Mathematics-II
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Module - 2
Lecture - 14
Eigen values and Eigen vectors Part – 2

Welcome viewers, this lecture is in continuation to my earlier lecture on Eigen Values and Eigen Vectors.

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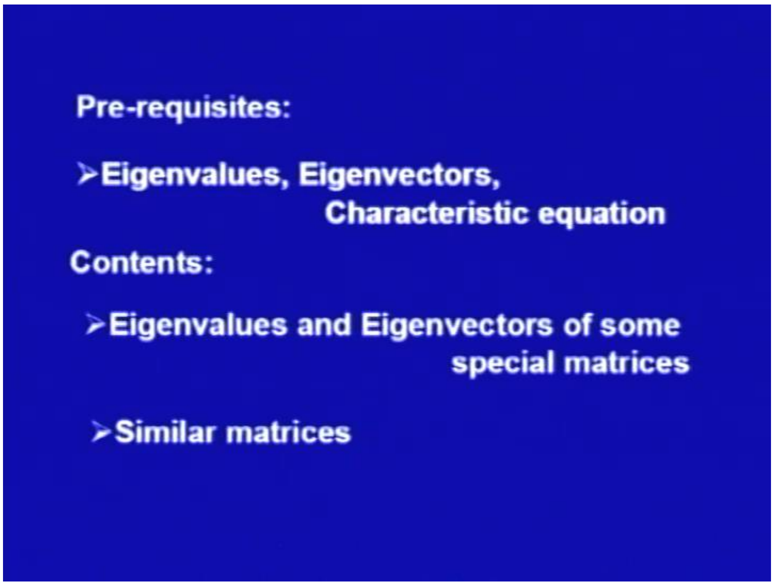

In this lectures, we will be discussing Eigen values and Eigen vectors, of some special matrices. And we will be discussing similar matrices. I assume that viewers have all ready gone through my 1st lecture on Eigen values and Eigen vectors and characteristic equation.

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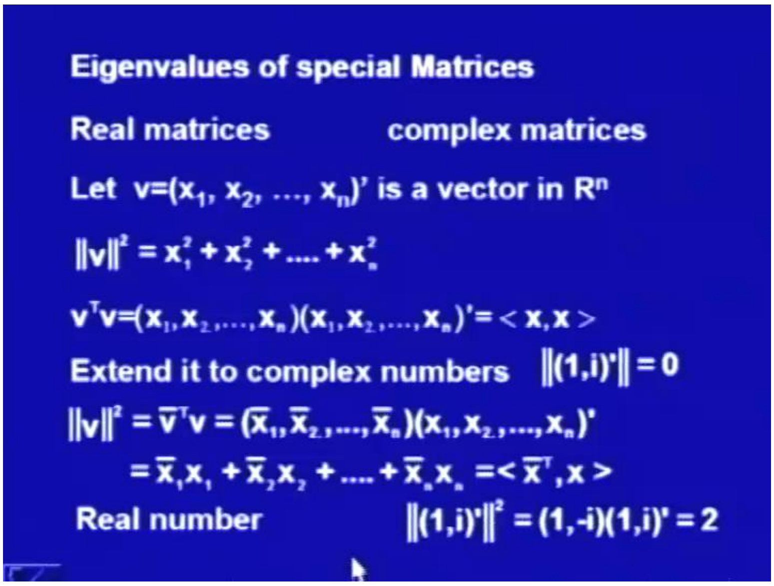

We start with Eigen values of some special matrices. We have all ready been discussing matrices, which are real matrices. That means, the elements of the matrices are real numbers, however in many situations the elements maybe complex numbers. And we say the matrices are complex matrices. Let us consider a vector V in R n, consisting of components X 1 X 2 X n. So, V is equal to X 1 X 2 X n, it is transpose is vector in R n.

Then, the length of the vector V is defines as X 1 square plus X 2 square plus X n square. Now, length as we all know is a positive number. And it is 0, only when the individual components X 1 X 2 X n are 0. So, you can verify this when X 1 X 2 X n are 0, then the length of V is 0. And, we defined length of V as V transpose V, which is the vector X 1 X 2 X n. The row vector multiplied by the column vector X 1 X 2 X n transpose.

This is also denoted by inner product of X and X. If we extend this concept to complex numbers and let us apply to complex vector 1 comma i transpose. Then, one can notice that its length will be 0. So, this vector is not 0 vector, but its length is 0. If we apply this definition of length which we have already developed for real matrices. So, we need to redefine this length in the relation to complex matrices. So, let us redefine it, that length V square is equal to V conjugate transpose times V.

V conjugate transpose means, we first take the conjugate of the numbers. These numbers are complex numbers. So, conjugate means the complex conjugates, and then multiplied by the vector X 1 X 2 X n transpose. And if you perform this multiplication, then this comes out to be X 1 bar X 1 plus X 2 bar X 2 and X n bar X n. And we define it as the inner product of X n and X. And if you have used this definition, this revised definition for length. Then, 1 plus i transpose square comes out to be 1 minus i. Because, a minus i is a conjugate of i multiplied by 1 i it is transpose.

And, if you multiply it, it is 1 and minus i square which is also 1, so the length is 2. So, length square is equal to 2. And this is not 0, when this vector is not 0. So, we know that this number is now a real number. And this has to be because length is a positive real number. So, we extend the definition of length in this particular manner.

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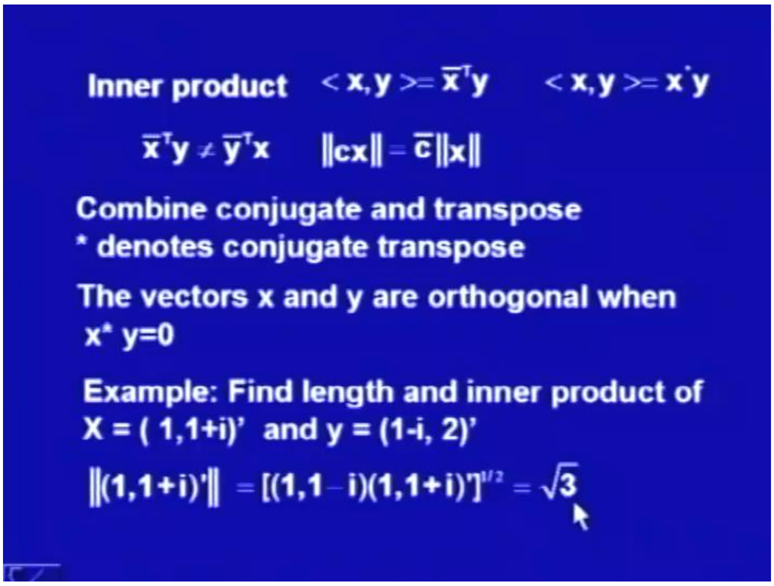

And accordingly, the inner product of two vectors x and y is defined as x bar transpose y. One may notice that, x bar transpose y is not the same as y bar transpose x. This is in contrast to what we have in real numbers. When length from x to y is the same as from y 2 x. Similarly c x, the length of c x is equal to c bar x, it is not c it is c bar. So, if we have a complex number, we have a complex that then c of x is c bar x. Of course, when we apply to real number and c happens to be real number, then c bar and c are equal.

Then, we write down, we combine the conjugate and transpose. And we write it as star denoting the conjugate transpose. And accordingly, we can say that inner product x, y is equal to x star y. So, with this notation, we have introduced x star y as the inner product of x and y. The vectors x and y are orthogonal, when x star y is equal to 0. So, this is true for this is the definition we apply for complex vectors x and y.

Now, let us illustrate these concepts with the help of an example. So, find length and inner product of the vector x, which is 1 comma 1 plus i transpose. And y is equal to 1 minus i comma 2 transpose. So, length of vector x will be equal to 1 comma 1 plus i square 1 comma 1 plus i, it is length will be equal to under root of this product. And if you work it out, it is 1 into 1 plus1 minus i into 1 plus i, that gives me under root 3.

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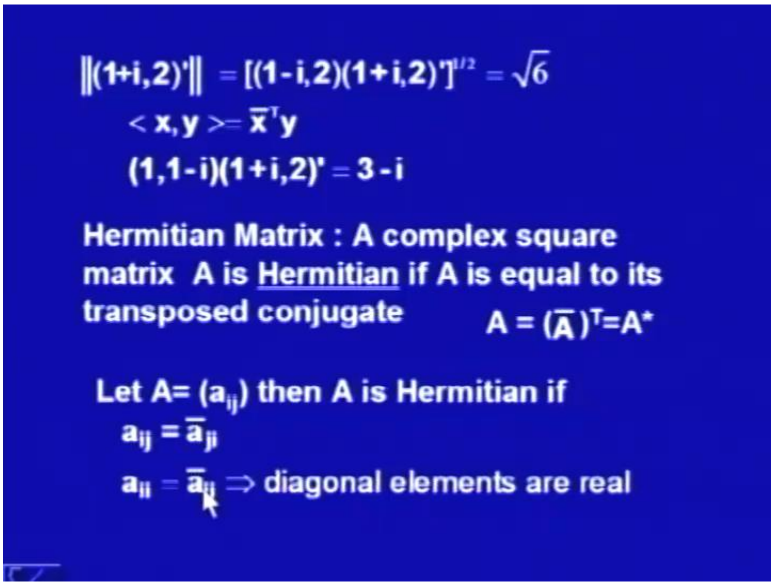

While the length of the second vector 1 plus i 2 is equal to 1 minus i 2. That is the conjugate of this number multiplied by 1 plus i 2. And then it is transpose we take the multiplication of these two matrices. In fact, these two vectors and this comes out to be under root 6. And if we have to calculate the inner product of x and y, it is defined as x bar transpose y.

Then, x bar transpose is 1 comma 1 minus i. And y is 1 plus i comma 2, that is given to us. And if you multiply it, it comes out to be 1 into 1 plus i plus 1 minus i into and that simplifies to 3 minus i. Now, this is about vector. Now, we go to a special matrix is a Hermitian matrix, it happens to be a complex matrix and in fact a square matrix. So, we say a complex square matrix A is Hermitian. If A is equal to it is transposed conjugate. That means, A is equal to transposed conjugate.

So, we have first take the conjugate and then transpose. In fact, it does not make much difference. Whether we first take the transpose, or we take the conjugates. And we denote it by A star. So, if A is equal to A star, then the matrix which is of complex elements is a Hermitian matrix. Now, if we have a square matrix A, then a typical element in this is a i j. Then according to this definition A is Hermitian. If a i j is equal to a bar j i, this bar denotes the conjugate.

And this index denotes that it is being transposed. So, a i j is equal to a j i bar, if this condition is satisfied, then the matrix is Hermitian. However is j is equal to I, then one can notice that, a i i is equal to a bar i i. That means, the diagonal elements have the property, that the element is equal to it is conjugate. And this will happen only when the number is a real number. That means, if we have a Hermitian matrix, then diagonal elements will always be real.

