Mathematics-II
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Lecture - 9
Linear Algebra Part – 04

Welcome viewers, this is 4th lecture on Linear Algebra. In earlier 3 lectures, we have discussed vector spaces, sub spaces and basis.

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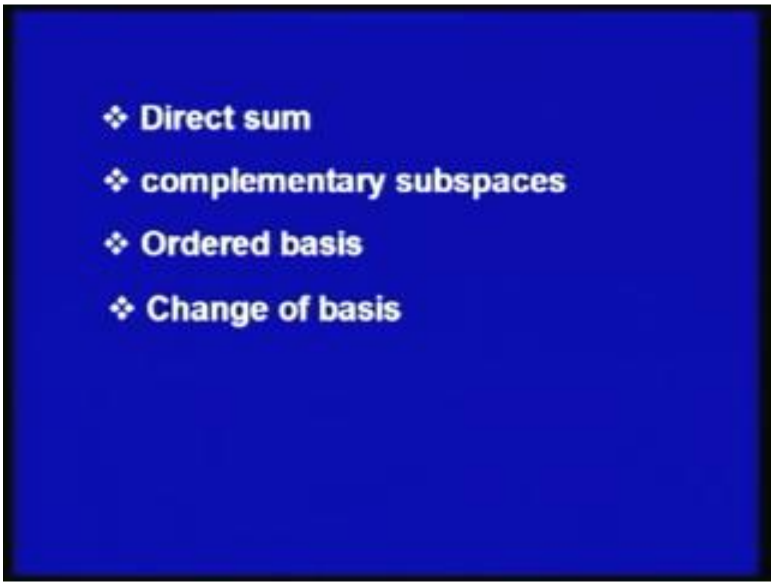

Today’s lecture has direct sum, complementary subspaces, ordered basis and change of basis.

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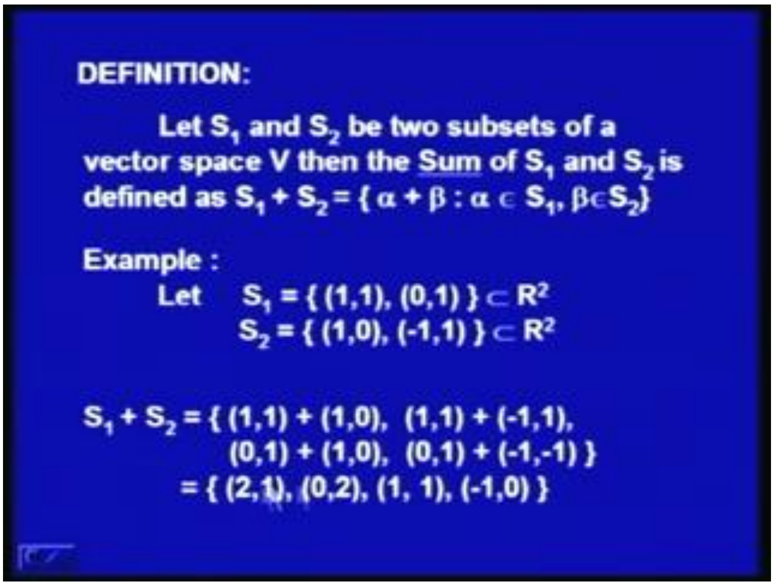

I will start with few definitions. The first is, the sum of two subsets S 1 and S 2 of a vector space V. So, if we have two subsets S 1 and S 2 of a vector space V. Then, the sum of S 1 and S 2 is defined as, S 1 plus S 2 equal to, alpha plus beta, such that, alpha belongs to S 1 and beta belongs to S 2. So, the set S 1 plus S 2 consist of all vectors of the form alpha plus beta in this vector space V, such that, alpha belong to S 1 and beta belongs to S 2.

For example, if we consider S 1 as a set consisting of 1 and 1 comma 0 and 1. Clearly, this is the subset of R 2 and another subset S 2 consisting of 2 vectors 1, 0 and minus 1, 1. This also a subset of R 2, one may notice that R 2 is a vector space. Now, the sum S 1 plus S 2, consist of sum of these two sets S 1 and S 2. So, to form a sum, I have to take of vector from this and a vector from this set S 2.

So, 1 comma 1 from S 1 and 1, 0 from S 2, then 1 comma 1 from S 1 and minus 1 comma 1 from S 2, 0 1 from S 1 and 1 0 from S 2. Finally, 0, 1 and minus 1, minus 1, you simplify it, it is 2, 1; 0, 2; 1, 1 and minus 1, 0. So, S 1 plus S 2 consist of the vectors of this.

