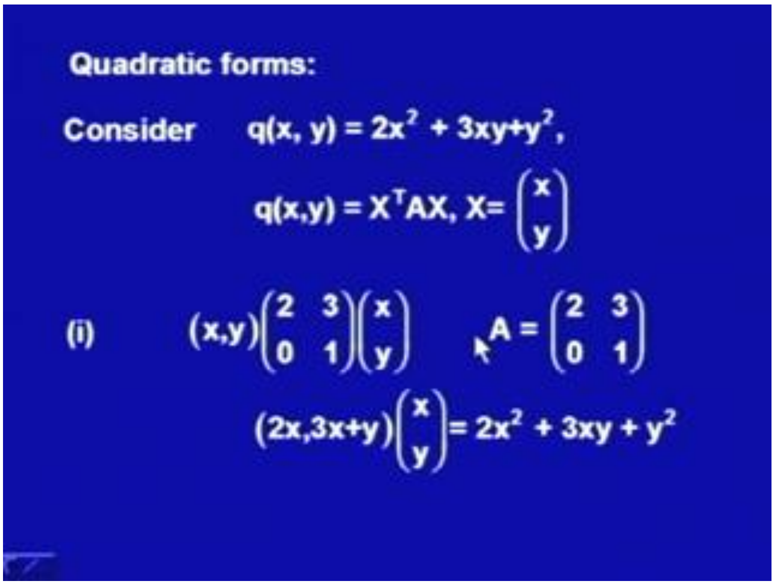


Now, the identification of conic section, requires rotation and translation of coordinate axis. So that, we can easily recognize by it is standard form. So, this rotation and translation are required, to bring the given quadratic into the standard form. And from there, one can easily recognize the quadratic. Now, these method can be better appreciated with matrices; and there eigenvalues and eigenvectors.

See what I have tried to explain here is, the case of two dimensions. But, if we are having three or more dimensions quadratic forms. Then, things will be difficult to visualize. But, with the help of matrices and eigenvalues and eigenvectors; one can very easily identify the quadratic forms. So, in general let us say we have three dimension surface. Or we call it a quadratic surface, which is a x square plus 2 b x y plus 2 c x z plus c y square plus 2 f y z plus c z square is equal to g.

So, this is a quadratic form in three dimensions, for a, b, c, d, e, f all the coefficients being real. Now, if not centered at origin the form of equation will be more complex. Now, there are no linear terms present in this. So, it is centered at the origin. But, if these terms are also there, then the surface will not be centered at the origin and form will be more difficult. In general form the quadratic is written as x transpose A x plus B x is equal to k. This is the matrix equation, where A is the square matrix and x is a column vector.

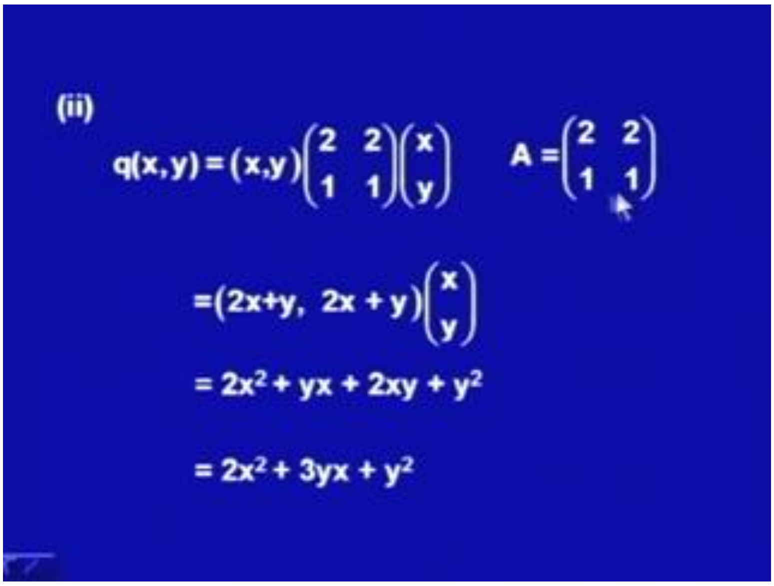
(Refer Slide Time: 09:24)



Now, we come to the quadratic forms, some examples. If we consider this quadratic form 2 x square plus 3 x y plus y square in two variables. Then, we can write this quadratic form in the matrix form as X transpose A X, where X is a column vector x and y. How we get the matrix A. A is a square matrix of order 2 by 2. So, we write this 2 and this one in the diagonals. And this quadratic cross term is written as 3 here and 0 here. And that makes the matrix A as 2 3 0 1. And let us see that this actually represents this quadratic.

So, let us multiply this, so we have 2 x. And this multiplied by this gives me 3 x plus y. When multiplied by x y, this gives me 2 x square plus 3 x y plus y square. So, this given quadratic is equivalent to this matrix representation, where A is this 2 by 2 matrix. But, this representation is not unique, you can check that.

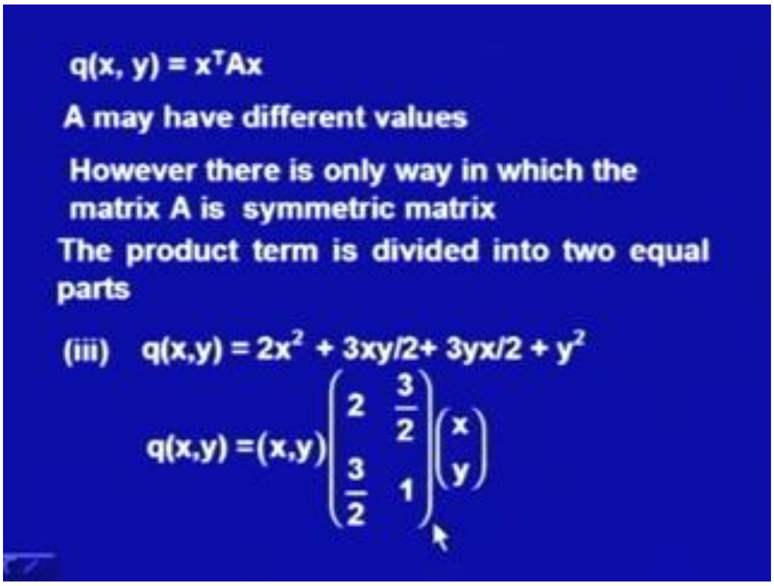
(Refer Slide Time: 10:55)



We can write down the same quadratic in this form. Like we have A as 2 2 1 and 1. What we have done is that, we have broken down that 3, the cross term in two parts. So, 2 is here and 1 is here. And accordingly, when we multiply, it is 2 x plus y. And this multiplied by this is 2 x plus y. Multiplied by this column vector, will give the same expression 2 x square plus y x plus 2 x y plus y square equal to 2 x square plus 3 y x plus y square.

So, if we breakdown this term as 2 and 1 will be getting this. You can break it as 1 and 2, again one can check that we will be getting the same expression for the quadratic. And this is not unique, you can breakdown in any number of ways. And you can have corresponding matrix A.

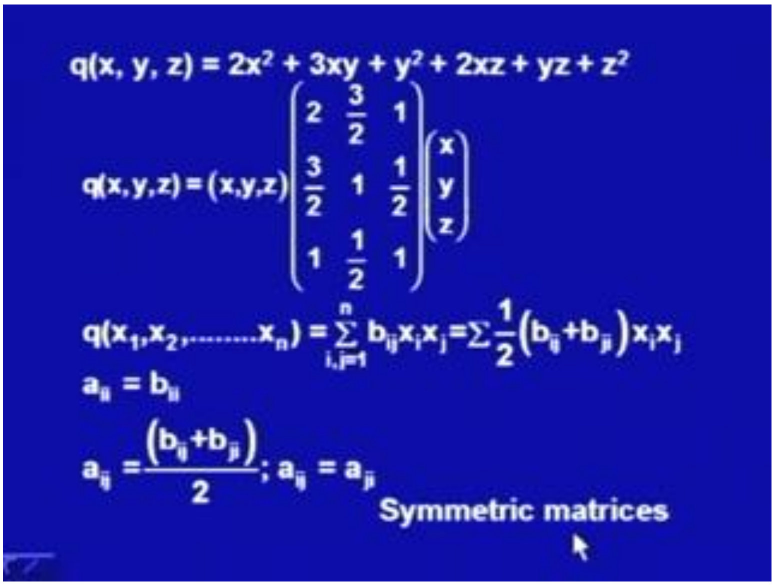
(Refer Slide Time: 11:59)



So, this q x y can be written in many different ways, as x transpose A x and A may have different values. However, there is only one way in which the matrix A is symmetric matrix. So, if we want to write down a matrix, which is symmetric. Then, we can have only one possible way to represent this quadratic as this matrix. So, let us see how we do this. For symmetric matrix, we divide the product term into two equal parts. That is if we have this quadratic. Then, 3 x y is divided into two parts, 3 x y by 2 here and 3 x y by 2 here.

And that makes the matrix A as 2 and 1, 2 is this square term. This one is this square term, and 3 by 2 corresponding to this and 3 by 2 corresponding to this. So, this matrix is the associated matrix A, with this given quadratic. And one may notice that, this is a symmetric matrix.

