Mathematics-II

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Lecture - 12
Linear Transformation Part – 2

Welcome viewers, today we are going to discuss Linear Transformation. I will start with the summary what we have done in my last lecture. In my last lecture I have started with the definition of linear transformation, I have discussed some properties of linear transformation. Such as, that if T is linear transformation and theta 1 and theta 2 being the identity elements of the vector space V and W, then T of theta 1 is equal to theta 2. That is theta 1 will map to theta 2.

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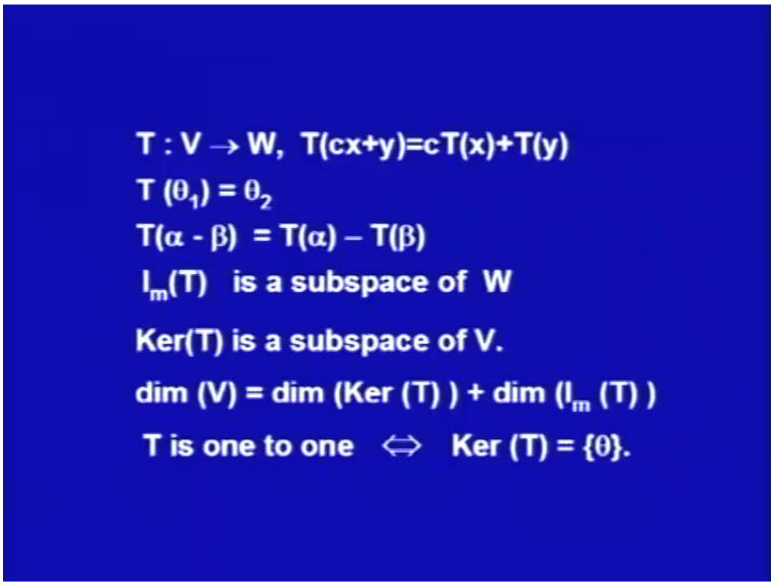

Then, T of alpha minus beta is equal to T of alpha minus T of beta, image of T is a subspace of W, image T being the range set for the transformation T. And kernel T subspace of V. We are further discuss that dimension of V is equal to dimension of kernel T plus dimension of image T. We have also discuss the T is one to one, then kernel T is theta itself.

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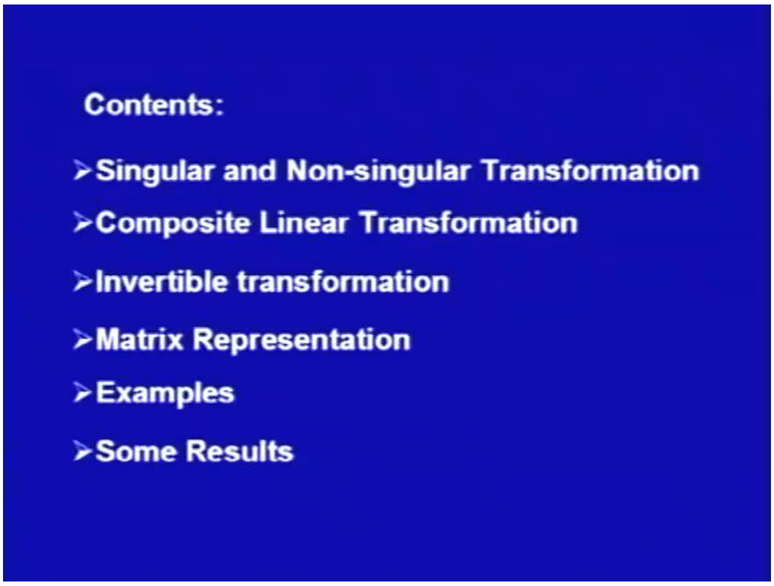

This lecture includes, singular and non-linear transformations, composite linear transformation, invertible transformation, matrix representation of transformation, some examples and some results.

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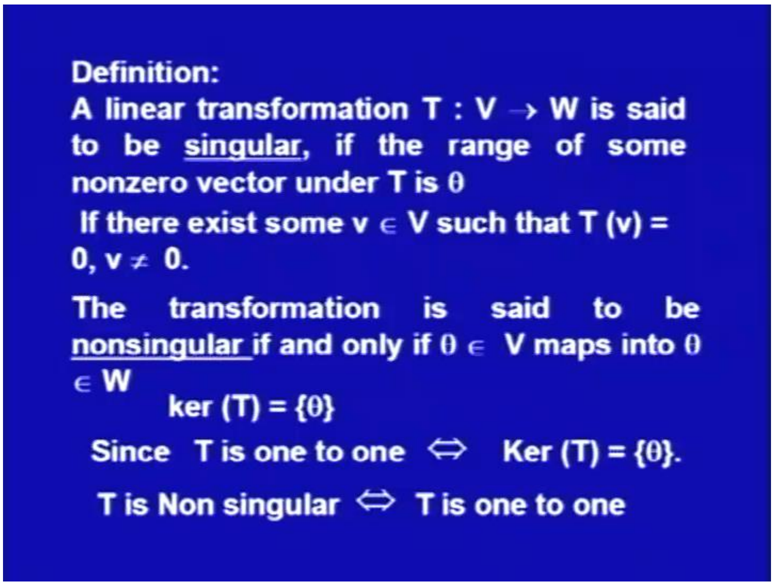

To start with I will define what do you mean by a singular, in linear transformation for this. Let us say T is a linear transformation from the vector space V into W. Then, this is singular if the range of some nonzero vector under this transformation is theta. That means, there exist some vector V belonging to V. Such that T v is equal to theta, then v is not 0.

And transformation is set to be non singular, if and only if there exist theta belonging to V which maps into theta belonging to W. That means kernel T is equal to theta. Now, we have earlier proved that T is one to one implies kernel T is equal to theta. And that means, the T is nonsingular when T is one to one. So, many times we take this as a definition for nonsingular transformation. That if T is nonsingular, then T is one to one linear transformation.

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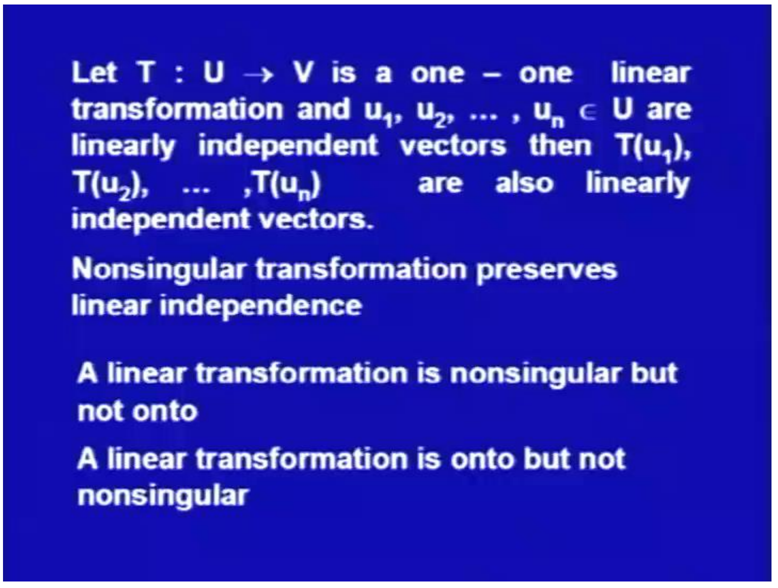

Now, the T is a linear transformation from U to V it is a one-one linear transformation. And the vectors u 1, u 2, u n belonging to U are linearly independent vectors. Then, that images T u 1, T u 2, T u n are also linearly independent vectors, this is a result which we have proved in my earlier lectures. Now, we will make use of this and then we can say that nonsingular transformation preserves linear independence. A linear transformation is nonsingular, but not onto one may notice that a linear transformation is onto, but not nonsingular. So, these two things are different.

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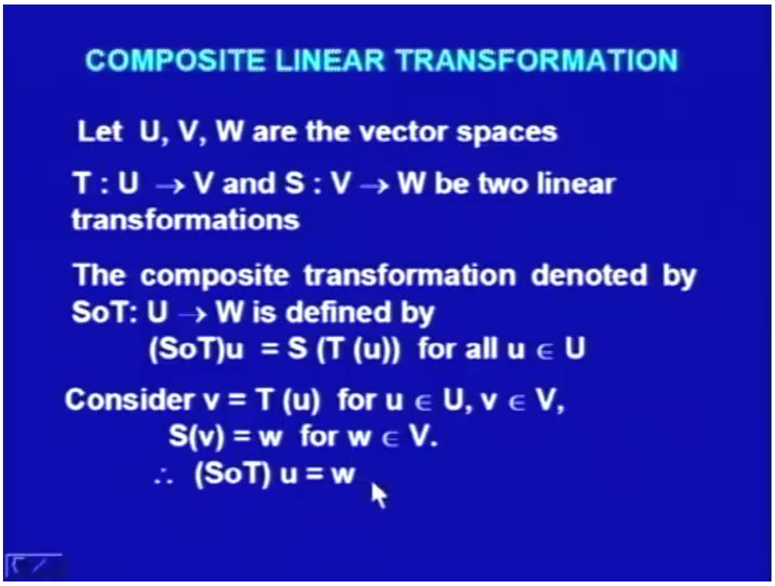

Then, we come to composite linear transformation for this we need three vector spaces U V and W. So, let U, V, W are the vector spaces and we have two transformations T from U to V and S from V into W. Then, the composite transformation which we denoted by S composite T, which is from U to W is defined as S o T or we call it S composite T u is equal to S of T u for all u belonging to U. That means, if I have a vector u and U, then T u will be in V. So, if V belonging to v such that it is equal to T u, then S of v is equal to w for some w belonging to V and then we say, S composite T of u is equal to w.