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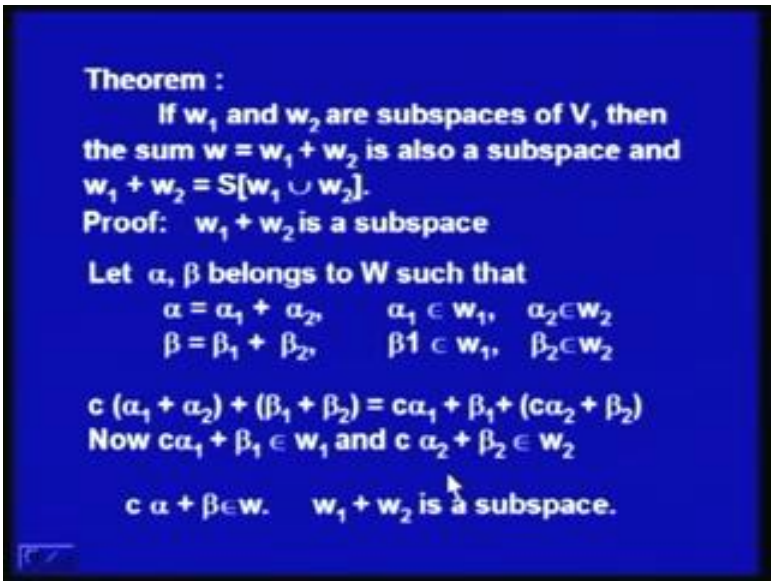

Then, the next is, the theorem, if w 1 and w 2 are subspaces of V. Then, the sum w is equal to w 1 plus w 2, is also subspace and w 1 plus w 2 is equal to span of w 1 union w 2. Now, to prove this, I will first prove, that w is equal to w 1 plus w 2. So, to prove this, let alpha beta belongs to W. Then, since W is w 1 plus w 2. So, this alpha can be written as, alpha 1 plus alpha 2, alpha 1 belonging to w 1 and alpha 2 belonging w 2.

Beta, also belongs to w, so this can also written as beta 1 plus beta 2, beta 1 belonging to w 1, beta 2 belonging w 2. So, their linear combinations, c alpha 1 plus alpha 2, plus beta 1 plus beta 2, must also belong to w 1 plus w 2, only then w 1 plus w 2 is a subspace. So, I write c times, alpha 1 plus alpha 2, plus beta 1 plus beta 2 is equal to simplifying it, it is c alpha 1 plus beta 1, plus c alpha 2 plus beta 2. But, alpha 1 belongs to w 1, beta 1, belong to w 1 and w 1 is a subspace.

So, c alpha 1 plus beta 1, belongs to w 1 and c alpha 2 plus beta 2, linear combination of 2 vectors of w 2, w 2 being a subspace. So, this also belongs to w 2. So, what we can say is, that c alpha plus beta also belongs to w. Because, the right hand side can be written as, sum of vectors one from w 1, another from w 2. And this proofs that, w 1 plus w 2 is a subspace.

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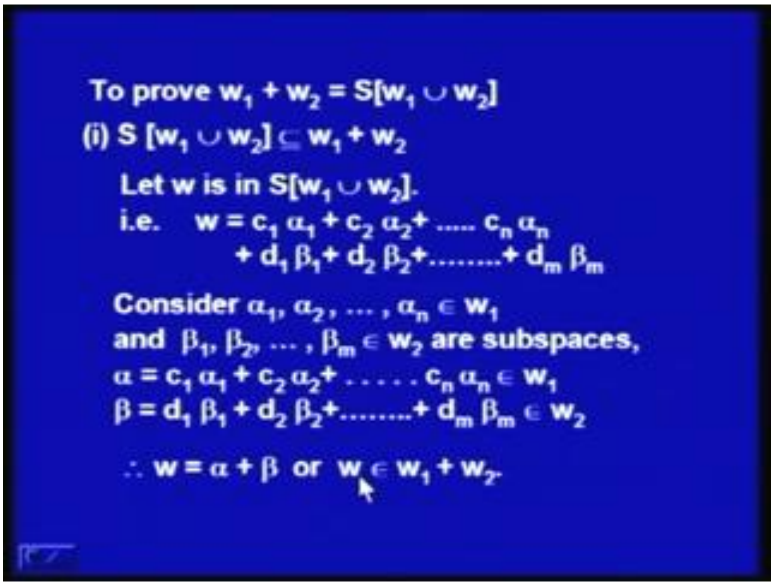

Now, the next part, we have to prove that, w 1 plus w 2 is equal to span of w 1 union w 2. So, this again, I prove in two parts. First, I will show that, S span of w 1 union w 2 is containing w 1 plus w 2. And later on, I will show that, w 1 plus w 2 is a subset of span of w 1 union w 2 and that proves, this equality. So, to prove the first part, I consider a span of w 1 union w 2. So, let us say, we have of vector w, in span of w 1 union w 2. Then, this can be written as a linear combination of vectors of w 1 union w 2.

