Mathematics II
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Lecture - 13
Eigen values and Eigen vectors Part – 1

Welcome viewers. This lecture is an Eigen values and Eigen vectors. The lecture includes Eigen values, Eigen vectors, methods for obtaining Eigen values and Eigen vectors, characteristic equation and Cayley Hamilton theorem.

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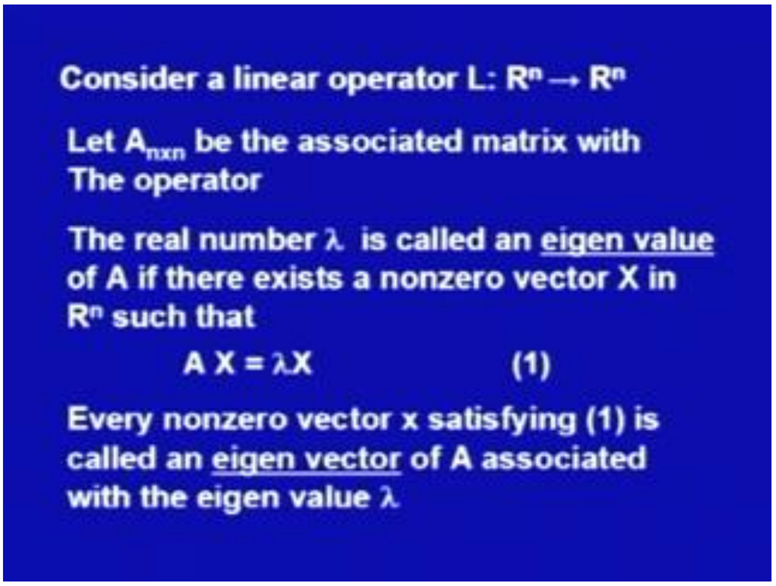

We start with the definition of Eigen values for this we consider a linear operator, which is a linear transformation from R n to R n. Let us say A which is n by n matrix associated with this transformation. Then, the real number lambda is called an Eigen value of the matrix A. If there exist along the nonzero vector X in this R n such that A X is equal to lambda X. So, every nonzero vector x satisfying this equation A X is equal to lambda L square an Eigen value of A associated with the Eigen value lambda.

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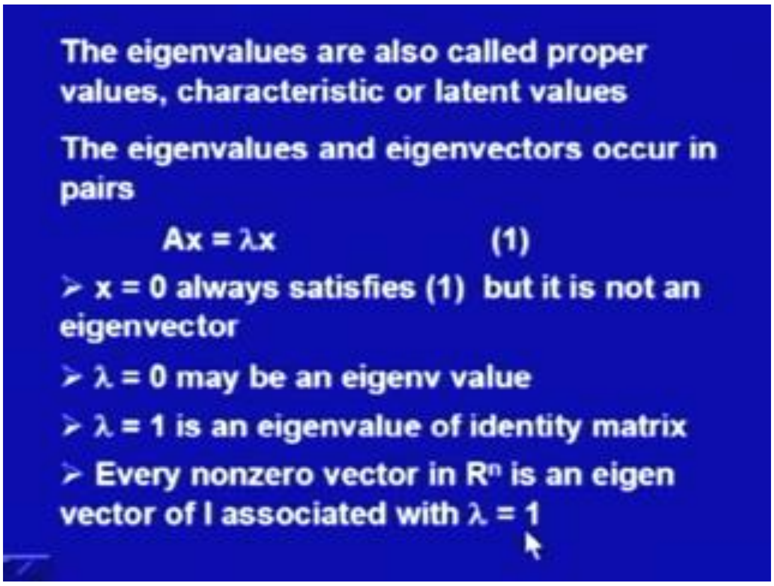

The Eigen values are also called proper values, sometimes we call them characteristic or some text call them latent values. The Eigen values and an Eigen vectors occur in pairs for a given lambda there are associated Eigen vectors. So, for a given lambda there exist a nonzero x satisfying this condition. Then, lambda is an Eigen value and x is the Eigenvector of the matrix or linear transformation or linear operator A.

From this, you can say that x is equal to 0 always satisfies 1. But, it is not an Eigen vector, because by the definition we have included those values of x which are nonzero. So, x is equal to 0 may this satisfying this equation, that it is not an Eigen vector. However, lambda is equal to 0 may be an Eigen value corresponding to this equation.

And lambda is equal to 1 is an Eigen value of identity matrix. Like, if you write down lambda is equal to 1 this equation becomes A x is equal to x. So, such a relationship is possible, then A is equal to I and that gives us the result that lambda is equal to 1 is an Eigen value of identity matrix. Further every nonzero vector in R n is an Eigen value of I associated with lambda is equal to 1. So, whenever I X is equal to X, then whatever value I assign to x this equation will be satisfied. So, I say every nonzero x will be an Eigen value of I.

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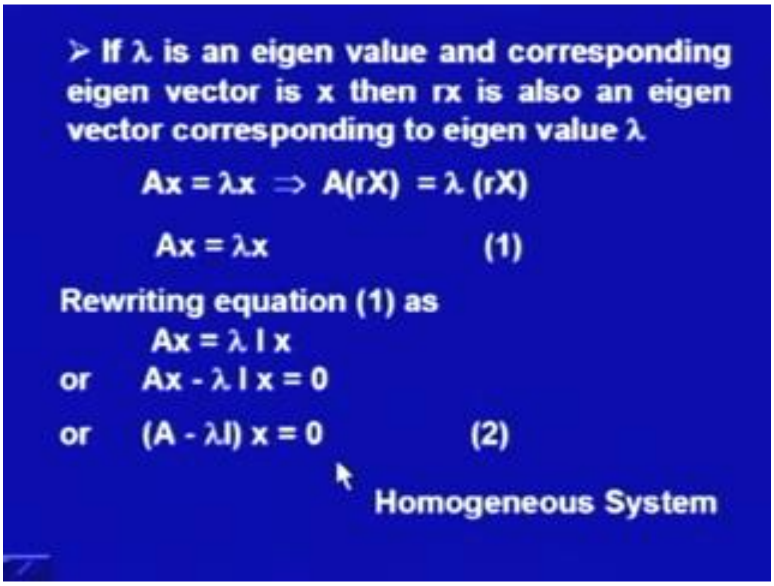

If lambda is an Eigen value and corresponding Eigen vector is x, then one can establish that r x is also an Eigen vector corresponding to the given Eigen value lambda. One can easily prove this, let us say lambda is an Eigen value that simply means, there exist x such that A x is equal to lambda x. And, this imply if I multiply both the sides by a scalar r, then r times A x is nothing but A times r X is equal to lambda times r X and that simply establishes this result.

Now, we start with the equation A x is equal to lambda x rewriting equation 1 as A x is equal to I am writing x as I x, so it becomes A x equal to lambda I x. Then, I take this on the other side, then it is A x minus lambda I x is equal to 0 or I can say it is A minus lambda I times x is equal to 0. So, this is nothing but a homogeneous system of n equations, because A is n dimensional, so this is the homogeneous system.

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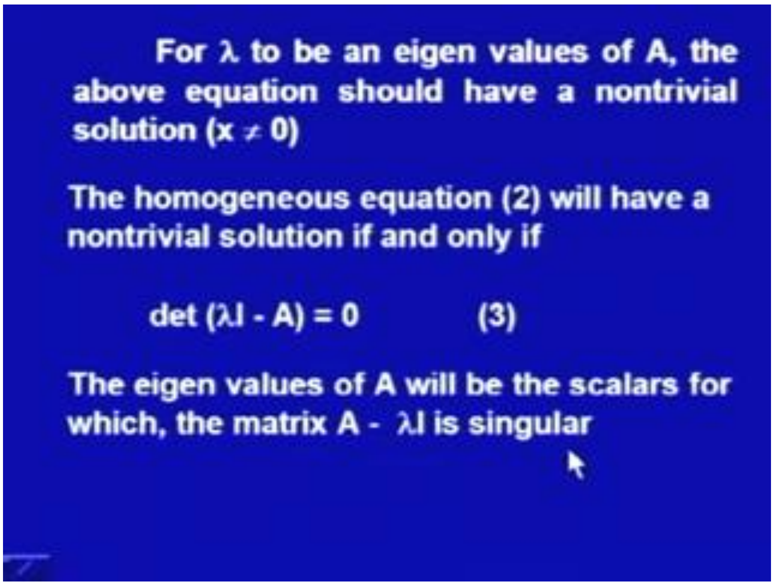

And this homogeneous system will have a nontrivial solution. Why I am talking about nontrivial solution? Because, I am interested in x is equal to 0, x is not equal to 0. So, lambda be an Eigen value, we should have a nontrivial solution, which is possible only when determinant of lambda I minus A is equal to 0. This is the result which we have discussed earlier. So, this homogeneous system will have a nontrivial solution, under this condition. So, this gives me the scalars lambda, which may satisfy this equation. So, the Eigen value of A will be the scalars for which the matrix A minus lambda I is singular.

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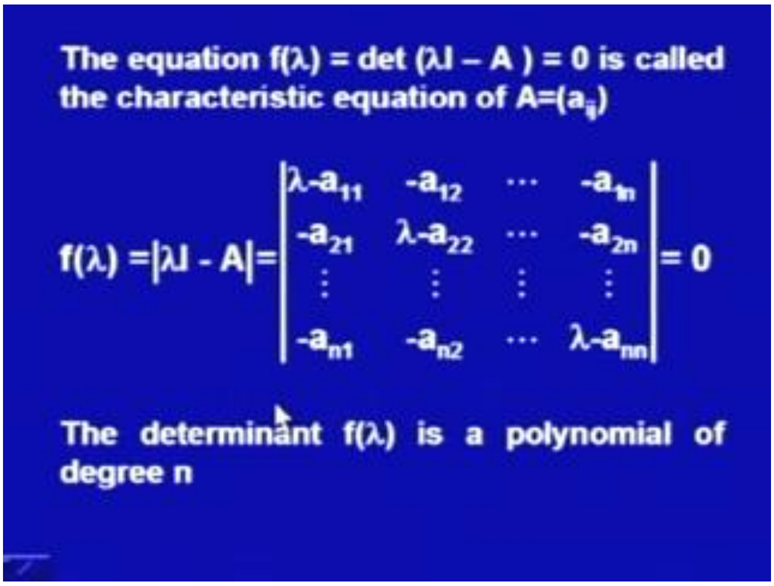

I can write down this determinant lambda I minus A is equal to 0, in this form one can notice that, in this determinant each and every term. The term in the diagonal they are of the form lambda minus a 1 1 lambda minus a 2 2 and lambda minus a n n. This is because of fact that I am writing lambda I minus A.

So, we need a diagonals are affected and because it is minus A, so all these elements will become negative. So, when I expand this determinant, then I can use any row or column, let us say I use this column. Then, it is lambda minus a 1 1 multiplied by this determinant, so 1 factor is lambda minus a 1 1.