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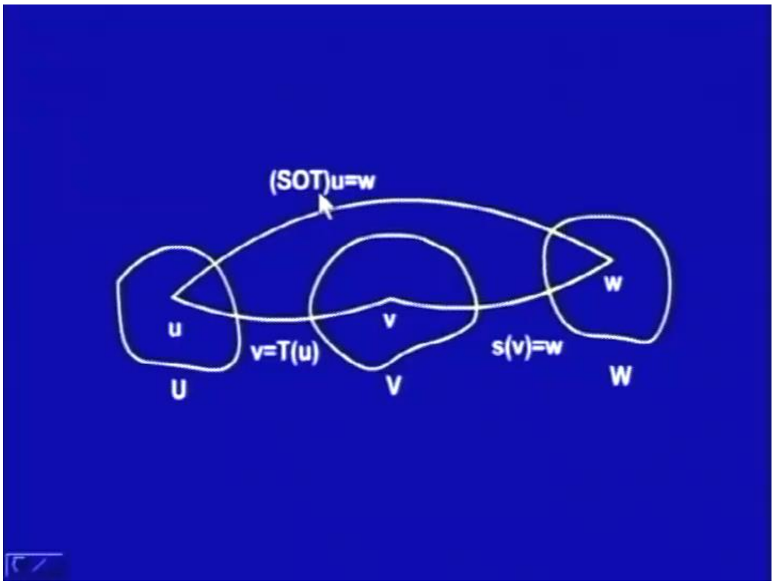

I illustrate this with the help of this diagram, in which I am showing three vector spaces U, V and W. And two linear transformations T from u to v and s from v to w, which means a member u in U will go to v in V under the transformation T. And the same v under is transformation s will go to w. And that means, this is a transformation which will take u to w and this transformation is called the composite transformation S T.

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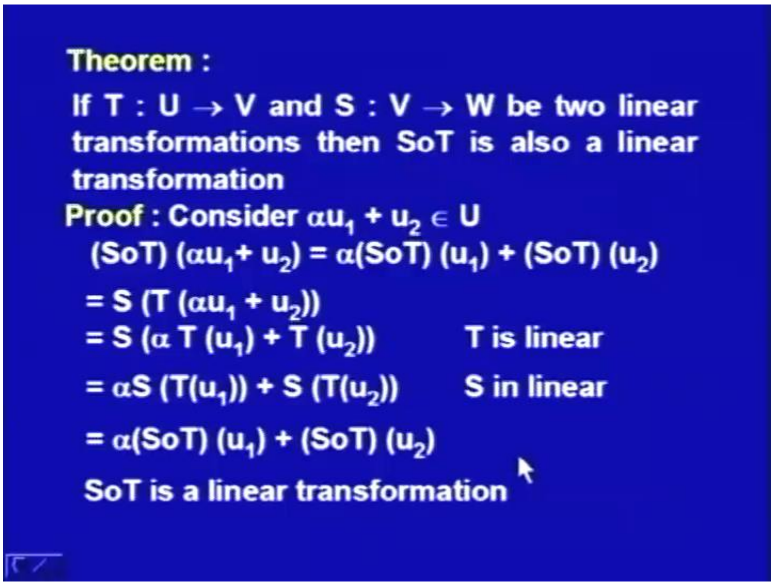

On the basis of this, I will establish certain results in the form of theorem. So, if we have a linear transformation T from the vector space U into V. And another transformation S from V into W, then S o T is also linear transformation. That means, if T s and T are linear transformation, then composite of these two will also be a linear transformation.

To prove this, we will consider alpha u 1 plus u 2 belonging to U. And we will prove that S o T of alpha u 1 plus u 2 is equal to alpha times S o T of u 1 plus S o T of u 2. We start with the left hand side S composite T means, S of T alpha u 1 plus u 2. So, if first operate this transformation T on this element, and since T happens to be a linear transformation. So, T of alpha u 1 plus u 2 in the next step will become alpha T u 1 plus T of u 2 these are basic definition of the linear transformation. And then this will become alpha S of T u 1 plus S of T u 2 because, S is also a linear transformation.

So, applying this linear transformation twice, once for T next time for S we will get left hand side equal to right hand side. And that proves the result that S o T is a linear transformation**.** So, given two linear transformation their composite is also a linear transformation.

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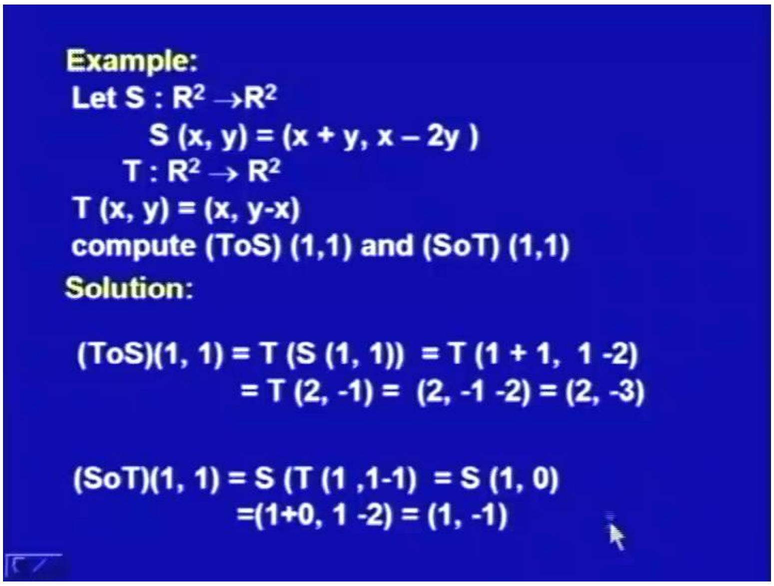

Now, let us see what do we mean by a composite transformation with the help of an example. So, we have a linear transformation S from R 2 to R 2 here, S x y is defined as x plus y and x minus 2 y. So, an element x y in R 2 will map to x plus y comma x minus 2 y under the transformation S. And another linear transformation T defined by a T of x y equal to x comma y minus x.

So, the question is to compute T composite S for the element1 comma 1 and S composite T for the element1 comma 1. So, let us see what happens I start with T o S of 1 comma 1, this means we first operate S. So, S linear transformation on the element 1 comma 1. So, S operated on to the vector 1 comma 1 gives me 1 plus 1, see the definition x plus y here x and y are 1, so it is 1 plus 1 comma x minus 2 y. So, it is 1 minus 2.

And that makes T of 2 minus 1, so this left hand side is equal to T of 2 minus 1. Then, I apply the definition for the second transformation T that is this definition. And this means x is equal to 2, so this is x and the second component will become y minus x y is equal to minus 1 and x is 2. So, it becomes minus 1 minus 2 and finally, it is 2 minus 3.

So, T o S of 1 comma 1 is 2 comma minus 3 by S composite T on 1 comma 1 is equal to S T on 1 comma 1 minus 1. I am first using this definition x comma y minus x. So, it is 1 comma 1 minus 1, so this reduces to S 1 comma 0. And now I apply this definition, so it is equal to 1 plus 0 x plus y comma 1 minus 2 this is equal to 1 comma minus 1. So, that is how we compute composite transformation. So, given two transformations one can use, one can obtain composite transformation.

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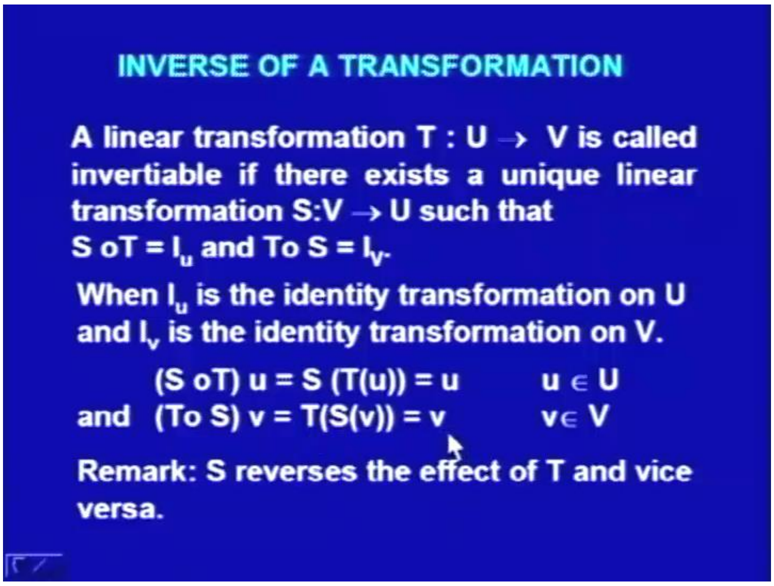

Now, we come to inverse of a transformation, a linear transformation T from the vector space U into V is called invertible. If there exist a uniquely linear transformation S form V into U. Such that, S composite T is identity of the vector space u and T composite S is the identity of the vector space v. Now, both these things should be satisfied.

If you can find such a transformation that these two composite transformation satisfy this. Then, we can say that the transformation is invertible. Here, I v is a identity transformation on U and this I v is the identity transformation on V. This was the first can be written as S composite T on u is equal to S T u is equal to u. So, u belonging to U and T composite S operated on v is equal to T S v is equal to v, so v belonging to V.

So, this is equivalent to this and the second one is equivalent to this. Now, one can see that S reverse is the effect of T and vice the versa. One can notice that u under this transformation goes T u, but S will take it back u itself. So, S reverse is the effect of T here also v becomes S v under the transformation S and T will take it back again to v. So, we can say that S reverse is the effect of T and vice versa.

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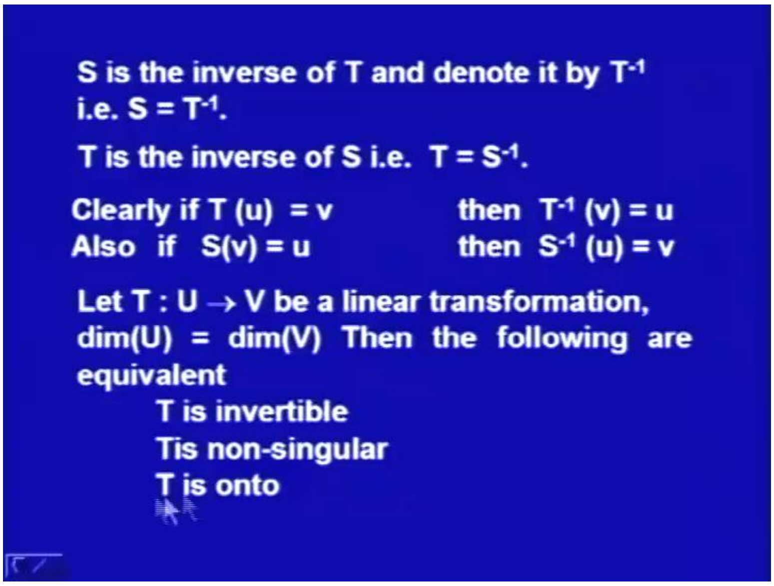

Further, if S is the inverse of T we denoted by T inverse and we say S is equal to T inverse. We say T is the inverse of S or we write T is equal to S inverse. Clearly, if T u is equal to v, then T inverse v is equal to u. As we said that, T inverse will all nullify the effect of this transformation, it will take it back. So, we apply T on u will become v, but T inverse will take it back v to u. Also if S v is equal to u then S inverse u is equal to v. Now, we have an important result it says that T is a linear transformation from the vector space U into V. And dimension U is equal to dimension V, then the following are equivalent that is T is invertible, T is nonsingular and T is onto.