So, what are the vectors of w 1 union w 2? They will be vectors belonging w 1 and vectors belonging to w 2. So, I write w as, c 1 alpha 1 plus c 2 alpha 2 plus c n alpha n, plus d 1 beta 1 plus d 2 beta 2 plus d m beta m. The idea is, that alpha 1 alpha 2 alpha n, they belongs to w 1 and remaining vectors, they belong to w 2. So, if these belong to w 1, then beta 1, beta 2, beta m, they belong to w 2, because they are from the set, w 1 union w 2.

So, now, we write down this part, which is consisting of vectors of w 1. So, linear combination of vectors of w 1, so c 1 alpha 1, plus c 2 alpha 2, c n alpha n must belongs to w 1. Because, linear combination also belongs to w 1, being a subspace. Similarly, this is a linear combination of vectors of w 2. So, w 2 being a subspace, so beta equal to d 1 beta 1, plus d 2 beta 2, plus d m beta m, must belong to w 2.

And that means, w can be expressed as, a vector from the set w 1 and the vector from the set w 2. And that means w is equal to; that means w belongs to w 1 plus w 2.

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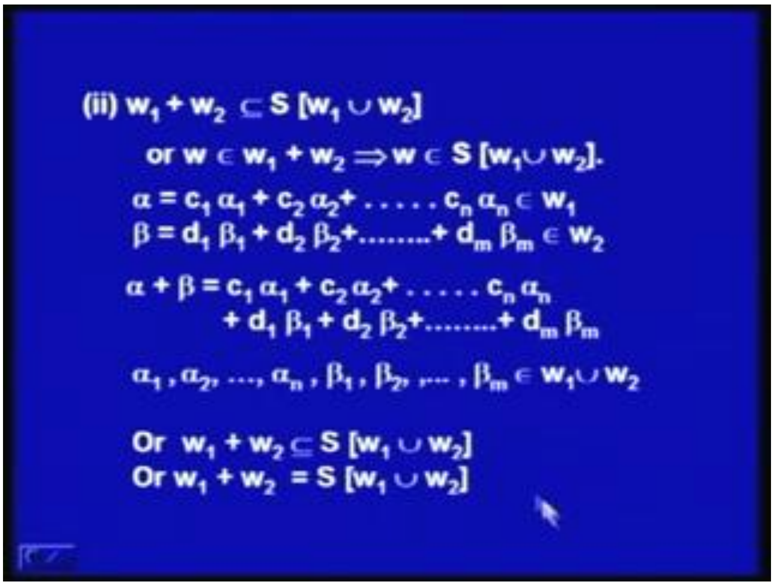

So, first part is proved. The second part is, that any vector in this set w 1 plus w 2, is also in the set, which is the span of w 1 union w 2. So, if w belongs to w 1 plus w 2, then we proved that, this also belongs to this. And that means, this is and w 1 plus w 2 is a subset of S w 1 union w 2. So, I start with, w belonging w 1 plus w 2. Since, w 1 plus w 2, this implies w belongs to span of w 1 union w 2.

Now, alpha is linear combination of these vectors and they belong to w 1. Beta is linear combination of vectors of w 2. So, they also belong to w 2, because they are subspaces. And then alpha plus beta, is c 1 alpha 1 plus c 2 alpha 2 plus c n alpha n and d 1 beta 1 plus d 2 beta 2, plus d m beta m. So, the sum of these two is alpha plus beta.

Now, what is this, this is a linear combination of vectors of w 1 union w 2. So, definitely this must be in the span of S, w 1 union w 2. So, w 1 plus w 2 is a subset of, span of w 1 union w 2 or combining the two things, what we can say is that w 1 plus w 2 is equal to span of w 1 union w 2.