Then, I can of course have terms corresponding to this. But, if I consider this particular term, then this is again a determinant of order n minus 1. So, one more term like lambda minus a 2 2 will come. And that way we will be having terms like lambda minus a 1 1 multiplied by lambda minus a 2 2 multiplied by lambda minus a n n. So, we will be having n such factors. And that means, this equation when expanded will be a polynomial of degrees n. So, we say this determinant which is a polynomial in lambda and it is of degree n. So, this characteristic equation is a polynomial in n, and since the polynomial of degree n.

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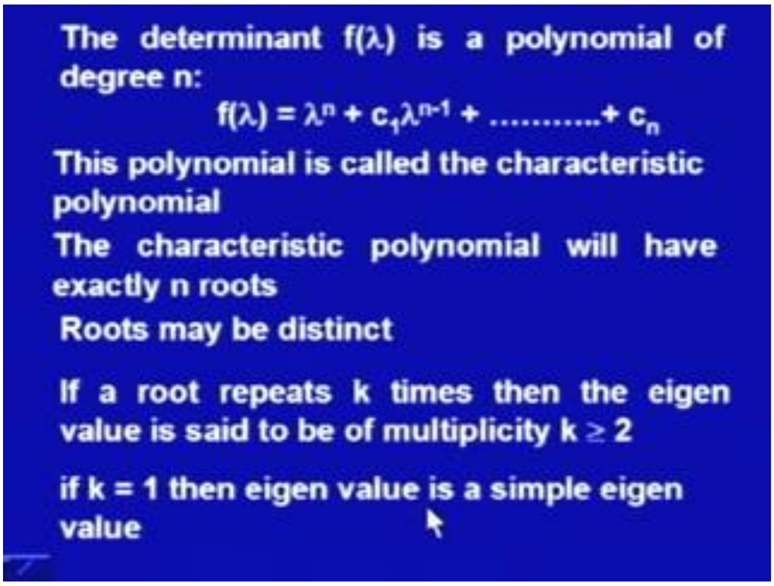

So, f lambda can be written this form lambda raise to power n plus c 1 lambda n minus 1 plus c n. This polynomial is called the characteristic polynomial. And see this polynomial of degree n, so it will have exactly n roots. The roots may be distinct, means that all of the roots may be different or some of the roots may be repeated, if a root is repeated k times. Then, the Eigen value is said to be of multiplicity k. So, we call it repeated with if k is less than equal to 2. So, if all the roots having multiplicity k is equal to 1, then the roots will be distinct. So, if k is equal to 1, then Eigen value is a simple Eigen value in that case.

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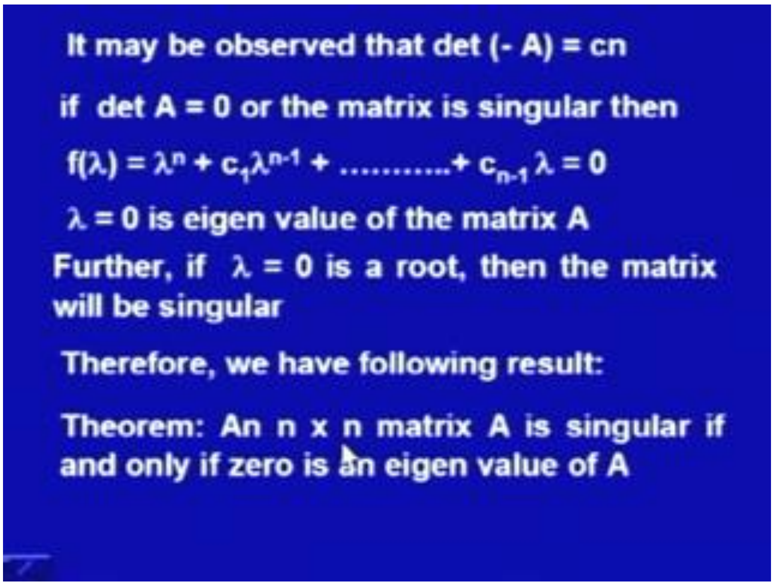

Now, one can observe that, if I put lambda is equal to 0, then determinant minus A is equal to coefficient c n. And if determinant A is 0 or the matrix is singular, then c n will be 0. And that means, my characteristic equation will come in this form, lambda raise to power n plus c 1 lambda m minus 1 plus n minus 1 lambda. The last one will be receive from here. And, this means I can take lambda outside and we can have lambda multiplied by polynomial of degree n minus 1.

And that simply needs, that lambda is equal to 0 is an Eigen value of the matrix A. So, determinant A is 0. Then lambda is equal to 0 is an Eigen value of the matrix A or lambda if the matrix is singular, then lambda is equal to 0, then be a root of the characteristic equation. Therefore, we have following result and we can write down in the form of a theorem, that an n by n matrix A is singular, if and only if 0 is an Eigen value of A. I have to proved one part, but the detail proof can be taken out later on.

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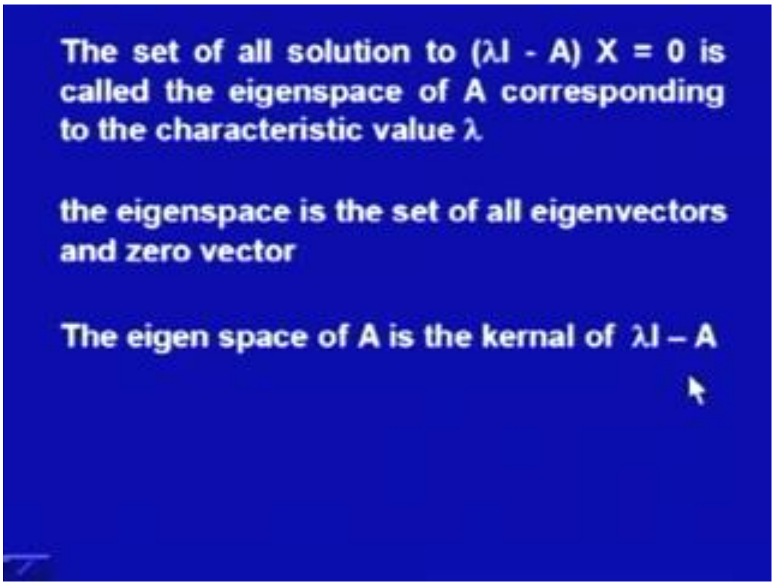

The set of all solutions to this equation is called the Eigen space of A corresponding to the characteristic value lambda. As I told you, that this will have number of solutions. So, they all the solutions will form an Eigen space corresponding to a value lambda. The Eigen space is a set of all Eigen vectors and zero vectors, mind here 0 vector is not an Eigen vector. And without zero vector it will not be a Eigen space or a vector space. So, it is important that Eigen space is a set of all Eigen vectors and a zero vector. The Eigen space of A is a kernel of this matrix lambda I minus A.

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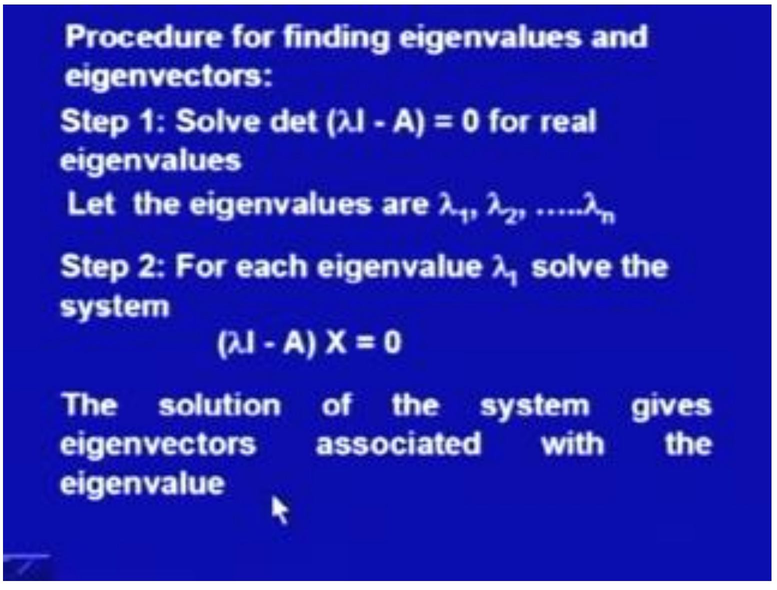

Now, we have discussed the meaning of Eigen values and Eigen vectors. But, now we will discuss of the procedure of finding Eigen values and Eigen vectors for a given matrix A. So, the first step towards this is like we form the matrix lambda I minus A for the given matrix A and then calculate its determinant equate it to 0.

And then we solve this equation as I told you, that this is polynomial of degree n. So, solution of this equation means finding the roots of this equation. And depending upon the value of n we will be having number of Eigen values. So, let us say the Eigen values are lambda 1 lambda 2 lambda n for A to B and n by n matrix it has to have n roots some of them may be repeated, but now the total number of roots will be n.

Now, for each lambda I we solve the system lambda I minus A X is equal to 0. So, we put this value lambda 1 here and we solve this equation. This as system of equation this X simple equation actually represents system of equation, so if A is n by n. So, there are n such equations, there will be n simultaneous equations. And this is the matrix equation and this equation is to be solved. The solution of the system gives the Eigen vector associated with the Eigen value lambda.

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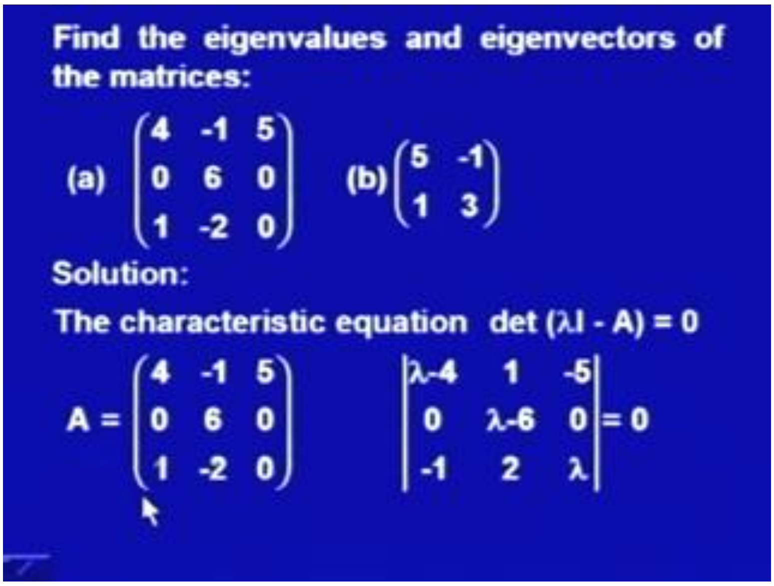

So, let us illustrate this procedure with the help of examples. So, in the first example I have I am considering a 3 by 3 matrix 4 minus 1 5 0 6 0 1 minus 2 0, while the second example I am considering a 2 by 2 matrix. So, first matrix A as this the first example, so according to the method which I have illustrated, I have to first obtain the solution of this characteristic equation.