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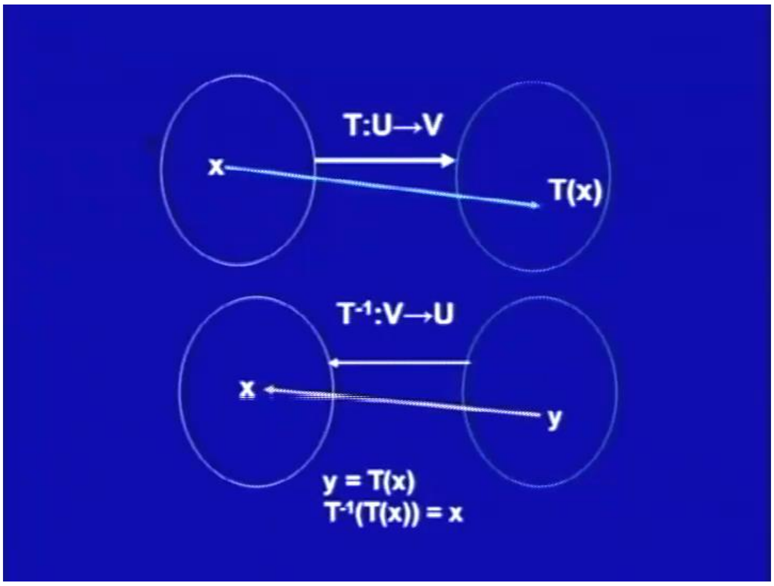

This I illustrate with the help of this diagram. Here, I have a two vector spaces U and V, T is a transformation which takes x 2 T x. Now, T inverse is a transformation, which will take y 2 x. Now, since T is a transformation we said T has to be nonsingular. That means, only one element if there are two different elements here, then they will map to one element here. And then this mapping has to be onto, because we have to the same element by a back x. So, that is how we have the result we say that invertible, nonsingular and onto they are equivalent.

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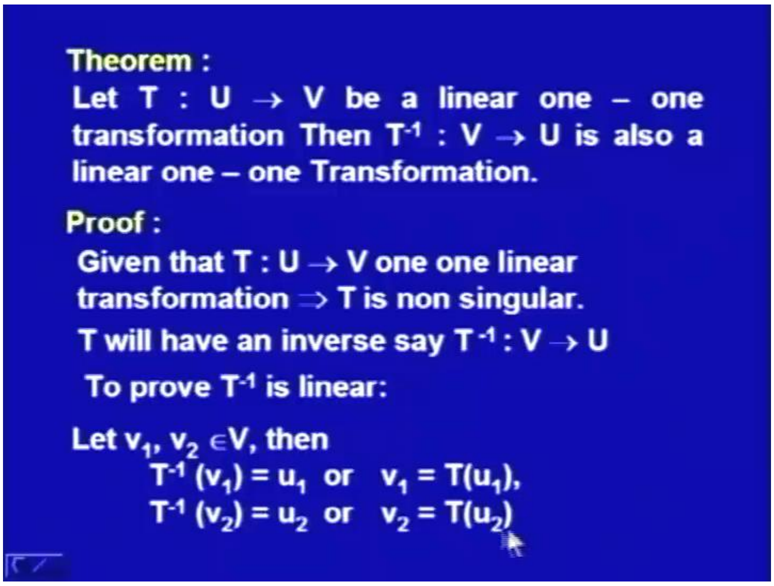

Now, we will prove this theorem, that if T is a linear transformation which is one-one. Then, T inverse from V into U is also a linear one-one transformation. To prove this, we first take a linear transformation T from U to V, which is given to be one-one linear transformation, we have earlier proved that such a linear one-one transformation will be nonsingular.

So, with this we further assume the existence of T inverse from V to U, then if this happens to be a linear transformation. Then, we should prove that it is this T inverse is additive and homogeneous. So, first we consider two vectors v 1 and v 2 in V, then T inverse v 1 is equal to u 1 and T inverse v 2 is equal to u 2, this is our assumption. Then, first means v 1 is equal to T of u 1 and second means v 2 is equal to T of u 2, this is because of the definition of T inverse.

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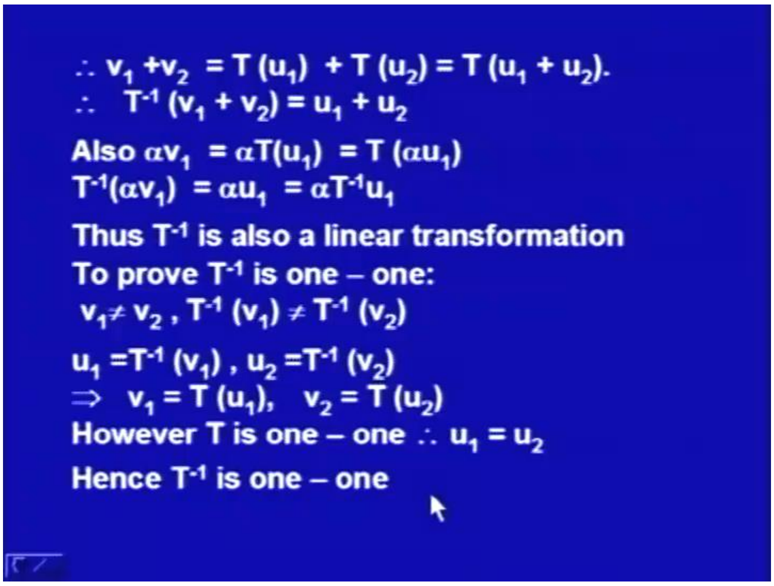

Then v 1 plus v 2 is equal to T of u 1 plus u 2. If you are convinced, then we can apply the linear property of linear transformation and what we can have that T is equal to u 1 plus u 2. And this simply means v 1 plus v 2 is equal to T of u 1 plus u 2 and by the definition of inverse T inverse of v 1 plus v 2 is equal to u 1 plus u 2. And that is what we need to prove that this is additive.

The second is alpha v 1 is equal to alpha T u 1, which is equal to T times alpha u 1 T being a linear transformation. So, from alpha v 1 is equal to T of alpha u 1 we can write down T inverse alpha v 1 is equal to alpha u 1 or it is equal to alpha times T inverse u 1. So, homogenous properties also satisfy. And hence, one can say that T inverse is also a linear transformation.

So, if T is linear then T inverse is also linear provided it is one. To prove that T inverse is one-one, we consider two different elements in v, say v 1 and v 2 which are not same. Then, we have to prove that T inverse v 1 their images are also be different in u. So, we start with v 1 not equal to v 2, then we will prove that T inverse v 1 is not equal to T inverse v 2.

Now, u 1 is equal to T inverse v 1 and u is equal to T inverse v 2 with this v 1 is equal to T u 1 and v 2 is equal to T u 2. Since, T is one-one, so v 1, v 2 will be different, but you want u 2 different. So, T is one-one, so u 1 is equal to u 2 and hence T inverse is one-

one. So, if we have one-one linear transformation T, then T inverse is also a one-one linear transformation.

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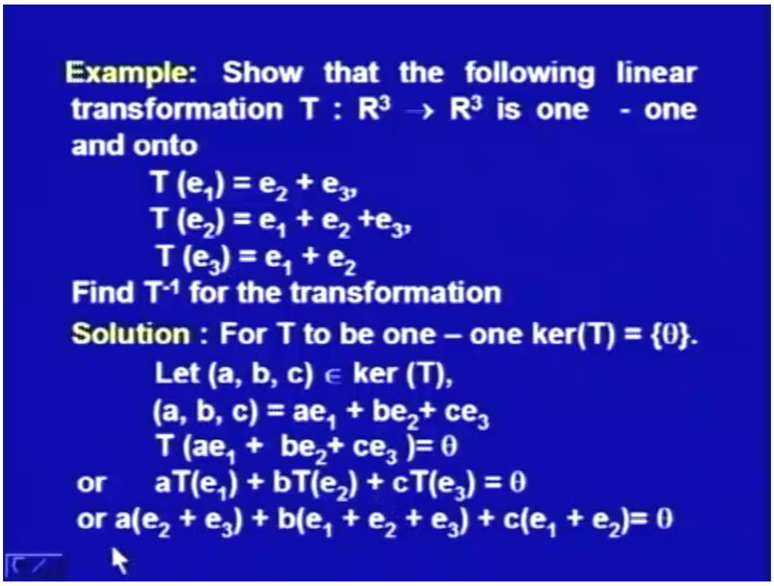

Now, in this example we will show that, the given linear transformation is one-one and onto. And if it is one-one and onto, then we will find it is inverse also. To start with T to be one-one it is penalty should be theta. That is how we proved? The T is one-one transformation. So, let us consider an element a, b, c belonging to kernel T, we will prove that a, b, c actually has to be 0 if it has to belong to kernel T and only then T inverse may exist.

So, let us have a, b, c belonging to kernel T. And since a, b, c belongs to kernel T, so it can be written as a e 1 plus b e 2 plus c e 3. That is any vector in R 3 can be represented as a linear combination of the base vectors e 1, e 2, e 3. Then, we apply linear transformation on this. So, T of a, b, c is equal to 1 T of a e 1 plus b e 2 plus c e 3 and this is equal to theta.

That is what we have assumed? That is what we have started with, so we have T of this is equal to theta. So, let us apply on this vector; that means, we can write down a times T e 1 plus b times T e 2 plus c times c 3 is equal to theta, because T is a linear transformation. Next, we can apply this definition, which is given to us T of e 1 according to definition T of e 1 is e 2 plus e 3. So, it is a times e 2 plus e 3 plus b times T of e 2 which is e 1 plus e 2 plus e 3 that is what has been given to us plus c times e 1 plus e 2 which is T 3.