(Refer Slide Time: 13:13)



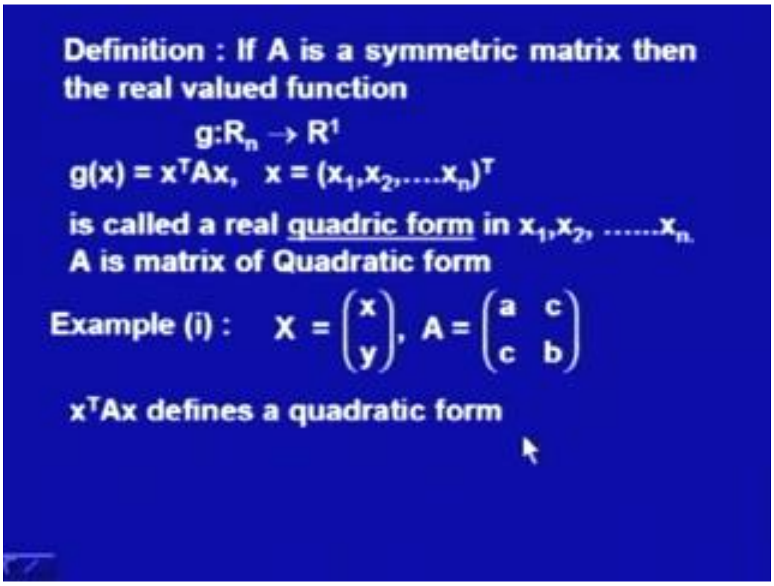
In the next example, I am considering a three dimensional surface. 2 x square plus 3 x y plus y square plus 2 x z plus y z plus z square. So, I need q x, y, z is equal to x, y, z. And a three dimensional 3 by 3 square matrix A, multiplied by the column matrix x, y, z. So, this 3 by 3 matrix is obtained as first term 2, corresponding to this term. Then, corresponding to this I have 1 here. From here, I will be having 1 here.

This 3 x y is divided into two equal parts. So, 3 by 2 here and 3 by 2 here, only x and y part. But, 2 x z will contribute, here and here, since it is 2. So, half of half it 1 is here and half of this is appearing here. And y z will appear in this part, this y z. So, half of y z is coming here and another half is here. So, that is how I obtain this matrix A. In general, if I have quadratic expression in n variables. Then, this can be written in this form, summation b i j x i x j i and j varies from 1 to n.

So, how to get b i j's, so I write b i j as b i j plus b j i by 2 multiplied by x i x j. So, I am writing this expression in this form. And from here, I can get the matrix a i j. So, a i j is a typical element of the matrix A; and this is equal to b i i, when i is equal to j. And when i is not equal to j, that means corresponding to the cross terms, I will write down this as b i j plus b j i by 2. And if this is what I am doing here, this is b i j and b j i is 3 here. So, divided by 2, that is a i j.

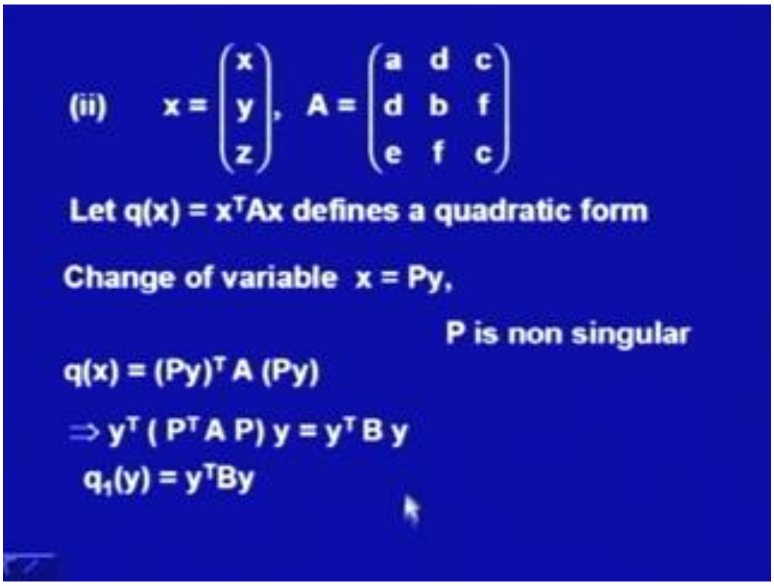
And if I define my matrix a i j in this manner. Then, one can easily observe, that a i j is equal to a j I, because if i is equal to j they are equal. And when i and j are not equal, then a i j is equal to a j i. So, for the matrix a i I have a i j is equal to a j i. And that simply means, that the matrix is symmetric. So, with this definition I can always represent a given quadratic in the matrix form, where the matrix is a symmetric matrix.

(Refer Slide Time: 16:04)



Now, I formally define quadratic form. We say that if A is symmetric matrix. Then, the real valued function g from R n to R 1 defined as g x is equal to x transpose A x, where x is a column vector; x 1, x 2, x n transpose. It is called a real quadratic form in the variables x 1, x 2, x n. So, if we can define such a function g x, represented as x transpose A x. Then, this g x is a quadratic form in variables x 1, x 2, x n. The matrix A is a matrix of quadratic form. For example, if I have a two dimensional vector x. So, we have x y is a column vector A is a symmetric matrix, a b in the diagonals and c c in the non-diagonal terms. Then, x transpose A x defines a quadratic form, one can check it easily.

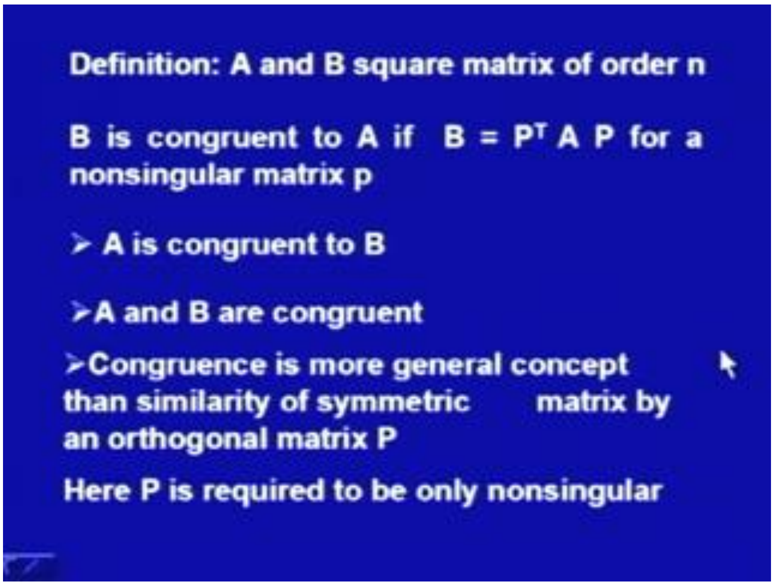
(Refer Slide Time: 17:12)



In the second example I have a three dimensional column vector x. And A is this symmetric form, d and d here c and c here f and f here. So, it is a quadratic form. By suitably changing the variable x is equal to P y; where P is a non-singular matrix. We can write down the quadratic expression P x. X is a column vector is equal to P y transpose. That what I am writing for x transpose A multiplied by P y, one can simplify it.

It is y transpose P transpose. So, P transpose into A and P multiplied by y, Because, multiplication is associative. So, you can regroup the terms and what we have is y transpose B y. Now, we can say this q x is not transform to another quadratic in y, we call it q 1 y, it is y transpose B y. So, quadratic can be transformed to another system. And this transformation will be x is equal to P y. So, we have given initially q x and then we have transform it into q 1 y.

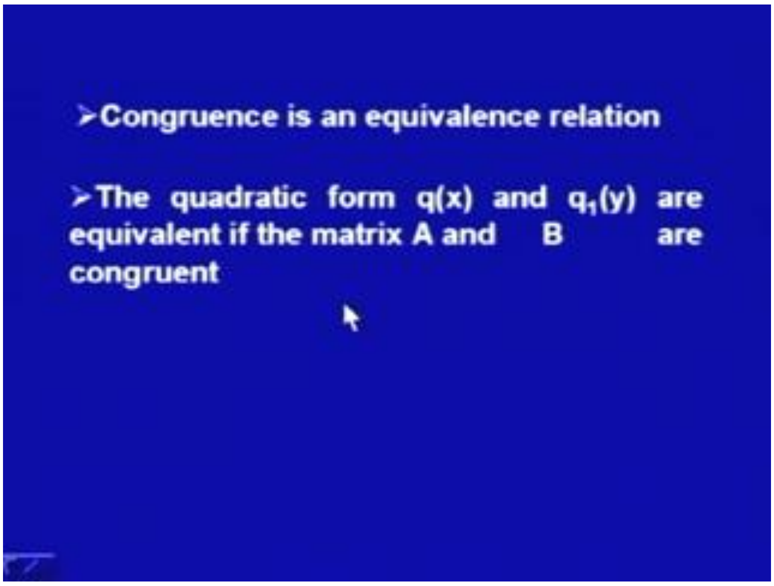
(Refer Slide Time: 18:48)



Now, we define that, if A and B are square matrices of order n. Then, B is congruent to A provided B is equal to P transpose A P, for a given nonsingular matrix P. So, the two matrices A and B are congruent, if they are related by this expression. So, B is equal to P transpose A P, for some non-singular matrix P. Then, B and A are congruent; and we can also say that A is congruent to B. If this condition is satisfied and maybe, we can in shot say that A and B are congruent.

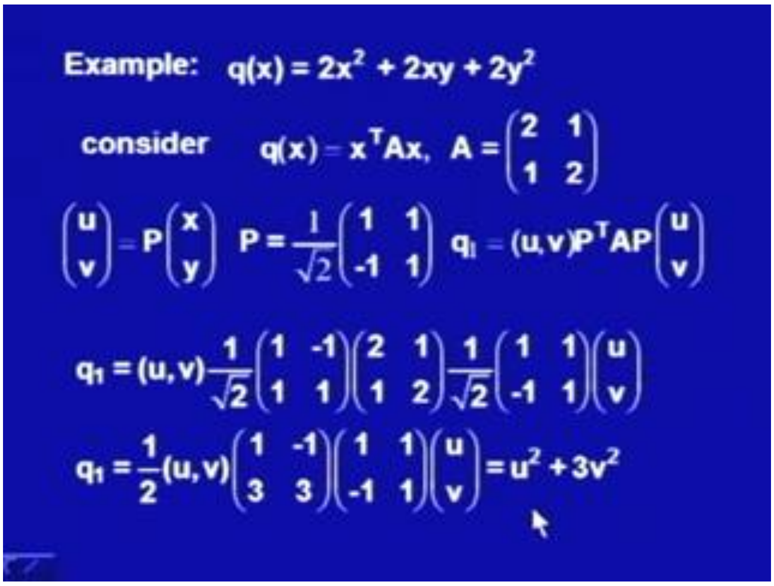
In fact, congruence is more general concept, then similarity of symmetric matrixes by an orthogonal matrix P. Since, we have already gone through this lecture, on similar matrices. We say that, B is equal to P inverse A P. Then, B is similar to A, and when P happens to be an orthogonal matrix. Then, P inverse and P transpose, they are same. And in that case, we can say that this is nothing but B is similar to A; where P happens to be an orthogonal matrix. But, congruence is a more generalized concept. Then, simply symmetric matrix, or it is more general concept, then similarity of symmetric matrices by an orthogonal matrix P. Hence, here for this concept, we require P has to be a nonsingular, which is not the case in this relationship.