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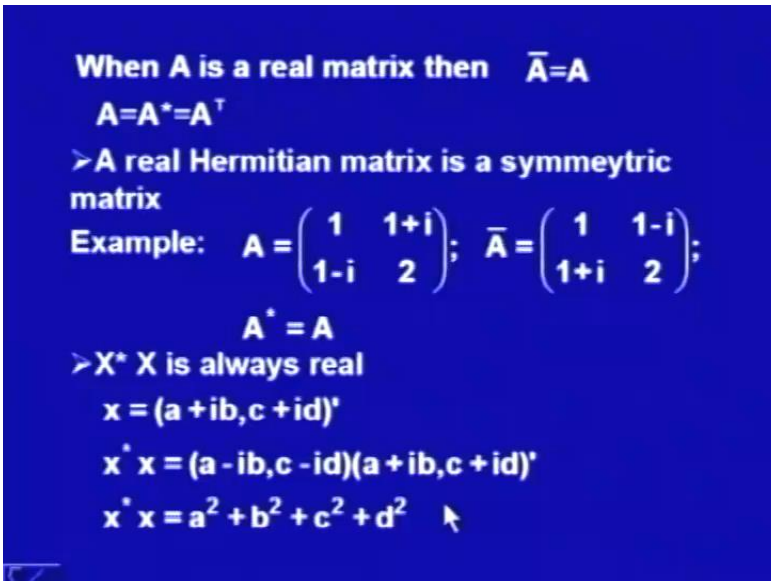

And from this one can also derive, that when A is a real matrix, then A bar is equal to A. The real matrix, so conjugate will be the number itself. So, A bar is equal to A. So, if we consider this A star, since A bar is equal to A. So, this A star will actually reduce to A transpose. So, then A happens to be a real matrix, then A is equal to A transpose for matrix to be Hermitian.

And this means, a real Hermitian matrix is a symmetric matrix. So, if the matrix happens to be real matrix, then it is nothing but a symmetric matrix. Let us take an example. We have complex matrix, the diagonal elements are 1 and 2. And 1 plus i and 1 minus i are non-diagonal elements. So, let us calculate it is conjugate. So, conjugate of 1 is 1, conjugate of 2 is 2. Conjugate of 1 plus i is 1 minus i, and conjugate of 1 minus i is 1 plus i.

And this means, if you take the transpose, then A star is equal to A. Or when you take the transpose, then this 1 plus i will go here. And this 1 minus i will come here. So, this A bar transpose is nothing but A. So, this matrix is a Hermitian matrix. Then, the result is that X star X is always real, whatever be x. So, let us consider x is a plus i b comma c plus i d. So, these are complex elements of a 2 by 2 matrix. I prove this result for 2 by 2, in fact, this can be done for n th order vector.

So, x star x is equal to a minus i b into c minus i d, that is the x star. And this is the column vector, so x star will become the row vector. And the corresponding elements will be the conjugate. So, a plus i b will become a minus i b. And c plus i d will become c minus i d multiplied by a plus i b and c plus i d transposed. And when you multiply it, it is a minus i b multiplied by a plus i b, that gives me a square plus b square. And the second element here is then multiplied by this, gives me c square plus d square. So, a b c d being real, because a plus i b and c plus id being complex. So, this is a real number, so x star x is always a real number.

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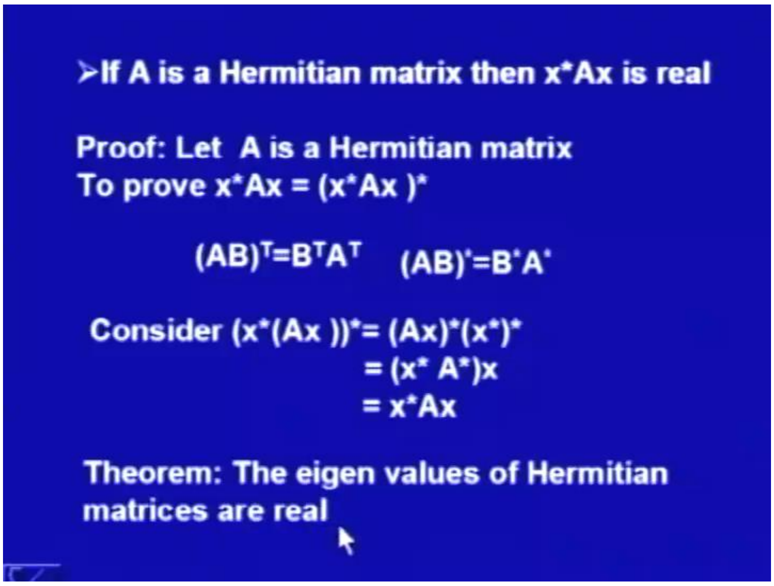

Then, if A is Hermitian matrix, then x star A x is real number. So, this is another result, let us try to prove this. Let A is a Hermitian matrix, we have prove that it, this number is real number. This number is a real number, then it is conjugate is a real number. And in fact, x star A x is nothing but a 1 by 1 matrix. Because, x star is a column vector and x is a column vector and x star is a row vector. So, this product will be 1 by 1 matrix. So, if it is transpose conjugate is a real number, then that means, that this is a real number.

So, to prove x star A x is equal to x star A x transpose. So, we are proving this result by proving this property, x star A x is equal to x star A x star. For this purpose, we will evaluate x star A x star, this can be evaluated using these properties. We know that, A B transpose is equal to B transpose A transpose. That means, when you take the transpose of a product the order will change. So, this means A B star is equal to B star A star.

Now, this result we will be using to establish this result. So, I start with the right hand side, x star A x star. So that means, this is the first matrix and this is the second matrix. So, second matrix will be started first. And this first matrix will be star next. So, it is A x star multiplied by x star star. Now, A x star again I am using this property. So, it becomes x star A star and when I take conjugate twice. Because, conjugate is a operation which will be negated, if we do it twice. So, x star star will become x.

And then this is nothing but x star A x, because this multiplication is associative. So, these brackets does not have any meaning. So, we have proved that x star A x star is equal to x star A x. And this is possible, only when this number is a real number. So, we have proved that if A is a Hermitian matrix, then x star A x is real. Now, on the basis of this result, we will prove that the Eigen values of Hermitian matrices are real. So, let us see how we prove this result.

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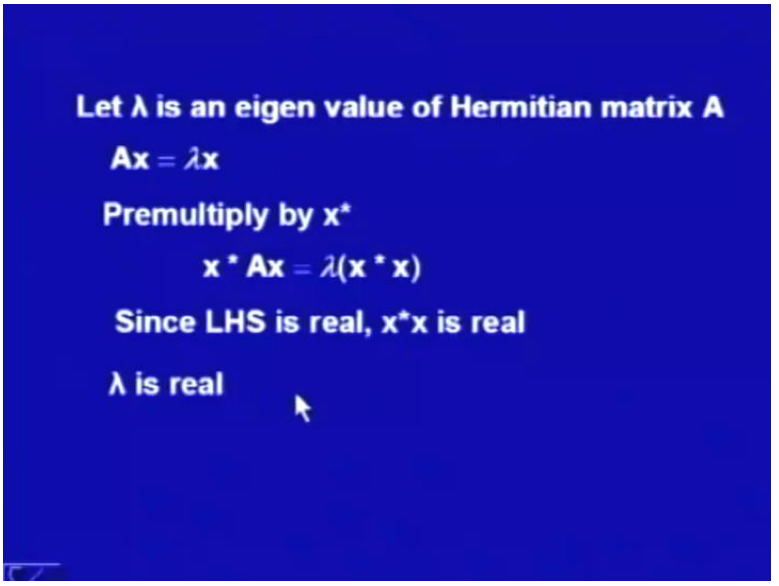

So, let us take lambda as an Eigen value of Hermitian matrix A. Then we can write A x is equal to lambda x. X being the Eigenvector corresponding to Eigen value lambda and A is Hermitian. We pre-multiply this equation by x star. So, we will have x star A x is equal to lambda x star x. Now, we have already proved the left hand side is real number in my earlier result. And also I have proved that x star x is a real number. So, left hand side is a real number right hand side is a real number. So, this lambda cannot be complex, because if this is complex. Then, this has to be complex, so lambda is real.

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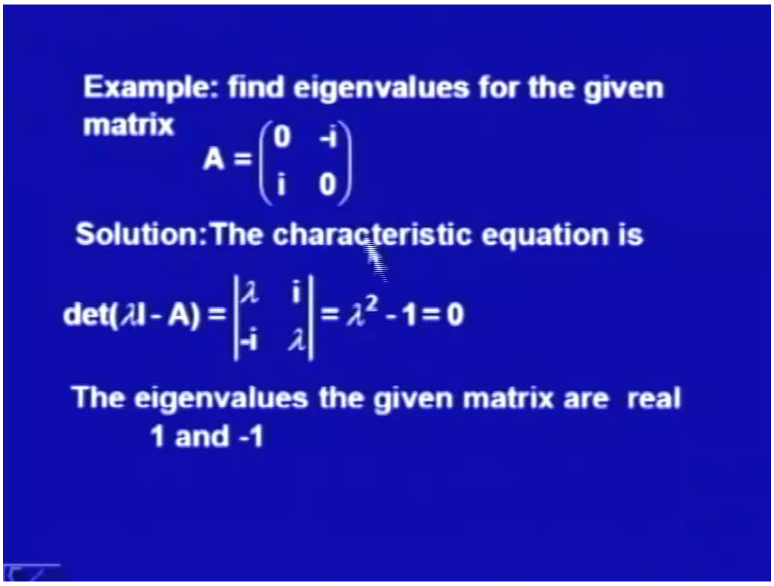

And that is how is say that Hermitian matrix matrices, will always have real Eigen values. We illustrate this with an example. So, let us consider an a matrix A consisting of 0 minus i, in the first row i 0 in the second row. So, it is a 2 by 2 Hermitian matrix. So, we see what are its Eigen values to calculate the Eigen value of this matrix. I have to compute lambda i minus A. So, determinant of lambda i minus A equal to 0 is the characteristic equation for this given matrix.

And for this lambda i minus A is to be computed. So, lambda into i, that is lambda A is not contributing here in the first element. So, it is simply lambda. Then, the second is lambda i. So, no contribution from i in this element. So, it is minus A, so it is i here then this i will become minus i and then we have this lambda. Now, if you evaluate this determinant it is lambda square minus. And this minus will make it plus, so it is plus i square. And plus i square becomes minus 1.

So, determinant lambda i minus A is lambda square minus 1 equal to 0. And, this is a second order equation in lambda, which can be easily solved. And we get the Eigen values of given matrix are real and they are 1 and minus 1. So, we have established a result and we have verified this in this example.