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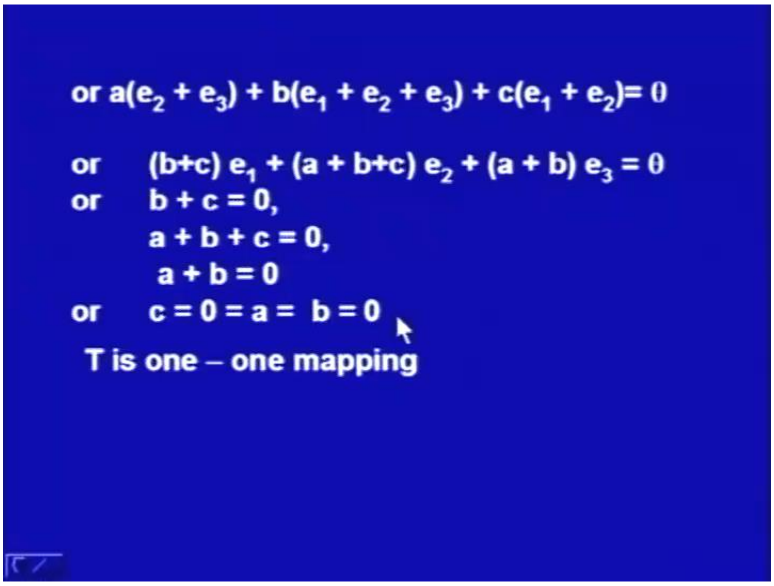

So, you simplify this and collecting different terms we will have b plus c e 1 plus a plus b plus c into e 2 plus a plus b into e 3 is equal to theta. So, we have collected different terms e 1 is coming from b as well as from c, e 2 is from a, b and c similarly e 3 is a plus b. So, we will have this expression and this is a vector expression. So, each of this component should be 0, so b plus c is equal to 0 a plus b plus c is to be 0 and a plus b equal to 0.

Now, we have three equations and three unknowns a, b and c from this a is equal to minus b and from this b is equal to minus c. If you substitute this in this equation will have a, b and c equal to 0. That means, we have started with a non zero vector a, b, c, but ultimately that vector comes out to be 0. That means, we will have only one vector in kernel T that is the identity vector and; that means, T is a one-one mapping.

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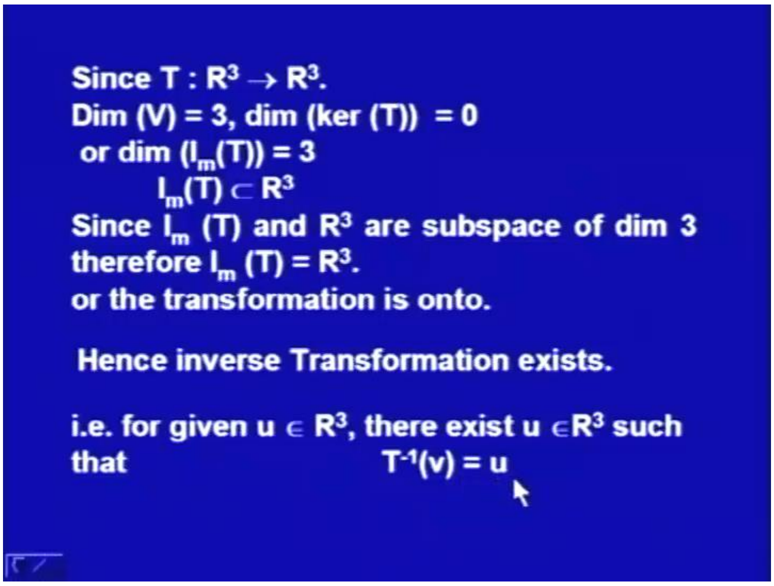

Now, the mapping T is this linear transformation T is R 3 to R 3. The dimension of V is 3 dimension of kernel T is 0 why because, it is one-one transformation. So, nullity is 0, so from nullity and right nullity theorem will have dimension of image T is equal to 3. Now, image T is a subset of R 3 and that means, image T is nothing but R 3. Therefore image T is the same as R 3. And that means, the transformation which is given to us is not only a one-one it is onto. And accordingly, the inverse transformation exist that is forgiven u belonging to R 3, there exist V belonging to R 3 such that T inverse v is equal to u.

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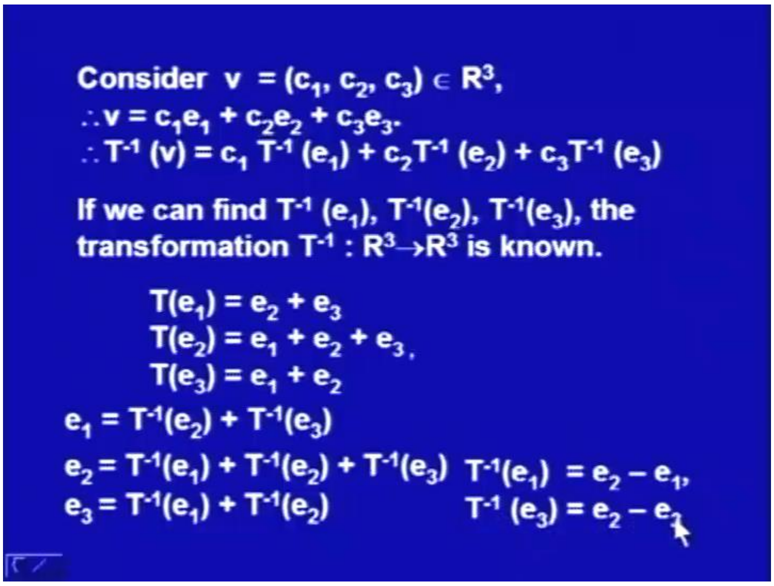

Now, for this I consider the vector v as c 1, c 2, c 3 belonging to R 3, which is a linear combination of the base vectors e 1, e 2, e 3. So, we write down v as c 1 e 1 plus c 2 e 2 plus c 3 e 3. So, applying T inverse we will have T inverse v is equal to c 1 or T inverse e 1 plus c 2 of T inverse e 2 plus c 3 of T inverse e 3, this we are getting, because T inverse is also a linear transformation, so applying T inverse on this means applying T inverse on these separately, so we have this expression. Now, we can find T inverse e 1 T inverse e 2 T inverse e 3, then the transformation will be known. So, if v is given to us, then from this we can find out the transformation if T inverse e 1 and T inverse e 2 and T inverse e 3 are known.

Now, what is given to us is T e 1 is equal to e 2 plus e 3 T of e 2 is equal to e 1 plus e 2 plus e 3 T of e 3 is equal to e 1 plus e 2. We are trying to find out T inverse e 1, T inverse e 2 and T inverse e 3 and if you can find that we can find out the inverse transformation. So, that is what we are going to do here, so we write down from this we can write down e 1 is equal to T inverse e 2 plus T inverse e 3.

So, I have applied T inverse here and using the property that T inverse is a linear transformation I can write down e 1 is equal to T inverse e 2 plus T inverse e 3 from this second expression I am writing e 2 is equal to T inverse e 1 plus T inverse e 2 plus T inverse e 3 and from the third e is equal to T inverse e 1 plus T inverse e 2. So, now we can solve these two equations from this we can write down T inverse e 1 is equal to e 2 minus e 1 and from the third T inverse e 3 is equal to e 2 minus e 3, here I am using the linear property of T inverse.

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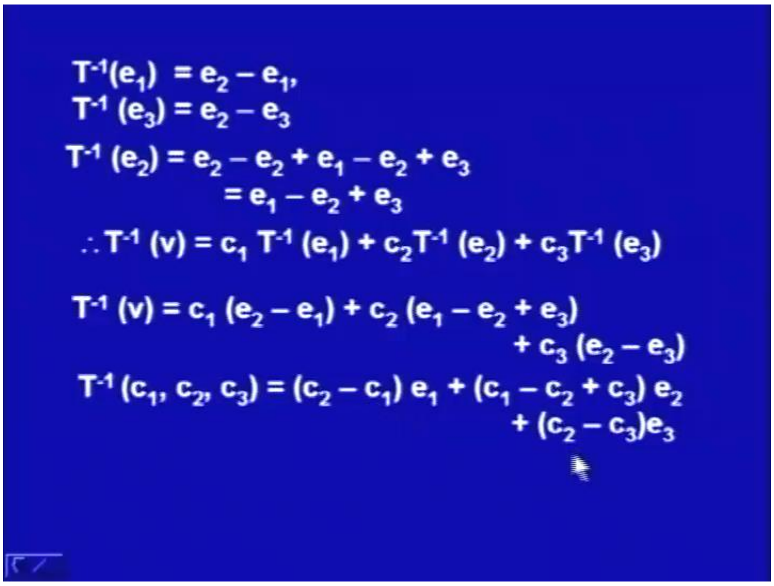

So, once we get T inverse e 1 is equal to e 2 minus e 1 and T inverse e 3 and e 2 minus e 3, then we can substitute in the second equation to get T inverse e 2. So, from this T inverse e 2 is e 2 minus e 2 plus e 1 minus e 2 plus e 3 and that simplifies to e 1 minus e 2 plus e 3. So, we get T inverse e 1 we know what is T inverse e 2 and we know what is T inverse e 3 and that means, we can find out what is the image for the vector v it is c 1, c 2, c 3.

So, we can write down T inverse v as linear combination c 1 T inverse e 1 plus c 2 T inverse e 2 plus c 3 T inverse e 3 and that means, T inverse v is determined. And substituting these values T inverse v is equal to c 1 e 2 minus e 1 plus c 2 e 1 minus e 2 plus e 3 plus c 3 e 2 minus e 3. Simplifying it, T inverse of c 1, c 2, c 3 it is a vector v is equal to c 2 minus c 1 e 1 plus c 1 minus c 2 plus c 3 e 2 plus c 2 minus c 3 e 3. So, given a vector c 1, c 2, c 3 we can find out its inverse transformation T as this.

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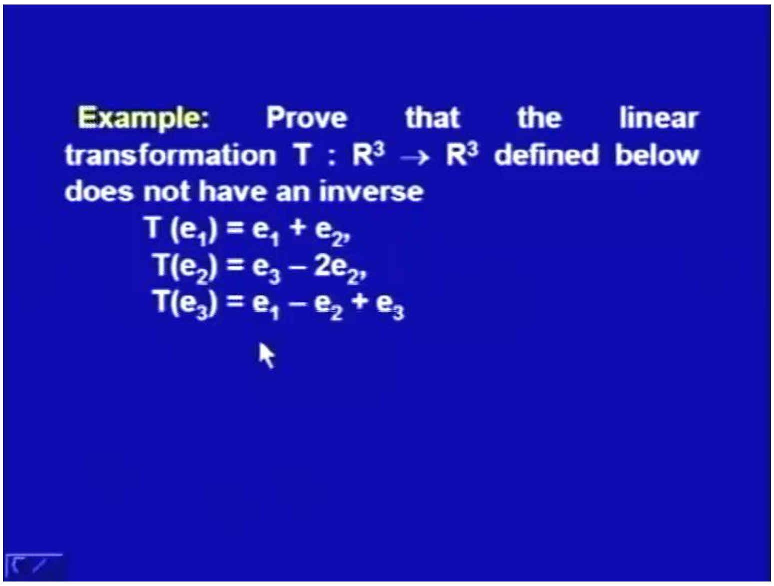

So, this transformation which was given to us is invertible and we can find the transformation also. Now, in the next example we will take a linear transformation defined as T of e 1 is equal to e 1 plus e 2 T of e 2 as e 3 minus 2 e 2 T of e 3 as e 1 minus e 2 plus e 3. And then we will see that, this is not this does not have an inverse.