So, the characteristic equation is determinant lambda I minus A is equal to 0. So, I substitute this value of A in this equation. So, the characteristic equation becomes lambda minus 4, because it is I means only diagonal terms will be effected. So, lambda will appear in the diagonal only and because of minus A, all these terms are negated here.

So, 4 plus, so it is minus 4 minus 1, so I have 1 here 5 the corresponding term is minus 5 0, this is minus 6 and 0 1 becomes minus 1 and we have 2 and lambda. So, this determinant equal to 0 is the characteristic equation for the given matrix A.

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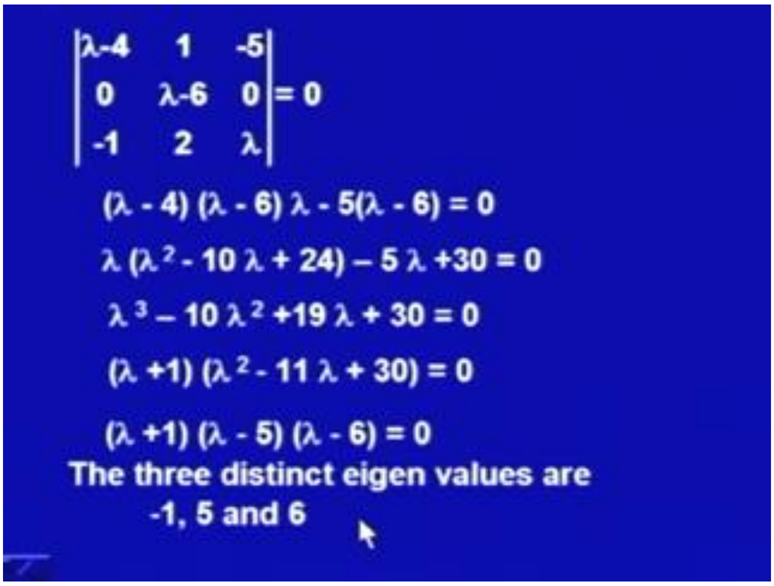

So, this is the latest equation this is to be solved. So, what I am doing here is let us consider that, we will expand this along this column. So, it is lambda minus 4 multiplied by lambda minus 6 into lambda multiplied by minus 5 into lambda minus 6. So, this is the characteristic equation.

So, when you expand this determinant we will have this characteristic equation, which can be simplified. So, these two terms are multiplied will have this expression. And then minus 5 lambda plus 30 is equal to 0 further simplification will give me a cubic polynomial, it is a 3 by 3 matrix. So, we expect characteristic polynomial to be cubic polynomial. So, we have lambda cube minus 10 lambda square plus 19 lambda by combining these two terms plus 30 is equal to 0.

So, this characteristic equation is to be solved, so we factorize this equation. So, lambda is equal to minus 1 is a factor of this equation. So, we take it out this factor and what remains is this. And simplifying this will have lambda plus 1 lambda minus 5 into lambda minus 6 is equal to 0. And this gives me three distinct Eigen values as minus 1 from this factor this 5 and from this factor 6. So, this 3 by 3 given matrix has 3 Eigen values minus 1 5 and 6. So, once we obtain Eigen values, the next step is to obtain Eigen vectors.

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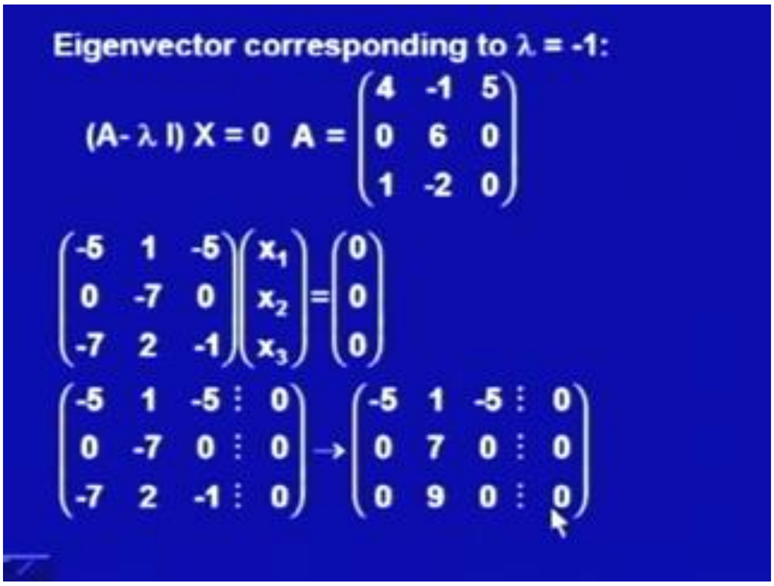

So, we have to obtain Eigen vectors corresponding to each of the Eigen values 1 by 1, so I start with lambda is equal to minus 1. So, for this I have to form this matrix equation A minus lambda I X is equal to 0 and lambda is equal to minus 1 in this and A is this given matrix.

So, if I substitute this A and this lambda is equal to minus 1 in the given equation I have minus 5 1 minus 5 this will be negative and 0 minus 7 0 minus 7 2 minus 1 multiplied by x 1 x 2 x 3 is equal to 0 0 0. This 3 by 3 system is to be solved and to solve this homogeneous system, I form the augmented matrix for this system and it is minus 5 1 minus 5 0 minus 7 0 minus 7 2 minus 1 the coefficient matrix are and this column is appended here.

And then I like to apply linear transformation, so that this can be reduced in a diagonal form or in an echelon form. So, this can be made 0 by suitable liner transformation and this matrix reduces to this matrix.

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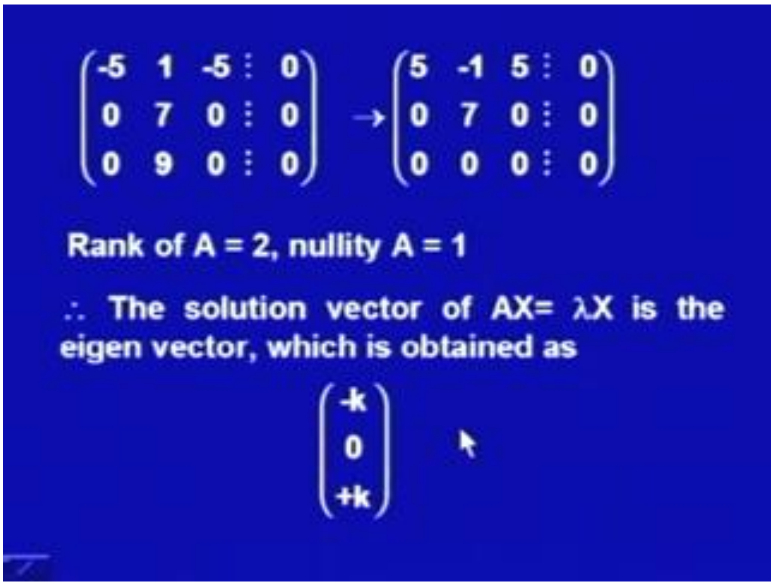

And from this I can further apply a linear transformation and what I have is 5 minus 1 5 0 0 7 0 0 and in fact I can have, but I can multiply this by 9 and this by 7. So, after subtracting this will become 0. So, one row becomes 0 in this augmented matrix and by this we can say that rank of the matrix A is 2 and from rank nullity theorem nullity A is equal to 1. And that means, the solution vector of A X equal to lambda X is the Eigen vector which can be obtained as minus k 0 and k, so nullity is equal to 1.

So, what can I say, what can I do is, I can write down the third it in the form of in the third equation I can write it write down the third variable as k. Then, this equation will automatically be satisfied for y is equal to 0 and by writing is y equal to 0 and z is equal to 5 will give me X is equal to minus 1.

So, this is a solution for the given equation and we will have in fact infinite many solutions of this equation and each solution will be represented in this form k can take infinite many values. So, we will be having all the solutions will be of this particular form, so we can say this is the basis for the Eigen vectors associated with lambda is equal to minus 1.

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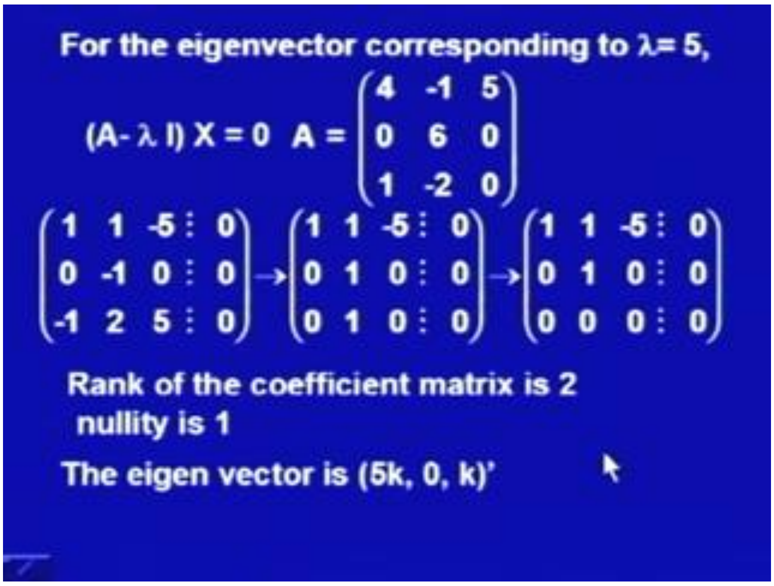

Once we have solve this problem, then we go to the next Eigen value lambda is equal to 5 the same X as steps can be repeated. That means, you again start with the matrix equation A minus lambda I times X is equal to 0. For the given matrix A and then we form the augmented matrix we apply linear transformation is series. And once, this matrix is reduced to be echelon form which is this particular example for lambda is equal to 5. And one can notice that the again this row becomes 0. And that means, rank of this coefficient matrix is 2 and nullity is equal to 1. And in that case, if you solve this equation the system of equation k can be taken as arbitrary value y can be 0. So, when we substitute it here, this is k is arbitrary, k and this is 0, so X becomes 5 k. So, the Eigen vector corresponding to the Eigen value lambda is equal to 5 will be of the form 5 k 0 k.