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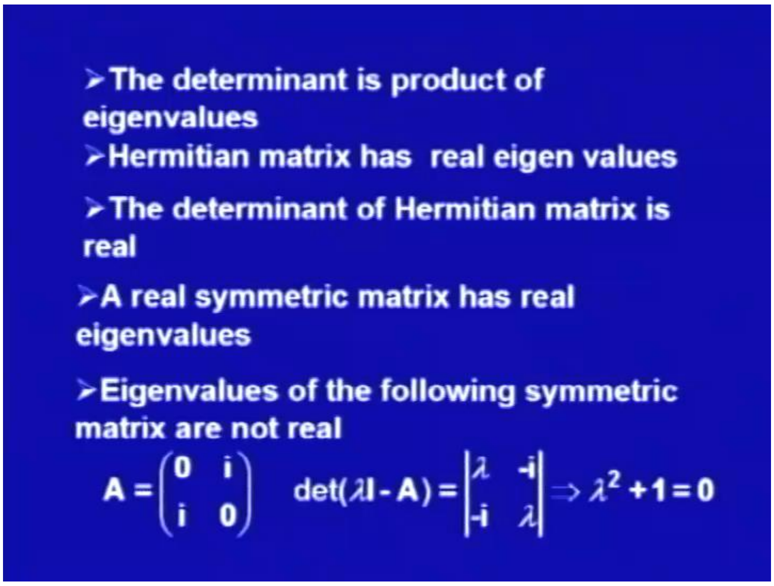

Now, the determinant is product of Eigen values. This is the result, which we have established in earlier lectures. So, here we have Hermitian matrix, which has real Eigen values. And if we combine these two results, we can say that the determinant of Hermitian matrix is real, because all the Eigen values are real. So, there product will also be real and product of Eigen values is determinant. So, the determinant of Hermitian matrix is real.

Now, this is an important result it says that, a real symmetric matrix has real Eigen values. Now, this result can easily be derived from what we have done so far. We have Hermitian matrix having real values, and a real Hermitian matrices, real symmetric matrix. So, if we use that result, we can easily arrive to the result that a real symmetric matrix has real Eigen values.

However, if you consider this matrix A it is symmetric, because this is a mirror image of this. So, symmetric matrix you can calculate its Eigen values. Determinant lambda i minus A is equal to lambda and this also lambda. Here, it is minus i and minus i simplified it is lambda square plus 1 equal to 0. Therefore, this characteristic equation lambda square plus 1 equal to 0, gives me Eigen values as i and minus I.

And that means, this real symmetric. This symmetric matrix gives me Eigen values as complex numbers i and minus i, where we have gone wrong. This is a symmetric matrix, but not a real symmetric matrix. So, the result which we have stated is related to real symmetric matrices. And the result is that a real symmetric matrix has real Eigen values. The symmetric matrix may have other Eigen values.

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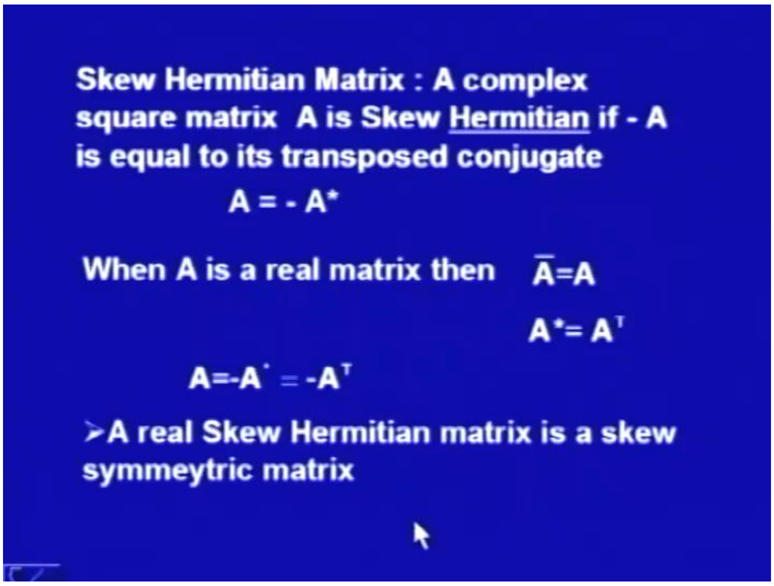

Now, we come to another concept Skew Hermitian Matrix. A complex square matrix A is Skew Hermitian if minus A is equal to it is transpose conjugate. That means, A is equal minus of A star. And when is a real matrix, then we know A bar is equal to A. And this reduces to A star is equal to A transpose. And that means, A is equal to minus A star means minus A transpose.

So, if we have a real matrix then it will be Skew Hermitian. If A is equal to minus A transpose or we say that a real Skew Hermitian matrix is a Skew symmetric matrix, because this is the definition of a Skew symmetric matrix. So, if a matrix is real and it is Skew Hermitian, then that matrices is going to be a Skew symmetric matrix.

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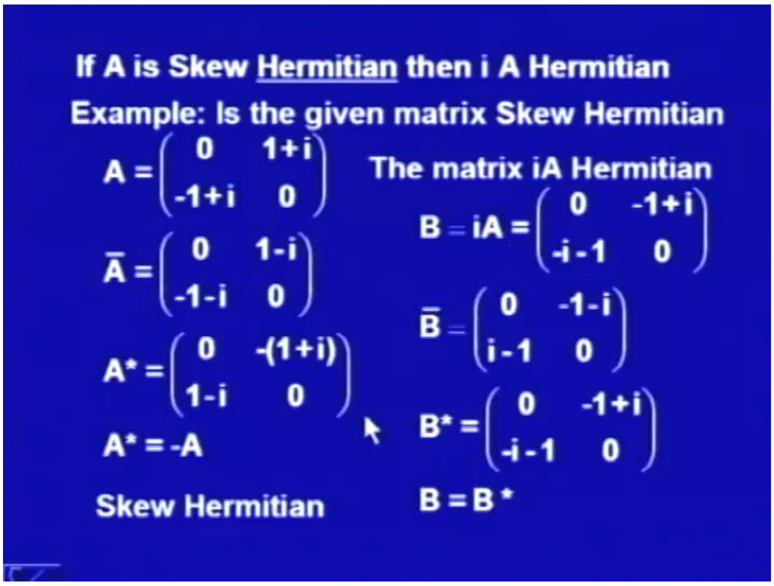

You may notice that, if A is Skew Hermitian then i times A is Hermitian. Let us, take an example we have been given a matrix A as 0 1 plus i minus 1 plus i 0. Let us see whether what is its conjugate A bar. So, these diagonal elements will not be effected. But, this 1 plus i will become 1 minus i, when we take the conjugate. And minus 1 plus i it is conjugate will be minus 1 minus i.

So, A star will be the transpose of this. So, this row becomes this column. And this row becomes this column. And from here, if we compare A and A star, one may notice that A star is nothing but minus times A. This is 0 and 0 does not make much difference. But, this element is negative of this. And this element is negative of this and hence, A star is equal to minus A. Or we can say that, this matrix is Skew Hermitian matrix.

So, given this Skew Hermitian matrix. We will see that, if we multiply this matrix by A this matrix by i. Then, what we have a resultant matrix as Hermitian matrix. So, let us do it here, the matrix i A is Hermitian. So, we multiply this matrix A by i. Let us call this matrix as B, 0 multiplied by i this remains as it is. So, diagonal elements are not affected, but this element will become minus 1. Why because, i in to i is minus 1 and 1 into i is i.

And this element will become minus i minus 1 and 0 remain as such. So, this B as i times A we calculate its conjugate. So, conjugate of minus 1 plus i is minus 1 minus i and conjugate of this is minus 1. And this minus i will become plus i conjugate, means the imaginary part will be negated. And then B star B transposed conjugate or conjugate transpose is equal to the transpose of this matrix. That is row becomes column and this row become this column and from here. If you compare B and B star this is nothing but B is equal to B star. And that proves, that B is Hermitian matrix. So, A given to be Skew Hermitian i times A is Hermitian.

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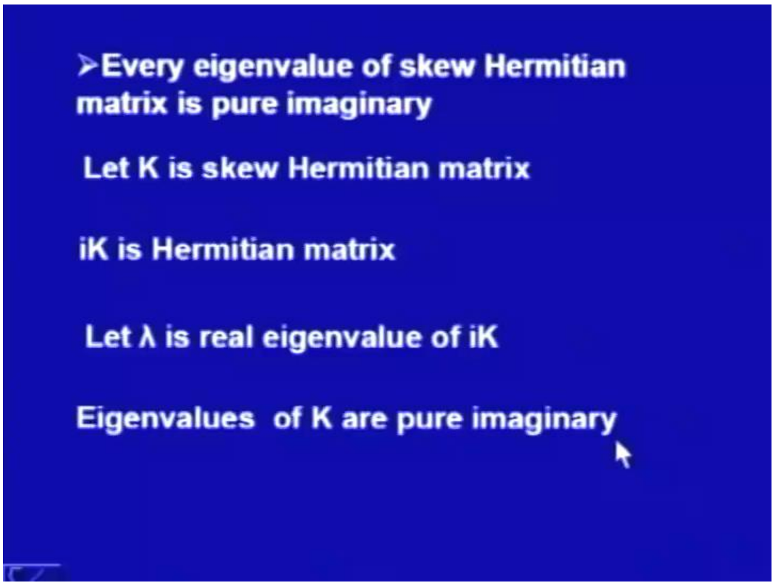

Then, every Eigen value of Skew Hermitian matrix is pure imaginary. We have proved a result related to Hermitian matrix. And we found that Hermitian matrix has real Eigen values. But, if the matrix happens to be Skew Hermitian then its Eigen values will be pure imaginary. So, to prove this let K is Skew Hermitian matrix. Then, i K is Hermitian matrix, let lambda is an Eigen value of real Hermitian matrix. That is lambda is a real Eigen value of i K, then Eigen values of K are pure imaginary. So, this what we get from this result that if K is Hermitian. Then, i K will be if K is Skew Hermitian then i K will be Hermitian.

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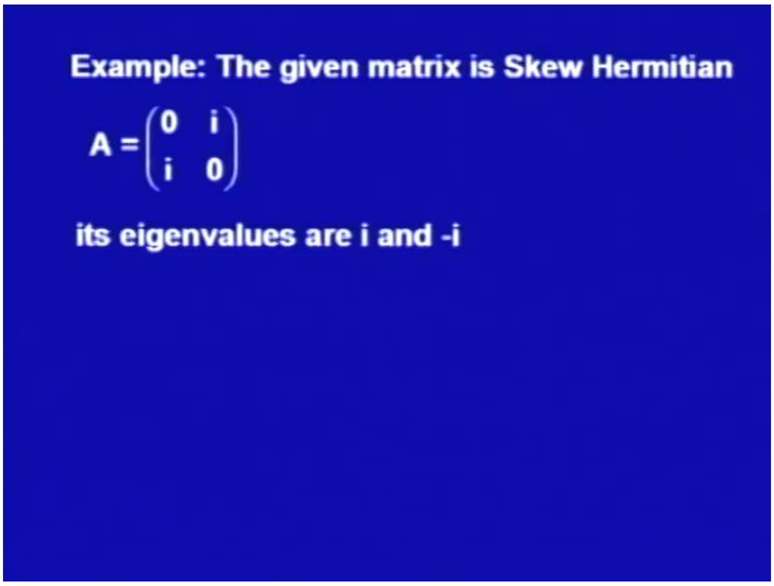

Let us check this, in this example the given matrix is Skew Hermitian. It is Eigen values are i and minus i. Therefore, this Skew Hermitian matrix A has pure imaginary Eigen values.