(Refer Slide Time: 20:41)



We further prove that, congruence is an equivalence relation. By equivalence relation, I mean to say that, A is congruent to A. And if A is congruent to B, then B is also congruent to A, that is symmetric relation. And finally, transitive, that is if A is congruent B and B is congruent to C. Then, A is congruent to C. So, these things can be very easily proved. And we conclude that, congruence is a equivalence relation. The quadratic forms q x and q 1 y are equivalent. If the matrices A and B are congruent, so this is what we derive.

(Refer Slide Time: 21:26)

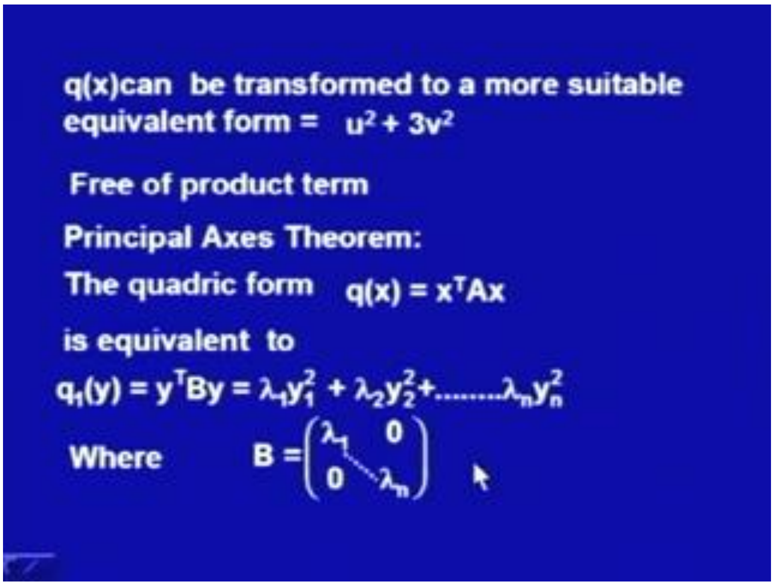


Let us check in the example, if q x is 2 x square plus 2 x y plus 2 y square. Then, q x can be written as x transpose A x, where A is the corresponding matrix. It is 2 2 and this term is 1 and 1. So, this is the representation for q x. Let us write down changed system u v is equal to P times the original system x y, where the matrix P is. Let us take this matrix P is 1 by root 2 1 1 minus 1 1.

How we arrive at this matrix, that I am not discussing at the movement. But, if we use this transformation, then q 1 comes out to be u v P transpose A P multiplied by u v. And that means, if you perform this multiplication, then q 1 comes out to be. We are substitute these values here, q 1 comes out to be u square plus 3 v square. The idea is I have started with this quadratic, in which some cross terms are present, but after this transformation, which I have applied here.

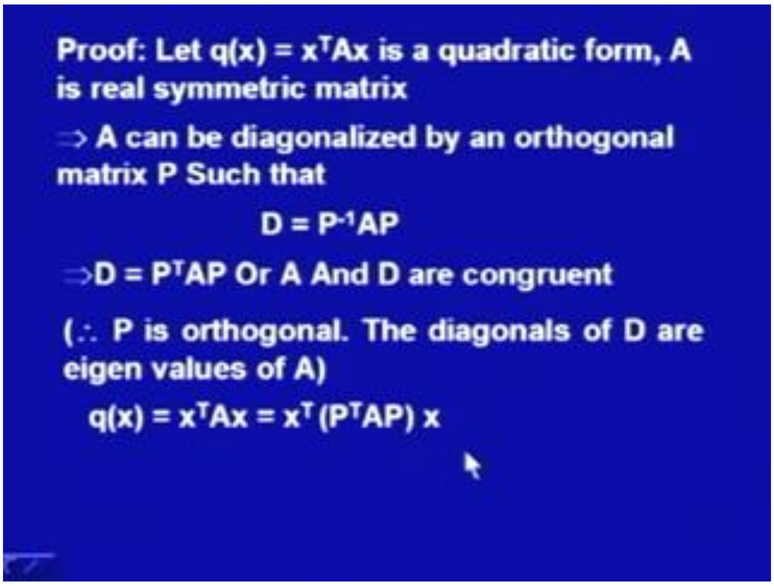
The congruent form is free of cross product term. So, it is now written as u square plus 3 v square. That means, given a quadratic, we can transform into a form, where cross term will not be present. And we know that handling, such things are simpler as compared to this.

(Refer Slide Time: 23:09)



So, q x can be transformed to a more suitable equivalent form, as u square plus 3 v square, it is free of product term. And this can be done for this is an example only. This can be done in general and for that, we will have principal axes theorem. According to this, the quadratic form q x is equal to x transpose A x is equivalent to this form. Q 1 y is equal to y transpose B y, which is lambda 1 by 1 square plus lambda 2 y 2 square plus lambda n y n square, where lambda 1 happens to be the eigenvalues of the matrix associated with this quadratic A. So, B is a diagonal matrix. And these terms are the eigenvalues of the matrix A. So, this is called principal axes theorem. So, for symmetric matrix, this is possible. So, let us try to prove this, here is the proof.

(Refer Slide Time: 24:21)

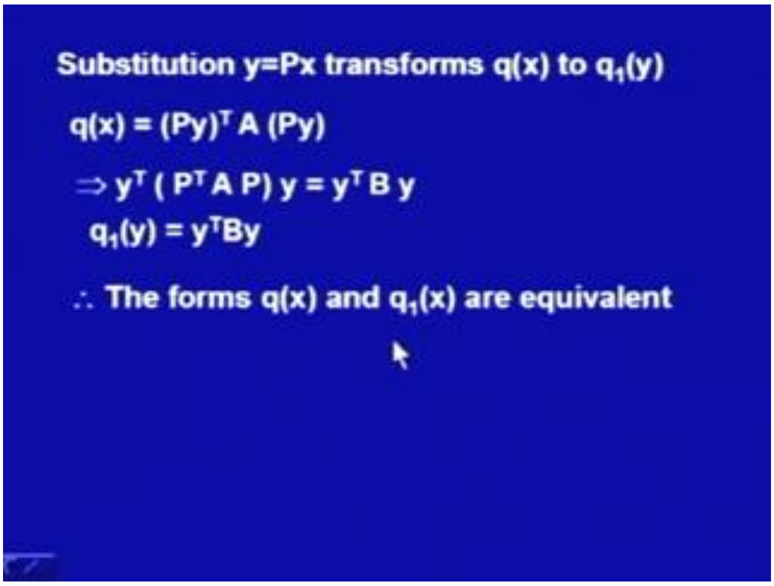


We consider q x is a quadratic represented, as x transpose A x, A is the associated matrix. And it is a real symmetric matrix. Now, A can be diagonalized by an orthogonal matrix P. Such that, D is equal to P inverse A P, this is the result, which we have already discussed in my earlier lectures. So, if A is the real symmetric matrix. Then, it can always be diagonalized by an orthogonal matrix P, which is the matrix of eigenvectors of this matrix A.

So, if P is the matrix of eigenvectors, then P is invertible. So, P inverse exist, in that case D is equal to P inverse A P will be a diagonal matrix. And since, it is this P inverse I can write it as P transpose. Because, it is a orthogonal matrix. So, P inverse is equal to P transpose. So, D will be P transpose A P or we can say, if we can write down D in this form, then by the definition of congruent. We can say that, A and D matrices are congruent.

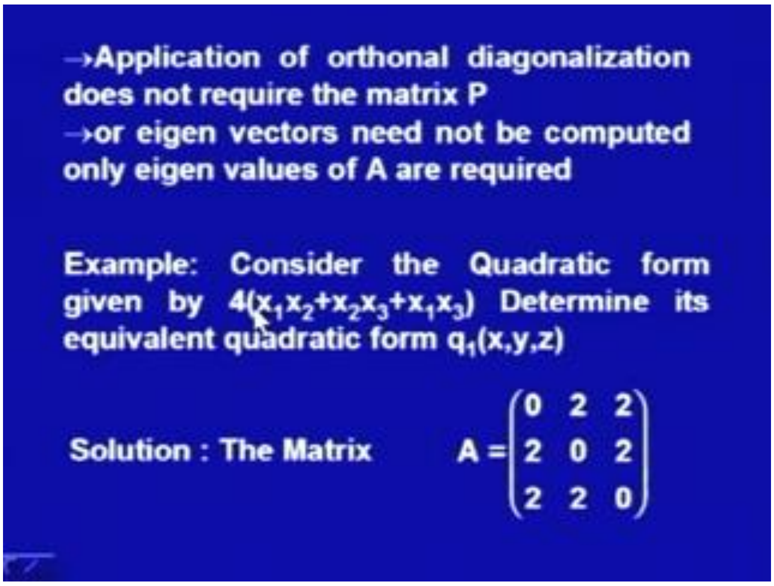
Now, this is possible, because P is orthogonal. That is I have already explained and the diagonals of D are eigenvalues of A. So, all this requires the earlier lectures, where on diagonalization. And with this q x is represented as x transpose A x equal to x transpose P transpose A P x.