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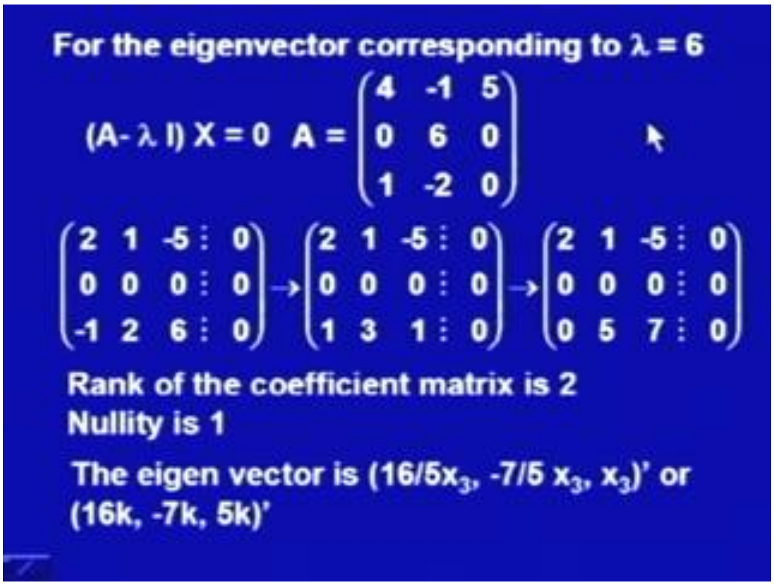

So, Eigen vector the third Eigen value was lambda is equal to 6. Again the whole thing is repeated A minus lambda I into X is equal to 0 for this given value of A. The augmented matrix is obtained first we apply linear when we apply a element transformations and this time these element transformations lead to this matrix here this row becomes 0.

So, again the rank of the coefficient matrix is 2, but the nullity is 1 when lambda is equal to 6. And that means, we can write down the solution of this matrix as if you say X 3 arbitrary. Then, y becomes minus 7 by 5 x 3 and we substitute x 3 as x 3 here and minus 7 by 5 X 3 here. Then, this can be X can be obtained as 16 by 5 x 3. That means, X y and z they are written in terms of the component x 3 or let us say x 3 is equal to k then this can be written in the form sixteen k minus 7 k and 5 k. So, this vector is an Eigen vector corresponding to Eigen value lambda is equal to 6, so we have three different Eigen values in this example and we have obtained three different Eigen vectors in this case.

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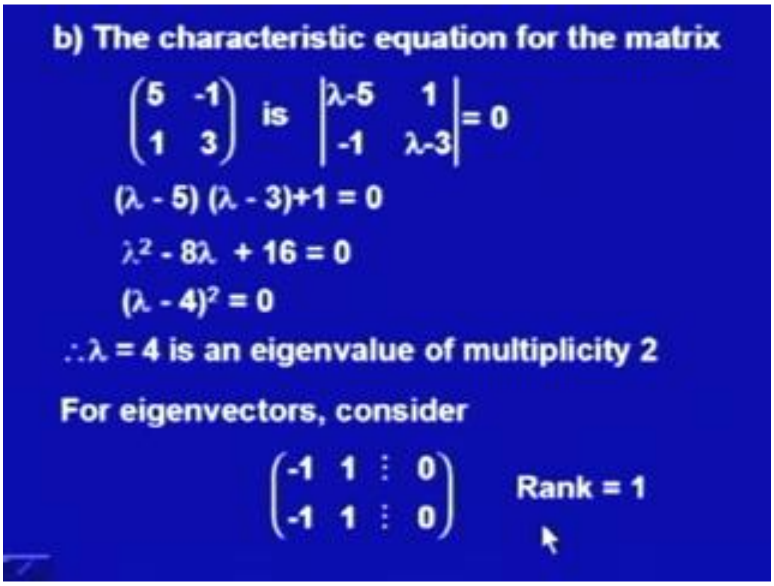

In the next part we have been given a 2 by 2 matrix. So, we apply the Eigen we have to first find the characteristic equation. So, we write down lambda minus 5 1 minus 1 lambda minus 3 determinant of this is equal to 0, we solve the determinant lambda minus 5 into lambda minus 3 plus 1 equal to 0.

And this gives me lambda square minus 8 lambda plus 15 and 1 16 equal to 0. This equation gives me lambda minus 5 lambda minus 4 whole square equal to 0. So, this case gives me lambda is equal 4 is an Eigen value and it is multiplicity is 2. So, it is a case when that Eigen values are repeated. So, when the Eigen values are repeated, then you have to solve this augmented matrix to obtain the Eigen vectors corresponding to lambda is equal to 4. And one can notice that this row is a same as this row, so rank of this matrix is 1 and nullity is also 1 from the rank nullity theorem.

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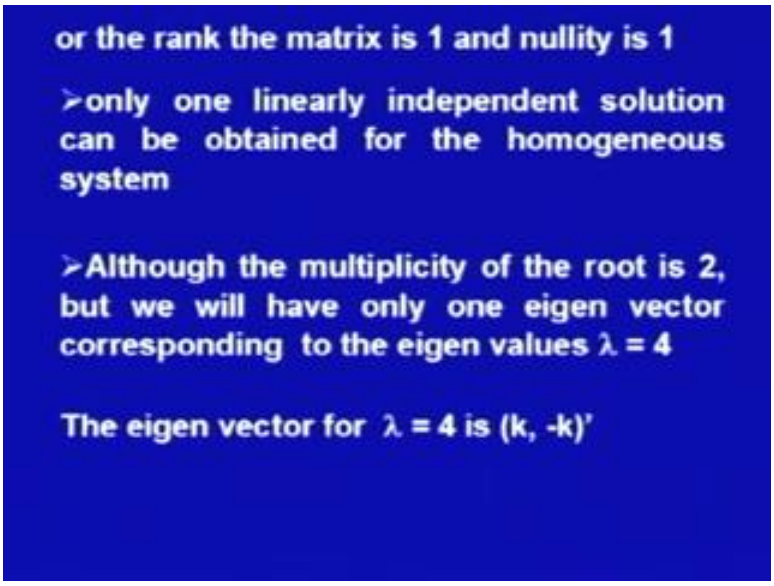

So, nullity is also 1 and that means, it has any one linearly independent solution, which can be obtained from the homogeneous system. So, although the multiplicity root is 2, but we have only one Eigen vector corresponding to the Eigen value lambda is equal to 4. So, in the earlier example was have three distinct values and we got three Eigen vectors. But, in this case we got two we got repeated Eigen values and we have only one Eigen value corresponding to that. So, what about Eigen value vector corresponding to the lambda is equal to 4 it is k and minus k.

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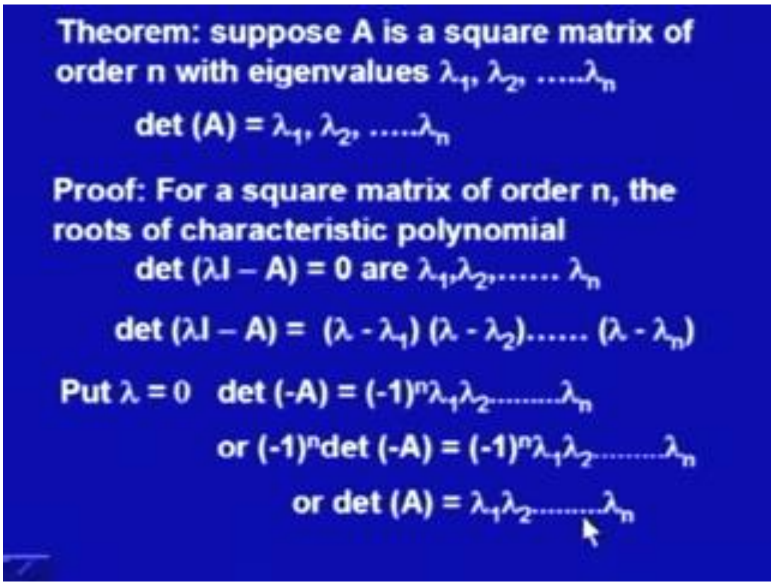

Now, we estimates some result related with Eigen value and Eigen vectors. So, let us say A is a square matrix of order n with Eigen values lambda 1 lambda 2 lambda n, then determinant of A is the product of Eigen values. Now to prove this, let us say square matrix of order n. Then, the roots of this characteristic equation the determinant lambda I minus A is equal to 0 are lambda 1 lambda 2 lambda n is been given to us. Then, since we have roots n this is the characteristic polynomial at degree n. So, we can write down that characteristic polynomial which has roots lambda 1 lambda 2 lambda n as the product of this.

So, we can say determinant lambda I minus A is equal to lambda minus lambda 1 lambda minus lambda 2 and the factor lambda minus lambda n. Now in this case, if we put lambda is equal to 0 in this, the determinant minus A is equal to this minus this minus this minus these lambdas are put to 0. So, they are n minus signs, so we will have minus 1 raise power n lambda 1 lambda 2 lambda n and this is equal to determinant of minus A, this is lambda is put to 0.

Now, we know that determinant of minus A is equal to minus 1 raise to power n determinant of A. So, this minus 1 and this minus 1 will get cancel and we have determinant A is equal to lambda 1 lambda 2 lambda n. So, we have proved the result.

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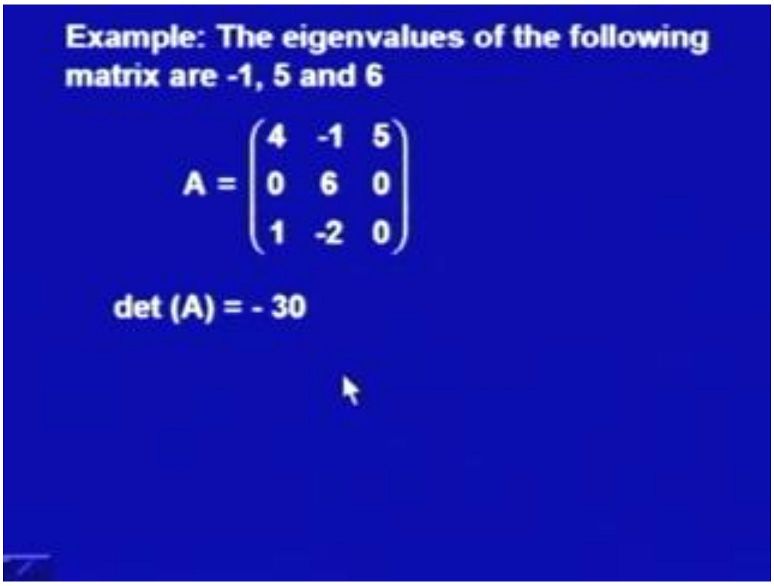

So, the example which we have done earlier, we have obtained the Eigen value as minus 1 5 and 6. So, this is this was the matrix in that example. So, if you substitute if you use this result then determinant of A is the product of this Eigen value. So, determinant of A is equal to minus 30.