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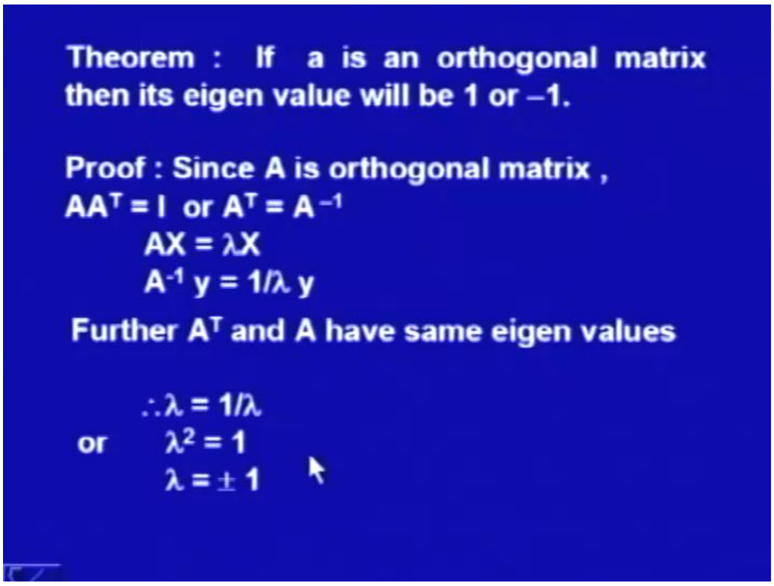

Now, we were talking about Skew Hermitian matrices. And Hermitian matrices we are talking about their Eigen values. Hermitian matrices have real Eigen values, while Skew Hermitian matrices have pure Eigen values purely imaginary Eigen values. Now, we will talk about orthogonal matrices. And we see that an orthogonal matrix, has Eigen values 1 or minus 1. So, let us try to prove this result.

It is being given that A is orthogonal matrix. Then, from the definition of orthogonal matrices, we know that AA transpose is identity or A transpose is nothing but it is inverse A inverse. So, let us say lambda is an Eigen value of this matrix A. Then, A X is equal to lambda X. And also A inverse y is equal to 1 upon lambda 1 upon lambda times y. And that means, A transpose and A have same Eigen values. And that means, lambda is equal to 1 upon lambda or lambda square is equal to 1. And that gives me lambda is equal to plus minus 1. So, if A is orthogonal matrix, then its Eigen values are 1 or minus 1.

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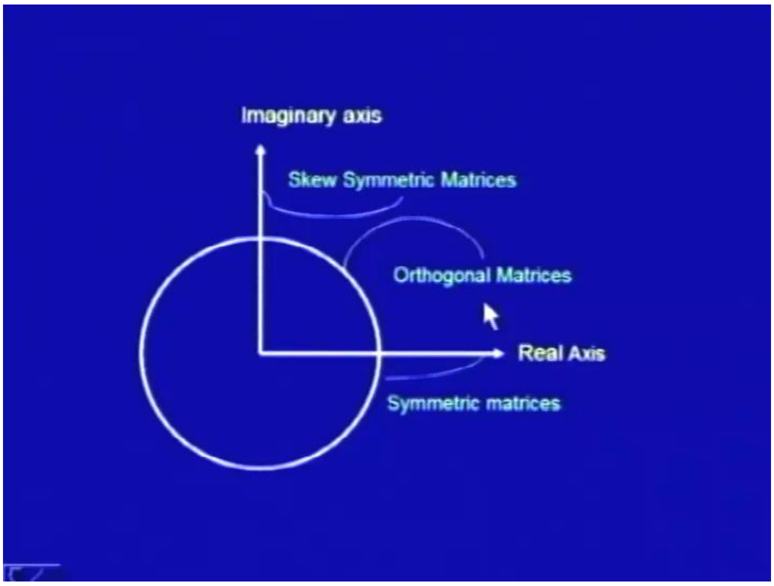

So, let us see how they look like on this plane. If I have a real axis here, and imaginary axis here, then all the Eigen values of symmetric matrices of real symmetric matrices will lie on the real axis. And the Eigen values of Skew symmetric matrices, will lie on the imaginary axis. While, this is a unit circle and then all the Eigen values of orthogonal matrices will lie on this unit circle.

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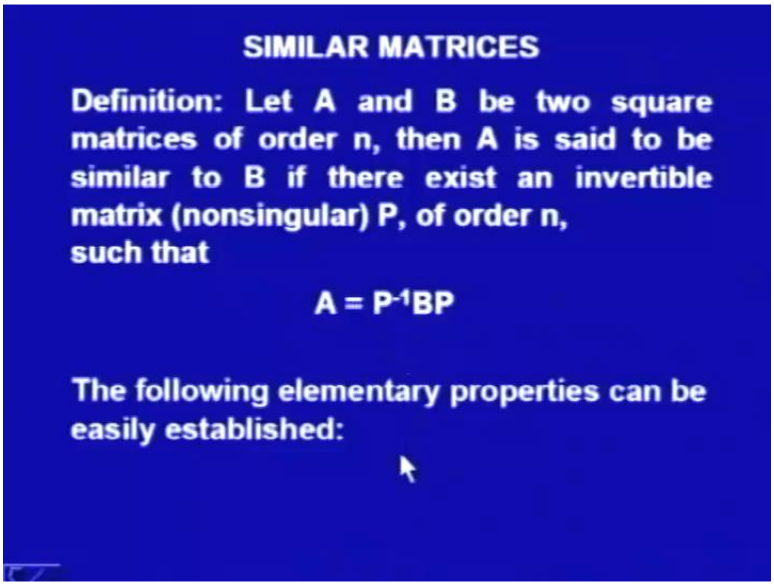

Now, we come to similar matrices. Let A and B be two square matrices of order n. Then, A is said to be similar to matrix B. If there exist an invertible matrix P of order n, such that A can be represented as P inverse B P. The following elementary properties can be easily established with respect to similar matrices.

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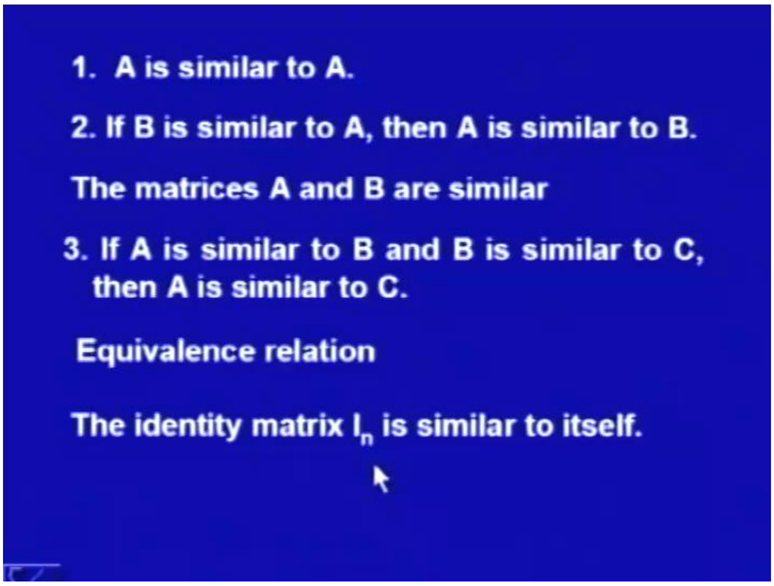

The first is that A is similar to A. So, if A is similar to A means there should exist some matrix P. Such that, A is represented as A times such that A is P inverse A P. So, in this case, if we take P inverse as equal to identity which is the same as P. Then, we can prove easily that A is similar A.

Secondly, if B is similar to A, then A is similar to B and this can we very easily established. And on the basis of this one can say the matrices A and B are similar. You do not have to say B is similar to A or A is similar to B, we can simply say that the matrices A and B are similar.

Thirdly, if A is similar to B and B is similar to a third matrix C. Then, A is similar to C. Now, one can see that the first property is reflexive property. Second is symmetric and third is transitive. And on this basis one can say that, the relationship. That is similar A is similar to B is an equivalence relation. And one can also see that the identity matrix of order n that is I n is similar to itself, because in that case P is I itself.

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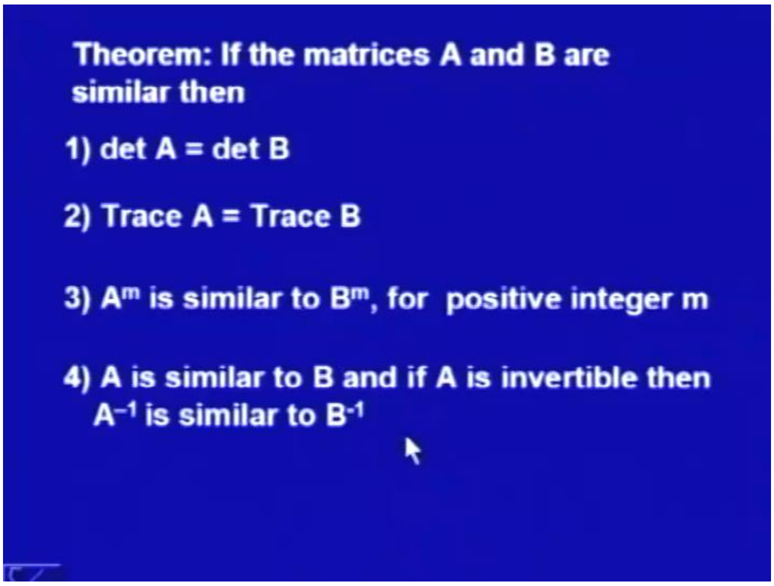

Now, if the matrices A and B are similar. Then, we can prove the determinant of A is equal to the determinant B. Or we can say similar matrices have same determinants. Not only this, but the trace of A and the trace of B are also equal. Further, if A m is similar to B m for if A and B are similar. Then, A m and B m will also be similar for any positive integer m. And finally, if A is similar to B and if A is invertible, then A inverse is similar to B inverse. These are some of the properties of similar matrices, let us prove them 1 by 1.