(Refer Slide Time: 26:12)



And then we can use this substitution y is equal to P x. And then we can easily check that, q x will transform to q 1 y. So, q x is written as P y transpose A P y, this simplifies to y transpose P transpose A P y. We can regroup the terms. And it is y transpose multiplied P transpose A P multiplied by y. And I write down P transpose A P as B. So, this expression becomes y transpose B y. So, this q x is now transformed to q 1 y, where this is a again a quadratic form. But, the associated matrix is B, so A and B are congruent matrices. So, the forms q x and q n x are equivalent.

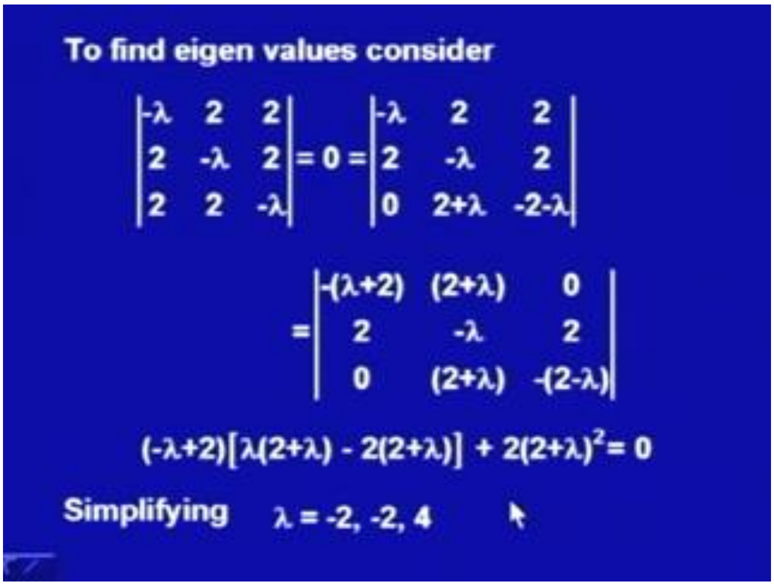
(Refer Slide Time: 27:15)



And application of orthogonal diagonalization, does not require the matrix P. Or eigenvectors need not be computed only eigenvalues of A are required. That is the advantage of this form. We do not have to require the matrix P,, only the eigenvalues are required. This is illustrated in the example. So, let us consider the quadratic form 4 multiplied by x 1 x 2 plus x 2 x 3 plus x 1 x 3, are quadratic in three dimensions.

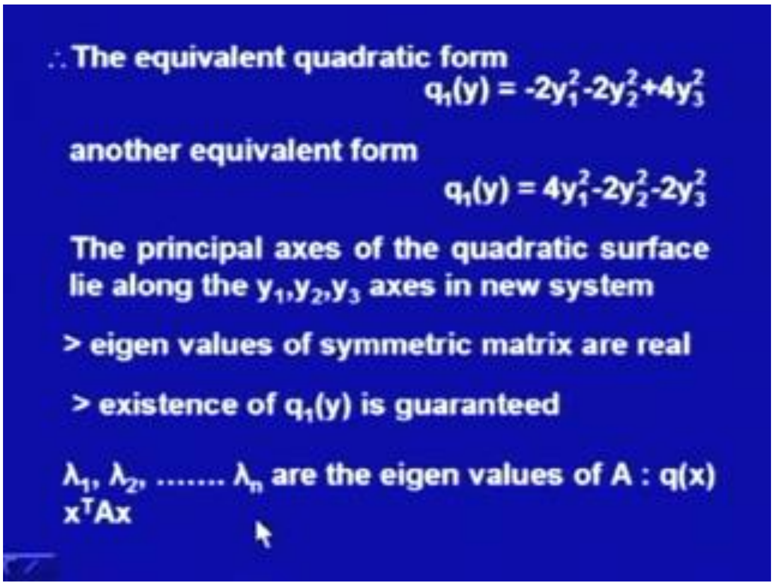
So, determinate it is equivalent quadratic form q 1 (x, y, z), in which the associated matrix is a diagonal matrix. Or this form is free of cross terms. So, let us consider the matrix corresponding to this. Since, all the diagonal term, all the square terms are absent, so diagonal will be 0. And it is 4 times, so it is 2 here and 2 here corresponding to x 1 x 2. And this 2 and 2 corresponding to x 1 x 3, and this 2 and this 2 corresponding to 4 times x 2 x 3. So, this is the associative matrix with this quadratic form.

(Refer Slide Time: 28:35)



Now, to find eigenvalues consider this characteristic equation. It is determinant of lambda i minus a. So, this is lambdas, minus lambda will appear in the diagonal. And this is rest of it is a, so it is a minus lambda i equal to 0. And you simplify it, the characteristic equation will come out to be, this is the simplification. And this gives me, this expression, which will be further simplified to give lambda is equal to minus 2. And minus 2 and one eigenvalue is 4. So, two roots are repeated and one is 4. We have three eigenvalues for this given matrix A.

(Refer Slide Time: 29:31)



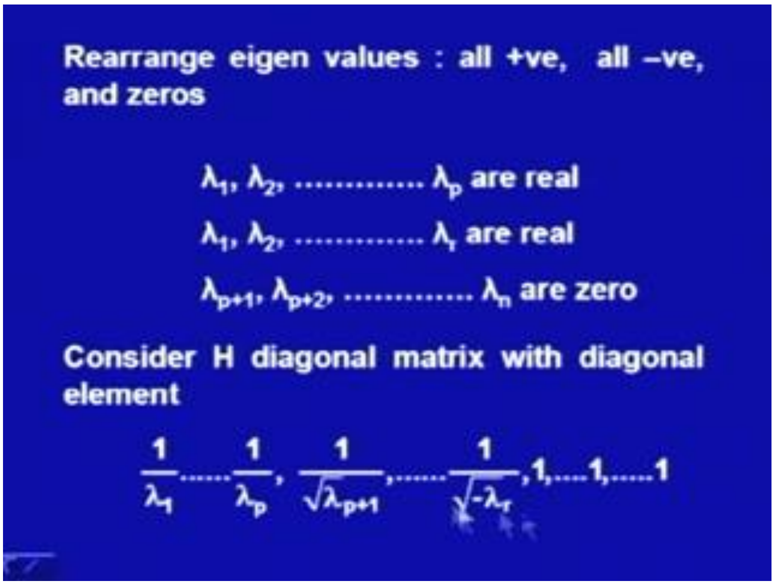
So, the equivalent quadratic form is simply minus 2 y 1 square minus 2 y 2 square plus 4 y 3 square. And we do not have to compute P and P transpose. And perform the multiplication to arrive at this quadratic form, only eigenvalues are needed. So, there are two eigenvalues minus 2 minus 2. So, we have two terms, minus 2 y 1 square minus 2 y 2 square corresponding to them. And third term is 4, because the eigenvalue is 4.

So, we have here 4 by 3 square, the another equivalent form maybe this. Here, again I am having two eigenvalues minus 2 minus 2 and one value 4. But, what I am doing here is, I am associating these two negative values with y 2 and y 3 and y 4, is associated with y 1. In this case 4 was associated with y 3. So, these are alternative forms and they are equivalent. Now, we say of this theorem as a principal axes theorem. Because, y 1, y 2, y 3 are actually the principal axes of the quadratic surface.

And these axis lie along y 1, y 2, y 3 axis, in the new system, and that is the reason, we say this, that this form is a standard form. And the theorem changes our coordinate axis to corresponding principal axis. And that is why, we say the theorem as principal axis theorem. Now, these are some of the results, which we have obtained earlier. And the actually, these forms the basis for finding out q 1 y.

We know that eigenvalues of symmetric matrices are real. So, these values are real, so we always have. We can always transform our given quadratic in this form, where all these coefficients are real. Because, eigenvalues are real, so this is always possible. If Eigen values of a matrix are complex. Then, this would not have been possible, but since the matrix is a symmetric matrix. So, the eigenvalues are always real, and we can always find out this form. And existence of q 1 y is guaranteed, due to the real eigenvalues. So, if lambda 1, lambda 2, lambda n are the eigenvalues of A.

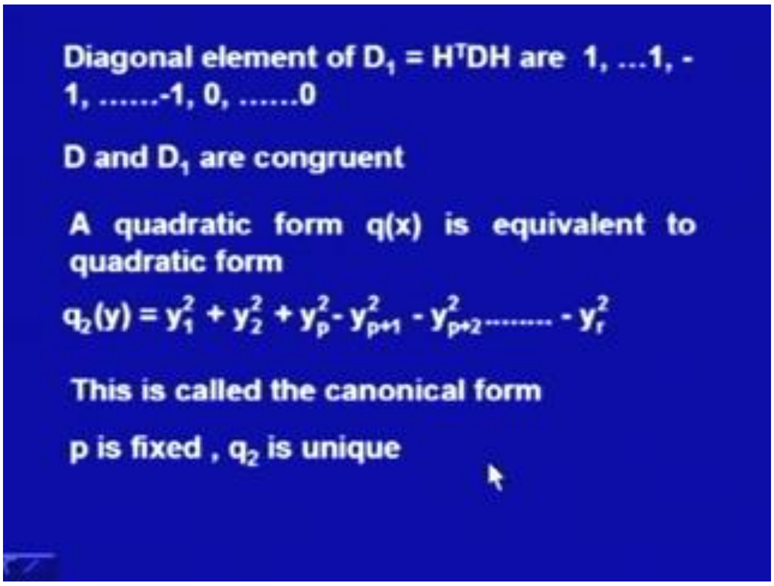
(Refer Slide Time: 32:25)



Then the q x, which is given as x transpose A x can be transformed to the real eigenvalues. Lambda 1, lambda 2, lambda n are arranged in a manner. So, that all positive eigenvalues appear first. And then all negative values and finally, the zeros eigenvalues. So, if lambda 1, lambda 2, lambda p are real positive values. And lambda 1, lambda 2, lambda r are real values. Then, lambda p plus 1, lambda p plus 2, lambda n are 0 values.