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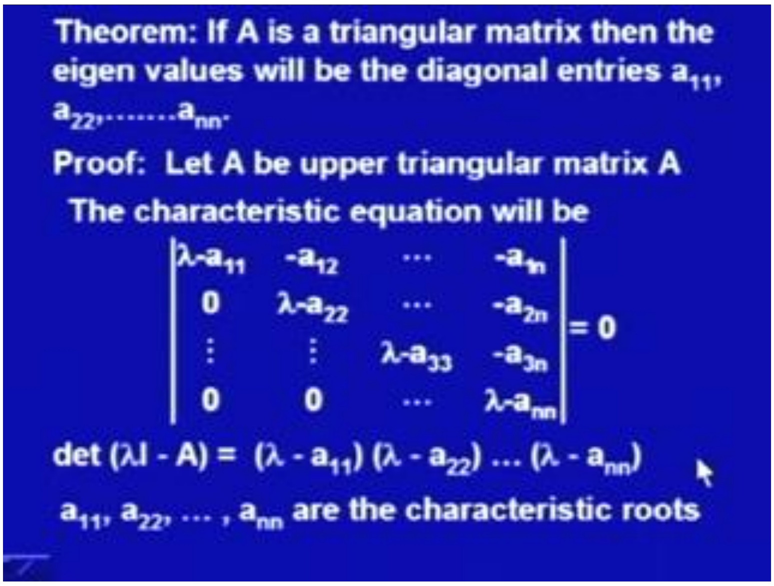

The next result is if A is a triangular matrix. Then, the Eigen values will be the diagonal entries a 1 1 a 2 2 a n n. Like, if A is a triangular matrix let us assume that it is an upper triangular matrix the similar result, similar way the result can be true follow a triangular matrices also. So, let us assume at the moment that A is a upper triangular matrix.

Then, the characteristic equation will be this. So, it say is the upper triangular matrix is upper triangular matrix. So, all the elements in the lower part is 0 and only these elements are nonzero. And then characteristic equation is lambda I minus A. So, will have minus a 1 1 minus a 2 2 and so on in the diagonal terms and will have lambda minus a 1 1 and all these terms will be simply negated.

So, if you expand this determinant it comes out to be lambda minus a 1 1. Now, we are expanding the determinant along this column. So, lambda minus a 1 1 into this determinant and lesser terms will not contribute anything, because they are 0. So, lambda minus a 1 1 into determinant this is lambda minus a 2 2 into this determinant and so on. And that means, it is lambda minus a 1 1 lambda minus a 2 2 lambda minus a n n.

So, this nth degree polynomial and all these are factors. So, if it is 0 means all these factors have to be 0. And that means, lambda is equal to a 1 1 is one Eigen value lambda is equal to a 2 2 is other Eigen value lambda is equal to a n n is the nth Eigen value of the given matrix A, what are a 1 1 a 2 2 a n n, they are nothing but the diagonal elements in the given matrix A and that was to be proved in this theorem.

So, this result is particularly important, because to solve an Eigen value problem 1 has 2 first obtain the characteristic equation. And that means, this determinant is to be obtained. Finding the nth degree polynomial corresponding to the given matrix, if you have to expand this determinant is any difficult, and then finding Eigen value and finding the solution of that characteristic equation is further complicated. But, if we can have a lower triangular matrix or upper triangular matrix, then we do not have to worry about the rest of the elements you can simply write down the Eigen values as a 1 1 a 2 2 a n n that Eigen elements. So, this result helps us in finding out the Eigen values of matrices.

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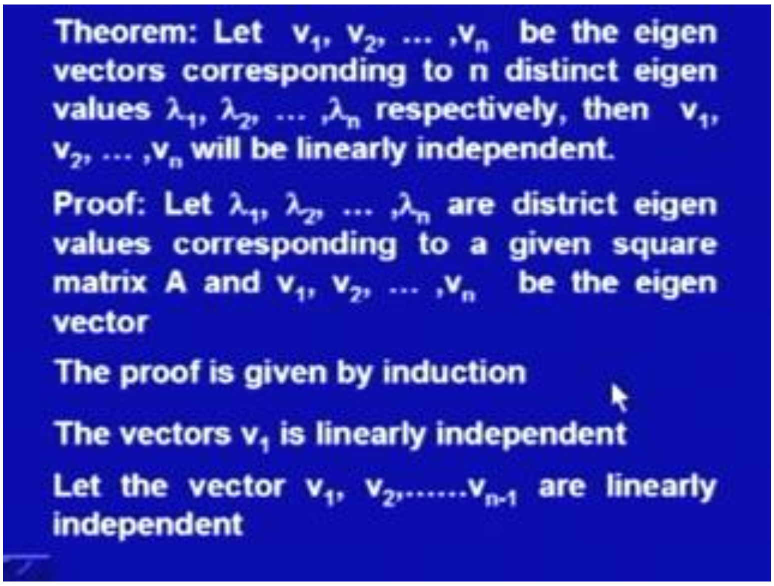

There is one more result it states that v 1 v 2 v n be the Eigen vectors corresponding to n distinct Eigen values of lambda 1 lambda 2 lambda n respectively. Then, v 1 v 2 v n will be linearly independent. So, if v 1 is an Eigenvector corresponding to lambda 1, v 2 is an Eigen vector corresponding to lambda 2 and v n is the Eigen vector corresponding to lambda n. Then, these vectors are distinct if these vectors are linearly independent if these Eigen values are distinct. So, this is the theorem.

So, let us prove this result. So, let us say that lambda 1 lambda 2 lambda n are distinct Eigen values corresponding to a given square matrix A and lambda 1 v and v 1 v 2 v n be the Eigen vectors. Then, we can prove this result by induction. So, we start with v 1 is an Eigen value corresponding to lambda 1, so only one vector, so it is always linearly independent. Next, we assume that v 1 v 2 v n minus 1 corresponding to different lambda 1 lambda 2 lambda n minus 1 are linearly independent.

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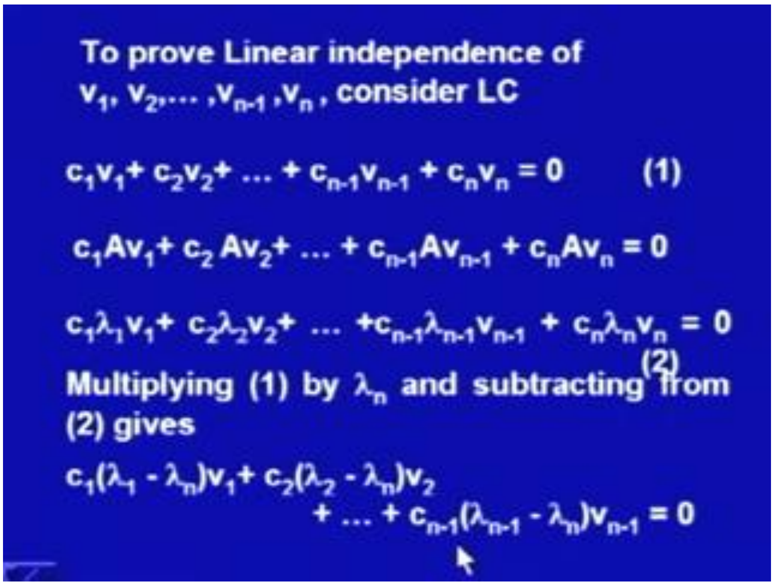

Then, we have to prove that when v n is added in this linearly independent set v 1 v 2 v n minus 1, then the complete set is linearly independent. So, to prove the linear independence of this set v 1 v 2 v n minus 1 v n. Let us consider the linear combination c 1 v 1 plus c 2 v 2 plus c n minus 1 v n minus 1 plus c n v n is equal to 0. And if we can prove that the c 1 c 2 c n minus 1 and c n are all 0, then we have a we can say that these vectors are linearly independent.

So we apply A, we cancelation A linear transformation A to this A is a matrix. Let us multiply this equation by matrix A. And since it is a linear combination and A is a matrix. So, we can say c 1 into A v 1 plus c 2 A v 2 plus n minus 1 A v n minus 1 plus c n A v n and A multiplied by 0 is 0.

Now, since v 1 is an Eigen value corresponding to Eigen we since v 1 is an Eigen vector corresponding to Eigen value lambda 1. So, A v 1 is equal to lambda 1 v 1, so that is we are replacing A v 1 by lambda 1 v 1. Similarly, A v 2 is to be replaced by lambda 2 v 2 as lambda 2 is the Eigen value and v 2 is its corresponding Eigen vector.

Similarly, this term will become lambda n minus 1 v n minus 1 and the last term will be c n lambda n v n equal to 0, let us call this equation as 2.

Then, multiplying the equation 1 by lambda n and subtracting from 2, what we get is this, is multiplied by lambda 1 and we multiplied with by lambda n subtract from this. So, what will have is this equation, c 1 v 1 is common and this is lambda 1. But, this is being multiplied by lambda n and subtraction is taking place. So, it is lambda 1 minus lambda n v 1 plus c 2 times lambda 2 is coming from here and lambda n from here. So, it is c 2 times lambda 2 minus lambda n v 2 and will have all factors up to these terms. But, when we see this last term it is c n lambda n v n and this is also c n lambda n v n. So, these 2 terms will be cancelled out. So, we will have terms c 1 up to c n minus 1.

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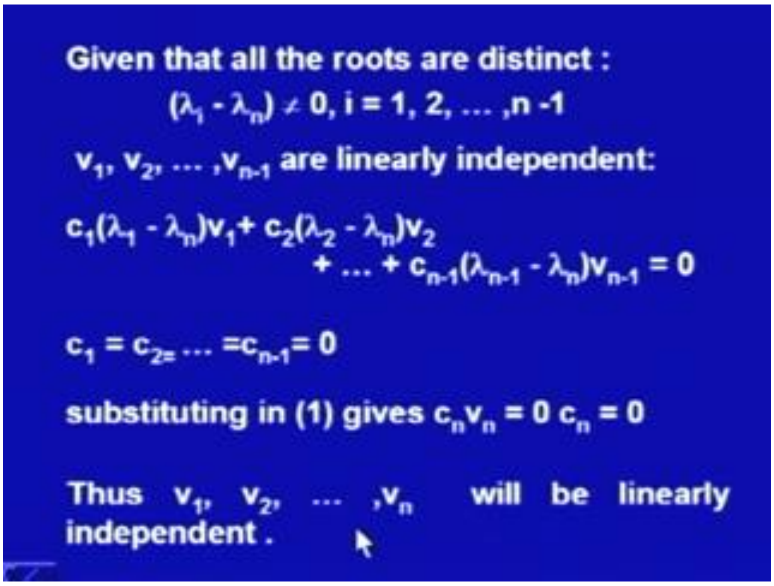

Now, given that all the roots are distinct. That means, lambda i is not the same as lambda n for any i 1 2 to n minus 1 lambda n is different, then all these lambda i is... So, they are not 0 and further v 1 v 2 v n minus 1 are linearly independent. And that means, this is equal to 0 this is not 0 this is not 0 this is not 0 this is not 0. So, sum of this means a combination is 0 simply means that c 1 c 2 and c n minus 1 has to be 0, because v 1 v 2 v n minus 1 are linearly independent.