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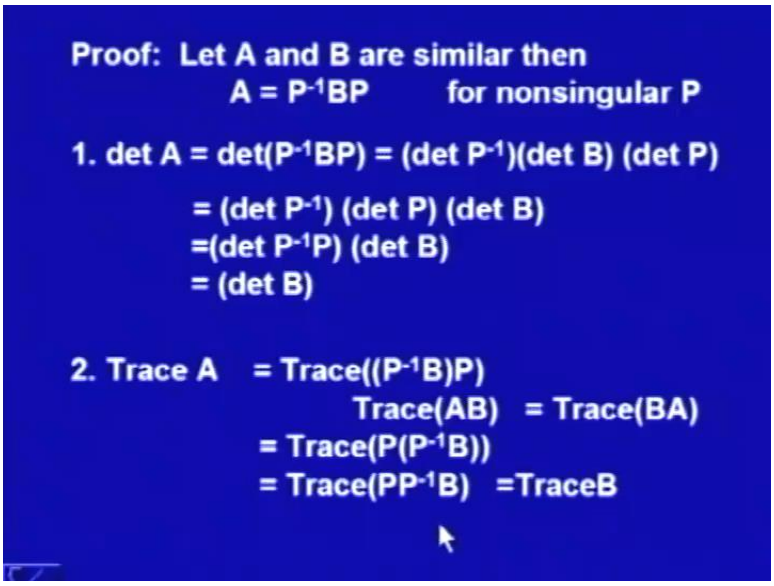

So, to prove the first property, let us say A and B are similar. Then, this statement means that there exist a matrix P which is nonsingular such that A is equal to P inverse B P. And from here, we can get determinant of both the sides. So, determinant A is equal to determinant of P inverse B P. And we know the property of determinants that, this determinant of product is equal to product determinants. So, determinant P inverse B P is equal to determinant P inverse multiplied by determinant B into determinant P.

Determinants happens to be real number so they can we they can change the order. And that means, we can bring this determinant P here. So, this determinant P inverse into determinant P into determinant B. And one can combine these two, it is determinant P inverse P into determinant B. And this is nothing but determinant of i which is 1. So, we can say determinant A is equal to determinant B. That is the first property, which we have listed.

The second is trace of A is equal to the trace of B. So, we can write down trace of A is equal trace inverse trace of B inverse B P, because B is similar to A. So, there exist such B there exist such P. So, to prove this result we will use this result with respect to product and trace, where we have just earlier established. That trace of A B is equal to trace of B A.

So, we can write down this trace of P inverse B P as trace of first matrix into second matrix. So, this means trace of P into trace of P inverse B. So, order has been changed.

And that means, this can also be using this product is associate. So, we can write trace of P in to P inverse B and P into P inverses identity. So, identity into B is B itself, so trace of A is equal to trace B. So, second result is established.

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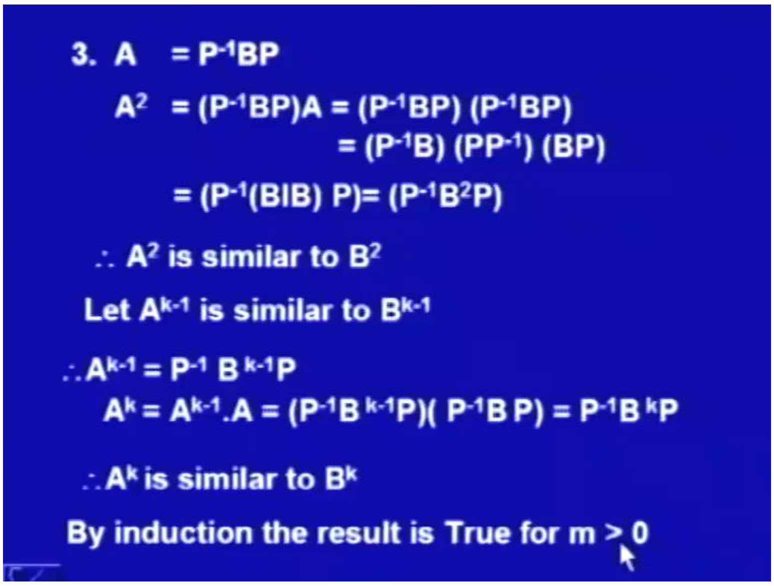

To prove the third result, let us consider A is equal to P inverse B P. We will prove this result by induction. So, we will let us see what happens to A square. So, if we write down if we multiply A with A. Then, it is A square which is equal to P inverse B P A, which is equal to P inverse B P into P inverse B P. I have written this is first A and this is second A. So, we multiply it this is multiplication is associate. So, you can change the brackets.

And that is why I write down this product. As P inverse B into P into P inverse into B P and that gives me P inverse B this is I and B P. So, it is P inverse B I B P and then one can write down this product as P inverse B square P. So, there we have written A is equal to P inverse B square P. And from the definition of similar matrices, one can say that A square is similar to B square.

So, now let us assume that A k minus 1 is similar to B k minus 1. Now, we write A k minus 1 as P inverse B k minus 1 in to P. Because, they are similar matrices and then A k is written A k minus 1 in to A. So, A k minus 1 is this which is given to us and A is similar to B. So, I write it as P inverse B P. If you simplify, then this expression comes out to be P inverse B k P and that proves that A k is similar to B k. So, by induction this result is true for any value of positive m, that proves the result.

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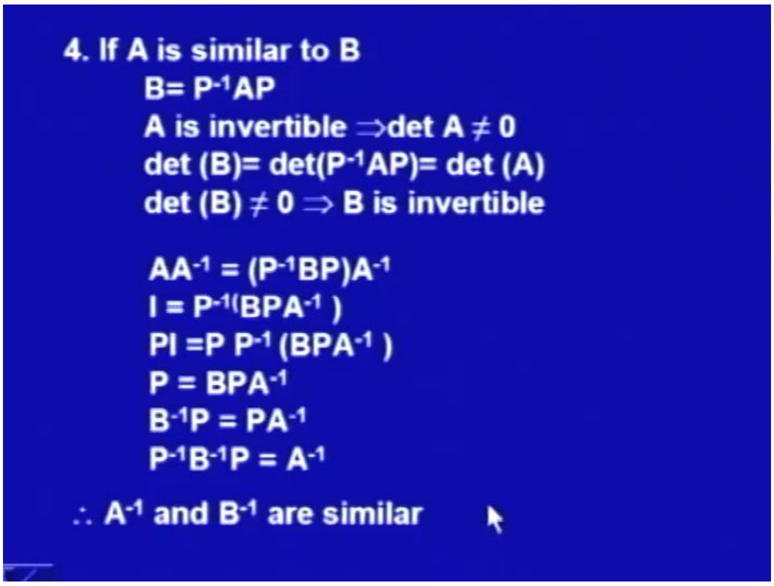

Then, the fourth according to fourth property if A is similar to B, then B can be written as P inverse AP. It is given that A is invertible, so determinant A is not 0 that is, what we have proved earlier, and; that means, determinant of B is equal to determinant P inverse A P which is equal to determinant of A. Since, determinant B is not 0, so B is also invertible.

Once B is invertible, then we consider A into A inverse A is similar to B, so I write it as P inverse BP into A inverse. Now, A into A inverse is identity, so this is simplified to P inverse B P A inverse or we can write it as P into I if I premultiply this expression by P. So, it is P into I is equal to P times P inverse B P A inverse or that from here we can get A k as A k minus 1 into A which is equal to A k minus 1 is replaced by this P minus 1 B k minus 1 P into P inverse B P. And, if you simplify this expression this comes out to P inverse B k P and this proves that A k is similar to B k

Now, by induction the result is true for m positive, (Refer Slide Time: 32:38) now, to prove the fourth result if A is similar to B, then B is expressed as P inverse A P. It is given that A is invertible; that means, determinant A is not 0, then determinant of B determinant of A are equal. So, determinant B is equal determinant P inverse A P which is determinant A and this means determinant B is not 0 and; that means, B is invertible also.

Then, A times A inverse is equal to A is similar to this expression A similar to B, so we write down A as P inverse BP multiplied by A inverse, now this is nothing but identity on the left hand side. And then P inverse we rearrange this term P inverse is take out B P A inverse and; that means, if i multiply this by P then P I is equal to P into P inverse into B P A inverse and; that means, P is equal to B P A inverse or B inverse P is equal to P A inverse from here. And finally, P inverse B inverse P is equal A inverse and; that means, A inverse is similar to B inverse. So, this is what we have to prove in the fourth property, that A inverse and B inverse are similar.

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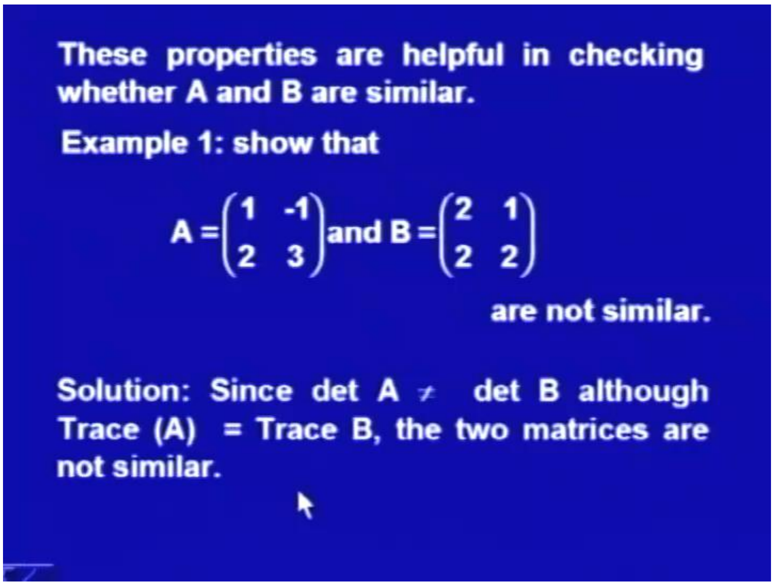

Now, these properties are helpful in checking whether A and B are similar, like if I have given these 2 by 2 simple matrices. Then one can very easily check that they are similar or not. We can first calculate the trace of this matrix A it is 4 and trace of B is 4, but this result is not conclusive. So, we check the determinant, determinant of A is 3 plus 2 and determinant of B is 4 minus 2, so determinant A is not equal to determinant B although trace A is equal trace B and this proves that the two matrices are not similar.

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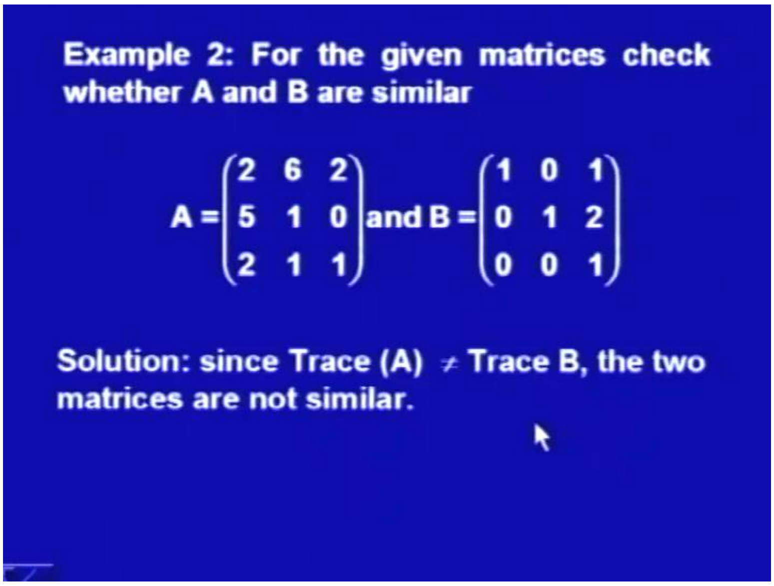

In this example, I have been given 3 by 3 matrices trace A is equal to trace B, let us check trace A is 2plus 1 plus 1 the sum of diagonal elements it is 4 and trace B is sum of diagonal elements as 3. So, trace A and trace B are not equal and one can conclude that the matrices are not similar.