So, r happens to be the rank of the matrix p and out of these r, p are positive and remaining will be negative, while lambda p plus 1 to lambda n or eigenvalues are 0. Now, we consider an another diagonal matrix, H with diagonal elements, as 1 over lambda 1, 1 over lambda p, 1 over under root lambda p plus 1. And then the negative terms are written with minus sing under root. So, lambda r is negative, so we write it as minus lambda r it is under root. And then we write down 1, 1, 1 corresponding to these zero value. So, I form a matrix H with these eigenvalues.

(Refer Slide Time: 33:54)



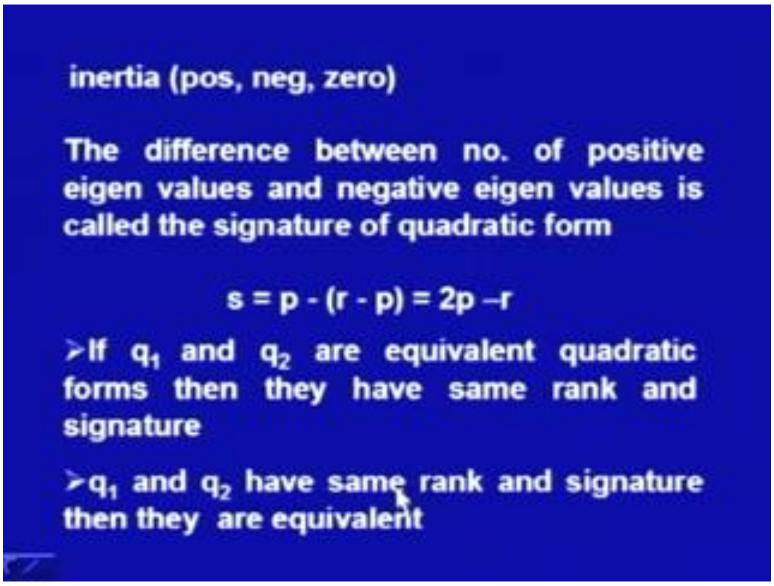
Now, let us consider this matrix D 1 as H transpose D H. Now, H, D and H are all diagonal matrices. And H is having term, positive terms as 1 over under root lambda. And H transpose is also having 1 over under root lambda, while, D is having under root lambda. So, when you multiply this every term in the diagonal of D 1, will be either having 1 or will be having minus 1; depending whether the corresponding eigenvalue is positive or negative.

And then the values corresponding to 0, will be having 0 over here. So, D 1 will either be having 1 or minus 1 or 0. So, this way we can write down the diagonal matrix D 1. And once we can express D 1 as this matrix, we can say that D and D 1 are congruent. And that means, a given quadratic q x is equivalent to a quadratic form q 2 y, which is equal to y 1 square plus y 2 square plus y p square positive terms minus y p plus 1 square minus y p plus 2 square up to y r square with negative terms.

The rank of this associated matrix was r. So, we will have at the most r terms some of them will be positive, some of them will be negative. The difference between q 1 y and q 2 y is in the coefficients. In q 1 y the coefficients are eigenvalues. But, in q 2 y the coefficients are plus 1, or minus 1 depending upon the sign of corresponding eigenvalue. So, for a given matrix q x, we can always transform it into a form; where q 2 y is y 1 square plus y 2 square plus y p square minus y p plus 1 square up to minus y r square.

So, this particular form is called canonical form. So, when p is fixed, q 2 is unique. So, this is the result, which we are having. And given polynomial can always be transformed to the corresponding canonical form; and it is a unique form.

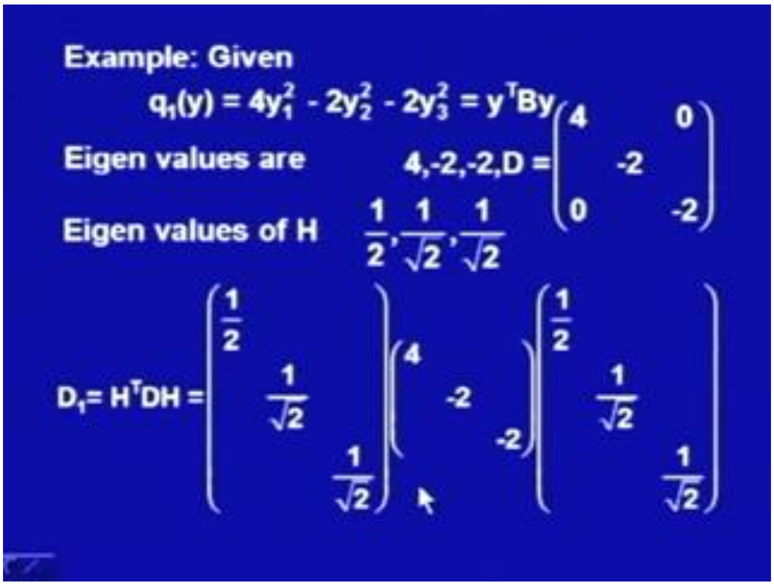
(Refer Slide Time: 36:32)



Now, I will introduce a term inertia, which is a trial consisting of three values. This first value represents, how many eigenvalues are positive. The second term represents the number of negative eigenvalues. And zero is the number of 0 eigenvalues. So, inertia of associated with the given quadratic is trial, indicating the number of positive eigenvalues. Negative eigenvalues and zero eigenvalues and from there, one can find out the canonical form.

Now, the difference between the number of positive eigenvalues, and negative eigenvalues is called the signature of quadratic form. So, if I am having p number of eigenvalues, which are positive and r being the rank. Then, r minus p will be the negative eigenvalues remaining will be 0. So, p minus r minus p is equal to 2 p minus r. So, we say that 2 p minus r is the signature of the quadratic form. We say, if q 1 and q 2 are equivalent quadratic forms. Then, they have same rank and same signature, this result is easy to prove. Then the next result is that q 1 and q 2 have same rank and signature. Then they are equivalent, it is just converse of this.

(Refer Slide Time: 38:06)

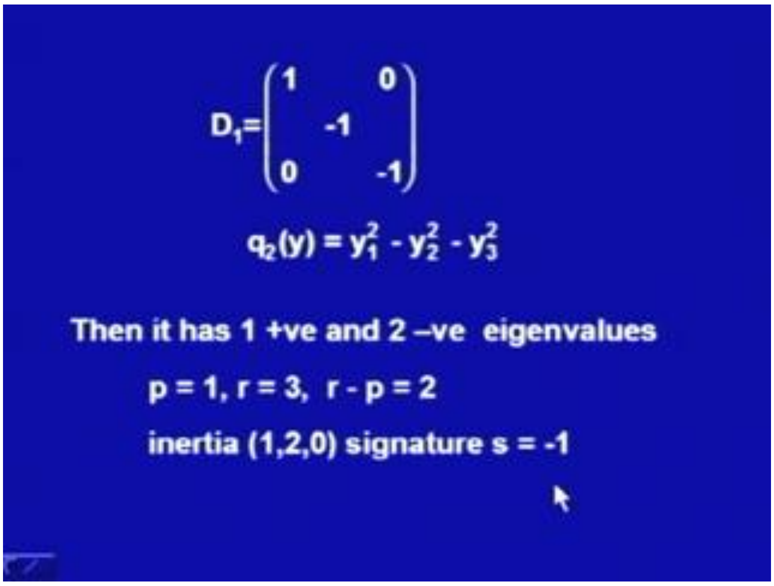


Let us illustrate this with an example, I have been given a quadratic 4 y 1 square minus 2 y 2 square minus 2 y 3 square. It is eigenvalues are 4 minus 2 minus 2 and D is 4 minus 2 minus 2. Actually this quadratic is been taken from earlier example. And we have already computed the eigenvalues of that matrix, as minus 2 and minus 2. So, for this quadratic eigenvalues are 4 minus 2 minus 2. And correspondingly, we have the diagonal matrix as 4 minus 2 minus 2 and of diagonal terms are all zeros.

Then, the eigenvalues of H will be 1 by 2 it is under root of 1 by lambda. So, it is 1 by 2 and then the next eigenvalue will be 1 by minus of the eigenvalue. So, that is why, it is coming out to be 1 by root 2. And another eigenvalue of H will be 1 by root 2. Now, let us now compute D 1 as a H transpose D H. So, I am writing H transpose, now H is a diagonal matrix. So, it is transpose will be the same as H.

And the values in the diagonal will be 1 by 2, which is under root of 1 by 4. Then, 1 over under root 2, it is under root of minus of minus 2. So, it is 1 by root 2 and we have 1 by root 2. Then, this is the matrix D diagonal elements, I have not shown here. But, they are zero elements and will have this diagonal matrix H. And you multiply it ((Refer Time: 39:54)), this multiplied by this and this multiplied by this is 1, this, this. And this multiplication gives me minus 1, this, this and this is minus 1.