So, c 1 c 2 and c n minus 1 they are all going to be 0, so when we substitute these values in 1, then we have the expression c n v n is equal to 0 v n is not 0 its given to as it is a vector which is nonzero. So, what we have is that, c n is equal to 0 is only alternative, so we have obtained a linear combination which is 0 only when c 1, c 2, cn's all are 0. And that proves that v 1, v 2, v n will be linearly independent. So, if we have n distinct value Eigen values then the corresponding Eigen vectors will be linearly independent. So, this was the theorem.

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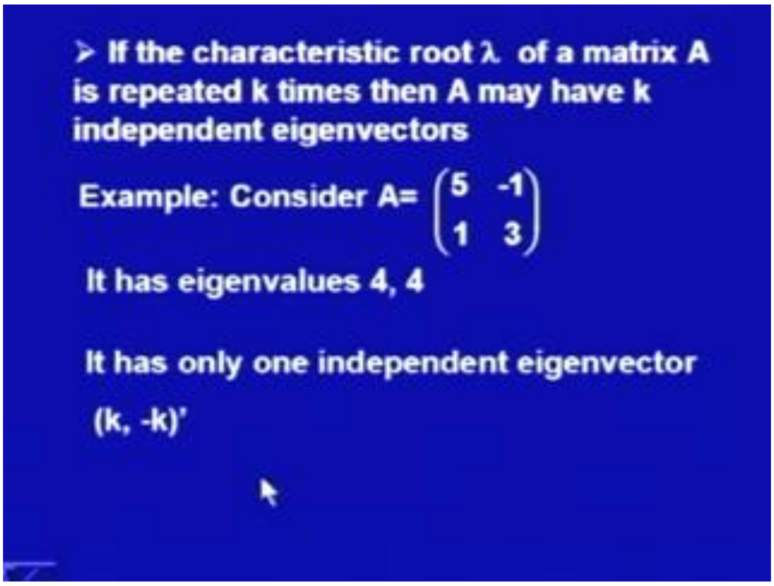

Now, if the characteristic root lambda of a matrix A is repeated k times. Then, A may have k independent Eigen vectors it may happen it may not. If all that values are distinct definitely we will have independent Eigen vectors. But, if they are not distinct they are repeated. Then, the corresponding Eigen values will be dependent or independent both the things are possible.

We have discussed an example in which A is a 2 by 2 matrix 5 minus 1 1 3 we have obtained its Eigen value as 4 and 4. So, the Eigen values are repeated and we could find only one independent Eigen vector in this example as k minus k, so for the roots repeated we got only one Eigen value.

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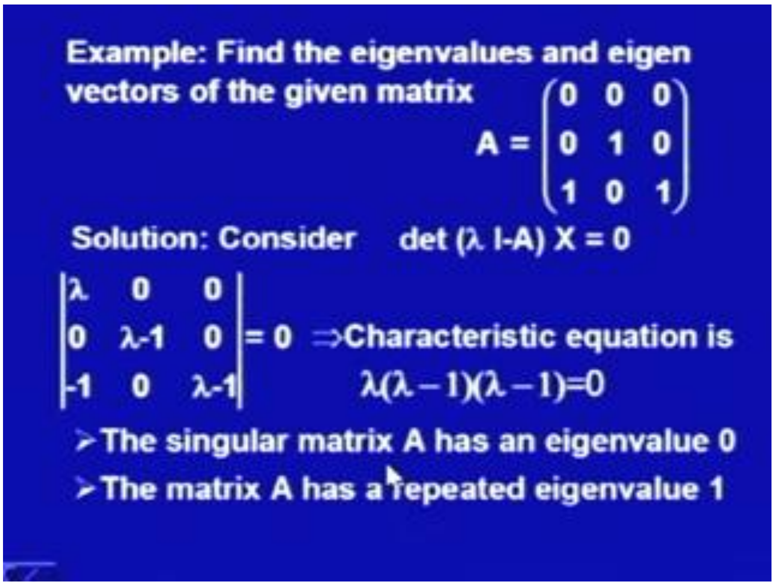

However, if we consider A as this matrix, then the characteristic equation is determinant lambda I minus A X is equal to 0 gives me this determinant equal to 0. And when you solve this determinant the characteristic equation comes out to be lambda lambda minus 1 lambda minus 1 equal to 0. That means, this is a singular matrix and it has as Eigen value 0, this result we have already established. And the matrix A has a repeated Eigen value 1, so it has 3 Eigen value 0 1 and 1.

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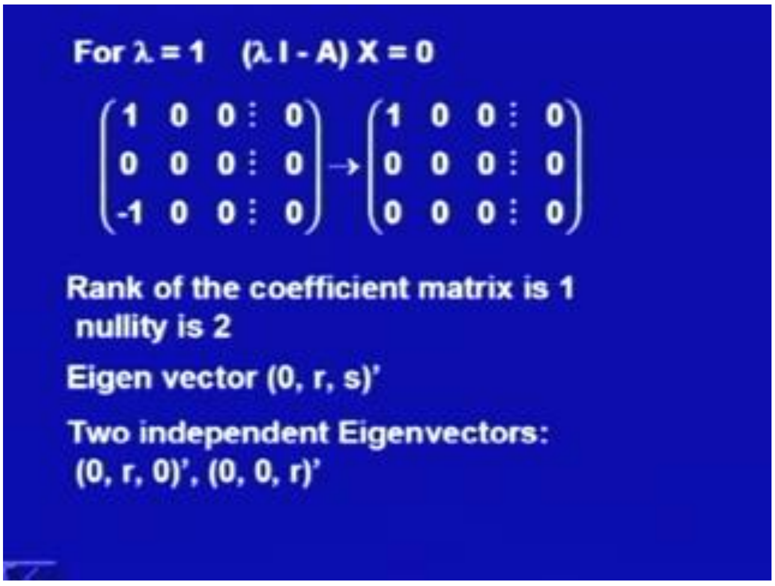

So, if we consider the repeated value lambda is equal to 1. Then, lambda I minus A X is equal to 0 is to be solved for lambda is equal to 1 and to solve this we consider augmented matrix. We apply elementary transmissions let us say this and this row has to be added. So, this is become 0 row, so this matrix has 2 0 rows.

So, the rank of the coefficient matrix comes out to be 1 and nullity is 2 and this means the solution of this system is 0 r and s see. Any value 0 r and s will satisfy this equation will satisfy this equation and we write down X is equal to 0, y is equal r and z is equal to s will also satisfy this equation. So, any solution will be of this form.

Any solution of this system will be of this form by this involves two arbitrary constant r are in s. That means, there are two independent vectors as a solution of this system they are 0 r 0 and 0 0 r. So, in this example we have I repeated Eigen value 1 with multiplicity 2. And when we solve for Eigen value Eigen vectors for corresponding to this Eigen value, we find that there are still two independent Eigen vectors associated with the given matrix.

So, I have compute different examples where the in one example there is one Eigen value which is repeated. But, we could get only one independent vector and in the another example there are two independent Eigen vectors associated with an Eigen value of multiplicity 2. So, if the roots are distinct we are sure that the vectors are going to be independent. But, if the vectors if the roots are not distinct. Then, we cannot say the vectors will be independent or not they may be independent in some examples and some other examples they may be dependent.

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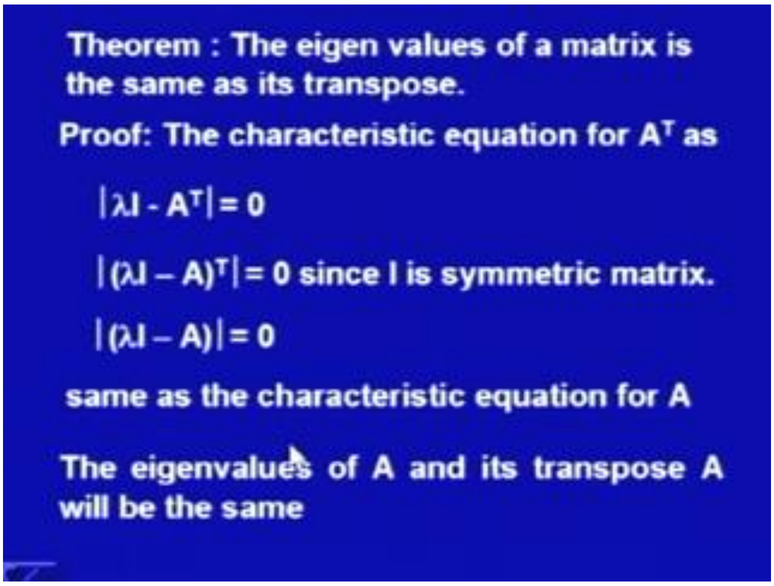

Now, another theorem is says that the Eigen values of a matrix is the same as its transpose. So; that means, if I have a matrix A having some Eigen values and if I take the Eigen values of the corresponding transpose matrix the Eigen values between matrices will be the same. So, let us consider the characteristic equation for A transpose. So, it is lambda I minus A transpose is equal to 0.

Then, lambda I minus lambda I minus A transpose can also be written as lambda I minus A transpose. Because, I is a symmetric matrix and determinant will not be affected. So, we can take determinant of lambda I minus A transpose is equal to 0 as simplification to this.

And this means lambda I minus A is equal to 0. Because, determinant of this and determinant of this they are same. And that means, you have same the characteristic equation for A as well as for A transpose. And that means, the solution will be the same and that implies that the Eigen values of A and it is transpose will be the same.

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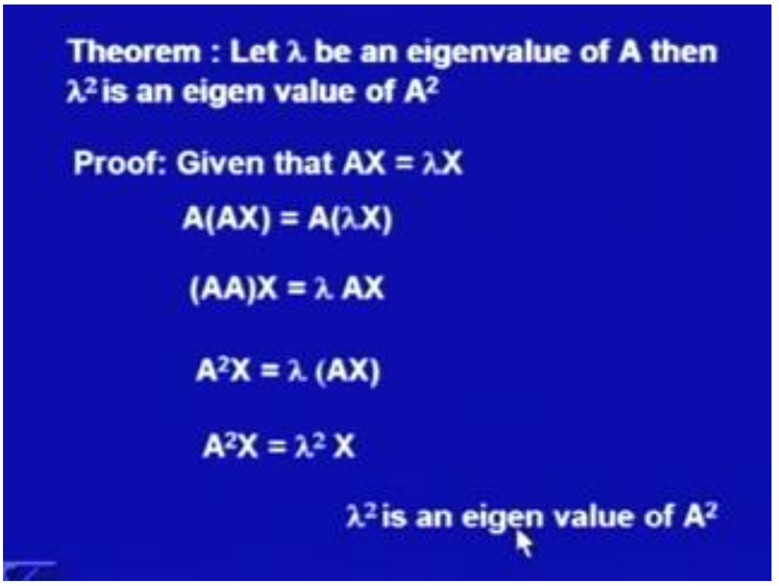

There is another property of Eigen values and Eigen vectors and according to this let lambda be an Eigen values of A. Then, lambda square is an Eigen values of A square. To prove this, let us say lambda is an Eigen values and X is this Eigen vector. So, A X is equal to lambda X is given to us. I pre multiply this equation by A, so it is A times A X is equal to A times lambda X. Since, lambda is a scalar and matrix multiplication is associated.