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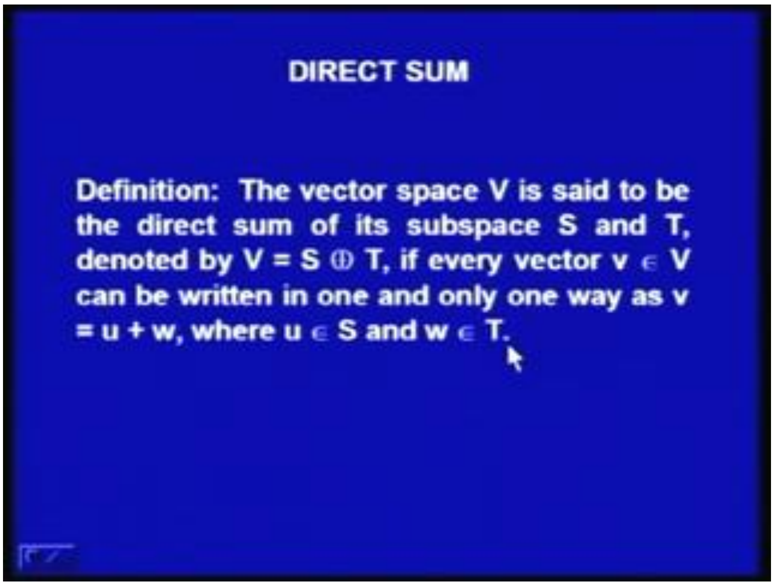

Now, we define direct sum, the vector space V is, said to be the direct sum of it is subspaces S and T and we denoted by V is equal to S plus T. Now, here we are using a different symbol, then plus just to differentiate between the sum and the direct sum. So, V is equal to S plus T. If every vector V belonging to V, can be written in only one way, as V is equal to u plus w, where, u belongs to S and w belongs to T.

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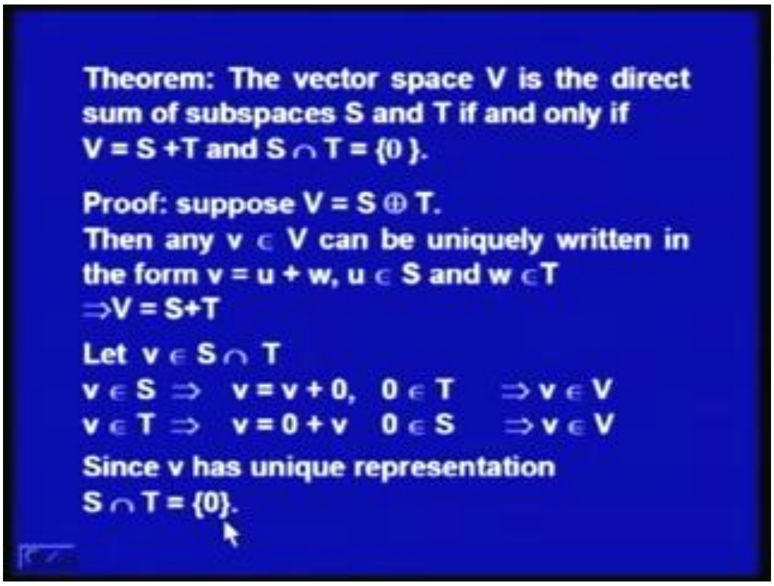

Now, we state theorem, according to this, the vector space V is the direct sum of subspaces S and T. If and only if V is equal to S plus T and S intersection T is equal to null vector 0. Now, the basic difference between direct sum and sum is that, while we can express V as S plus T, sum of S and T, where we have, this representation is unique. And this is guaranteed by, S intersection T is equal to 0. So, let us try to prove this.

Suppose, V is equal to S plus T, then any vector v belonging to V, can be uniquely written in the form, v is equal to u plus w, u belonging S and w belonging to T. That means, V is equal to S plus T. Now, we have to show that, if it is uniquely represented by the definition of direct sum. Then, this means, S intersection T is equal to 0. So, if v belongs to S, then we can write down v is equal to v plus 0, 0 belonging to T and v belonging to S.

So, now it is a sum of 2 vectors, one from S, another from T, so v belongs to V. Similarly v belongs to T, then we can write down v is equal to 0 plus v, is a null vector from the set S and v in T. So, this v will also belongs to V. So, now, we have two representations for the same vector in V. Now, this is possible, only when S intersection T is equal to 0. Since, v has unique representation. So, this has to be 0, otherwise, will be having two different representations, against our assumption that it is a direct sum. So, S intersection T has to be a null vector.