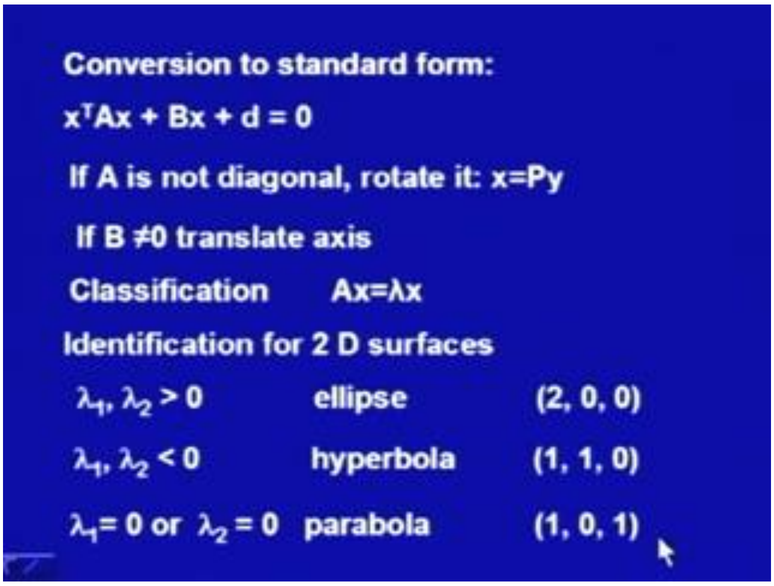
(Refer Slide Time: 40:06)



And correspondingly, we have D 1 is equal to a diagonal matrix with 1 minus 1 and minus 1. Accordingly q 2 y, comes out to be y 1 square minus y 2 square minus y 3 square. So, these are two negative eigenvalues. So, it is minus y 2 square minus y 3 square. And since, it has one positive eigenvalue and two negative eigenvalues. So, we can say p is equal to 1 rank of this matrix is 3. So, number of negative values will be r minus p that is 2.

And from here, inertia is positive value is 1, negative value is 2, 0 value is 0. So, number of 0 values is 0 here, so inertia is destroyed. And signature of this quadratic form is s is equal to minus 1. Because we are having more negative eigenvalues compared to positive eigenvalue; so signature come out to be negative.

(Refer Slide Time: 41:08)



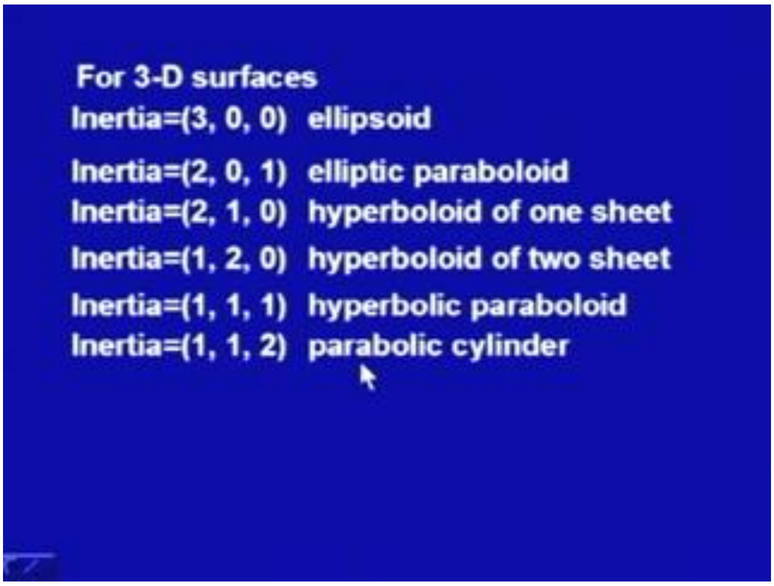
Now, we start with the general form of the quadratic equation, x transpose A x plus B x plus d equal to 0. Of course, we have till now concentrated on this part x transpose A x. So, let us consider, we have a general form x transpose A x plus B x plus D is equal to 0. Then, if A is not diagonal matrix. Then, we rotate it by this transformation, x is equal to P y. And it will become diagonal matrix and if B is not 0. Then, we translate the axis, so that we can get rid of this.

And we finally, have x transpose B x, or we can say y transpose B y plus D equal to 0. So, this can be easily converted into a standard form. And the classification of this standard form, actually depends upon this eigenvalue problem A x is equal to lambda x. And that is how, we identify or classify 2 D surfaces. See this eigenvalue problem, if it has 2 eigenvalues which are positive. Then, the corresponding standard form will be an ellipse, it is inertia will be 2. Because, number of eigenvalues are 2, it is two dimensions.

So, negative values will be 0 and no eigenvalue is 0 in this case, it is hyperbola when both the eigenvalues are negative, when the product lambda 1 and lambda 2 is negative. So, one eigenvalue is positive and one eigenvalue is negative in this case. So, one for number of positive values, one negative eigenvalues no value which is 0, and it is parabola, when one eigenvalue is 0.

So, either lambda 1 is equal to 0 or lambda 2 is equal to 0. Then, it will represent a parabola, it has one positive value, it has no negative value and 1, 0 value. So, depending upon the inertia corresponding to given matrix in the given representation. We can classify the quadratic as ellipse hyperbola or parabola; and corresponding inertia is given as this.

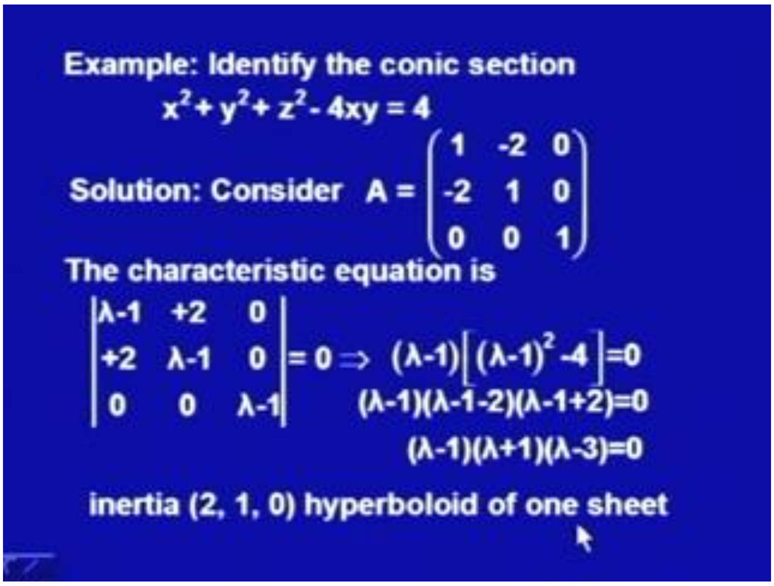
(Refer Slide Time: 43:47)



Now, for 3 D surfaces we can similarly classify a quadratic surfaces. We say if inertia is 3, 0, 0, then it is ellipsoid. That means, all the 3 eigenvalues are positive. No eigenvalue is 0 and nothing is 0, no eigenvalue is 0 in this case. Then, we have ellipsoid. And when inertia is 2, 0, 1. Then, we call it elliptic parabolic. So, 2 eigenvalues are positive, one value is 0, no eigenvalue is negative, we will have elliptic parabolic.

Then, inertia is 2, 1, 0 that means, 2 positive eigenvalues, 1 negative eigenvalue. Then, we say it is hyperboloid of one sheet. Then, 1, 2, 0 is inertia, we have 1 positive value, 2 negative eigenvalues. Then, we will have hyperboloid of 2 sheets. And when inertia is 1, 1, 1. That means, we will have 1 positive, 1 negative and one 0 eigenvalue. Then, we call that surface as hyperbolic parabolic 1, 1, 2 represents a parabolic cylinder.

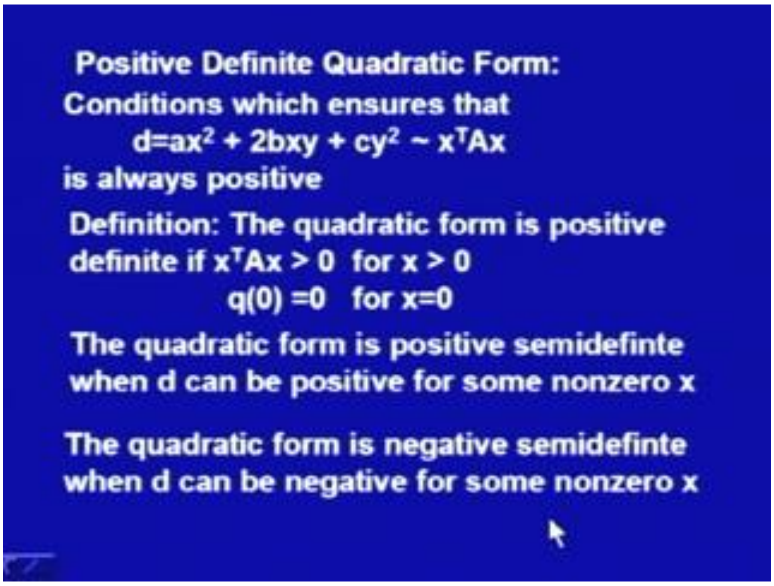
(Refer Slide Time: 44:58)



So, that is how we classify a different surface. Now, in this example you have to identify the conic section. So, the conic section is given, as x square plus y square plus z square minus 4 x y is equal to 4. It is a three dimension surface. So, to identity what it represent, we have to calculate it is inertia. For that, we have to first write down the corresponding matrix A. And terms in the diagonal are 1, 1, 1. And this minus 4 corresponds to minus 2 and minus 2, in this position in the matrix A.

Then, we calculate the eigenvalues. For this, we consider the characteristic equation as this. And this simplifies to this equation characteristic equation, lambda minus 1 multiplied by lambda minus 1 square minus 4. And further simplification will give me lambda, minus 1 minus 2 and lambda minus 1 plus 2. So, then eigenvalues are given from this expression, and they are 1 minus 1 and 3. So, we will have two positive eigenvalues namely 1 and 3. And one negative value minus 1. So, the inertia for this given matrix is 2, 1, 0. And according to the classification, this surface is hyperboloid of one sheet.

