Mathematics II
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Module - 2
Lecture - 10
Inner Product

A welcome viewer, today's topic is Inner Product. But, before I start inner product, I would like to give an example on the theory, which we have developed in the last lecture regarding change of basis.

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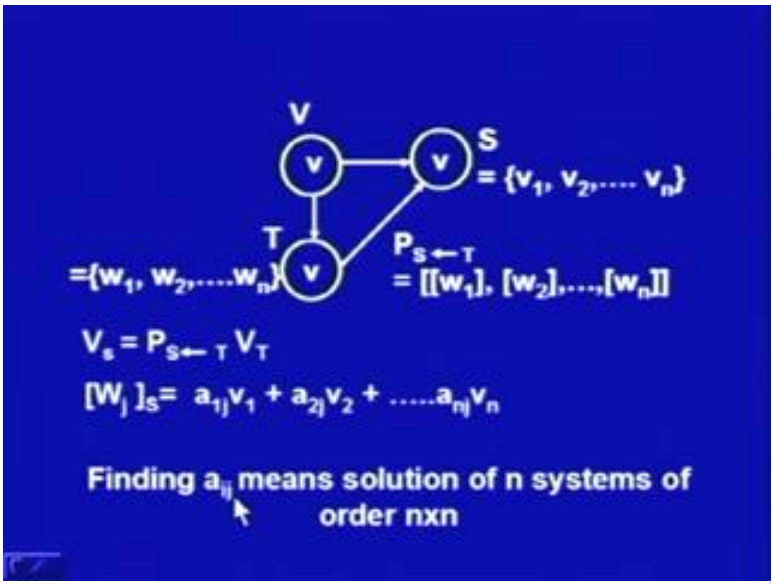

We have developed some results like V S is equal to P T S, V T. But, we could not take up examples to illustrate this result. Today, I will start with this example first illustrate this and then I will go to the inner products. Now, to start with, let us say V is a vector space of dimension n, S and T, be it is two subspaces. S consisting of v 1, v 2, v n is a basis for S and T is w 1, w 2, w n. This is for V.

Now, let us say the vector v can be expressed as V s in terms of the vectors v 1, v 2, v n. And V T in terms of bases w 1, w 2, w n. Then, relationship between V s and V T, we this we have derived in my last lecture as V s is equal to P S T a matrix, multiplied by a column vector V T. This matrix is called transition matrix. And it is called transition matrix from T to S.

Now, to illustrate this, we have to find out a i j. What is a i j? A i j is a typical element of this matrix. Now, to get a i j, we have to solve n systems of order n by n. So, let me illustrate this with an example.

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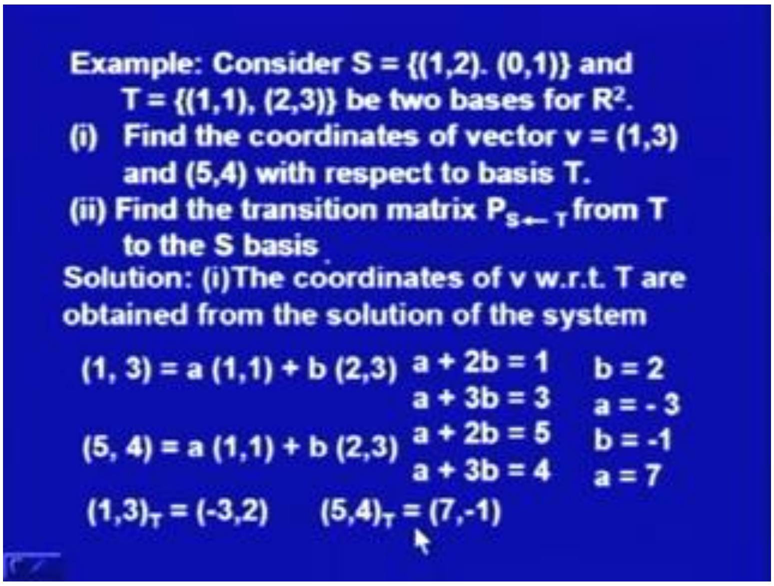

So, let us consider S a subspace of R 2 consisting of bases 1 comma 2 and 0 comma 1. And another bases T 1 comma 1, 2 comma 3, these are two bases for R 2. We have to find the coordinates of vector V, which is 1 comma 3 and 5, 4 with respect to basis T. This is first part. In the second part, we have to find the transition matrix P S T from T to S to the basis S.

Now, in the first part, you have to find the coordinates of V with respect to T. Now, this can be obtained from the solution of the system, which is obtained as 1 comma 3 is equal to a 1 comma 1 plus b 2 comma 3. That means, I want to express this vector 1, 3 as a linear combination of bases vectors 1 comma 1 and 2 comma 3 of the basis T.

Now, to this, to get a and b, I have to solve this system. And the system is a plus 2 b is equal to 1, which is coming from the left hand side. And a plus 3 b, a plus 3 b is equal to 3 from the right hand side. When, we solve this equation, this you can very easily solve it, subtract this from this. That will give me b is equal to 2. And substituting b is equal to 2 in one of these equations, we get a is equal to minus 3.

So, 1 comma 3 can be expressed as a linear combination of 2 vectors or we can say the representation of 1, 3 in the basis T is 2 comma minus 3. Similarly, the vector 5 comma 4 is represented as a linear combination of basis vectors of T, a 1 comma 1 plus b 2 comma 3. And we again solve the system of equations to get b is equal to minus 1 and a is equal to 7. And that means, 1 comma 3 in T basis representation is minus 3 comma 2, 5 comma 4 is 7 minus 1 in the basis T.

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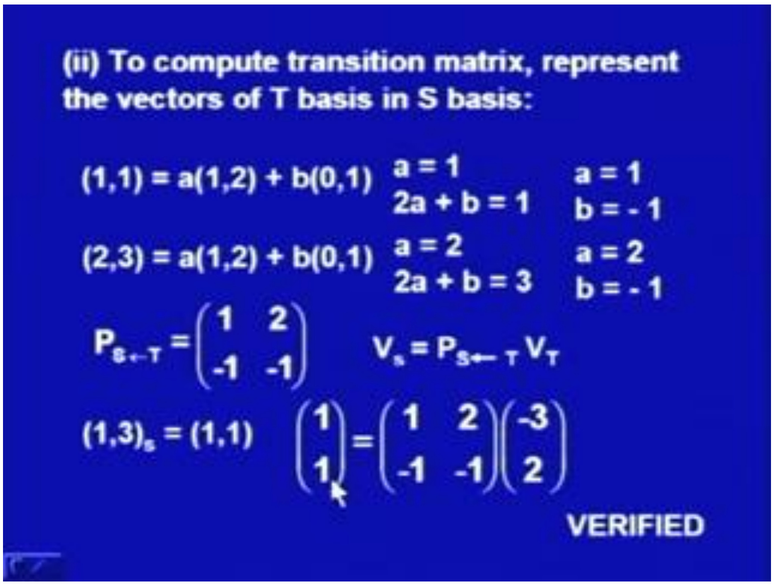

Now, the second part is, to compute the transition matrix, you have to represent the vectors of T basis in S basis. That means, first, we consider the vector of S basis 1 comma 1. We write down it as, a 1 comma 2 plus b 0 comma 1. And we solve it as we have done in early in the earlier part. So, a is equal to 1 and b is equal to minus 1 is the solution of this.

Then, the second part 2 comma 3, this is the second vector in the basis. So, we express it again as a linear combination of 1, 2 and 0, 1. And, we form a system of equations as this. They are solved to get a is equal to 2 and b is equal to minus 1. Solving this, we write down the transition matrix. The first vector is written as the column of this transition matrix. This is w 1 and this is the second vector corresponding to 2 comma 3. So, this is the transition matrix.

Now, let us try to verify the result, which we have developed. That is Vs is equal to P S T, V T. For this, we say 1 comma 3 S is equal to 1 comma 1. This can be obtained by solving as we have done here. And then we say 1 comma 1, this is V S is a transition matrix, which we have obtained here multiplied by V T, V T is the vector 1, 3 in T. This we have all ready obtained as minus 3, 2.

And, if you really perform this multiplication, you can see that, 1 comma 1 multiplied by minus 3 plus 2 into 2 is 1. And, this 1 is equal to minus 1 into minus 3 minus 2. That comes out to be 1. So, the result is verified. Now, with this, we have completed the example and now, we go to the inner product.

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So, the contents of this lecture consist of the definition of inner product. The inner product spaces, norm, distances, orthogonal vectors, orthonormal vectors, Gram Schmidt process.

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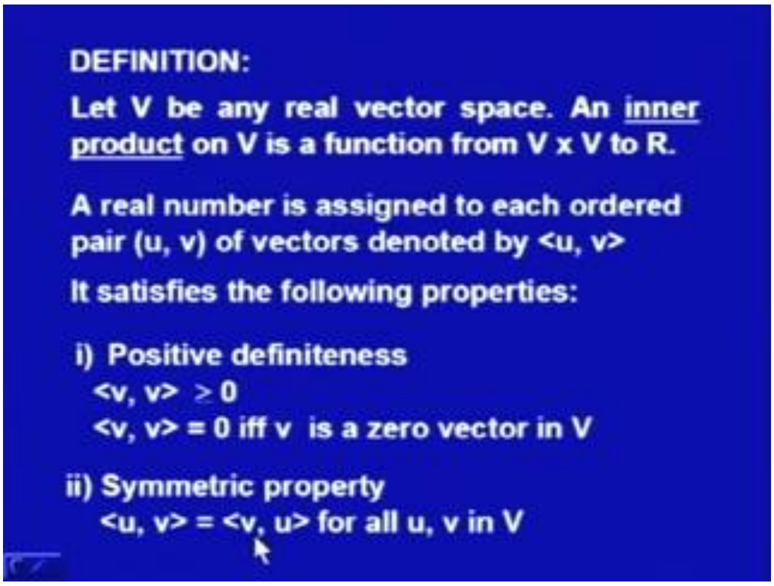

To define inner product, let us consider V be any real vector space. An inner product on V is a function from V cross V 2 R. That means, a real number is assigned to each ordered pair u v of vectors denoted by u comma v. So, this notation, we use for inner product of u and v, provided it satisfies the following properties. The first property, we call it as positive definiteness. This means that v comma v, the inner product of v comma v is greater than equal to 0 and v comma v is 0. If and only, if v is a 0 vector in V. So, v comma v is 0, only when v comma v is 0. When, V is a 0 vector. This is the positive definiteness. And the second is the symmetric property. This means that inner product of u and v is the same as inner product of v and u, for all pairs of u and v in V.