So, left hand side becomes A into A X is equal to right hand side becomes lambda is lambda can be taken outside from the right hand side and right hand side becomes lambda times A X. And that means, A square X is equal to lambda time A X and X into the lambda X is been given to us we can have A square X is equal to lambda X and from here one can say that lambda square is an Eigen values of A square. So, lambda square is an Eigen values of the matrix A square, this is most this is to be proved.

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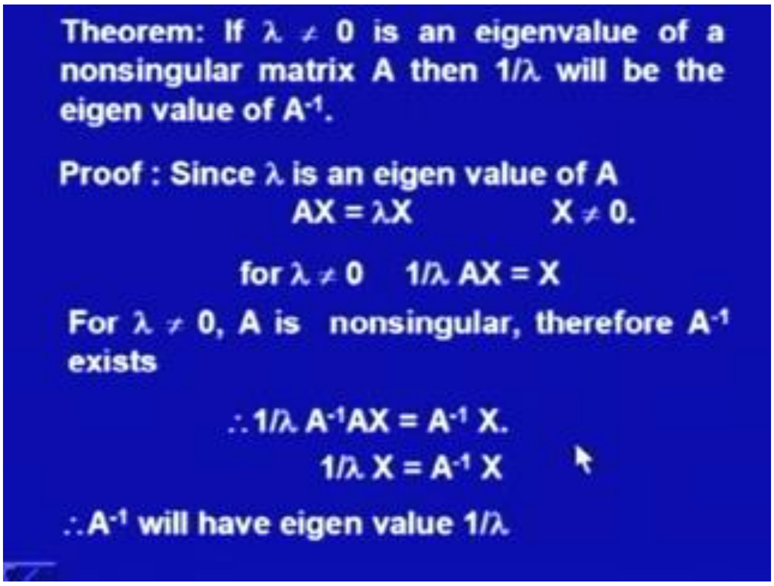

Then, the next theorem says that if lambda is not equal to 0 is an Eigen value of a nonsingular matrix A, then 1 upon lambda will be the Eigen value of A inverse. The same lines we can prove this result. So, we say that lambda is an Eigen value of A; that means, A X is equal to lambda X and X is not equal to 0. Because, this is the equation for determining lambda Eigen value lambda and Eigen vector X.

So, X is not to be 0. For lambda not 0, we can write down this equation as 1 upon lambda A X times X. For lambda not 0, A is nonsingular this is been given to us A is nonsingular. So, lambda is not zero. Therefore, A inverse exist for nonsingular matrix A inverse exist and we can multiply this equation by A inverse. So, will have 1 upon lambda multiplied by A inverse A X is equal to A inverse X. And A inverse A is identity, so 1 upon lambda into I X is X is equal to A inverse X and from here one can conclude that A inverse has an Eigen value 1 upon lambda.

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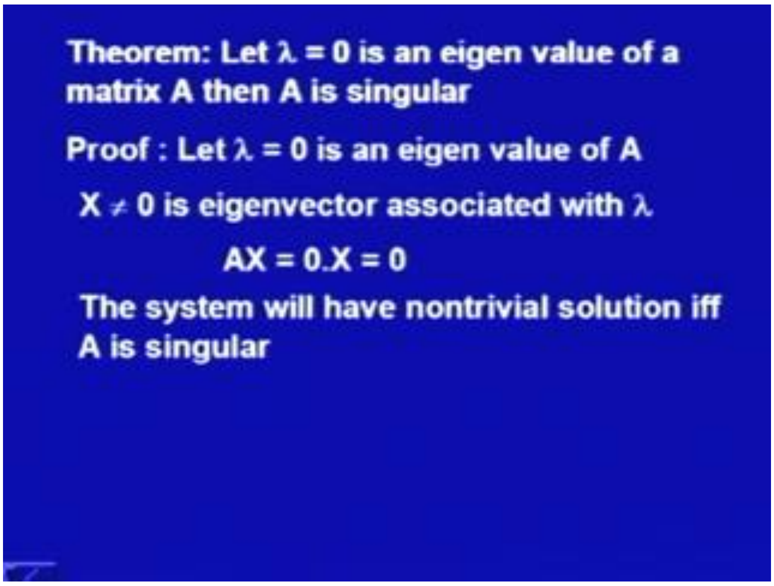

Now, we say that if lambda is equal to 0 is an Eigen value of a matrix A, then A is singular. To prove this, let lambda is equal to 0 is an Eigen value of A and X is not equal to 0 is an Eigen vector associated with lambda. That means, A X is equal to 0 into X is equal to 0. So, this system A X is equal to 0 will have a nontrivial solution if and only if A is singular. This is the result which we already developed and this proves that lambda is equal to 0 is an Eigen value of a matrix A, then A is singular.

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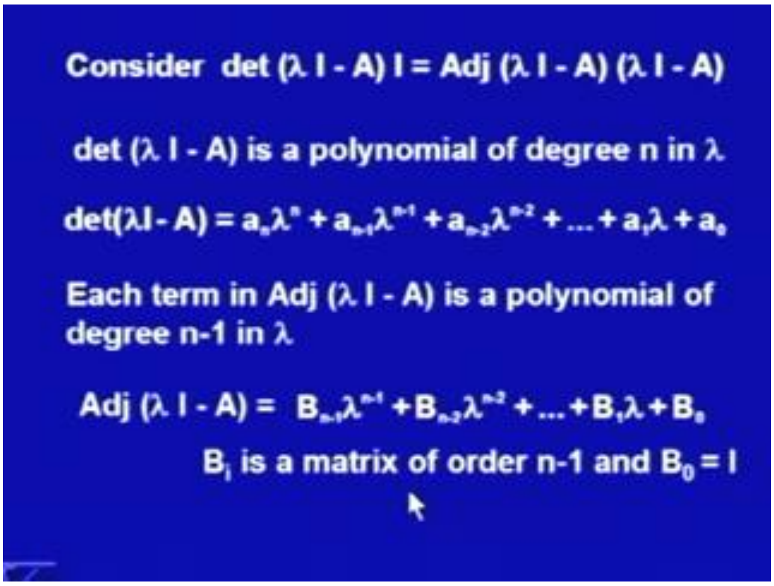

Now, let us consider this expression determinant lambda I minus A times identity matrix is equal to ad joint of lambda I minus A multiplied by lambda I minus A. This result we have already developed. Now, this lambda I minus A is determinant is a polynomial of degree n. Then, this multiplied by I is a matrix equation is a matrix expression.

And we can write down this determinant lambda I minus A as an lambda n plus an minus 1 lambda n minus 1 plus an minus 2 lambda n minus 2 plus a 1 lambda plus a naught you swap polynomial of degree n. Each term in ad joint lambda I minus A this 1 the right hand side is the polynomial of degree n minus 1 in lambda. Why, because in the ad joint matrix each term is a determinant and determinant of 1 or the lower. So, it is a polynomial of degree n minus 1 in lambda.

So, we can write down ad joint of lambda I minus A is equal to B n minus 1 lambda n minus 1 plus B n minus 2 lambda n minus 2 plus B 1 lambda plus B naught. What we have done is that? We have written B n minus 1 as a n minus 1 order square matrix. B n minus 2 is a n minus 1 order square matrix. So, each term is written separately, so we write down this ad joint lambda I minus A as this expression here B naught is identity.

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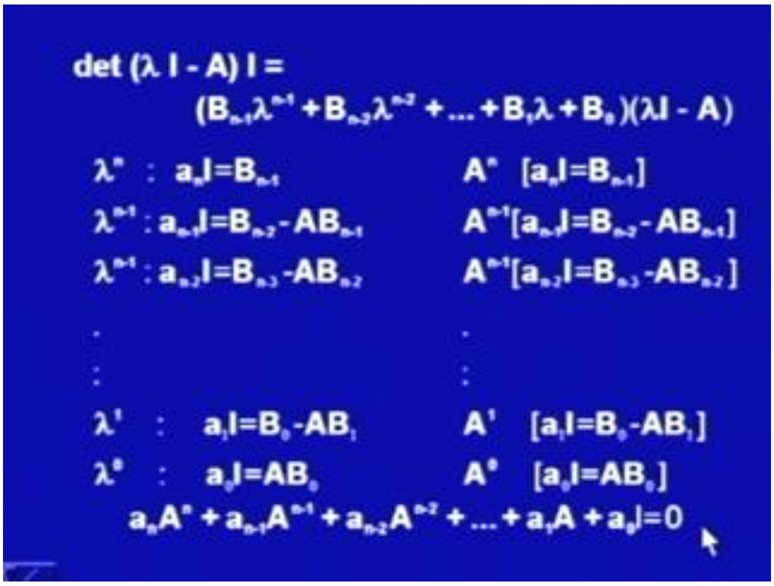

Then, determinant lambda I minus A I is equal to B naught lambda n minus 1 plus B n minus 2 lambda n minus 2, last term has to be B naught multiplied by lambda I minus A. What we can do is, on this side we have and polynomial expression lambda I minus A.

This is multiplied by matrix I and here also we have polynomial terms lambda n minus 1 lambda n minus 2 and they are multiplied by matrices.

So, let us equate terms related to lambda raise to power n and it is various powers. So, if I compare coefficients of lambda n, then on this side it is a n I and on the right hand side it is B n minus 1, B n minus 1 and this is multiplied by lambda I. So, this becomes lambda raise to power n. So, on the left hand side I have a naught I, there is the coefficient of lambda n, in this characteristic equation. And in the right hand side we have B n minus 1. Secondly, lambda n minus 1, so on the left hand side I have a n minus I is equal to from here, I will be having two terms one corresponding to this, another corresponding to this.

So, we will have this term, similarly other powers. So, we have this expression. Now, if I multiply the first equation by A n and this equation by A n minus 1 and then I add. So, what will happen, this is A naught n this is A raise to power n and here I have B n minus 1. B n minus 1 this is A n minus 1, this B multiplied by this, so this is A n minus 1 and here also have A, so this becomes A n.

So, if this way if I multiply each and every term and add, then this term will get cancel with this, this term will get cancel with the other terms and this term will get cancel with this. So, if I add these expressions, then we will have a naught A plus a n minus 1 A n minus 1 plus a n minus 2 a n minus 2 and so on, a 1 A plus a naught multiplied by I in the left hand side. But on the right hand side will not any term. In fact, all the term will get cancel, so this equation is obtained and this is equal to 0. This is the matrix equation equal to 0.