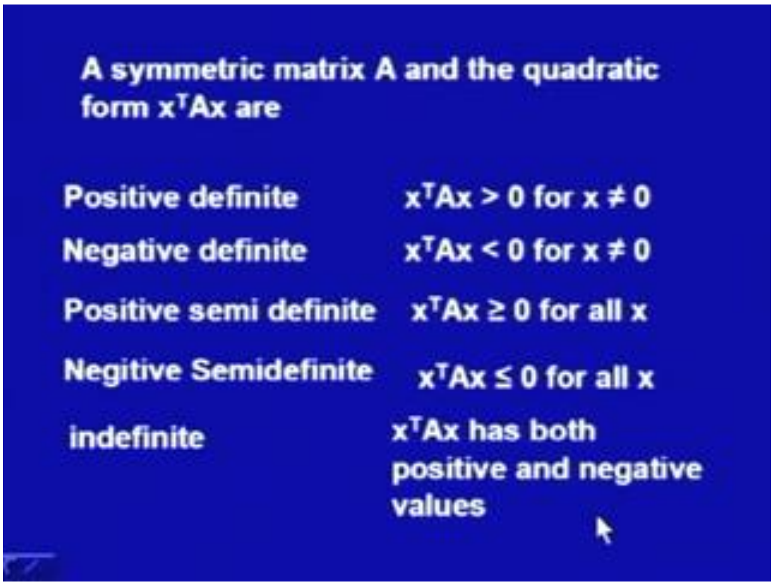
(Refer Slide Time: 46:31)



Now, we will discuss positive definite quadratic forms. Many times, we are interested to know the sign of this expression a x square plus 2 b x y plus c y square, it is a quadratic expression. Let us say is equal to d and in matrix location, we write it as x transpose A x. So, we want to know, whether this expression is positive or negative. So, we want to know under, what condition this expression is positive. Under what condition it may be negative and under what conditions, it is always positive.

So, those conditions refer to positive definite quadratic form. So, I will define the quadratic form is positive definite. If x transpose A x is always positive, for positive x and it is 0 only when x is equal to 0. So, expression x transpose A x, if it is always positive. Then, that expression is or that quadratic form is positive definite. The quadratic form is positive semi definite when d can be positive for some nonzero x. It may be positive for other values. But, there may be some value of x which is not 0, but d is positive. The quadratic form is negative semi definite, when d can be negative for some nonzero x also.

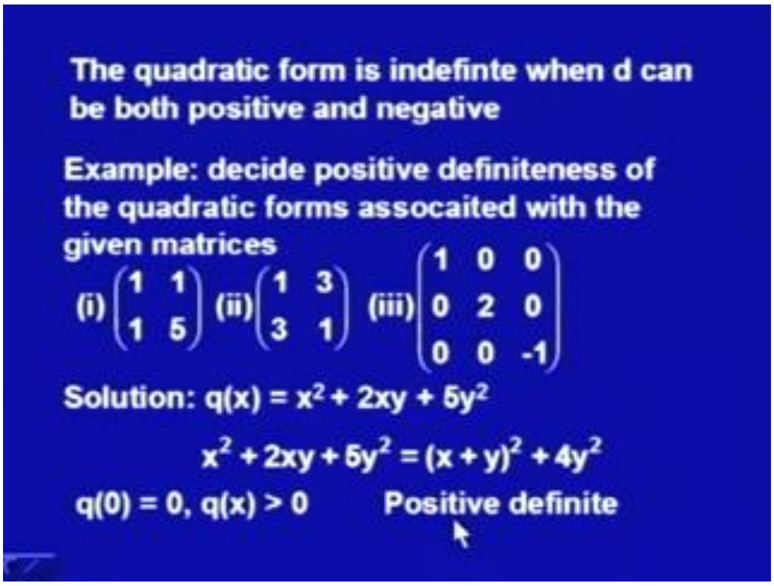
(Refer Slide Time: 48:08)



So, a symmetric matrix A and the quadratic form x transpose A x are positive definite is x transpose A x is positive for x not 0. We call then the form as negative definite, when x transpose A x is negative for some for x not 0. And positive semi definite is x transpose, A x maybe positive or maybe 0 for all x. Negative semi definite, when x transpose A x is negative for all x, negative or 0 for all x. And we say the form is indefinite, if we cannot determine the sign of x transpose A x.

If it is positive always, we call it positive definite. If it is negative always, then we call it negative definite for nonzero values of x. And a positive semi definite or semi indefinite, negative semi definite, when we will have this expressions, and if the sign cannot be determined, then it maybe indefinite.

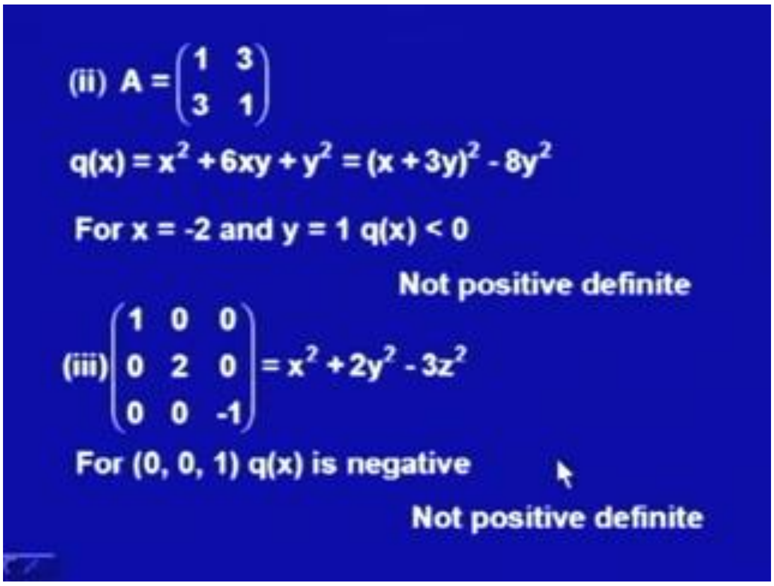
(Refer Slide Time: 49:22)



The quadratic form is indefinite. When d can be both positive and negative, we can take any value. So, in this example, we try to decide the positive definiteness of the given quadratic form. So, in the first we have the associated matrix as 1 1 1 5. In the second, we will have the associated matrix 1 3 3 1. And in the third, we will have this matrix, we will go one by one.

This matrix is associated, with this quadratic expression x square plus 5 y square plus 2 x y. We can write down this form as x plus y whole square, I am rewriting this expression plus 4 y square. See x plus y square is equal to x square plus y square plus 2 x y. So, 2 x y is coming here and y square I have taken from this. So, what remains here is 4 y square, now this expression is 0 when x is equal to 0 and y is equal to 0. And for all values of x, what whether positive or negative this expression will always be 0 or we can say that this expression is positive definite. Or this quadratic form is positive definite quadratic form is positive definite.

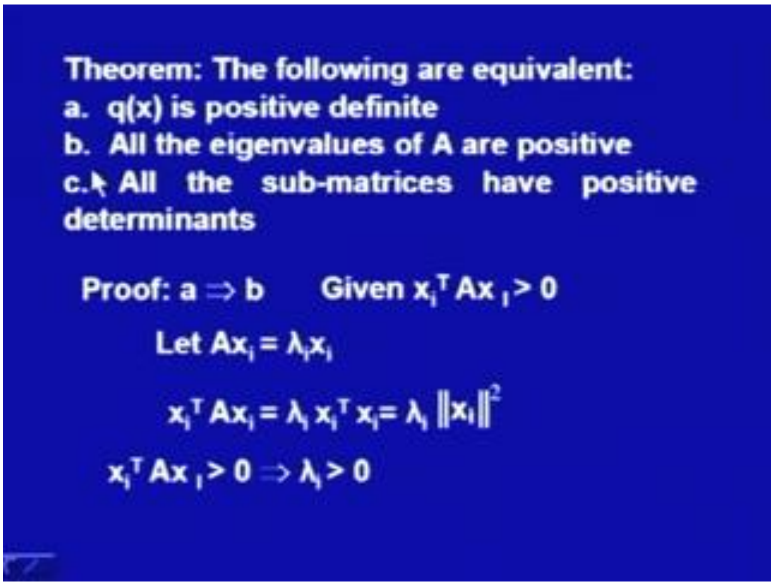
(Refer Slide Time: 50:53)



In the second example, I have this matrix. And corresponding to this matrix, I have this form x square plus 6 x y plus y square. And this form, I write it as x plus 3 y whole square minus 8 y square. And one can notice that, for x is equal to minus 2 and y is equal to 1. Then, this term will be more than this term; and that makes q x negative. So, this is not positive definite, because it is sign will be q x is negative for value x is equal to minus 2 and y is equal to 1. So, this is not positive definite.

In the other example, the 3 by 3 matrix it is a diagonal matrix. It comes out to be x square plus 2 y square minus 3 z square. One can notice that, if x is equal to 0 y is equal to 0. Then if z is, whatever be the values of z, then this is negative. So, it is again not positive definite, so 0, 0, 1, q x is negative and it is not positive definite.

(Refer Slide Time: 52:20)

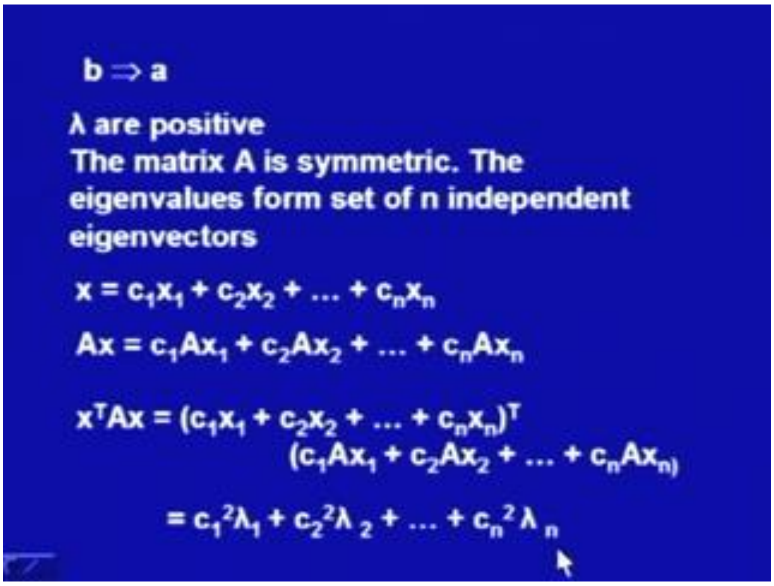


Now, in this theorem we will like to state the conditions under, which we can ensure that a given quadratic is always positive definite. So, the theorem states that, the three conditions, the following three conditions are equivalent. One condition is if q x is positive definite. Second is all the eigenvalues of A are positive, third is all the sub-matrices have positive determinants. So, I will start with a given quadratic x i transpose A x i is positive.