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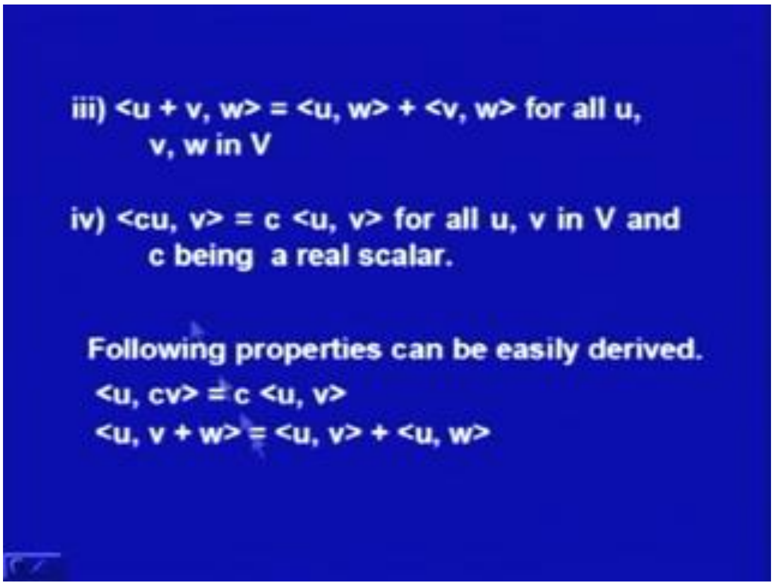

According to the third property, u plus v comma w, the inner product is equal to inner product of u and w plus inner product of v and w for all u v and w in V. And the last one, c u comma v, inner product of c u and v is equal to c times, inner product of u comma v. For all u v in V and c being a real scalar. So, if it satisfies these properties, then an inner product is defined.

On the basis of these four properties, the following properties can easily be derived. That is u comma c v is equal to c times, inner product of u comma v, because it is symmetric. So, c v comma u is the same as u comma c v. And then c can be taken out. So, u comma c v is equal to c times u comma v. And the third one the next property, which can be easily derived is u comma v plus w is equal to u comma v plus u comma w. And again, it is derived from the symmetric property. And then using the third property, one can easily arrive at this result.

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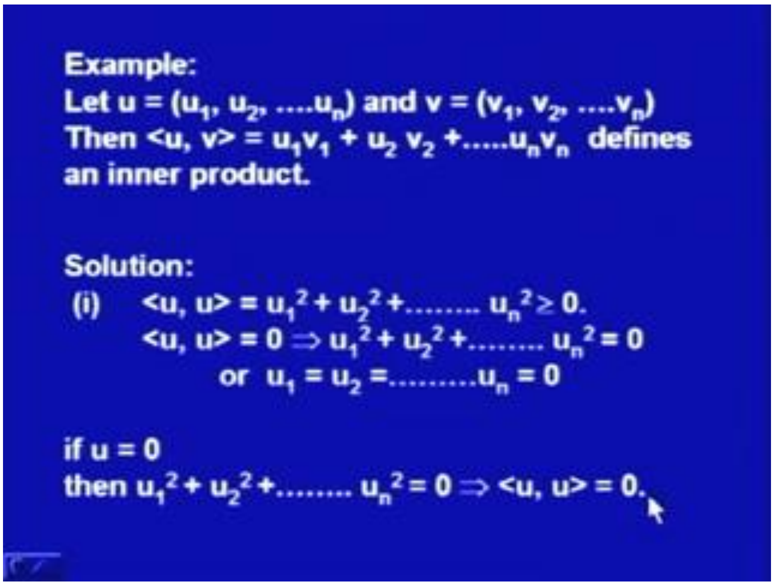

Let us taken as an example. So, if we have a vector u in R n, u 1, u 2, u n and v as v 1, v 2, v n. Then, if you define u comma v as u 1, v 1 plus u 2, v 2 plus u n, v n, then it defines a vector. Then, it defines an inner product. It defines inner product means, we have to verify, that it satisfies all the four properties, which we have described just now. So, to start with u comma u, u comma u is u is 1 square plus u 2 square plus u n square, because we are taking u at both the places.

So, this is the inner product according to this definition. And this is a square of real numbers, sum of squares of real number, this always be positive. And this is 0, only when u 1 square plus u 2 square plus u n square is equal to 0. Since, all these numbers are positive. So, there sum will be 0, only when individual terms are 0, u 1, u 2, u n is 0. That means, u comma u is 0, simply means that, u is 0. So, that is the first part. And, if you u is equal to 0. Then, u 1 square plus u 2 square plus u n square is equal to 0. So, u comma u is 0 for 0 vectors. That proves the first part.

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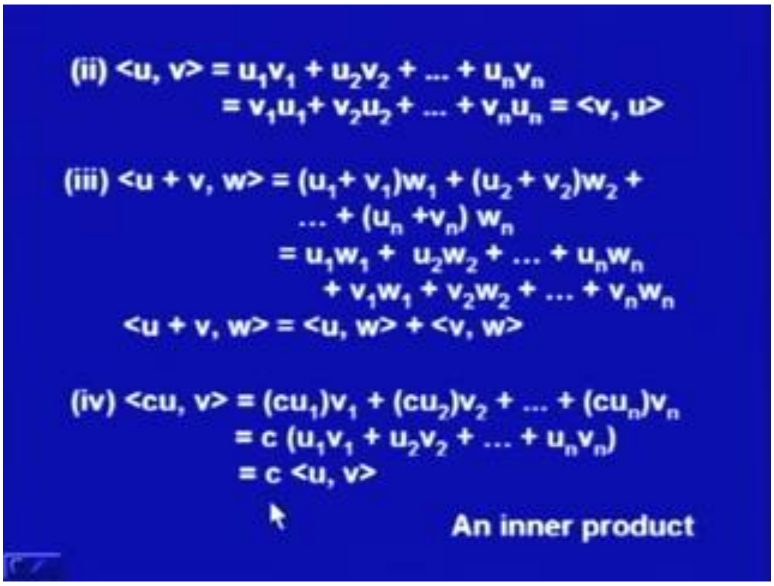

The second part is, that u comma v and v comma u are same. So, according to definition u comma v is u 1, v 1 plus u 2, v 2 plus u n, v n. And since u 1 and v 1 are real number. So, they can be commuted. So, u 1, v1 can be written as v 1, u 1. Similarly, the next term v 2, u 2 and the last term u n, v n is expressed as v n, u n. And again, the planning, the definition, we can see that, this is nothing but v comma u. And that means, the symmetric property is also proved.

Now, the third property u plus v comma w. So, if we consider u plus v as u 1 plus v 1, u 2 plus v 2, u n plus v n and w as w 1, w 2, w n. Then, by applying the definition of inner product, it is u 1 plus v 1 multiplied by w 1 plus u 2 plus v 2 multiplied by w 2 and so on. And the last term will be u n plus v n multiplied by w n. And since, these are nothing but simply real numbers. So, u 1 plus v 1 multiplied by w 1 is nothing but u 1, w 1 plus v 1, w 1. And u 2, v 2 multiplied by w 2 is u 2, w 2 and here we are writing v 2, w 2 and so on.

This is possible, because these numbers are real numbers. And we have collected terms first for u 1, w 1 and second for v 1, w 1 etcetera. And that means, u plus v comma w, this inner product is equal to inner product of u w plus inner product of v w. And that proves the third property. The fourth property is c u comma v. So, we consider the right hand side by applying the definition c u 1, v 1 plus c u 2, v 2 plus c u n, v n, c can be taken outside.

So, what we have is u 1, v1 plus u 2, v 2 plus u n v n. And this is nothing but c times the inner product of u comma v and that proves the last property. And this means, the definition which we have given, that is the definition of an inner product.

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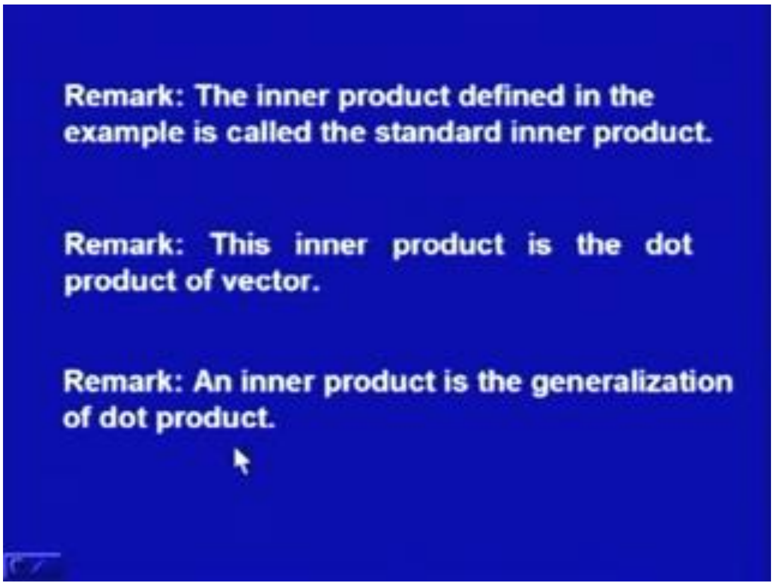

Now, the inner product defined in this example is called the standard inner product. And if you realize, then this inner product is the dot product of vectors. So, inner product is a generalization of dot product for vectors. This is, what we like to conclude and then inner product is actually generalization of dot product. And we can, in fact, it can be defined in many different ways.

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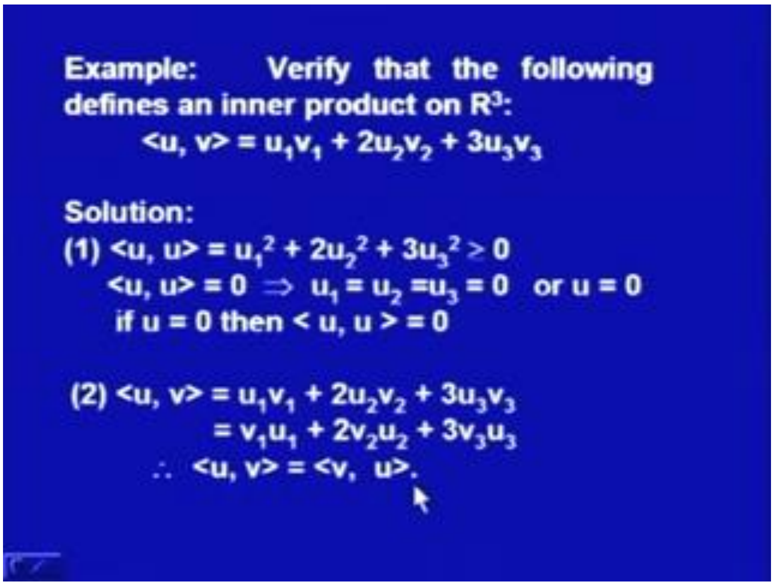

So, in this example, we say that, this definition u comma v is equal to u 1 plus v 1 plus twice u 2, v 2 plus thrice u 3; v 3 also defines an inner product. So, the same way, as we have done for the earlier definition, we apply this. So, we again check the first property u comma u is equal to, according to the second definition; it is u 1 square u 1 and v 1. They are equal to u 1 square.