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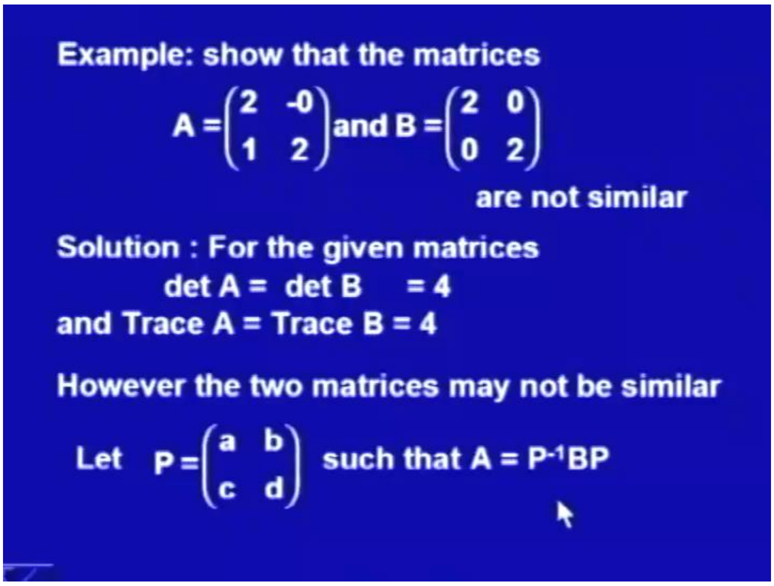

In this example, I have been given 2 by 2 matrices there trace 2 and 2 4 and 2 and 2 4. So, trace A is equal to trace B one can calculate their determinants this determinant is 4 and this determinant is 4, so there determinants are equal. Can be conclude that A and B are similar.

So, my claim is that these two matrices are not similar, although there determinants and traces are equal. Now, to show this result let me have an a square matrix P which is taken a general matrix consisting of elements a b c d such that one can express A in terms of the matrix B as P inverse B P. So, let us say there exist some P for which this is possible, so if we can find out a b c d in such a manner making P invertible, then we say that A and B are similar. If we cannot then of course, A and B will be not be similar.

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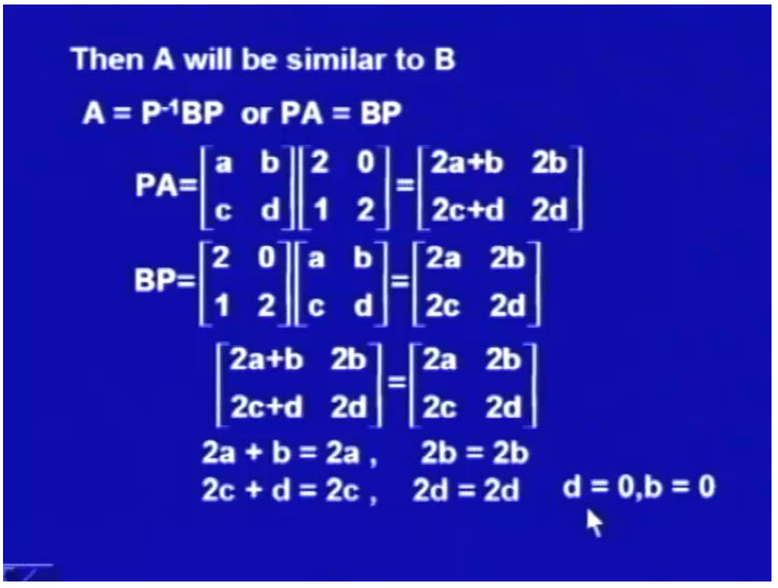

So to do this, we write down A is equal to P inverse B P or this can be simplified to PA is equal to B P. So, we first calculate, what is P A, I have assumed P as a b c d a b is given to me as this.

So, let us multiplier them, so it is 2 a plus b and 2 b and this multiplication will give me this row and P A is this 2 by 2 matrix. Similarly, I calculate B P, so B is given to me as this matrix and a b c d. I have assumed as this get this product and this product comes out to be this, so if A and B are similar, then P A should be equal to B P as I have shown here.

So, let us try to prove, let us try to show that this matrix is equal to this matrix, so we have 2 a plus b into 2 a plus b 2 b 2 c plus d 2 d is equal to 2 a 2 b 2 c 2 d. So, if there are these are equal element wise then the result is proved. So, this gives me four equation 2 a plus b is equal to 2 a this means 2 b is equal to 2 b this means 2 c plus d is equal to 2 c and this means 2 d is equal to 2 d, so these are four equations, since from here, this are identically satisfied, but from here.

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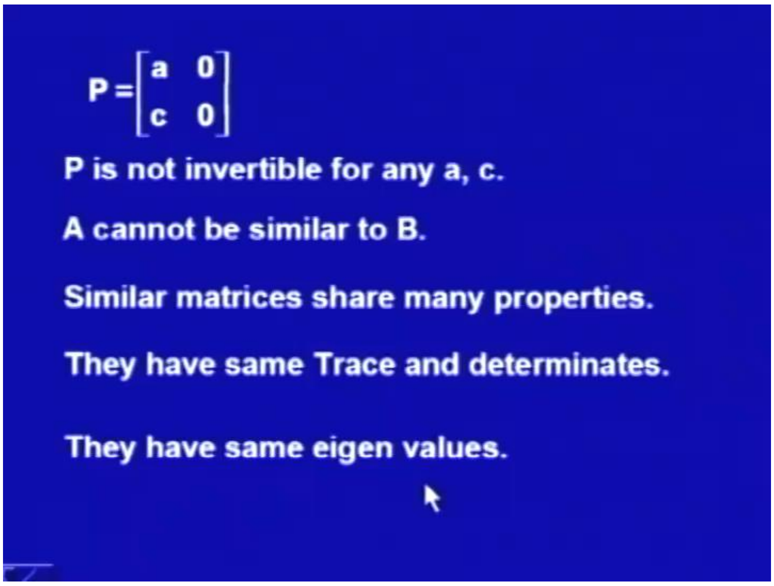

And from this, one can show that d is equal to 0 and b is equal to 0; that means the matrix P will be of the form a c 0 0 and that simply means that determinant of P, whatever be the values of a and c will be 0 and that proves that P is not invertible for any combination of a and c and; that means, we cannot find a matrix P which is invertible, such that A can be expressed as P inverse B P or finally we can say that A cannot be similar to B.

So in this example, we have proved that two matrices which have same determinant same trace, but still there they cannot be similar. So, we it is not conclusive that determinant A is equal to determinant B trace A is equal to trace B then the matrices will be similar. Now, similar matrices share many other properties we have seen that they have same trace and determinants, now I will show that they have same Eigen values

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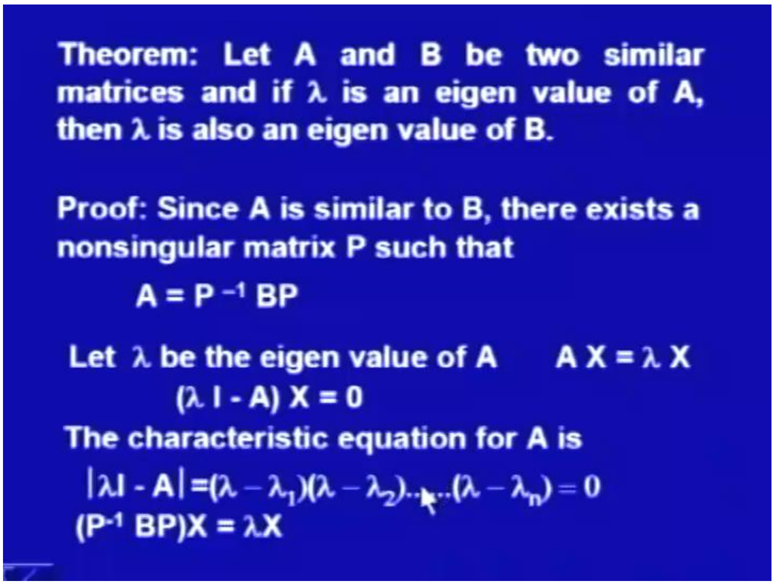

So let us I have this theorem, let A and B be two similar matrices and if lambda is an Eigen value of A, then lambda is also an Eigen value B. So, to prove this result it is it is being given to us that A and B are similar, so we can say that there exists nonsingular matrix P. So, that A is equal to P inverse B P by the definition of similar matrices. Let lambda be an Eigen value A; that means, there are nonzero vectors X such that A X is equal to lambda X, so its Eigen its characteristic equation will be lambda I minus A into X is equal to 0.

And, lambda I minus A, let us say it has the matrix A has Eigen values lambda 1 lambda 2 lambda n. Then, lambda I minus A is equal to lambda minus lambda 1 lambda minus lambda 2 into lambda minus lambda n equal to 0, because we have assumed that lambda one lambda two lambda n are the roots of this characteristic equation. Now, since A and B are similar, so we can say P inverse B P X is equal to lambda X, so I have written the value of A in this equation.

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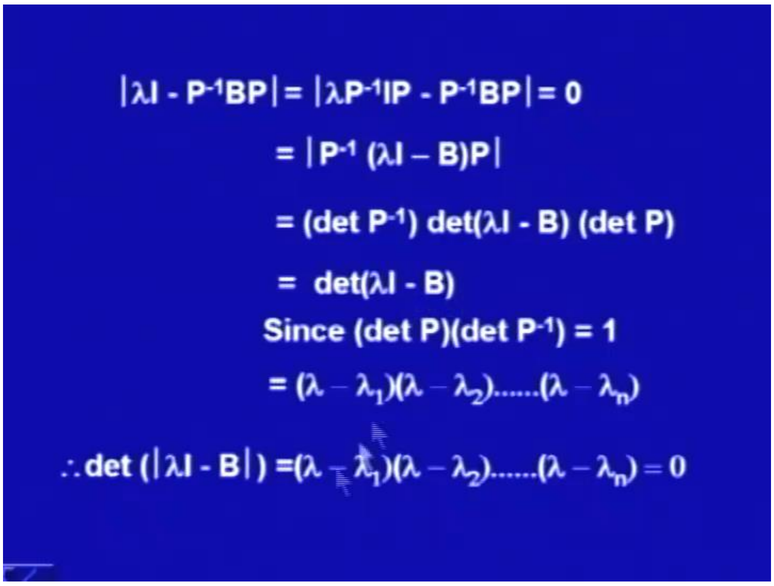

And, this gives me lambda I minus P inverse B P is equal to lambda P inverse I P minus P inverse B P is equal to 0, so this lambda I, I have written in this manner. So, lambda P inverse I P minus P inverse B P equal to 0. And, that gives me that P inverse lambda I minus BP its determinant equal to 0 and this simply means that determinant P inverse into determinant of lambda I minus B into determinant of P and that gives me determinant of lambda I minus B. So, I have started with lambda I minus A and this comes out be determinant of lambda I minus B.