I will first prove that a implies b means, if q x is positive definite. Then, it implies all the eigenvalues of A are positive. So, with this given positive definite form, let us consider an eigenvalue problem A x i is equal to lambda i x i. Now, in this form A is a symmetric matrix. So, for this form we can multiply this matrix by x i transpose from left, as well as on the right. So, x i transpose A x i is equal to lambda x i transpose x i. And we know x i transpose x i is nothing but the norm of x i which is always positive.

So, x transpose A x i is always positive provided lambda i is positive. So, if a given form is quadratic definite, then all the eigenvalues of A should be positive. So, I have started with positive definite form. And I have proved that it has all eigenvalues, which are positive, so a implies b.

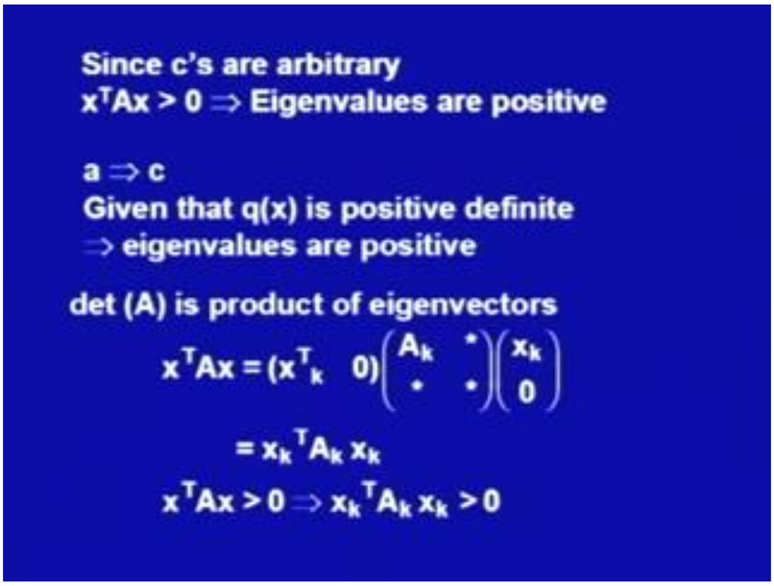
(Refer Slide Time: 54:14)



Now, to consider b implies a, I will start with all positive eigenvalues. And I will show that the corresponding matrix is positive definite. So, assuming all lambdas are positive and the matrix A is symmetric, because this matrix is related with the quadratic. Then, the eigenvalues form set of n independent eigenvectors. This is the result, which I am deriving from my earlier lectures on eigenvalues and eigenvectors. So, they form n independent eigenvectors.

So, we can write down x, any vector x as a linear combination of these n independent eigenvectors. So, will have x is equal to c 1 x 1 plus c 2 x 2 plus c n x n I multiply it by A. So, A x is equal to c 1 A x 1 plus c 2 A x 2 plus c n A x n; then I multiply it by x transpose pre-multiplication. So, it is x transpose A x is equal to c 1 x 1 plus c 2 x 2 plus c n x n, this is x transpose. X transpose is x is nothing but this is x transpose multiplied by c 1 A x 1 plus c 2 A x 2 plus c n A x n, whatever I have obtained here. And then if you perform this multiplication it is column this it is transpose. So, if we multiply this will have c 1 square lambda 1 plus c 2 square lambda 2 plus c n square lambda n square.

(Refer Slide Time: 56:00)



Since c's are arbitrary, so x transpose A x is positive. And that means, eigenvalues are positive, because the eigenvalues are positive will have x transpose A x is positive. And that means, we will have positive definite quadratic form. So, we have started with positive eigenvalues. And we could prove that, we have positive definite quadratic form. So, b implies c is proved, now let us try a implies c.

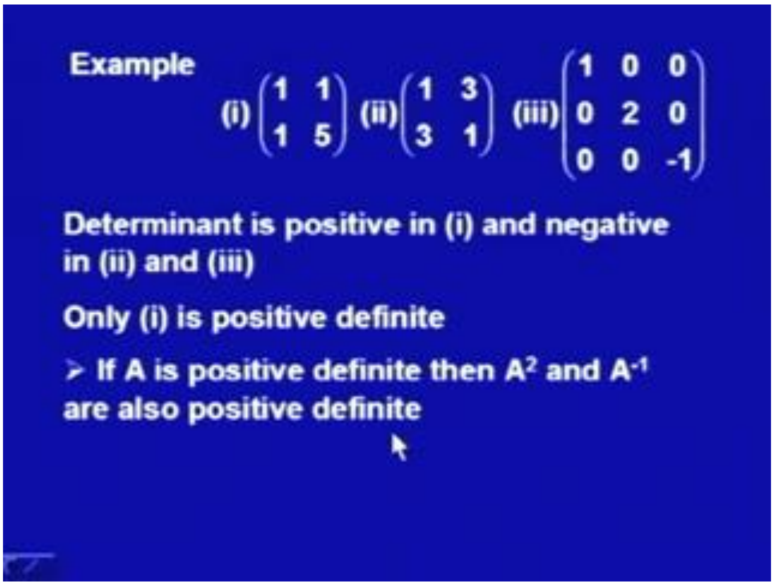
So, given that q x is a positive definite, this that is the meaning of a. We assume we are having eigenvalues are positive, because q x is positive definite. So, b means eigenvalues are positive, so we have this. Then, what is determinant of A, determinant A is product of eigenvectors. Now, we consider x transpose A x as x k 0. What is x k transpose is a vector, which is having nonzero values and rest of them are zeros.

So, I reduce this matrix A into k by k matrix, that is A k and rest of them are these elements in A. So, I write it as x k transpose and rest of the elements are 0. So, I am considering only those values, which are corresponding to A k they are nonzero and rest of them are 0. So, this particular x transpose is chosen. So, accordingly we will have this column vector. So, when we perform this multiplication, I will have x k transpose A k x transpose, but x transpose A x is positive.

So, x k transpose A k multiplied by x k transpose is positive provided determination of A k is positive. This means the sub matrix A k has positive determinant. But, what is A k, A k is a sub matrix of A. And this can be any order matrix, it can be order 1, order 2, order there. And so on, in fact this result should be true for all values of k. And that means, if we have a positive definite form. Then, all sub matrices will have positive determinant.

Now, that proves c that means, starting from a we can prove the c, the reverse of this that is if we have a matrix having all sub matrices determinant 0. Then, we can prove that it is a positive definite form that is c implies a. I leave this is an exercise for the viewer. In this example, we are using the theorem, which I have just now stated. And with the help of that theorem, we like to see whether a given quadratic form is positive definite or not.

(Refer Slide Time: 59:13)

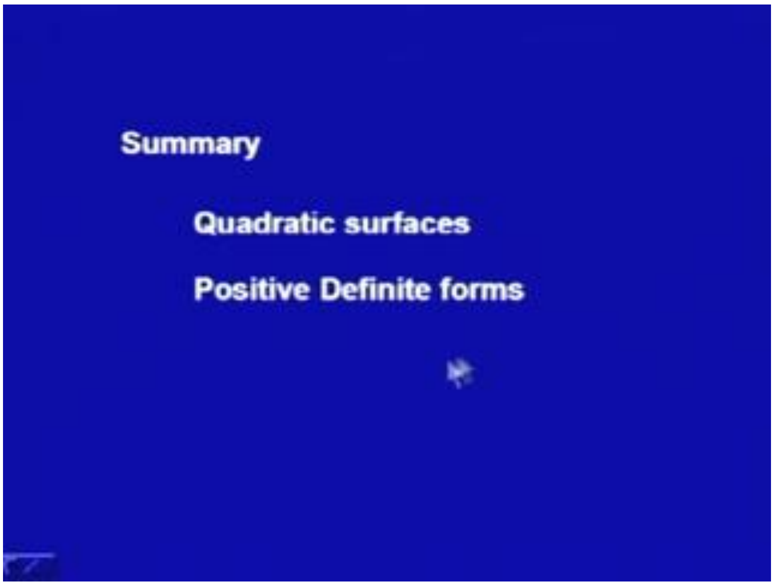


So, if in the first example if I have a 2 by 2 matrix, then one can notice its determinant is 1 into 5 minus 1 is positive. And since, it has a positive determinant and all sub matrices have positive determinant. So, it has all positive eigenvalues and it is a positive definite form. So, first is positive definite form, but in the second case first sub matrix is having determinant positive.

But, if you consider this sub matrix, then it is determinant is 1 minus 9 is negative. So, it is not positive definite and in the third case again one of the eigenvalue is negative. So, again this is also not positive definite form. So, in this example, we will have only one positive definite form. In the second example determinant is negative, so will have will not have a positive definite form.

In this example, one eigenvalue is negative, then it is this is also not positive definite form. So, only one is positive further if A is positive definite matrix. Then, A square is also positive definite, I leave this as an exercise for the viewer to prove this. Further, if A inverse is also positive definite, if A is positive definite. So, these are two exercises user can try easily.

(Refer Slide Time: 1:00:48)



That gives that comes with this we come to the end of the lecture. Today, we have covered quadratic surfaces. We have discussed, when under what conditions a given quadratic form is positive definite quadratic form; that is all.

Thank you.