The second term is twice u 2, v 2, u 2 and v 2 are equal. So, it is twice u 2 square the second term. And third term is 3, u 3, v 3. So, as u 3 and v 3 are equal, here we have u comma u. So, it is thrice u 3 square and again this is greater than equal to 0. And u comma u is 0, implies that u 1 is equal to u 2 is equal to u 3 is equal to 0. Because, all these terms are positive terms, they can be 0. When, individually each of them is 0. And that means, u comma u is 0 only when u is equal to 0.

And of course, if u is equal to 0, that means u 1, u 2, u 3 are 0. Then, u comma u, u the inner product u comma u is also 0. So, that proves the first property for this definition. Now, comes the second property u comma v is equal to u 1, v1 plus twice u 2 ,v 2 plus 3v u 3, v 3, according to the definition. Then, we can again commute individual terms. They are real numbers. So, we can have v 1, u 1 plus twice v 2, u 2 plus thrice v 3, u 3. And this simply means; that u comma v inner product is equal to inner product of v comma u. And that proves the symmetric property.

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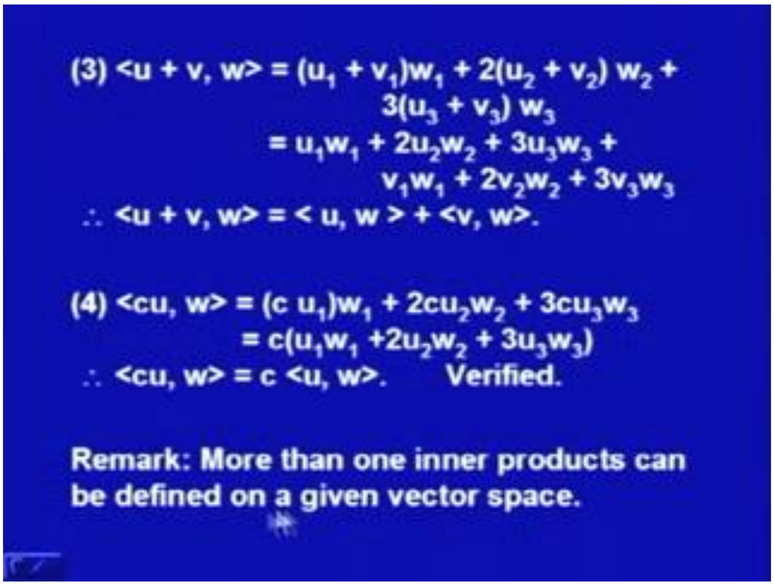

Third is, u plus v comma w is equal to u 1 plus v 1 w 1 plus 2 times u second component u 2 plus v 2 multiplied by w 2, plus 3 times the third component u 3 plus v 3 into w 3. Or, we can write it as u 1, w 1 plus twice u 2, w 2 plus thrice u 3, w 3 is plus v 1, w 1 plus twice v 2, w 2 plus thrice v 3 w 3. That means, we are reorganizing terms in this particular manner. And then we can easily see that u plus v comma w is equal to inner product of u comma w plus inner product of v comma w. That proves the third property.

And the fourth is c u comma w is equal to c u 1, w 1 plus twice c u 2 second term multiplied by w 2 plus thrice c u 3 multiplied by w 3. And then c can be taken out from each of these and that proves the final results c u comma w is equal to c times, the inner product of u comma w. And that establishes, that the definition, which is given to us is a definition for inner product.

So, we can for R 3, we can have two types of inner product defined. One the way, we have the standard inner product, which we have defined earlier example, n is equal to 3. And this is another definition for inner product. So, for a vector space, we can define inner product in many different ways. So, more than one inner product can be defined on a given vector space. And this is the remark, which can be obtained from this example.

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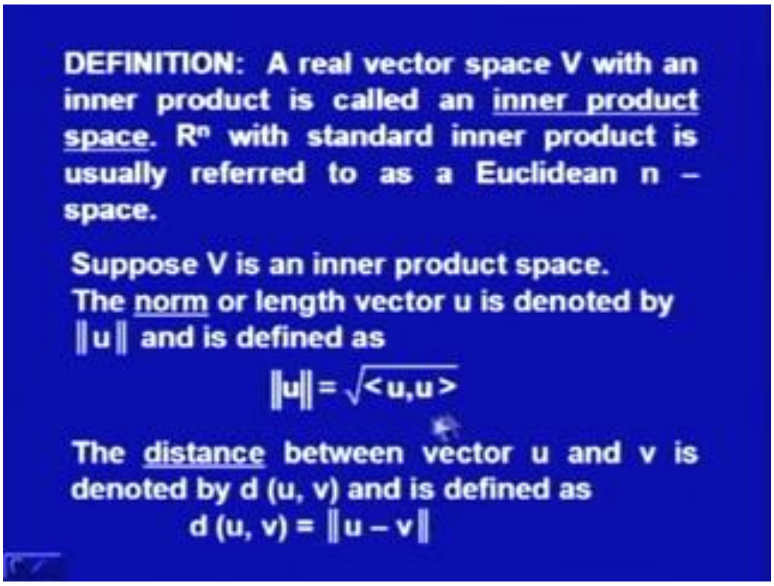

Now, we define inner product space. So, we say a real vector space V with an inner product is called an inner product space. So, to define an inner product space, what we need is, we have to define an inner product on a real vector space. Normally, R n with standard inner product is called as Euclidean n space.

Now, few more definitions, suppose V is an inner product space. Then, the norm or length of vector u is denoted by this symbol. And is define as, we call it as norm of u as under root of inner product u comma u. This is one definition. And the another definition is, for the distance between 2 vectors u and v. And it is denoted by d u comma v and is defined as d u comma v is equal to norm of u minus v.

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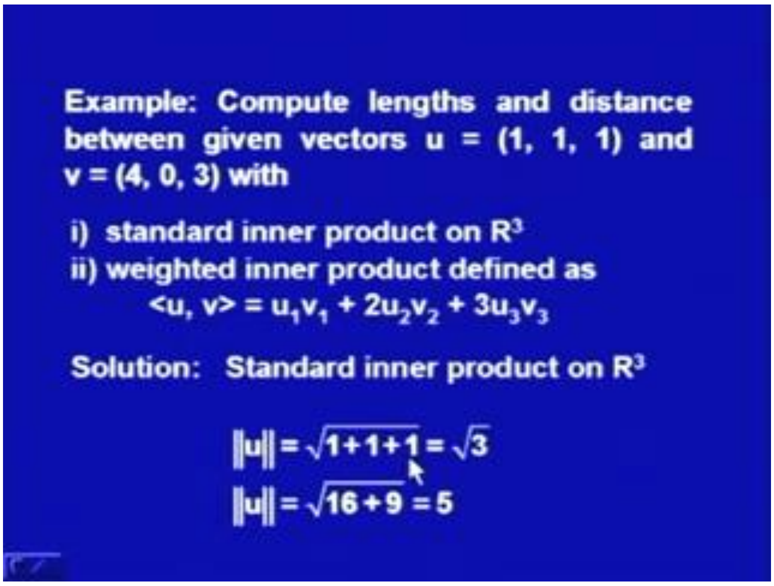

Now, let us illustrate this with example. So, compute lengths and distance between given vectors u 1 comma 1 comma 1 and v as 4, 0 and 3 with the standard inner product on R 3. And weighted inner product, defined as u comma v is equal to u 1, v1 plus twice u 2, v 2 plus thrice u 3, v 3. So, we solve it, we start with standard inner product on R 3. So, first, you have to find the length.

So, length of the vector u, which is 1 comma 1 comma 1, it is defined as under root of inner product of u comma u, the standard inner product of u comma u. It is 1 plus 1 plus 1. So, it comes out to be under root 3 by for the other vector 4, 0, 3. It is 16 plus 9; it is coming out to be 5. So, this is the length of 4 comma 0 comma 3.

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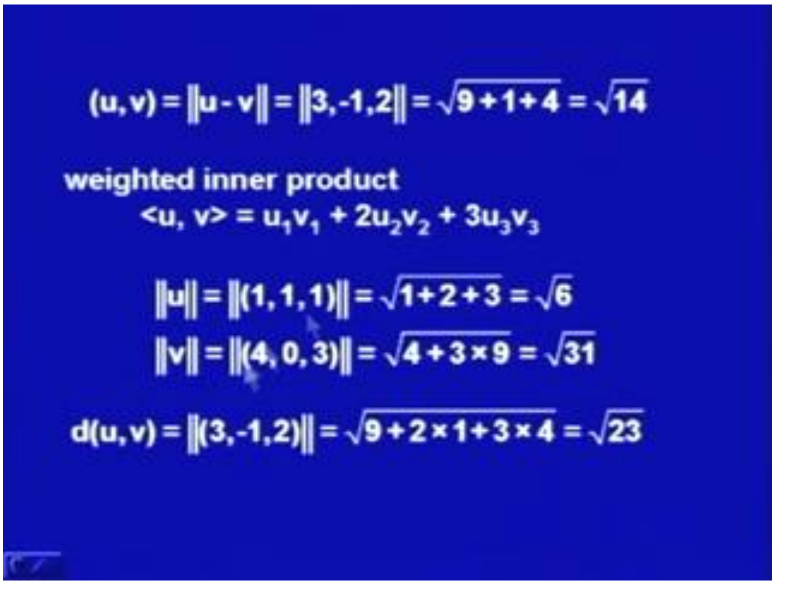

The distance u comma v is norm of u minus v. And if will given u and v, it comes out to be norm of 3 comma 1, 3 comma minus 1 comma 2. And, it is inner product will be 9 plus 1 plus 4. That comes out to be under root 14. So, distance between u and v on the standard inner product is under root 14. While, if we consider the weighted inner product defined as this. Then, the norm of u is 1 plus 2 plus 3 and this comes out to be under root 6.

While, for the second vector, it is 4 plus 3 comma 9, this is 3 and u 3 square is 9. So, it is 4 plus 3 into 9 that comes out to be under root 31. So, norm of v on weighted inner product comes out to be under root 31. At the distance between u and v on this inner product, weighted inner product is 3 norms of 3 minus 1 comma 2. And, this comes out to be 3 u comma v is 3 into 3 plus 2 times minus 1 into minus 1. That is 1 plus 3 times u 3, v 3, that is 3 times 2 comma 2. So, it is 4 and this comes out to be under root 23.