And; that means the characteristic equation for A and for B they are same, so we can write down determinant of lambda I minus B as lambda minus lambda 1 lambda minus lambda 2 into lambda minus lambda n. And, since lambda 1 lambda 2 lambda are n are other roots of this characteristic equation lambda I minus A equal to 0, so; that means, they are roots of this equation determinant lambda I minus B also.

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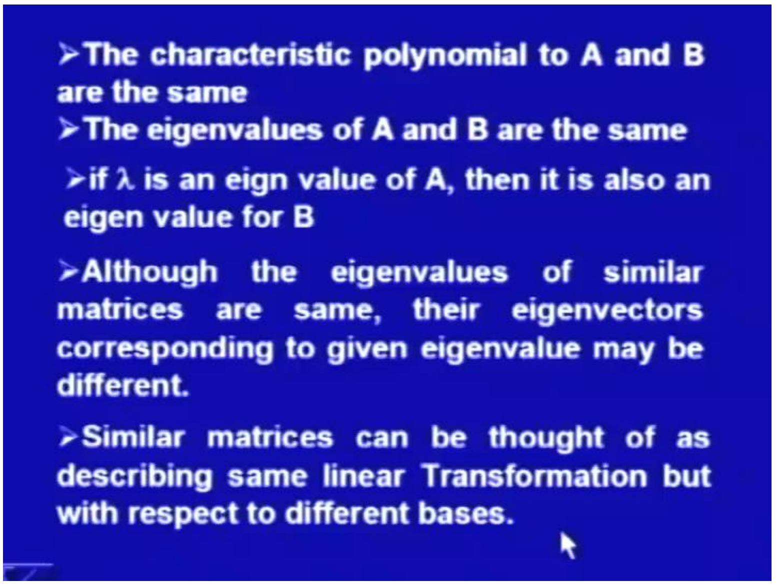

And; that means, the characteristics polynomial of A and B are same and therefore, the Eigen values of A and B are also same. So, that proves the result and this means that if lambda is an Eigen value of A, then it is also an Eigen value for B, now although the Eigen values of similar matrices are same their Eigen vectors corresponding to given Eigen values maybe different. So, this is an observation that Eigen values maybe same we have proved in the form of a theorem that the Eigen values of similar matrices are same, but the Eigen vectors need not be the same. In fact, we say that similar matrices can be thought of as describing same linear transformation, but with respect to different bases.

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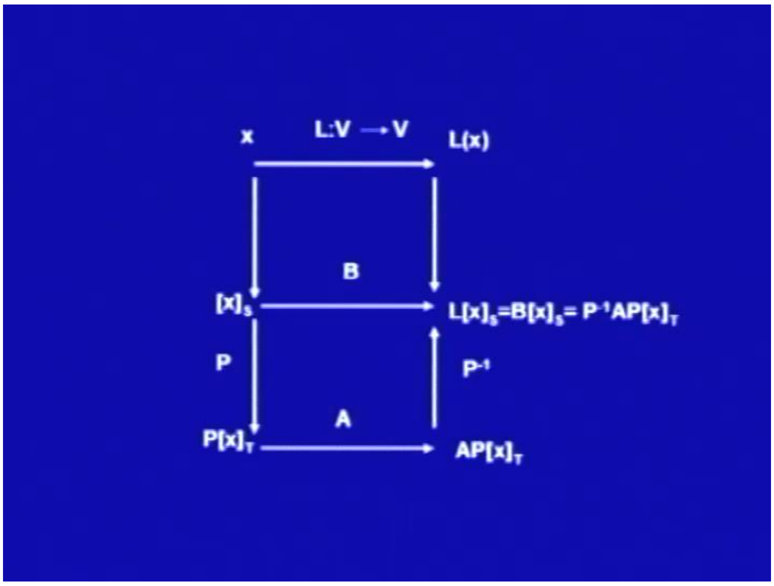

Let me illustrate, this with this diagram, let us say we have a linear transformation in fact linear operator from V to V, V may be R n. Then, any vector X under this operator will map to L x. Let us, say there are two bases for V the s and t, so x s denotes the coordinates of this vector x with respect to basis s. And x s, so this is L x the x will map to Lx., so Lx with respect to basis s can be written as B times x s. So, this vector x s will map to this vectors L of x s. So, B is the transformation matrix, so x s is written as B times x s.

Now, his vectors the coordinates of this vector x in s is x s, but with respect to y the relationship between with respect to be another basis t, the relationship will be x s is equal to t times x t, so these are transformation matrix which takes the vectors of which takes the vectors I mean coordinates x s to x s x t. So, this vector then maps to this P of x t is A times A of P x t. And then you can again comeback from here you can come back to this, so transformation P inverse is to be applied matrix P inverse is to be multiplied to get this matrix. So; that means, it is B P is P inverse A P x t, so we can say it is the same vector, but in a different bases.

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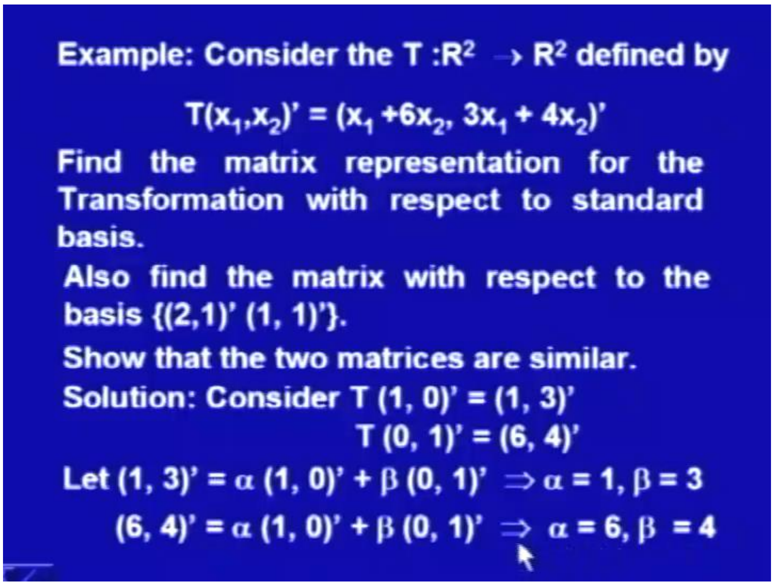

Now, we consider different we consider an example in which T is a transformation from R 2 to R 2 defined by this transformation. We have to find the matrix representation for the transformation related with the standard basis, and then we will also find the matrix with respect to the basis 2 1 and 1 1 and we will show that the two matrices are similar. This will illustrate what we have discuss just now in a diagrammatical representation

So, we start with the vector 1 0 and 0 1 as the base vector in the standard basis, so 1 0 will map to 1 3 under this transformation. One can check x 1 is equal to one and x 2 is equal to 0, so this comes out to be 1 and the second element will map to 3. Similarly, T 0 1 will be 6 0. Then, the vector 1 3 will be represented as a linear combination of the base vectors 1 0 and 0 1; that means, alpha is equal to 1 and beta is equal to 3, so these are the coordinates of this vector. Similarly, 6 4 will be represented as alpha times 1 0 plus beta times 0 1 and that gives me alpha is equal to 6 beta is equal to 4.

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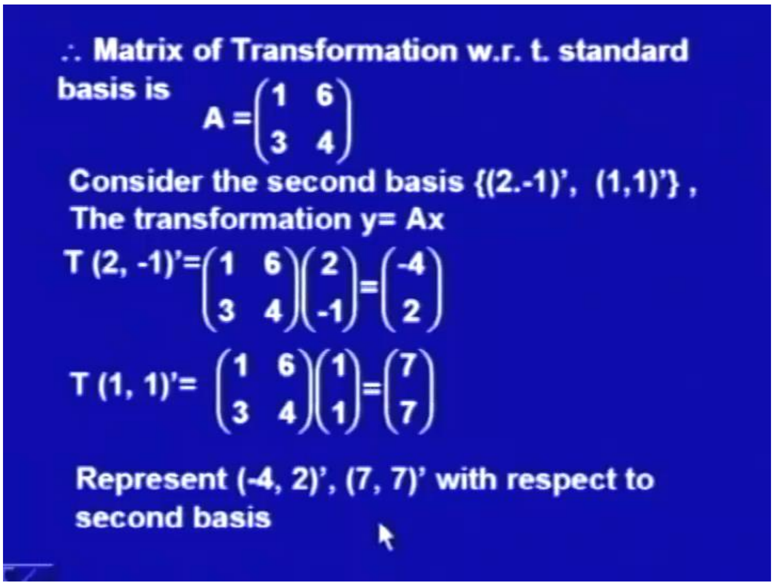

Now, matrix of transformation with respect to standard basis will now be written as 1 3 6 and 4. So, this was the first column vector and this is the second column vector which we have obtained earlier. And accordingly, if we consider the second basis as 2 minus 1 and 1 1, then the transformation y is equal to A x will give me T is equal to 2 minus 1 is equal to this matrix A. And then 2 minus 1 and this gives me 2 and minus 6 is 4, 3 into 4 multiplied by 2 minus 1 gives me 2, so T 2 0 1 will become 4 minus 2. Similarly, T 1 1 can be expressed as this matrix A multiplied by 1 1 and that give me 7 7, so representation of minus 4 2 and 7 7.