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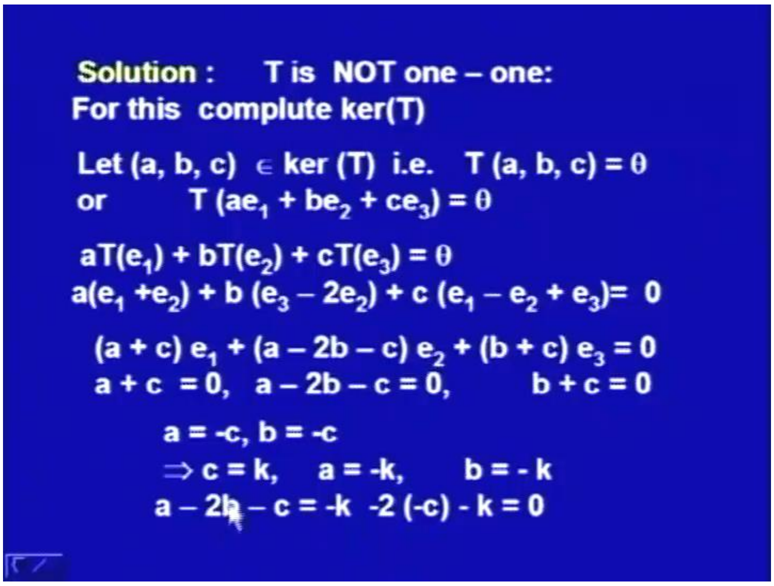

To prove this, we will see that it is not a one-one transformation and for this purpose. We will compute first the kernel of T, we will see that the kernel of T has some member a, b, c which is not 0, but still T, b, c is theta. So, let us assume that a, b, c is a non zero vector belonging to kernel T and T a, b, c is theta or T we write down the vector a, b, c is a linear combination of base vectors e 1, e 2, e 3. So, T of a e 1 plus b e 2 plus c of e 3 is equal to theta.

And we know that T is a linear transformation. So, we can write it as a times T of e 1 plus b times T of e 2 plus c time T of e 3 equal to theta. And then we will make use of definition T e 1, T e 2, T e 3 has been given to us, we write down a times e 1 plus e 2 that is the value for T e 1 plus b times e 3 minus to e 2 expression for T e 2 plus c times T e 3. That is e 1 minus e 2 plus e 3 and this is equal to 0.

Then, we collect different terms e 1 first, so it is from here it is a no contribution from this and here we have c. So, we will have a plus c times e 1 plus the next is e 2, so a is coming from the first term a minus 2 b minus c times e 2 plus b plus c no contribution from this. So, we will have this expression equal to 0. And that means, a plus c is equal to 0 plus a minus 2 b minus c equal to 0 and b plus c equal to 0. So, the three components are going to be 0, so the we have three questions in three unknowns a b c.

So, from first equation we can say a is equal to minus c from the third equation we say b is equal to minus c. And when we substitute these values in the second equation, we can say c is equal to k and a is equal to minus k and b is equal to minus k. So, this value is coming from this equation. That is what we are getting from the second equation a minus 2 b minus c is equal to minus k minus 2 times minus c minus k is equal to 0 and that gives me the value of c.

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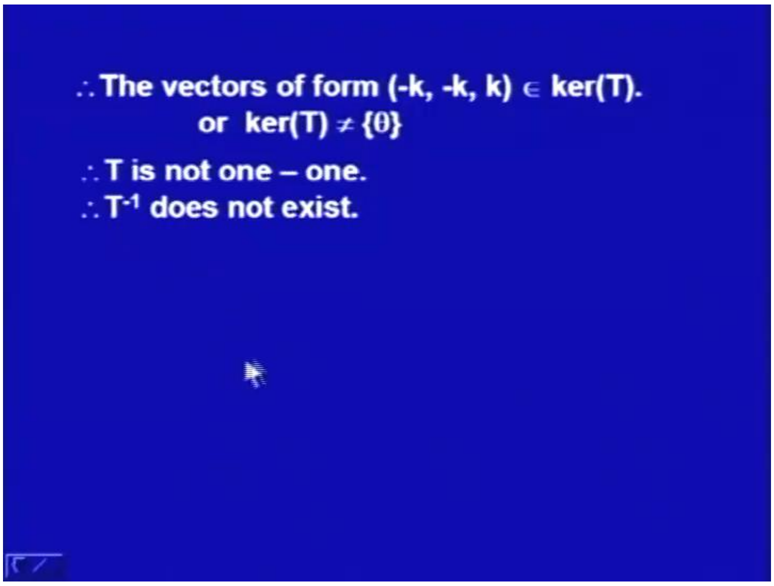

And accordingly the vector form minus k minus k, k belongs to kernel T k can be any scalar and it may be 0 also, but it may be non zero. That means, there exist a non zero vector in kernel T. So, kernel T is not nearly 0 it has some more vectors and that means, it is not a one-one transformation. And of course, according T inverse does not exist in such a case.

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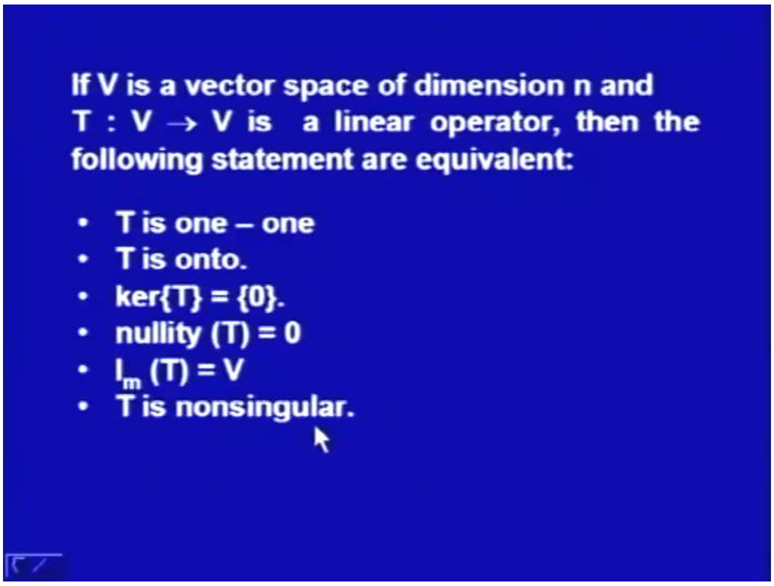

So, we have proved that T inverse does not exist in such a case. And then in this result we are summarizing what we have done, so far. We have a vector space V of dimension n and we have an operator T from V to V, we call it a linear operator. Then, the following statements are equivalent T is one-one, T is onto kernel T is equal to theta and only in identity vector, nullity T is equal to 0 in this case, image T is equal to V and T is nonsingular. So, this is what we have done for the linear transformation.

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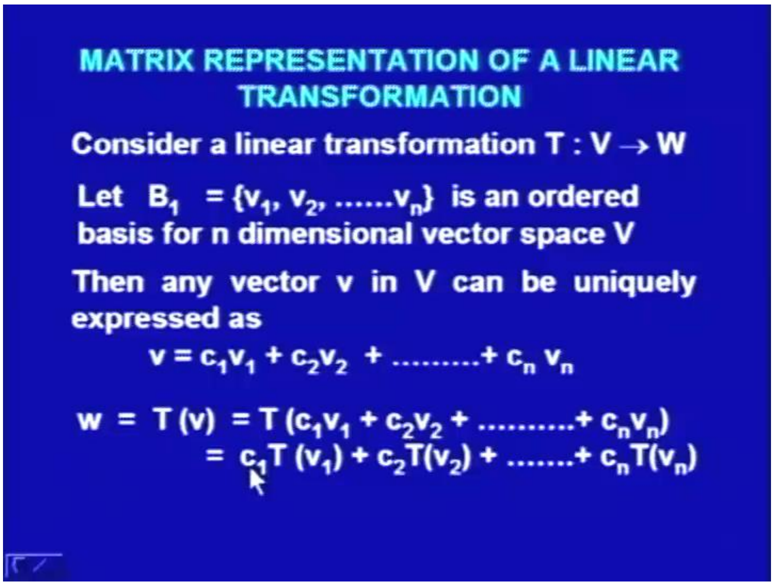

In the next part of my lecture, I will be discussing matrix representation of a linear transformation. Now, let us consider T is a linear transformation from the vector space V into W, let B 1 is a order basis for V vector space V consisting of v 1, v 2, v n is an n dimensional vector space. Then, any vector v in V can uniquely be expressed as v is equal to c 1 v 1 plus c 2 v 2 plus c n v n, this is true, because this is a basis for the vector space v.

So, any vector in V can be expressed in terms of it is base vectors. So, c 1, c 2, c n will then be called as a coordinates of v with respect to this is B 1. Now, let us say w is equal to T of v that is, this is equal to T times c 1 v 1 plus c 2 v 2 plus c n v n. So, T being the linear transformation one can write down this as c 1 times T of v 1 plus c 2 times T of v 2 plus v 1 times T of v n. So, this transformation T of v is expressed in terms of images of the base vectors v 1 v 2 and v n. So, this vector w is represented as a linear combination of T v 1, T v 2 and T of v n.