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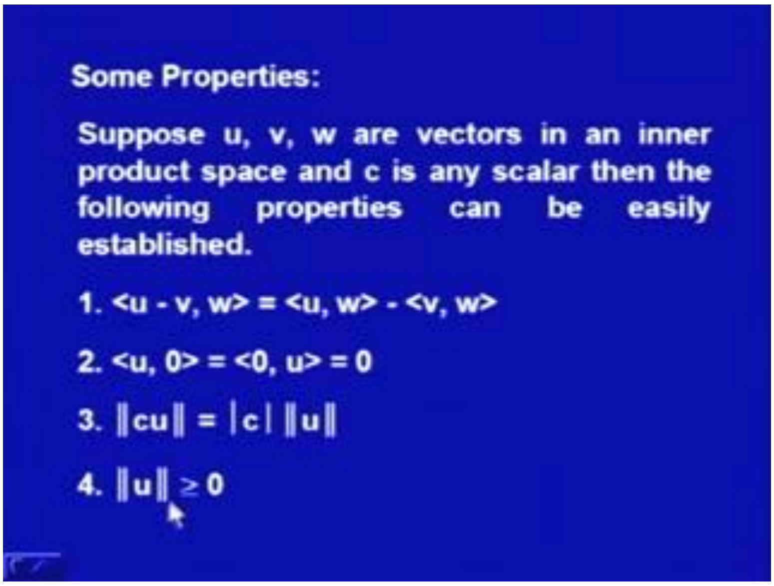

Now, we discus some properties of norms and distances. So, suppose u v and w are vectors in an inner product space and c is any scalar. Then, the following properties can be easily established. The first property is the inner product of u minus v and w is u comma w minus v comma w. The second property is previously, we have addition here. Now, we have subtraction here. The second is u inner product of u comma 0 is the same as inner product of 0 comma u. And this is always 0. The third is inner norm of c u is equal to the magnitude of the scalar c multiplied by norm of u. Then, norm of u is always greater than equal to 0. These are the properties, which can easily be derived.

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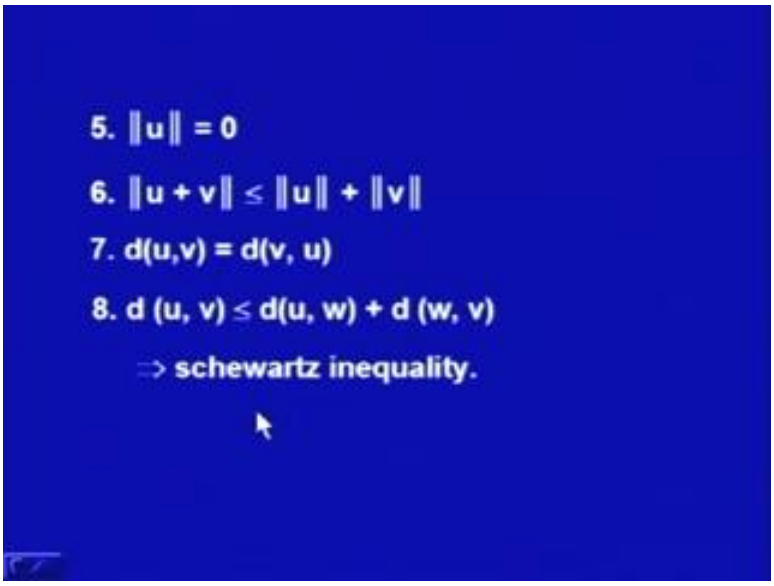

And fifth is norm of u, if u is equal to 0, when u is equal to 0. Next, is norm of u plus v is less than equal to norm of u plus norm of v. And then we have d u v is equal d v u. That means, distance between u and v or distance between u and v, v and u. It is the same. And 8th property is distance between u and v is less than equal to distance between u and w plus distance between w and v. This is called Schwartz inequality.

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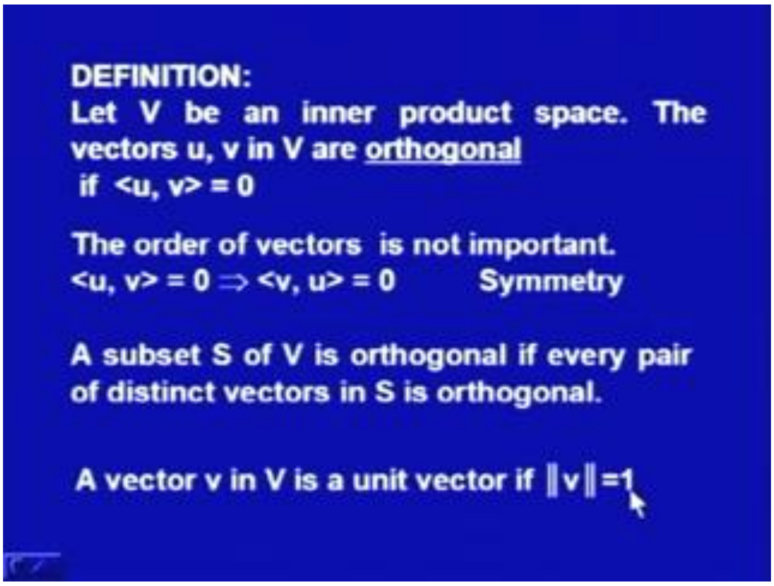

Now, we discuss orthogonality of vectors. So, let us define it first, let V be an inner product space. The vectors u and v in V are orthogonal. If the inner product of u and v is 0, so this is the definition for orthogonality of 2 vectors. Here, the order of vectors is not important, because if u comma v is equal to 0 than inner product of v comma u is also 0, due to symmetry. So, if u is orthogonal to v. Then, v is also orthogonal to u.

Now, a subset S of V is orthogonal, if every pair of distinct vectors in S is orthogonal. Here, we have defined orthogonality between 2 vectors. Now, we are defining orthogonality of a set of vectors. So, a subset S of V is orthogonal, if every pair of distinct vectors in S is orthogonal. And then a vector v in V is a unit vector, if norm of this vector is one. That is, how we defined unit vector.

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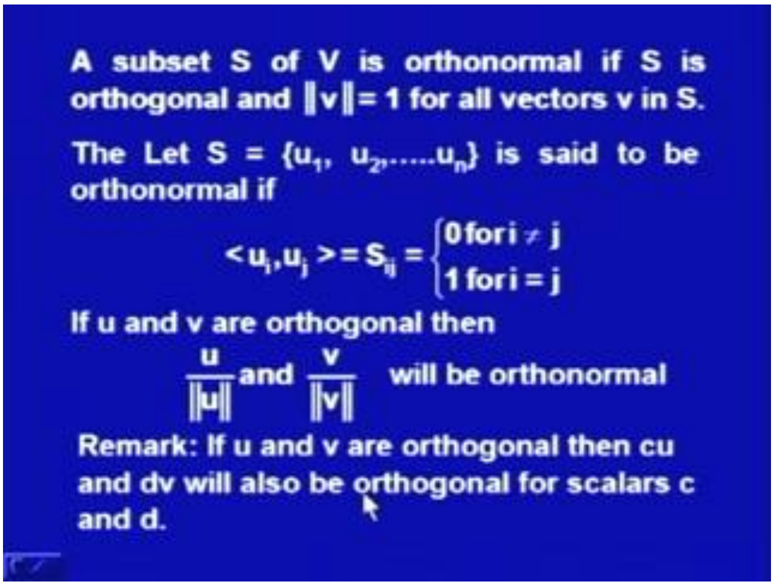

A subset S of V is orthogonal. If S is orthogonal and norm of v is equal to 1, for all vectors v in S. That means, if these two properties are satisfied for subset S. Then, this is called orthonormal. The first property is that S is orthogonal set and norm of each vector in S is 1. That is, how we define orthonormal set. The set S consisting of u 1, u 2, u n is said to be orthonormal.

If u i, u j is equal to S i j or we call it delta i j. If it is 0, when i is not equal to j and it is 1 for i is equal to j. So, this definition is actually means this. For example, if u 1 and u 2, i is equal to 1 and i j j is equal to 2. So, this is orthonormal. If these 2 vectors are orthogonal, that means u i, u 1 and u 2. This is equal to 0, when i is not equal to j. This should be true for all i and j, when they are not equal.

But, if they are equal, the inner product of u i comma u i when i and j are equal. That is norm and this has to be 1, because it is an orthonormal vector. So, that is how, we define orthonormal set. And mathematically, we say that, inner product of u i comma u j is equal to 0, for i is not equal to j and one for i is equal to j. So, if this property is satisfied for all i and j for a set S. Then, we say that set S orthonormal.

If u and v are orthogonal, then u divided by norm of u and v divided by norm of v. These vectors will be orthonormal, because orthogonality does not mean orthonormality. Because, normality wants that, there norm should be 1. Hence, this additional property is required. So, if u and v are orthogonal, then to make it orthonormal, you have to divide individual vectors by their norms. So, u and norm of u and v and norm of v will be orthonormal. So, this way we are actually getting the norm of the vectors as 1.

If u and v are orthogonal, then c u and d v will also be orthogonal for scalar c and d. This can be easily proved, because if u and v is inner product of u and v is 0. Then, inner product of cu comma v is also 0 and inner product of c u comma d v will also be 0. So, this is what we have.

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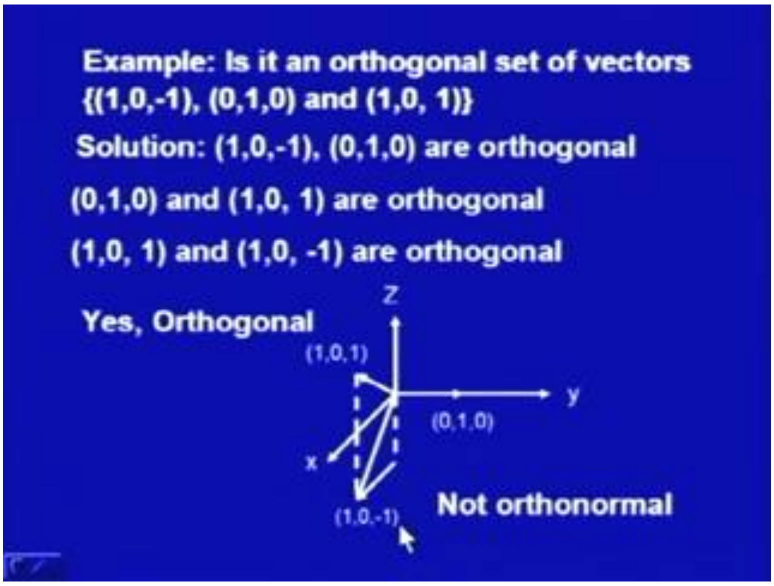

Now, as an example, I have given a set of 3 vectors 1, 0 minus 1, 0, 1, 0 and 1, 0, 1. We have to find, whether this is an orthogonal set of vectors or not. Now, to check this, we have to see, whether they are orthogonal or not, whether, this pair is orthogonal or not. So, let us consider 1, 0 minus 1 and 0, 1 comma 0. Let us form, it is inner product and one can see that, 1 comma 1 multiplied by 0, 0 multiplied by 1 minus 1 multiplied by 0. On the standard inner product, this is 0.