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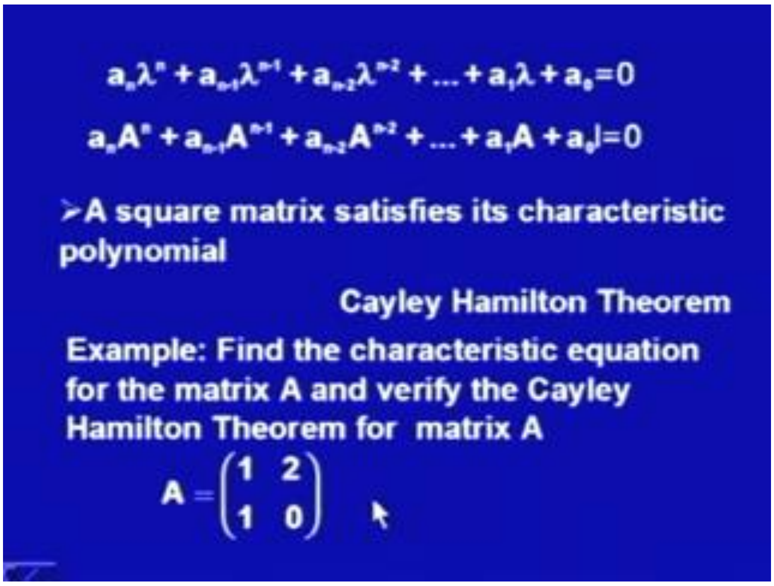

So, this is the characteristic equation which we have a n lambda n plus a n minus 1 lambda n minus 1 plus a 1 lambda plus a naught equal to 0. And what we have a obtain is, a n A n plus n minus 1 multiplied by the matrix A raise to power n minus 1 and so on. So, if you compare these two expressions what we can say that this lambda is replaced by A.

So, if this is the characteristic equation, then the matrix A will satisfy it is own characteristic polynomial. So, this is the result which we have that a square matrix satisfies it is characteristic polynomial. Now, this is an important result, we call it Cayley Hamilton theorem. Let us illustrate this result with the help of this example. So, find the characteristic equation for the given matrix A and verify the Cayley Hamilton theorem for the matrix.

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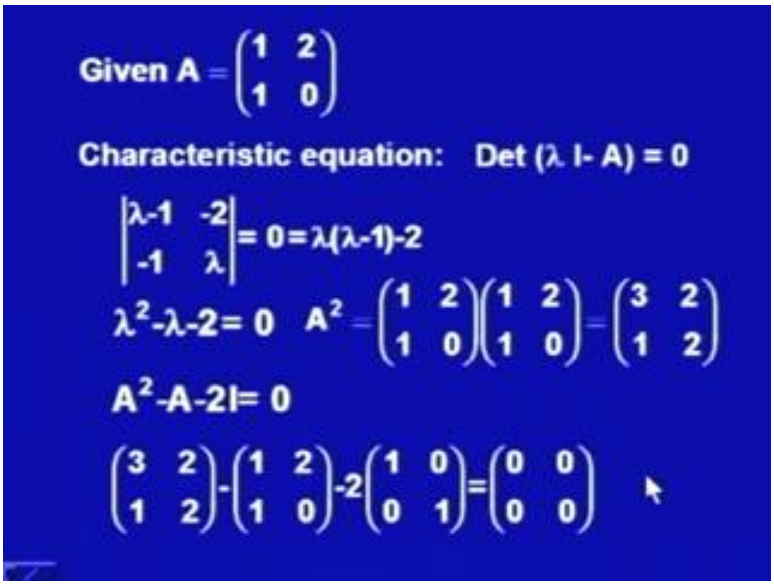

So, let us say A happens to be 1 2 1 0 simple 2 by 2 matrix for this we can find the characteristic equation determinant lambda I minus A equal to 0. As this determinant equal to 0, simplify this determinant it is lambda times lambda minus 1 minus 2 equal to 0. And lambda square minus lambda minus 2 equal to 0 is the characteristic equation for this given matrix.

So, according to Cayley Hamilton theorem, this matrix A satisfies it is characteristic polynomial. That means, A square minus A minus 2 is equal to 0. So, to prove this let us compute A square, A square is this matrix A multiplied by A. So, if you simplify it is 1 plus 2 that is 3 1 into 2 is 2 this is 0. And similarly, when we multiply this row by this we will have 1 this row multiplied by column is this 2.

Substituting the values of A square and A in the matrix equation A square minus A minus 2 I gives 3 2 1 2 minus the matrix A minus 2 times identity is equal to 0. And check, 3 minus 1 minus 2 is 0. And similarly other terms and will have A square minus A minus 2 I equal to 0. So, the we have proved in this example, that matrix A satisfies it is characteristic polynomial.

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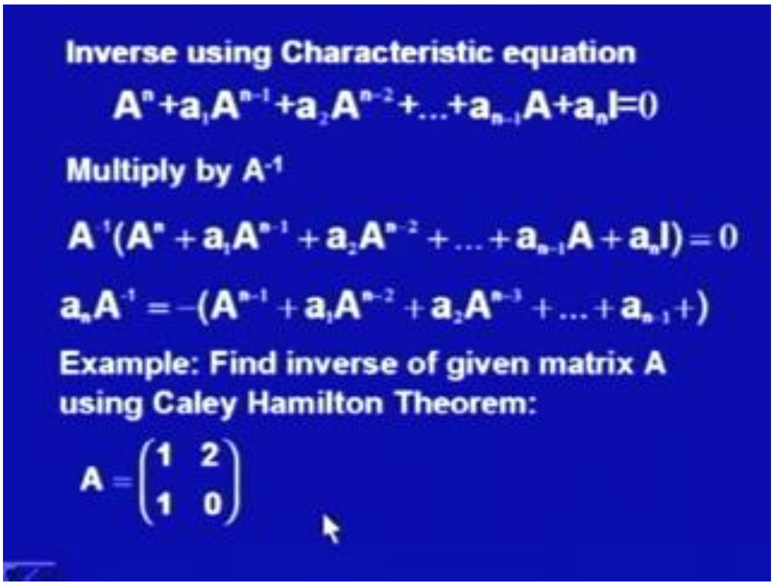

We can use this result for finding the inverse of a given matrix. So, if you start with the characteristic equation lambda n plus a 1 lambda n minus 1 plus a 2 lambda n minus 2 and so on. And we write down the corresponding matrix equation according to Cayley Hamilton theorem will have A n plus a 1 A n minus 1 plus a 2 n minus 2 and so on equal to 0.

Then, multiply this equation by A inverse if it exist. Then, it is A inverse is pre multiplied equal to 0. And then this a naught multiplied by A inverse I is this term on the left hand side rest of the terms are taken on the other side. So, we have an expression involving A inverse in terms of powers of A. This will can be used for finding A inverse. Let us illustrate this with an example. So, find inverse of given matrix A using Cayley Hamilton theorem A is 1 2 1 0 which is given to us it is a 2 by 2 matrix.

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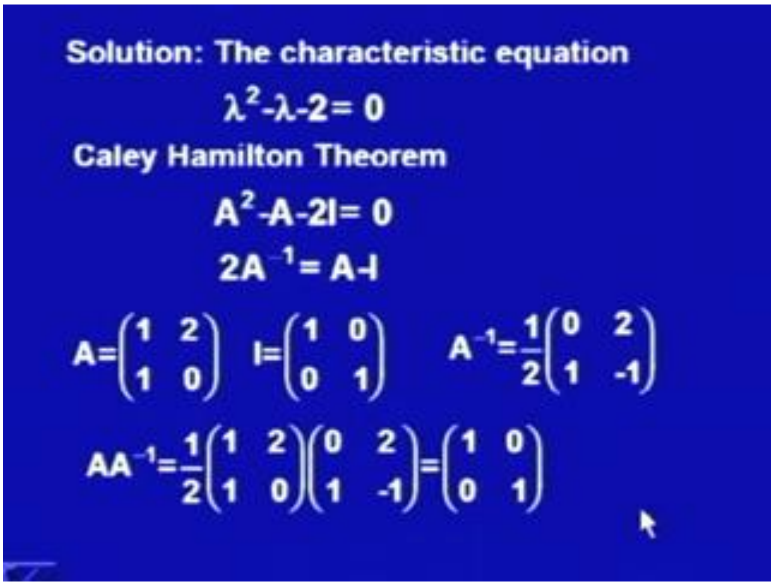

So, we first find it is characteristic equation, which we have obtained as in the case of earlier example, lambda square minus lambda minus 2 equal to 0. Then, the corresponding matrix equation according to Cayley Hamilton theorem is A square minus A minus 2 I equal to 0. We substitute the value of as a pre multiply this equation by A inverse. So, will have 2 A inverse is equal to this two terms can be taken on the other side is A minus I.

So, we can substitute the value of A and I in this equation. And this gives me A inverse as half, this minus this 0 2 minus 0 is 2, 1 minus 0 is 1 and 0 minus 1 is minus 1. So, this is A inverse. Let us, verify that we have got the right result. So, we say A into A inverse is equal to half into this 1 2 1 0 multiplied by the inverse 0 2 1 minus 1 is equal to 1 0 0 1. So, A A inverse comes out to be identity I have tried this example with pre multiplying this equation by A inverse. But, the same result can be obtained. If you multiply post if you perform post multiplication, then again we will get the same result.

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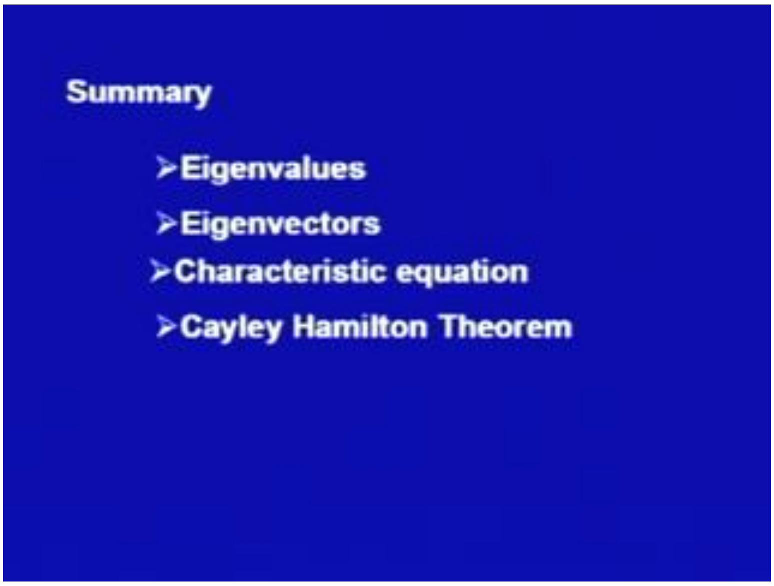

To summarize, we have discuss Eigen values, Eigen vectors, characteristic equations. We have discussed the method for finding Eigen values and Eigen vectors of a given matrix. This method depends upon in the evaluation of determinants finding determinants is not a simple problem. In fact, we have to develop more methods of finding Eigen values and Eigen vectors for a given problem. We have already discussed Cayley Hamilton theorem. We have discussed how to use Cayley Hamilton theorem for finding inverse of a given matrix, with this we come to the end of the lecture.

Thank you.