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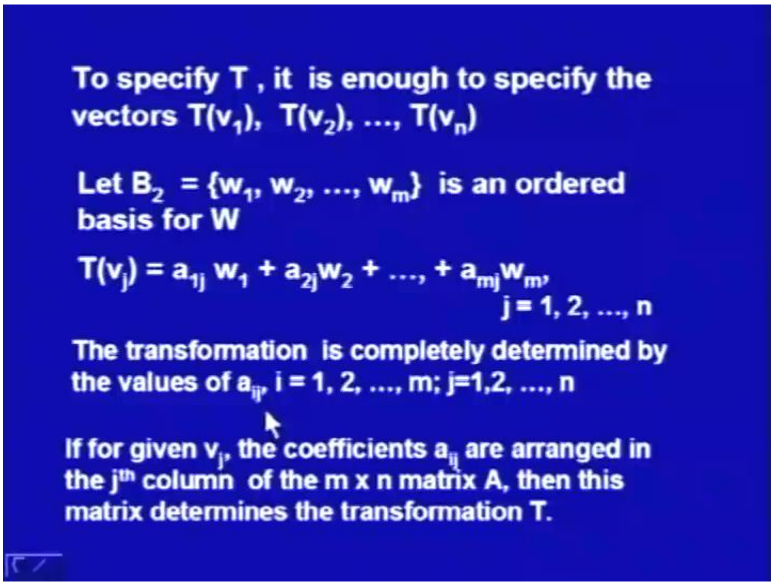

That means, to specify T it is enough to specify the vectors T v 1, T v 2 and T v n. But, what is T v 1, T v 2, T v n these are vectors in W. And let us say W has a basis B 2 consisting of w 1, w 2, w m is an ordered basis for w, this happens to be an m dimensional vector space. So, we will have only m vectors. Then, any vector v j, v j is a vector in the basis B 1.

So, T of v j is equal to a 1 j w 1 plus a 2 j w 2 plus a m j wm, what I am going to do here is I am trying to express T v j in terms of the base vectors w 1, w 2, w m these are the scalars, so T v j is equal to this for all j is equal to 1 to n. Now, the transformation is completely determined with the vectors a i j, if I know these coefficients a 1 to a 2 j a m j for all j is then I know the transformation. So, that is how we write down T v j, so if forgiven v j the coefficient a i j are arranged in the j'th column of the m by n matrix a. Then, this matrix determines a transformation T, it has j'th v j is a j'th vector. So, what we do is we try to arrange these coefficients in the j'th column for T v j.

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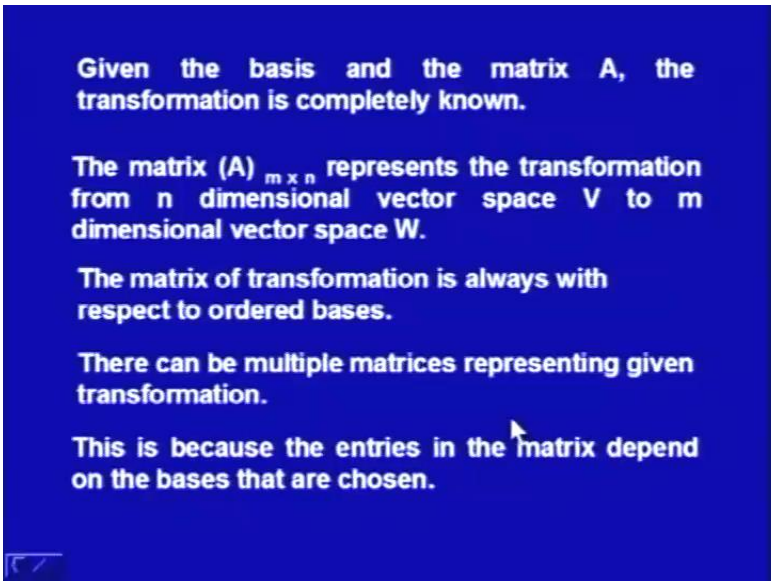

And that is we have the matrix A. So, given the basis and the matrix A, the transformation is completely known. The matrix A will be an m by n matrix why because, we are having n vector. So, we are going to have n columns and in each column there will be m vector, so we will be having A m into n m rows n columns.

So, this represents the transformation from an n dimensional vector space v to n dimensional vector space W. The matrix of transformation is always with respect to order basis. This is the point which you have to remember and we change basis the matrix of transformation will be different. And that is why, they can be multiple matrices representing the given transformation depending upon the basis you consider. This is because, entries in the matrix depend on the basis they are being chosen. So, for different basis we may have different representation for the matrix A the transformation remains the same.

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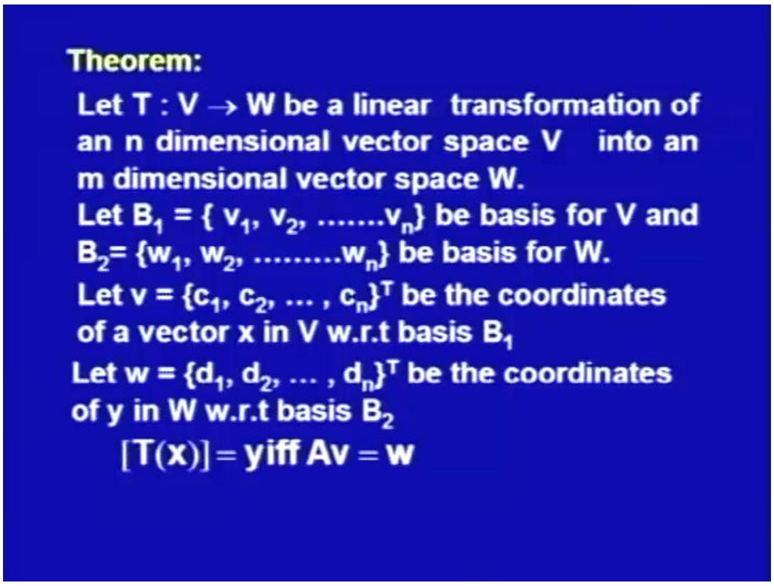

Now, we have a theorem if we have a linear transformation T from a n dimensional vector space V into an m dimensional vector space W and B 1 is a basis for B and B 2 is a basis for W and B is equal to c 1, c 2, c n transpose is a column vector with the coordinate vector x in v with respect to this is B 1. And w is equal to equal to d 1, d 2, d n transpose with the coordinates of y in w with respect to basis B 2, then T of x is equal to y means A v is equal to W. So, with the transformation T we can associate a matrix a such that A v is equal to w. So, for a given transformation T x is equal to y, we can associate a linear transformation A, such that Av is equal to w.

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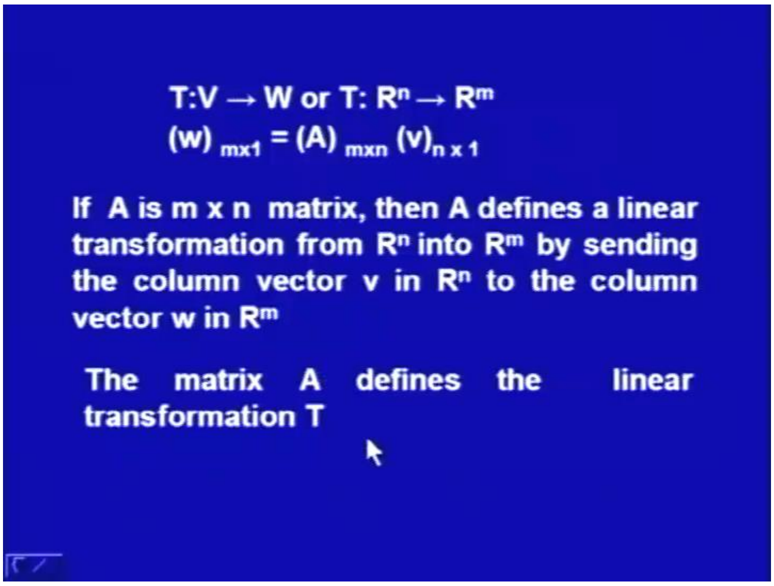

Now, if we have T linear transformation from V to W or we can say T is a transformation from R n to R m. Then, w which is a column vector m by 1 is equal to the matrix A and m by n matrix multiplied by the column vector n by 1, so this is what we have. So, if A is an m by n matrix, then A defines a linear transformation from R n into R m by sending the column vector v in R n to the column vector w in R m. So, the matrix A defines the linear transformation T. So, with every linear transformation there is an associated matrix A.

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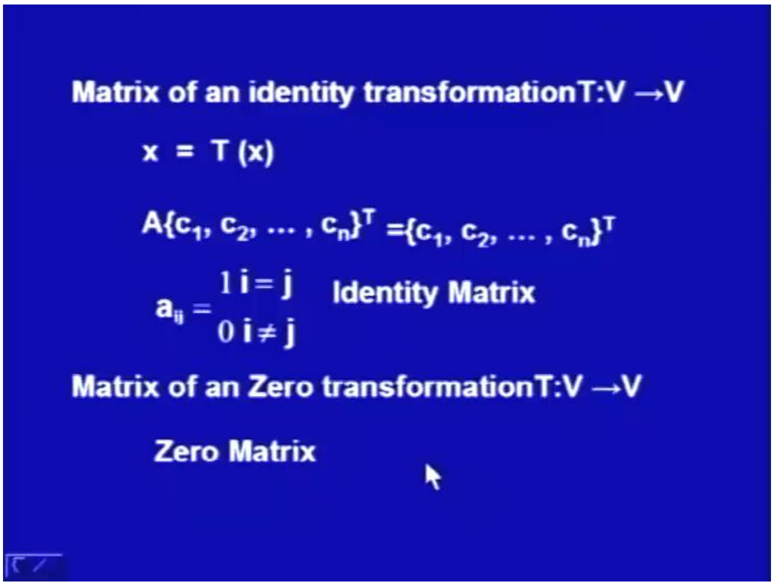

Now, matrix of an identity transformation, so if we have a transformation T from V to V. Then, the identity transformation means every x in V will map to T of x or T of x will be x itself. So, if I have a vector c 1, c 2, c n that is that will map to A c 1, c 2, c n. Then, this relationship is possible provided a i j is equal to 1, whenever i is equal to j and 0 when i not equal to j.

So, if we can have such a i j's then this is possible and this will be such a this will be such a matrix for the given linear transformation T. And if we now require the definition for the identity matrix this is nothing but, the definition of identity matrix a i j is equal to 1, then i is equal to j. That means, in columns we have in the diagonals we have 1 and rest of the elements are 0. So, such a matrix is an identity matrix or we can say this identity matrix is a associated with the identity transformation T. Similarly, the 0 transformation from V to V the associated matrix is the 0 matrix.