If nothing is said, then we consider the standard inner product. So, we are considering standard inner product and this inner product is 0. Then, consider a second pair 0, 1, 0 and 1, 0, 1. Again, this is this set this pair is orthogonal, because 0 comma 0 multiplied by 1 plus 1 multiplied by 0 and 0 multiplied by 1 all are 0. So, it is, this is also orthogonal. And there is third set 1, 0 minus 1 and 1, 0, 1. If this is also this pair is also set of orthogonal vectors. Then, you can say that this set is a set of orthogonal vector.

So, let us check this 1 comma 1 is 1. This is 0 1 comma minus 1 is minus 1. So, the sum is 0. So, this also is an orthogonal pair and that means, the set S consist of orthogonal set of vector. Now, let us see, what are these vectors the first vector 0, 1, 0, 0, 1, 0 is a vector on y axis. Then, another vector 1, 0 minus 1, 1, 0 minus 1 is this vector, which is on y x plane. And this is on x z plane, x and z plane.

The third vector is 1, 0, 1 this is also on x z plane. And so these 2 vectors, they are lying on x z plane. And this is third vector is y vector. So, x z plane is y axis is perpendicular to x z plane. So, these 3 vectors are orthogonal, then these 2 vectors whether they are perpendicular or not. So, let us check, whether they are perpendicular or not.

So, we have to compute the inner product. It is 1, 0, 1, 1, 0, 1 is this 45 degree line with this. And this is again on the negative side. So, this is also 45. So, that means, this angle between these 2 vectors is 90 degree. So, that is, how these 3 vectors are orthogonal, this is a geometrical interpretation. But, these 3 vectors are orthogonal. But, they are not orthonormal. Although the norm of this vector is one, but the norm of this vector is not 1. Similarly, the norm of this vector is not 1. So, this set is not a set of orthonormal vectors. They are simply orthogonal vectors.

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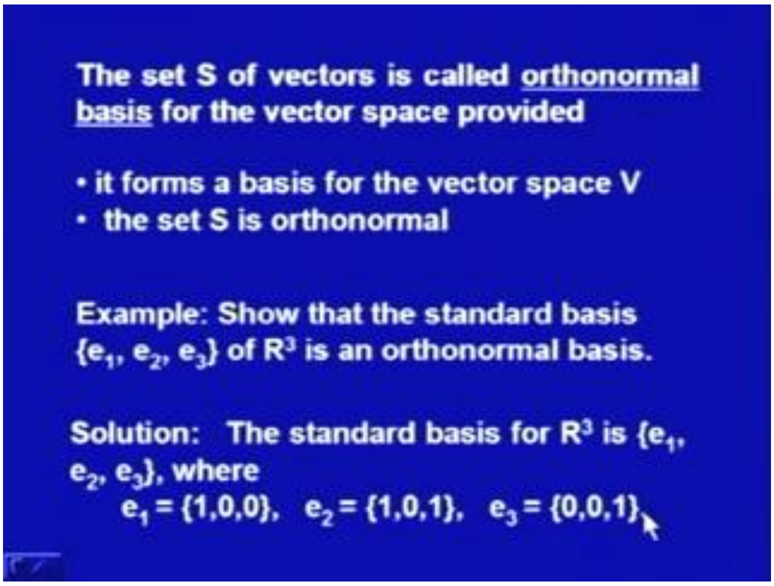

The set S of vectors is called orthonormal basis for the vector space, provided it satisfies two properties. One is, it forms a basis for the vector space V and the second is the set S orthonormal. Now, see, we are defining a vector space. We are defining a basis for a vector space. But, now we are interested in orthonormal basis for the vector space. So, to have an orthonormal basis for a vector space, first thing is that, it should form a basis for the vector space. And then it should have an additional property that the set is orthonormal.

Now, this is example, we have to show that the standard basis e 1, e 2 and e 3 of R 3 is an orthonormal basis. You have all ready seen that e 1, e 2, e 3 forms a basis for the vector space. Now, we have to show that these 3 vectors in the basis. They are actually orthonormal. So, e 1 is 1, 0 comma 0, e 2 is 1, 0, 1 and e 3 is 0, 0, 1.

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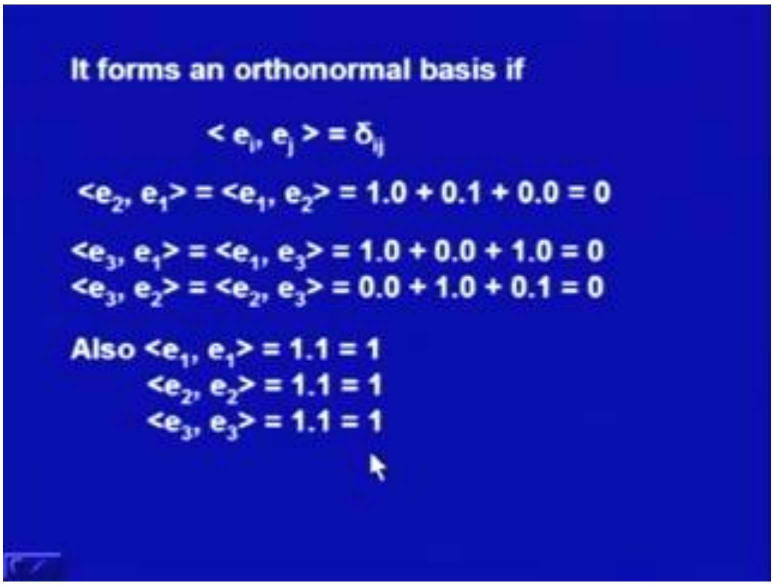

So, if you consider the product e i comma e j, this is equal to delta i j. You can check it for each pair e 2, e 1 or e 1, e 2 is equal to 1 comma multiplied by 0 plus 0 multiplied by one plus 0 multiplied by 0. That is 0. Similarly, e 3 comma e 1 or is the same as inner product of e 1 and e 3. This is also 0 and e 3 inner product of e 3 and e 2 is the same as inner product of e 2 and e 3. This also comes out to be 1.

And further norm of e 1 is inner product e 1 comma e 1 is 1 comma 1, 1 multiplied by one that comes out to be 1. Similarly, a norm of e 2 is also one norm of e 3 is equal to 1. So, we have shown that, e 1, e 2, e 3 forms a basis. They are orthogonal pairs they form orthogonal pars. And then each vector is of magnitude or is of norm one. So, this basis is orthonormal basis.

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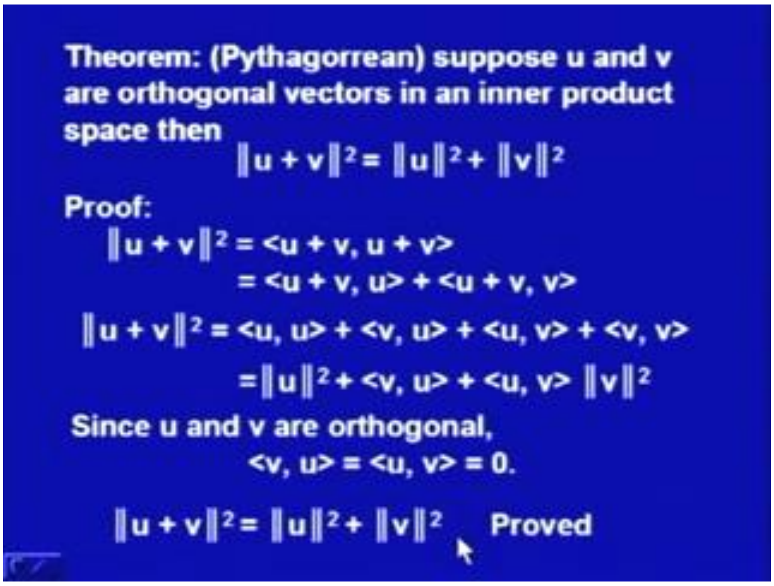

Now, we have some more results on inner products. First result is, that norm of u plus v square is equal to norm of u square plus norm of v square. Now, to prove this, we consider u plus v square is u plus v comma u plus v. So, we apply the property of inner products, u plus v comma u plus u plus v comma v. And that means u plus v square is equal to, again we apply the same property, here u comma u plus v comma u. Here, again we apply the same property u comma v plus v comma v.

And that means, this is nothing but norm of u square plus v comma u plus u comma v. And finally, plus v square and since u and v are orthogonal, this is being given to us. So, v comma u is 0 and u comma v is equal to 0 and that proves our result. That norm of u plus v square is equal to norm of u square plus norm of v square. So, this result is proved.

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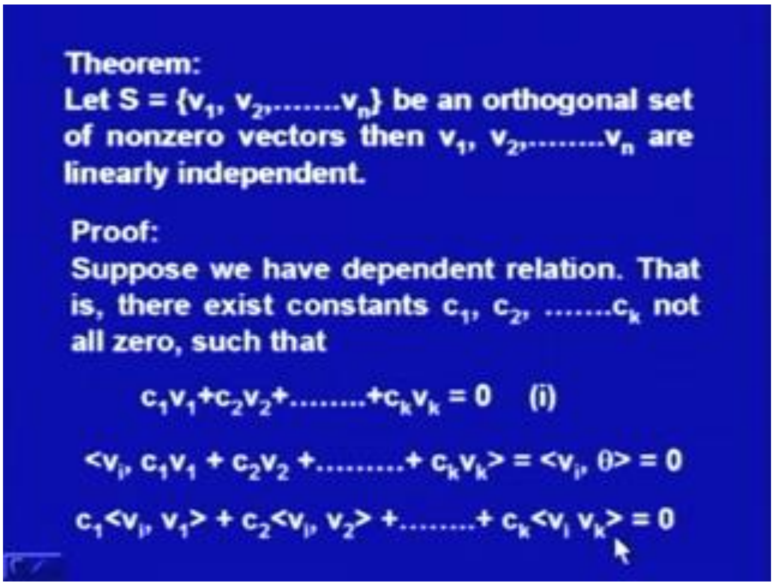

Now, we have an interesting result. It states that, if S consisting of n vectors as v 1, v 2, v n. And since, it is an orthogonal set of nonzero vectors. Then, v 1, v 2, v n are linearly independent. Now, this is an important result. Let us see, how we prove this. Suppose, we have dependent relationship between v 1, v 2, v n. That means, they are not independent, let us say, they are dependent vectors.

Then, there exist constant c 1, c 2, c n not all 0. Such that, c 1, v 1 plus c 2, v 2 plus c k, v k is equal to 0, there must be some k not necessary. The all n, may be independent, but some k vectors maybe linearly dependent. So, c 1, v 1 plus c 2, v 2 plus c k, v k is equal to 0. So, we can say here, that it is, if we take the inner product of this with respect to v i. So, I am taking inner product here.

So, on the right hand side is inner product v i comma 0, this is 0. So, that is not a problem. So, we will take v i comma c 1, v 1 plus c 2, v 2 plus c k, v k. So, this can be simplified as c 1 into v i comma v 1 plus c 2, v i comma v 2 plus c k, v i comma v k is equal to 0. Now, this is the property of inner products. That is why; we are taking c 1 out, so addition property and scalar multiplication property. Now, v i comma v 1, this is 0. If it is not 1, it is going to be 0, because they are orthogonal. Similarly, this is going to be 0. So, all these will be going to 0.