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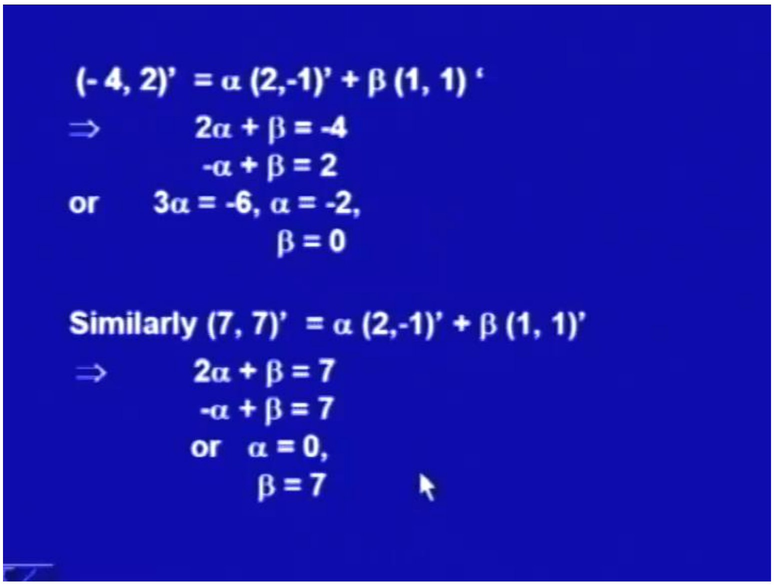

With respect to second basis comes out to be a linear combination of this, so minus 4 comma 2 is equal to alpha times, the base vector 2 minus 1 plus beta times 1 comma 1. And then one can is simplify this equation, so it is 2 alpha plus beta is equal to minus 4 and minus alpha plus beta is equal to 2. So, when you solve these two equations the coefficient alpha comes out to be minus 2 and beta comes out to be 0. Similarly, if we write down 7 7 as a linear combination of base vectors, then again we will have system of equations 2 alpha plus beta is equal to 7 minus alpha plus beta is equal to 7 solve them, we will get alpha is equal to 0 and beta is equal to 7.

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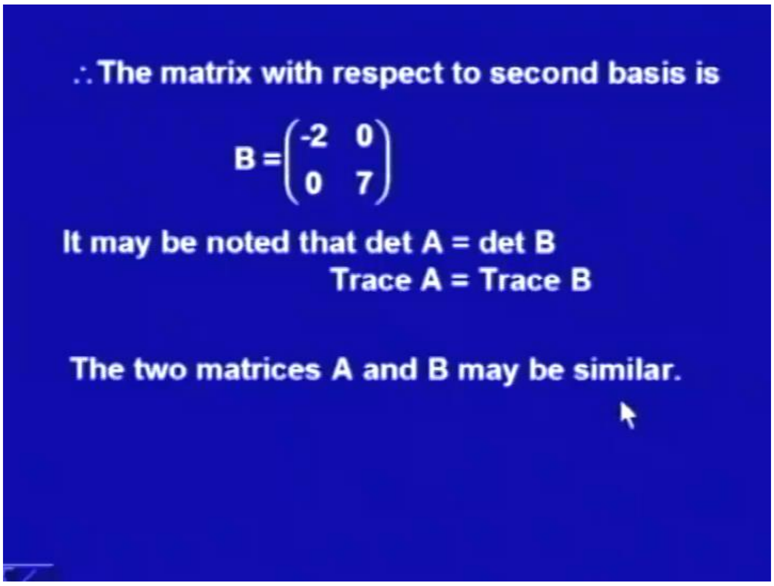

And; that means, the matrix with respect to second basis comes out to be minus 2 0 and 0 7. So, we have now we have two different matrices with respect to two different basis and one can note that determinant A is equal to determinant B and trace A is equal to trace B for these two matrices. Now, the two matrices A and B may be similar, so simply that these two results will not mean that A and B will be similar we can say that A and B may be similar.

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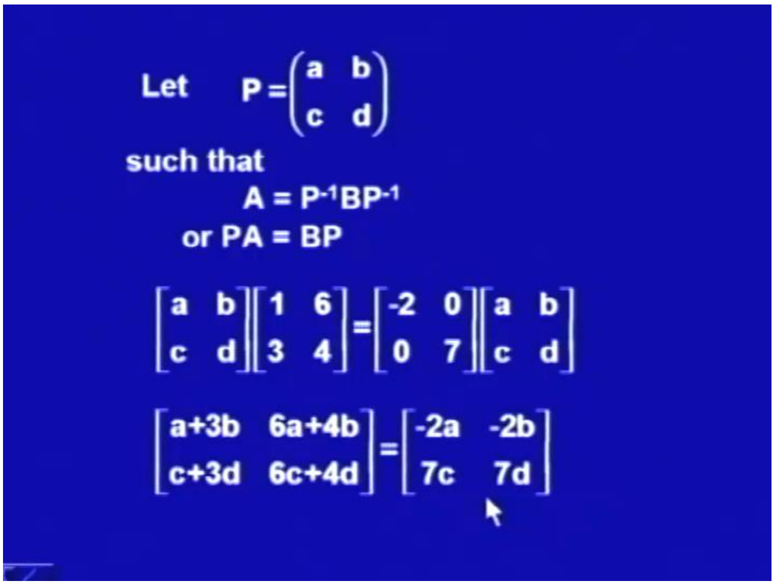

Now, for this let us consider, there exists a nonsingular transformation P which is a b c d. So, that we can write down A is equal to P inverse B P, and then P A is equal to B P and from here we can get AB into the first matrix is equal to second matrix multiplied by P. And from here, if we simplify we will have left hand side as a plus 3 b 6 a plus 2 b 6 c plus 3 d and 6 c plus 4 d and the right hand side will have minus 2 a minus 2 b 7 c and 7 d. and from here.

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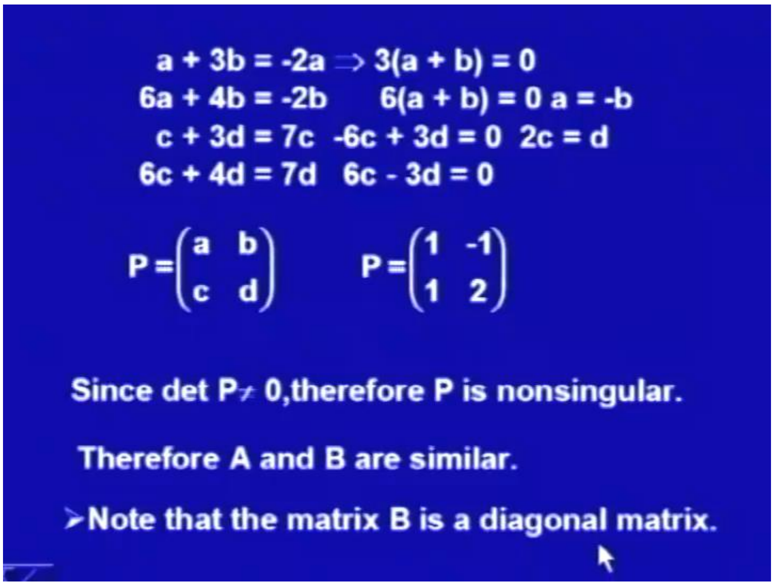

If we solve for a and b, then we will have a plus 3 b is equal to minus 2 a, 6 a plus 4 b is equal to minus 2 b, c plus 3 d is equal to 7 c and lastly 6 c plus 4 d is equal to 7 d and from here one can get that 3 times a plus b is equal to 0 from these two equations.

And that means, if you solve these four equations in four unknowns we will get a is equal to 1 b is equal to minus 1 c is 1 and d is equal to 2,.so this is one such combination. And since, determinant P is not 0; that means, P is nonsingular, so inverse exist. So, we can say that a and b are similar, because we could be able to find out such a matrix a b c d which is nonsingular and which gives as A is equal to P inverse B P. So, one can notice from this system that the matrix B is a diagonal matrix.

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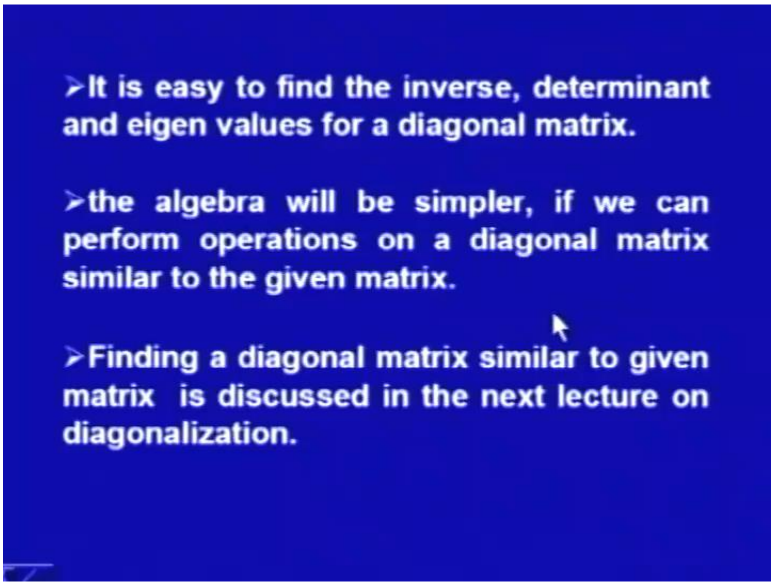

Now on the basis of this, we can say that it is easy to find the inverse determinant and Eigen values for a diagonal matrix, because if it is a diagonal matrix the determinant will be simply product Eigen value. Then, diagonal elements are the Eigen values, so if and the inverse will be simply the inverse of diagonal elements, so if the matrix happens to be a diagonal matrix then number of things can be simplified.

So, algebra will be simpler if we can perform operations on a diagonal matrix similar to the given matrix. So, that is how we say that, finding a similar matrix which is a diagonal matrix we simplify the algebra. So, finding a diagonal matrix similar to given matrix is discussed in the next lecture on diagonalization, so with this remark I close this lecture.

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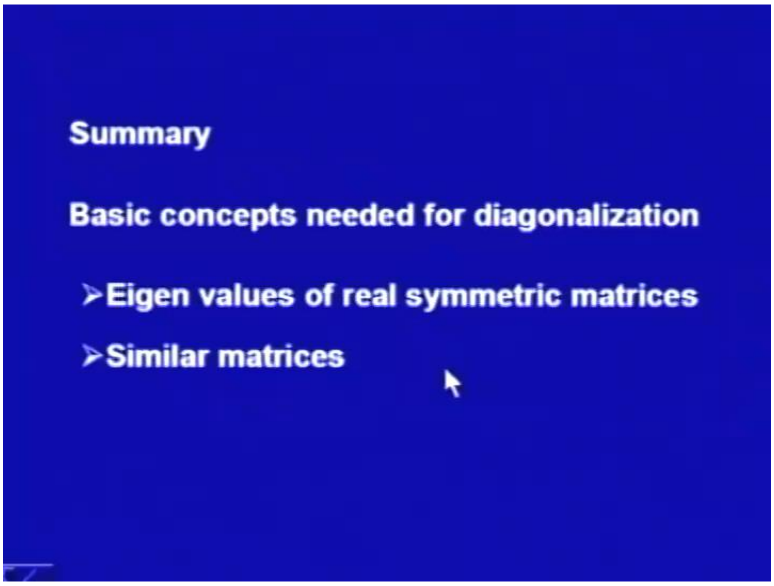

And to summarize, what I have done in this lecture, I have given the basic concepts needed for diagonalization. I have discussed Eigen values of real symmetric matrices and similar matrices and with this we are ready to discuss the diagonalization that will be the content of my next lecture.

Thank you.