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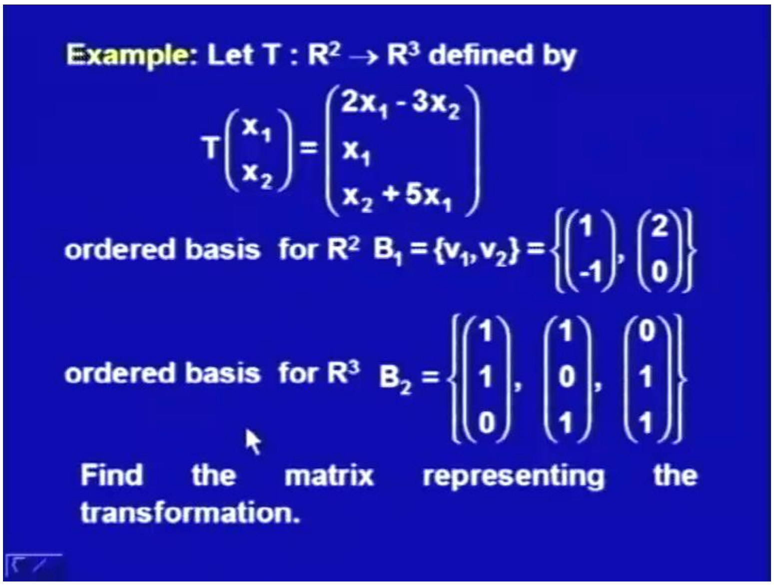

Now, in the next example a transformation has been given to us from R 2 to R 3, we have to find out the associated matrix for this the basis for R 2 is given as B 1 consisting of two vectors v 1, v 2 as 1 minus 1 and 2 0. While the ordered basis for R 3 is provided as 1 1 0 1 0 1 and 0 1 1, these basis have to be specified. Because, the matrix depend on the basic if we change the matrix, if we change the basis the matrix will be different, we have to find the matrix representing the transformation.

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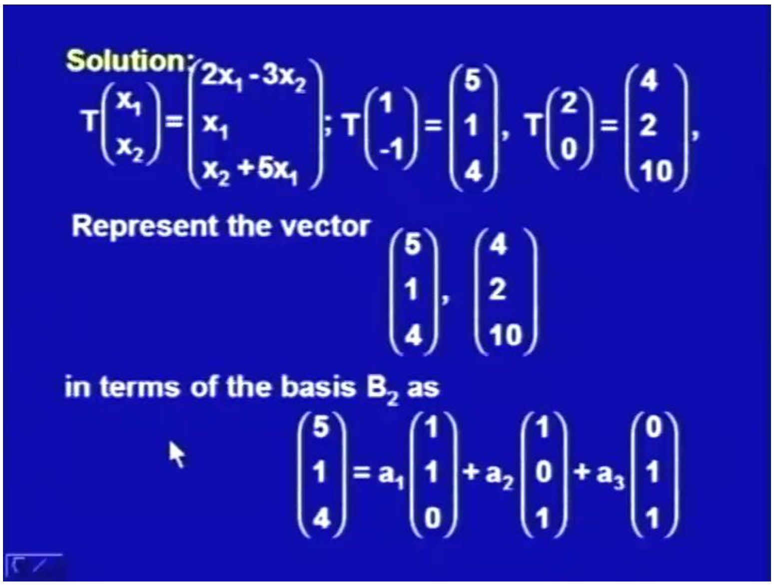

So, for the solution we start with the definition T x 1 x 2 is given to us as 2 x 1 minus 3 x 2, second component is x 1 and third component is x 2 plus 5 x 1. So, we first see how these images, how these basis elements map to the image set. So, T 1 minus 1 is equal to 5 1 4. Let us see how x 1 is 1 and x 2 is equal to minus 1, so 2 x 1 minus 3 x 2 is 5 x 1 is 1 and x 2 plus 5 x 1 that comes out to be 4.

The second element second vector 2 0 will map to 4 to 10 in R 3 like x 1 is 1 x 2 is 0, so when we substitute it here it will be 4 2 and 10. Now, we try to represent the vectors 5 1 4, 4 to 10 in terms of the basis B 2, so 5 1 4 is represented as a linear combination of basis 1 1 0, 1 0 1 and 0 1 1. So, we have to determine a 1, a 2, a 3, so that this matrix equation is satisfied.

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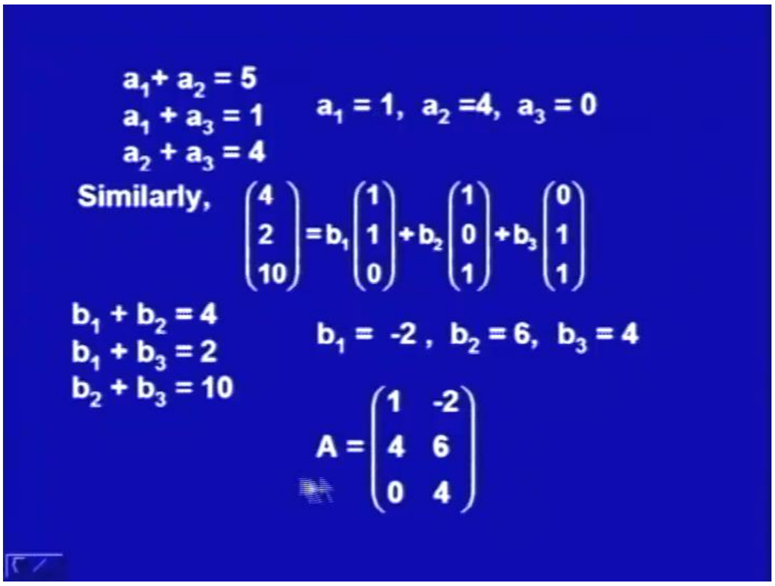

For this purpose I will say a 1 plus a 2 is equal to 5 a 1 plus a 3 is equal to 1 a 2 plus a 3 is equal to 4. And then we solve these three equations in three unknowns a 1, a 2, a 3 we will get a 1 is equal to 1 a 2 is equal to 4 and a 3 is equal to 0. That means, we have obtained the image of the first vector in terms of the base elements. So, this is for the second 4 2 10 we will express it as b 1 1 1 0 plus b 2 1 0 1 plus b 3 0 1 1.

So, it is a linear combination of these vectors and again we have three equations 4 is equal to b 1 plus b 2 2 equal to b 1 plus b 3 and 10 equal to b 2 plus b 3. These three equations are solved simultaneously and the solution comes out to be b 1 is equal to minus 2, b 2 is equal to 6, b 3 is equal to 4. So, we can say the matrix A is 1 4 0 and minus 2 6 and 4, how we get this matrix, this first is this is the representation the first vector T e 1 and this is the representation of the second vector 4 2 10.

So, second matrix and second vector that we say it is an ordered basis. That is why we first write down the first vector, and then we write down the second vector. So, corresponding to 4 2 10 we have this column and corresponding to the earlier vector we have this column 1 4 0. So, this is the matrix associated with the given linear transformation. That is how we obtain the matrix for the given linear transformation.

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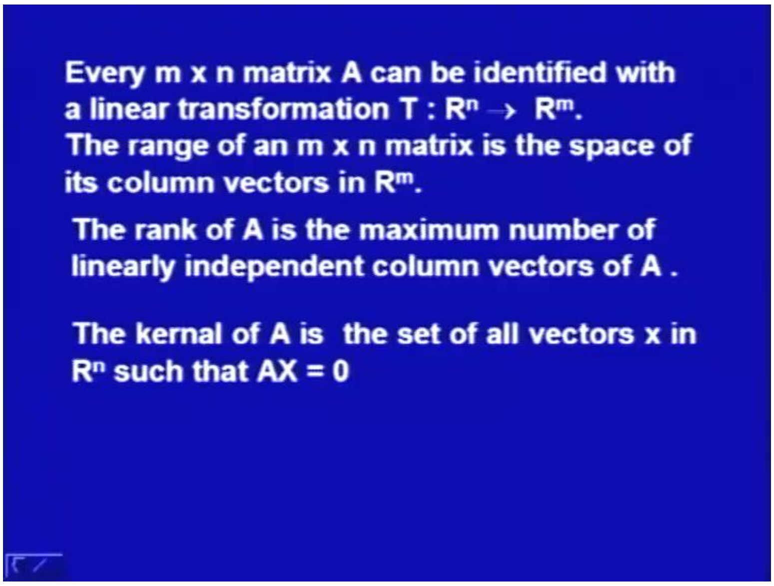

Now, we say that every m by n matrix A can be identified with the linear transformation T from R n to R m and the range of an m by n matrix is the space of it is column vectors in R m, we are having in the matrix A we are will be having vectors in R m. So, the range space of that will be the range of the matrix. The rank of A is a maximum number of linearly independent column vectors of A, that is how we defined rank of the matrix A.

A will have number of column vectors. So, then independent column vectors will constitute the base vectors for the matrix a for the space. And that is why we say that rank of A is a maximum number of linearly independent column vectors of A. Then, the kernel of A is a set of all vectors x in R n such that A X is equal to 0. All these definitions are actually taken from the definition of linear transformations, only thing once we have identified the linear transformation and matrix representation one can very easily write down these things.

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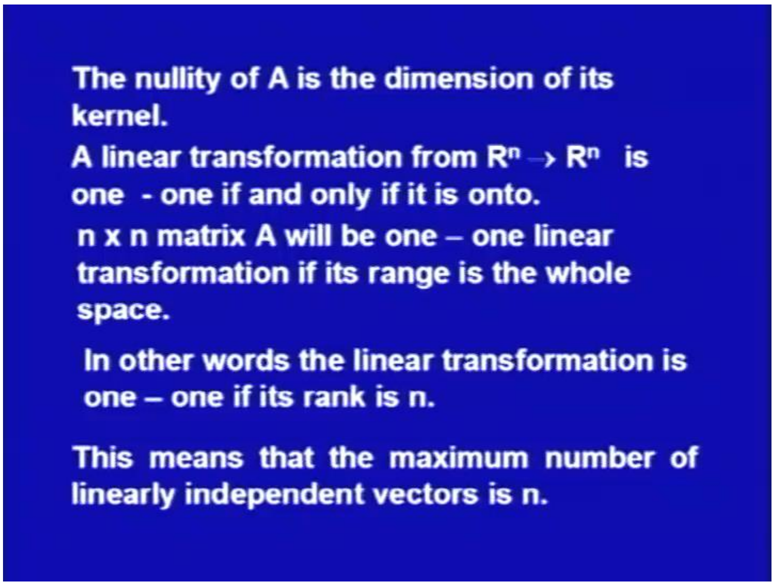

And accordingly, then nullity of A is the dimension of it is kernel. A linear transformation from R n to R n 1 is 1 to 1 if and only if it is onto this is the result, which we have obtained earlier, only thing is we have written for the matrix n by n matrix A will be one-one linear transformation if it is range is the whole space. In other words, the linear transformation is one-one if it is rank is n and this means that the maximum number of linearly independent vectors is n.