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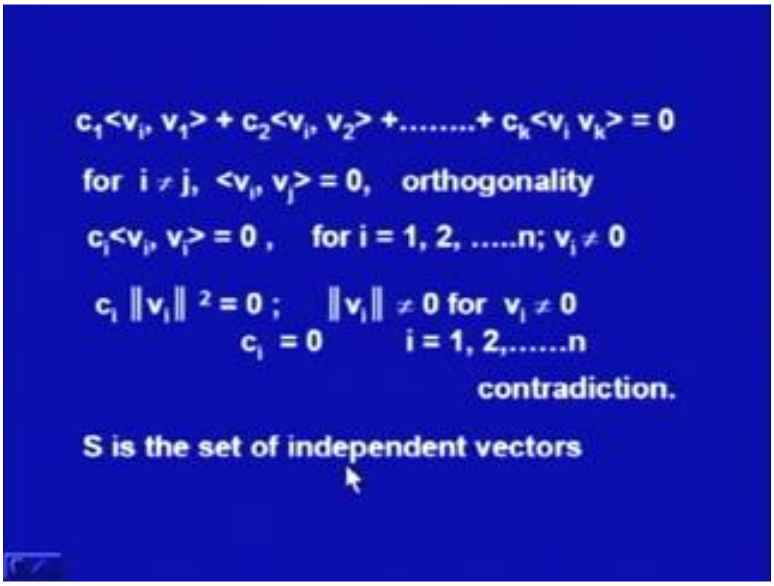

So, if i is not equal to j, inner product of v i and v j is 0. Due to orthogonality, that means, we have only c i multiplied by v i comma v i is 0. For all i is equal to 1, 2 to n and whatever i, you consider this is going to happen provided. And, we know that, v i comma v i is not 0. So, what we can say is, c i, v i is this is v i comma v i is v i square, norm of v i square and this is not 0. So, that means, c i is equal to 0 and this is the contradiction.

We have said that, there are nonzero c i is that is why a dependent relationship is formed. But, we have arrived with the contradiction. That simply means that, the vectors are nearly independent. So, that proves the result. So, S is the set of independent vectors. So, if the set if set S consisting of orthogonal vectors. Then, this set is actually the set of independent vectors. So, this is an important result.

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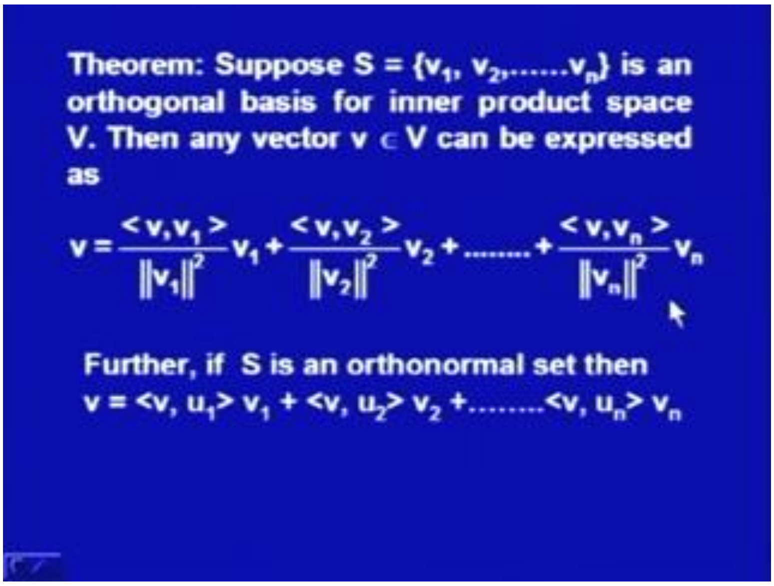

Now, another result, suppose S is equal to v 1, v 2, v n is an orthogonal basis for inner product space V. Then, any vector v belonging to V can be expressed as this. So, let us try to prove this, there is one more result in this. Further, if S is an orthogonal set. Then, orthonormal set, then all this will be 0, all this will be 1. So, it is nothing but v is equal to v comma u 1 into v 1 plus inner product of v comma u 2 multiplied by v 2 plus inner product of v comma un multiplied by v n. So, let us try to prove this.

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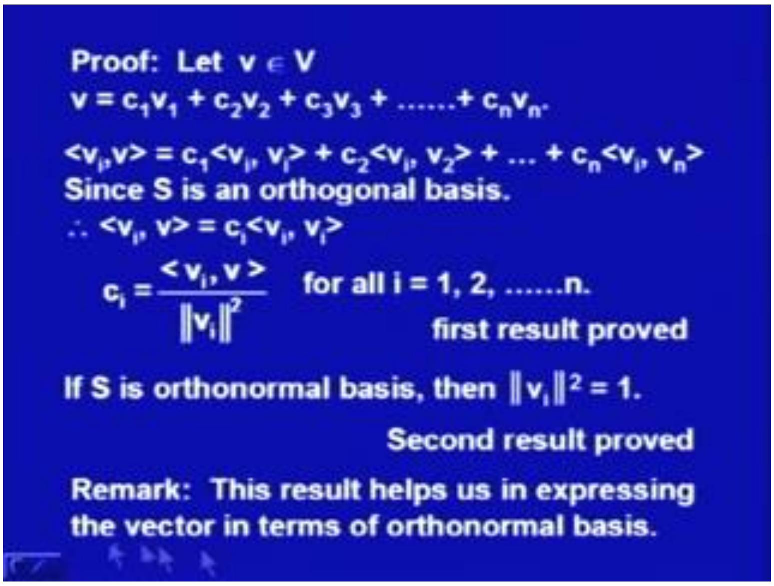

So, let us say v belongs to V. So, v is a vector belonging to v and v 1, v 2, v 3, v n. They form a basis. So, v can be expressed as a linear combination of this vector. So, v is equal to c 1, v 1 plus c 2, v 2 plus c 3, v 3 plus c n, v n. And that means, if we take inner product of this with respect to v i on both the sides. Then, v i comma v inner product of v i comma v is equal to c 1, v i comma v 1. The first term plus c 2 times v i comma v 2, the second term and the last term will be c n and inner product of v i comma v n.

Now, since S is an orthogonal basis. So, v i and v j is equal to 0 and v i comma v is equal to c i multiplied by inner product of v i comma v i rest of the terms will be not contributing anything. Only, the ith term in which, we are taking inner product of v i with v i only, that will be nonzero. That we multiplied by c i, but rest the terms will be 0. So, on the left hand side, we have v i comma v and on the right hand side, we have only single term c i multiplied by v i comma v i. But, this is nothing but norm of v i.

And that means, c i is equal to v i inner product of v i comma v divided by v i square. And this can be done for all i's. So, this we have done for v i, but this can be done for v 1, v 2, v n. So, this we can find all ci's and that means, our first result is proved. And of course, if S is orthonormal basis, then norm of v i square is equal to 1. For all i's and that is why; we have the second result. So, this is another interesting result. Now, why this is interesting, because this result helps us in expressing the vector in terms of orthonormal basis.

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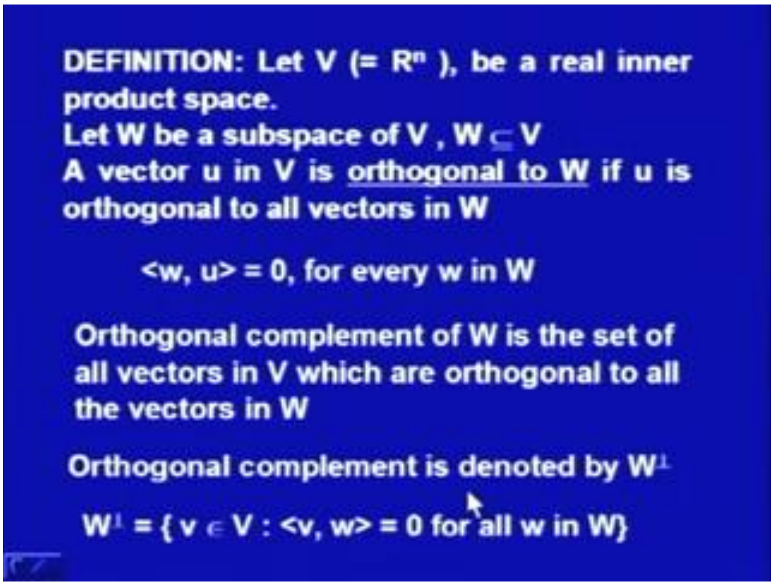

Now, we have another definition. Let V be a real inner product space. That means, we have a vector space and an inner product is defined on it. Let W be a subspace of V, then a vector u in v is orthogonal to W. If u is orthogonal to all vectors in W, see, we were talking about orthogonality between pair of vectors. Now, we are talking orthogonality of a vector with respect to a set of vectors. So, a vector u and v is orthogonal to W.

If the vector u is orthogonal to all vectors in W, that means inner product of w and u is 0, for every w in W. So, for this is true for all w's in this set W, if this is true, then we say the vector u is orthogonal to the set W. Then, we talk about orthogonal compliment of W is a set of all vectors in V, which are orthogonal to all the vectors in W. We will come back to this again and we say that orthogonal compliment is denoted by W perpendicular. We will come back to this again.

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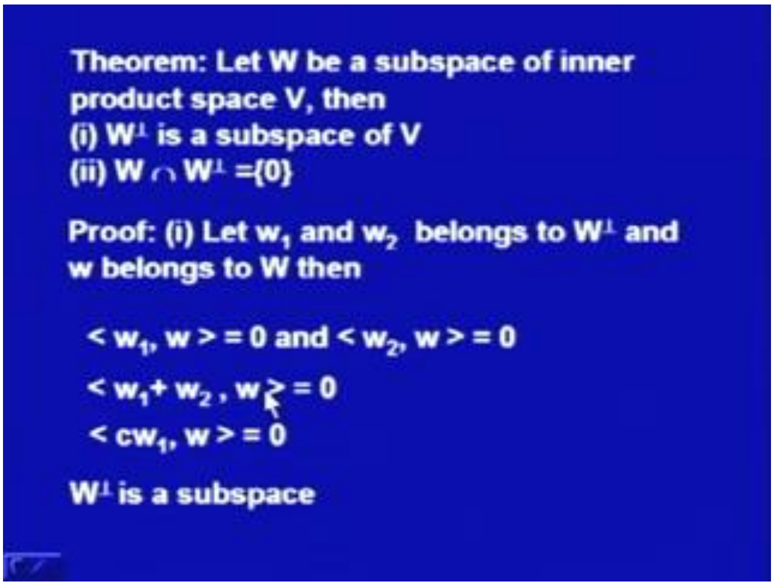

Now, let W be a subspace of inner product space V. Then, W perpendicular is a subspace of V and W intersection of W perpendicular is 0. This is really an interesting result. That W intersection W perpendicular is 0. Now, to prove these results, let us say w 1 and w 2 belongs to W perpendicular and w belongs to W. Then, w 1 comma w is 0 and w 2 comma w is equal to 0, because this belongs to W perpendicular.

So, W perpendicular is a set of all vectors, which are perpendicular to the vectors of W. So, that is the idea of W perpendicular. So, w 1 plus w 2 comma w, this will also be 0. And that means, this also belongs to W perpendicular. Further, the inner perpendicular w

1 comma w is equal to 0. That means, c times w 1 comma w, this inner product will also be 0.

So, we have sum of 2 vectors belongs to W perpendicular. And sum with a scalar product will also belong to W perpendicular. And that means, this is actually the definition of subspace. So, we say W perpendicular is a subspace of V.

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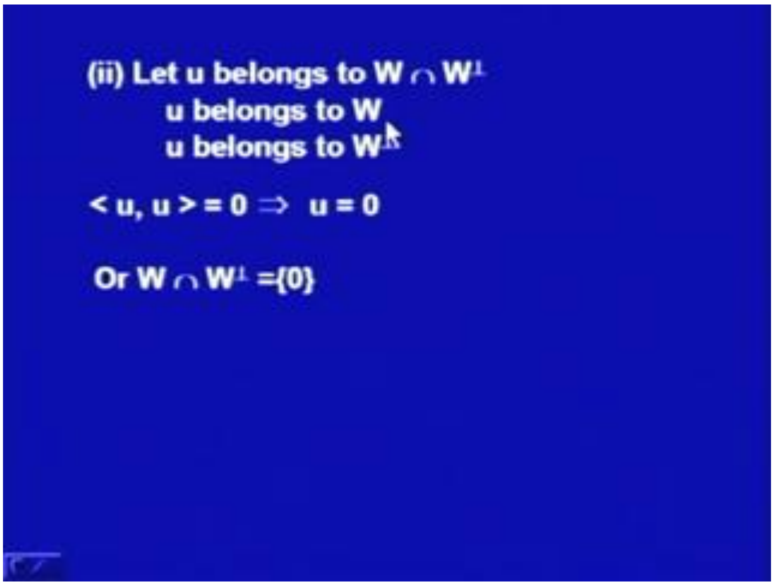

To prove this second property, let u belongs to both this W intersection W perpendicular. That means, u belongs to as W as u belongs to W perpendicular. And since, u belongs to W as well as W perpendicular. So, vectors of W and W perpendicular are orthogonal. So, we take a product; take a vector from W perpendicular. And then there inner product will be 0. So, we take u from W and we take u from W perpendicular. So, inner product of u comma u is 0 and that means, when this happens, when u is equal to 0. So, if u belongs to W intersection W perpendicular. Then, that vector has to be 0. So, that simply means that, W intersection W perpendicular is nothing but the 0 vector. Now, we discuss Gram Schmidt process for orthonormalization.

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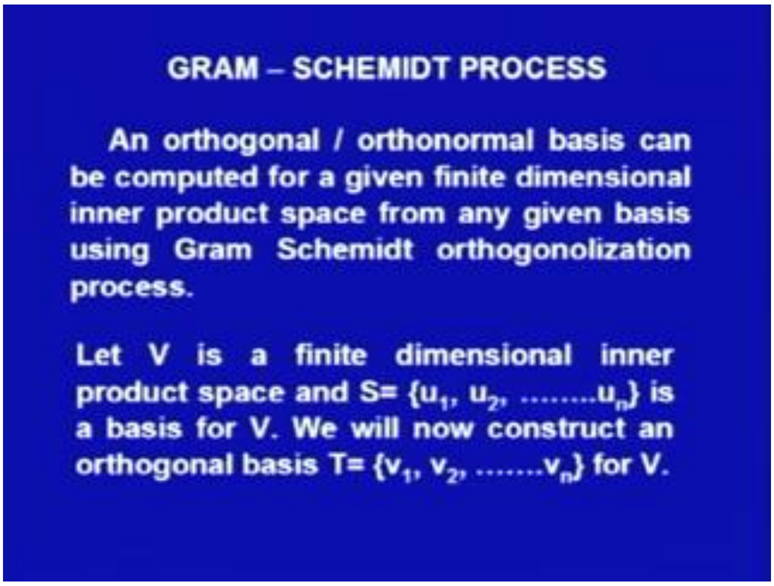

Let us say a vector space V is given to us. It is a finite dimension inner product space and we have been given a basis S consisting of n vectors u 1, u 2, u n. Then, with the help of this process, we can obtain another basis v 1, v 2, v n for the inner product space V, which is orthogonal. And then from that orthogonal basis, we can divide each vector with it is norm and we can have an orthonormal basis. So, we will first describe this all algorithm.

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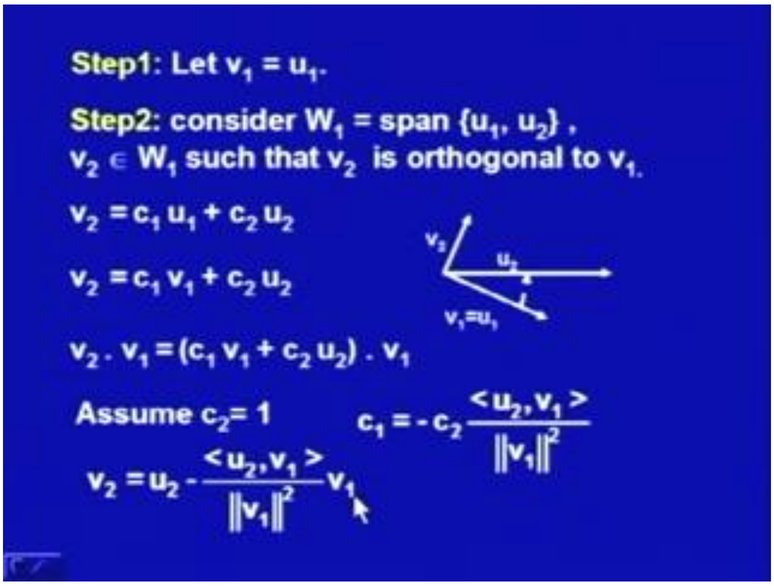

So, the step 1 is, we consider any vector u 1 from the set S. And let us call this vector as V 1. In the second step, we consider another vector u 2 from the set S and consider span of these 2 vectors. And let us call that span as w 1. Now, we consider a vector v 2 in the span w 1v, such that, v 2 is orthogonal to v 1. So, if we can get this, then v 2 is a linear combination of c 1, u 1 plus c 2, u 2, u 1 is nothing but v 1.

So, we have this v 2 is equal to c 1, v 1 plus c 2, u 2. Then, we take the inner product with v 1 and we try to check that, v 2 v 1 comes out to be 0. So, this equation is solved by equating it to 0. And this is what the geometrical interpretation is, we have u 1. We have u 2 vector, they are not necessarily orthogonal. But, from this linear combination of u 1 and u 2, we can find out a vector v 2 which is orthogonal to u 1. So, this equation helps us in obtaining v 2.

So, what we do is, we solve this c 1 is equal to minus c 2 inner product of u 2, v 1 divided by v 1 square. And if we take c 2 is equal to 1. Then, this vector v 2 can be obtained as u 2, u 2 minus inner product u 2, v 1 divided by v 1 square into v 1. So, we have obtained vector v 2, which is perpendicular to v 1. So, for given 2 vectors, we have orthogonal 2 vectors.

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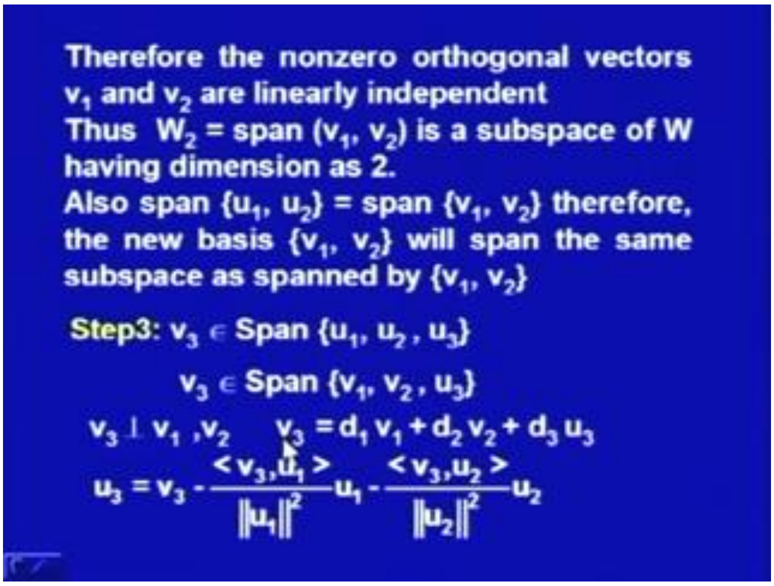

Now, therefore, the nonzero orthogonal vectors v 1 and v 2 are linearly independent. This we have just now proved. Now, the span of v 1, v 2 is a subspace of W having dimension 2, because these vectors are independent. Further, span of u 1, u 2 is a same as span of v 1, v 2. Therefore, the new basis v 1, v 2 will span the same subspace as spanned by v 1, and v 2. So, we have got 2 vectors. Now, we try to extend this to 3 vectors for this similar procedure is applied.

We consider span of 3 vectors u 1, u 2, u 3 from the given set S and we consider v 3 in this. And we apply the same philosophy, what we have done in earlier step. So, v 3 belongs to span of v 1, v 2, u 3. And we choose v 3 in such a manner, that it is perpendicular to v 1 and v 2. So, we have two equations. We have this equation v 3 is equal to d 1, v 1 plus d 2, v 2 plus d 3, u 3 and we apply to it, this concept.

And, this will give us u 3 as v 3 minus v 3, u 1 inner product of this divided by u 1 square multiplied by u 1 minus inner product of v 3, u 2 divided by u 2 square into u 2. So, one can obtain this equation with the help of this equation. So, we got third vector.