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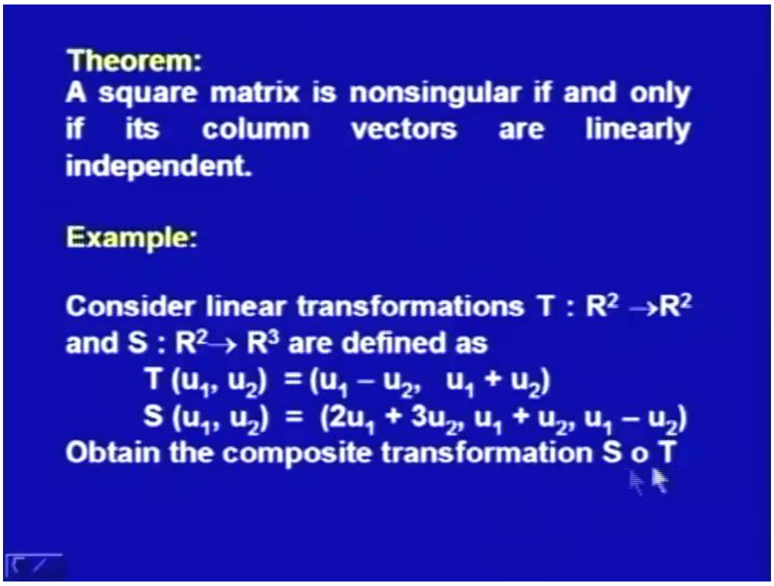

And accordingly a square matrix is nonsingular if and only if it is column vectors are linearly independent. This is the result which we have started a square matrix is a linear transformation is a nonsingular if it is a linear transformation we have defined has to be nonsingular if kernel T is equal to theta. And then with the help of the results, which we have establish just now one can very easily say that a square matrix is nonsingular if and only if it is column vectors are linearly independent.

We will consider a linear transformation T from R 2 to R 2. And other transformation S from R 2 to R 2, R 2 to R 3 which I defined as T u 1 u 2 will map to u 1 minus u 2 comma u 1 plus u 2 and S u 1 comma u 2 to twice u 1 plus thrice u 2 u 1 plus u 2 and u 1 minus u 2, now we have to obtain the composite transformation S composite T.

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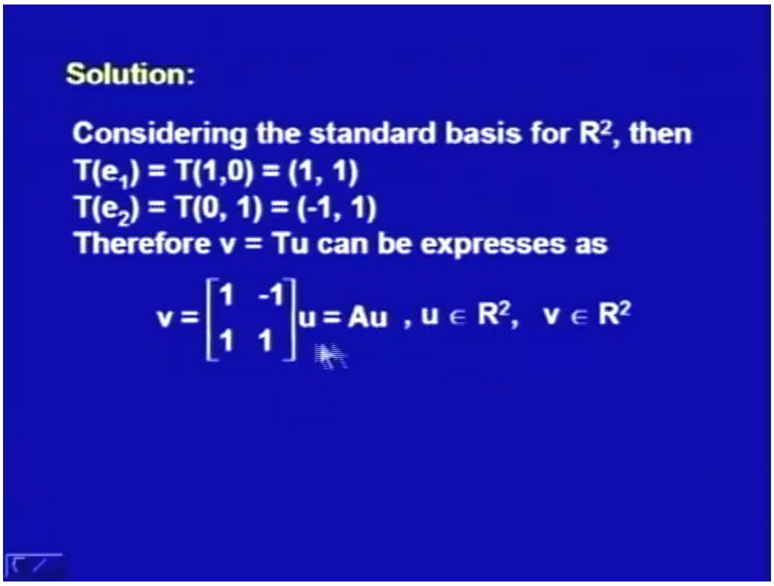

We start with the standard basis for R 2 then T of e 1 is equal to T of 1 0 is 1 1, T of e 2 equal to T of 0 comma 1 is equal to minus 1 and 1 according the definition being given to us. Therefore, v is equal to T u can be expressed as this matrix 1 1, this first column and the second image of second vector is minus 1 1, this is I express as this column. So, this is the matrix representation for the given transformation.

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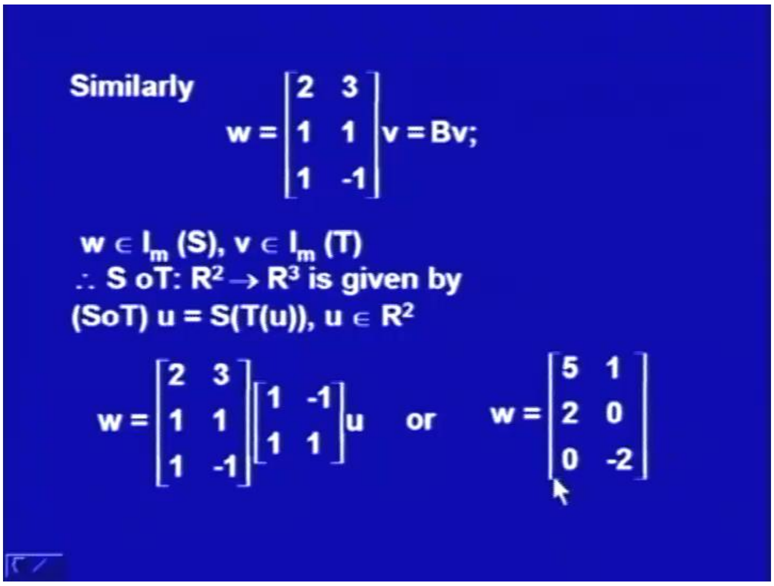

Similarly, for the second we say w is equal to 2 1 1 3 1 minus 1 is the second matrix associated with the transformation. Now, let us say w belongs to image set of S and v belongs to image set of T, then S o T is a transformation from R 2 to R 3 and it is given as S composite T u is S of T u u belonging to R 2.

And since this is being given to us this w is equal to B v being given to us. So, I simply substitute the matrices. So, w is equal to 2 1 1 3 1 minus 1 and this v is 1 minus 1 1 1 u or if you multiply these two matrices, what we can get w as product of 2 matrices 5 1 2 0 0 minus 2.

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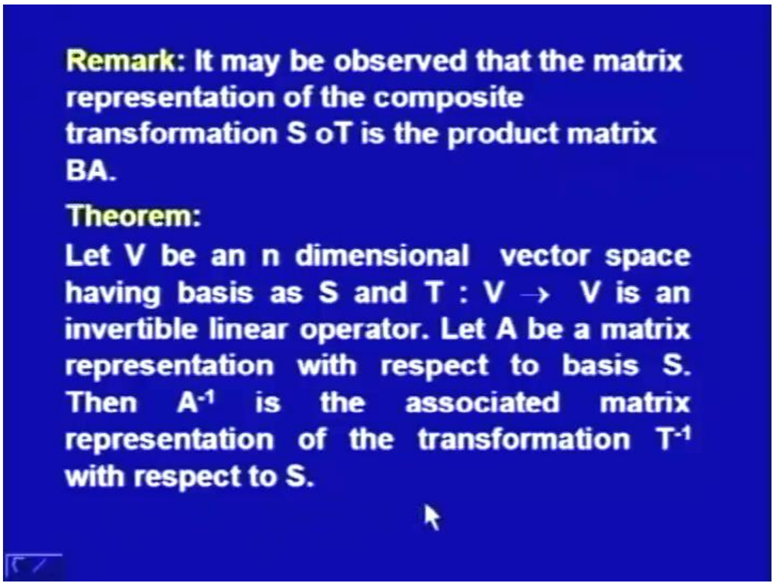

So, that is how we can say that the matrix representation of the composite transformation S o T is the product matrix B A. Now, let V be an n dimensional vector space having basis as S and T is an invertible linear operator. Then, A be the matrix representation with respect to basis of S. Then, A inverse is the associated matrix representation of the transformation T inverse with respect to S.

So, a vector space being given to us a transformation is given to us associated with this transformation we have a matrix A given to us. Then A inverse is the associated matrix representation of the transformation T inverse with respect to the basis S.

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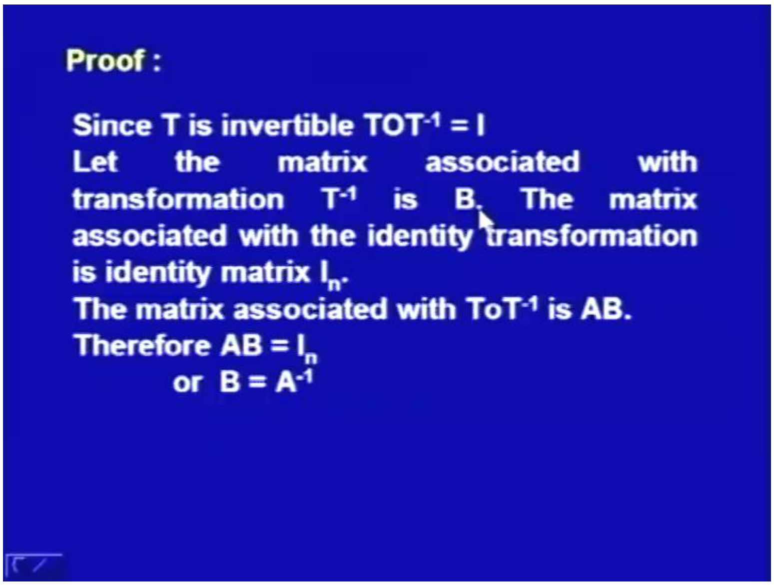

So, this is what we have to prove. So, to proof this we consider that T is invertible and since it is invertible. So, we can say T composite T inverse is equal to I. So, there exist T inverse for the given transformation T such that T composite T inverse is equal to I. Let the matrix associated with the transformation T inverse is B, the matrix associated with the identity transformation is we know is an identity matrix.

So, the matrix associated with T composite inverse is A B. Therefore, A B is identity matrix and from here we conclude that B is equal to A inverse by this I mean to say that if a given transformation to us as T it is associated matrix is A if the transformation is invertible. Then, we can very easily find out the matrix associated with the inverse transformation that is A inverse.

So, the two things are related, now with this we come to an end of this lecture. This lecture is related with the linear transformation, we have continued with our earlier lecture. So, in this set of two lectures we have covered concept related to linear transformation I have started with the definition, some basic results are given. And finally, I have to represent a linear transformation as a matrix. And then we relate certain more results and the next time we will discuss rank and other related concepts.

Thank you.