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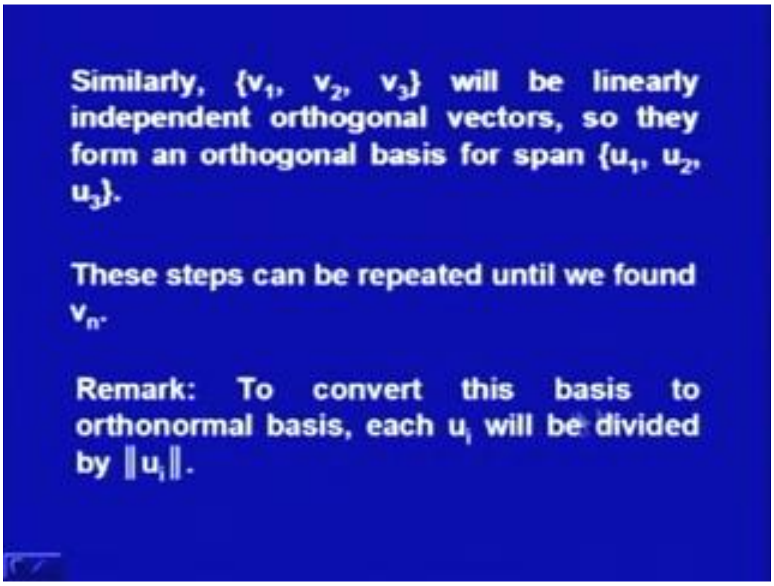

Similarly, v 1, v 2, v 3 will be linearly independent orthogonal vectors. So, that they form in orthogonal basis for span of u 1, u 2, u 3. Similarly, we can apply more number of steps till we get all vectors v n. And to convert this basis to orthonormal basis, each vector u i will be divided by it is norm.

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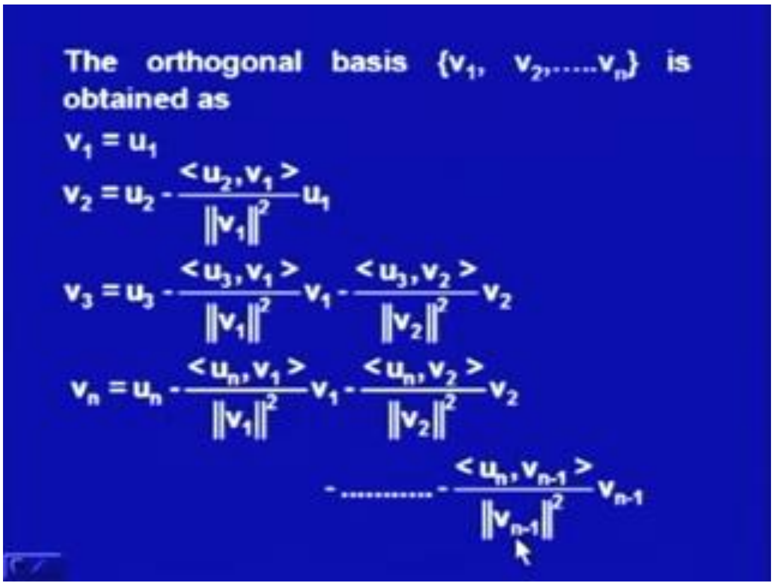

So, this way, we will get the orthonormal basis and the formula is first from u 1. We will get v 1, u 2. After getting v 1, we will apply this formula to get v 2. This left hand side involves u 2 and v 1. Then, we will have v 3, v 3 is obtained in terms of v 1, v 2, u 1, u 2 and u 3.

So, this is known on the left hand side. So, from here we get v 3 and we can go on applying this. This is general expression for v n. And then we can orthonormalize it and then we illustrate this procedure with an example.

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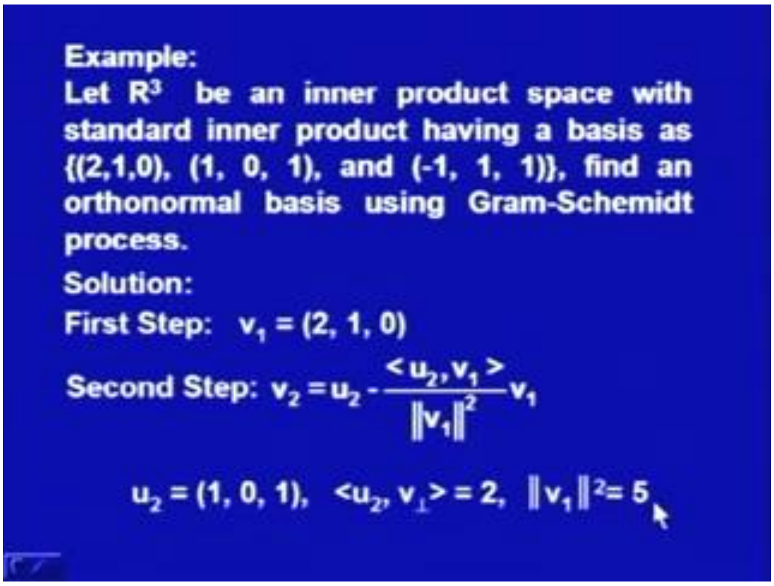

So, we consider R 3 instead of R n consisting of this 3 vectors 2, 1, 0, 1, 0, 1 and minus 1, 1 as a basis for R 3. We have to find an orthonormal basis using Gram Schmidt process. So, what we have done is, we first consider u 1 as 2, 1 and 0. So, u 1 is 2, 1 and 0, so same as v 1. The seconds step gives us this formula. So, we consider u 2 as this vector 1, 0, 1. We can compute, we can substitute the values u 2 is 1, 0, 1, u 2 comma v 2 is 2 and v 1 square comes out to be 5. We substitute the values here and what we get is v 2.

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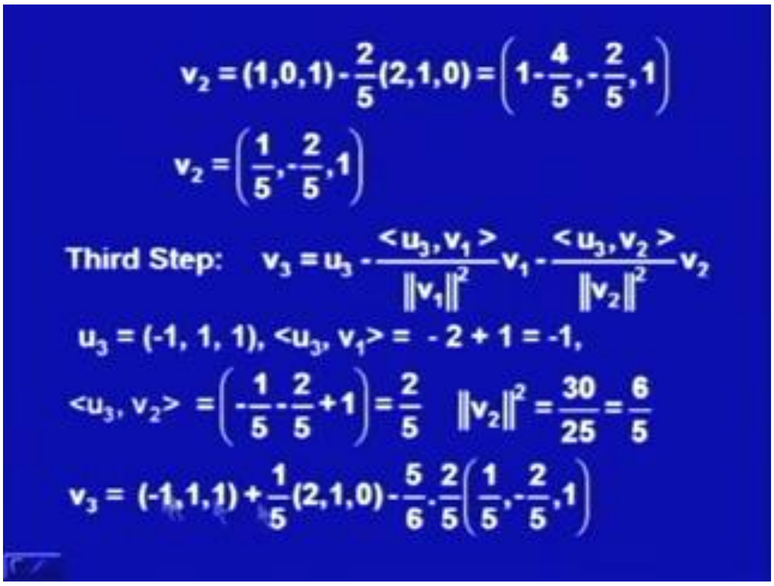

And v 2 is obtained as this 1, 0, 1 minus 2 by 5, 2, 1, 0 is this vector. So, that is how, we get v 2 and the third step is this formula. We have all ready obtained. We substitute the values u 3 as minus 1, 1. The third vector from the given set, we compute u 3 comma v 1 as minus v 1. We compute u 3 comma v 2, which comes out to be 2 by 5 and then v 2 square is computed as 6 by 5.

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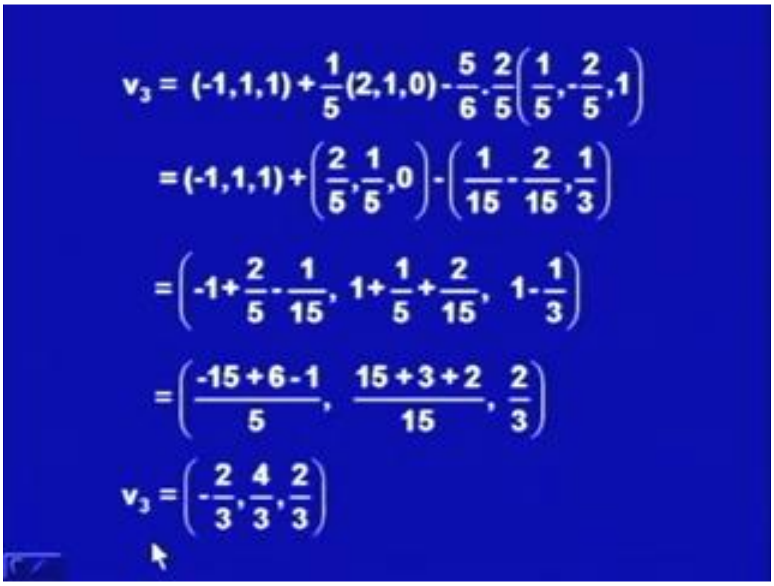

These values are substituted here and what we get is the vector v 3, which is finally obtained as through series of expressions. We finally arrive at v 3 as minus 2 by 3, 4 by 3 and 2 by 3. So, all the 3 vectors are obtained.

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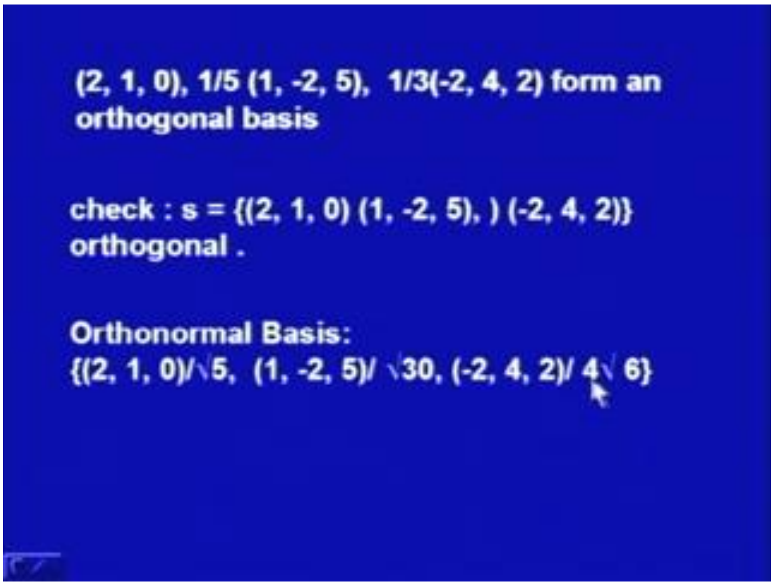

Now, 2, 1, 0, 1 by 5, 1 minus 2, 5 and 1 by 3 minus 2, 4, 2 form an orthogonal basis. And if you have to obtain the orthonormal basis, then we simply divide these vectors by their norm. So, this is 1 minus 2, 5 is divided by under root 30. It is norm minus 2, 4 comma 2 is divided by it is norm. So, this is the orthonormal basis, one can check that this is orthogonal set and this is orthonormal set.

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So, that is how, we apply Gram Schmidt process to arrive at the orthonormal basis from the given basis. So, today viewer at the end, I will summarize, what I have done today, I have given the definition for inner product. I have defined inner product spaces, the norm distances, we have taken examples to illustrate this and we have taken. So, we have discussed some properties of inner products. We have discussed orthogonal vectors orthonormal vectors. At the end, we have discussed Gram Schmidt process for orthonormalization, which I have illustrated with the help of example.

Thank you.