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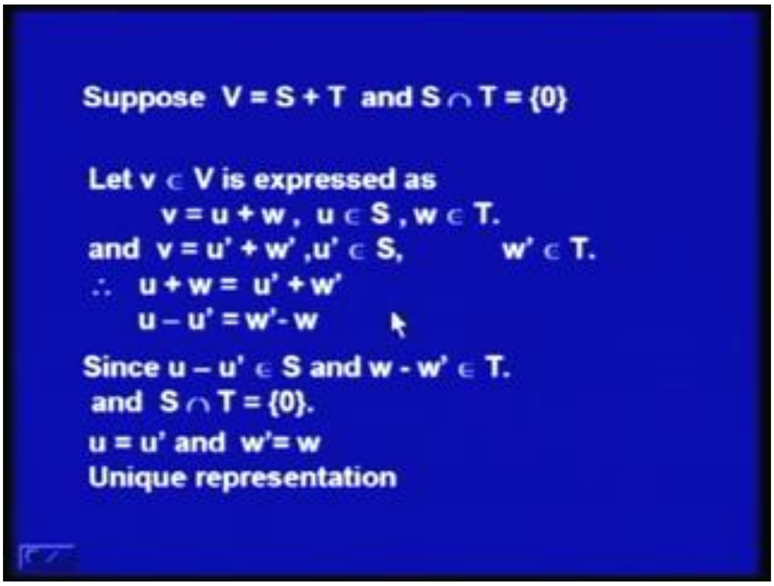

Now, in the second case, we assume that, v is equal to S plus T and S intersection T is equal to 0. Then, we say that, this representation will be unique and V, this plus will now be, not be the sum, but the direct sum. So, we start with this and we proved that V is the direct sum of S plus T. Now, to do this, let v belongs to V is expressed as v is equal to u plus w, u belonging to S and w belonging to T. At the same time, the same vector v, can also be expressed as u dash plus w dash, that u dash belonging to S and w dash belonging T.

So, since the 2 vectors are same. So, I can write down, u plus w is equal to u dash plus w dash or I can simplify, u minus u dash is equal to w dash minus w. Now, u minus u dash belongs to S and w dash minus w belongs to T and they are equal. Now, since, S intersection T is equal to 0 vectors. So, this, they are equal only when, they are, u minus u dash is 0 and w dash minus w is equal to 0. And that means, S intersection T is equal to 0.

So, since u minus u dash is 0, implies u is equal to u dash. And w dash minus w is equal to 0 implies, w dash is equal to w. So, we have started with two different representations for the same vector. By ultimately, we arrive at the unique representation. And that means, V is equal to S plus T and S intersection T is equal to null vector. Actually, means that a vector can be expressed uniquely in V. And that means, that it is not sum, it is the direct sum.

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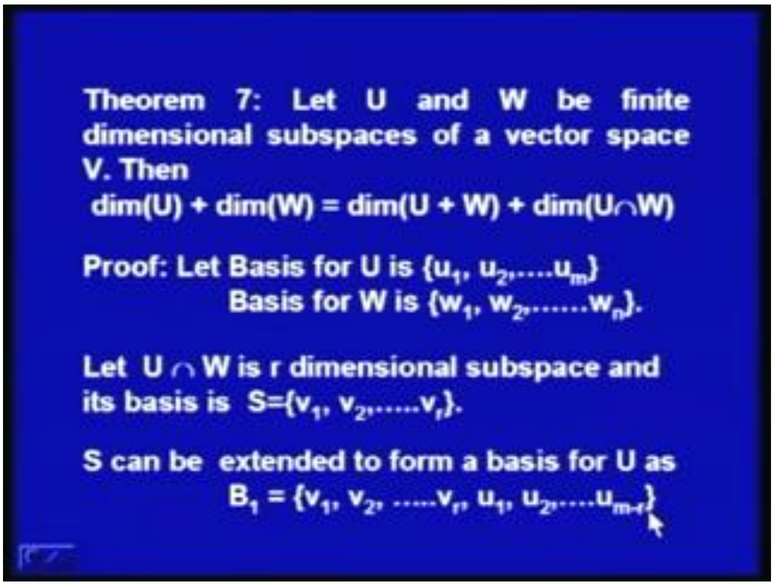

So, this can be use as a definition for direct sum also. The next theorem says, that if, U and W, be finite dimensional subspaces of a vector space V. Then, dimension of U, plus dimension of W is equal to dimension of U plus W plus dimension of U intersection W.

This is an important result. Let us try to prove this. Now, U and W, be finite dimension subspaces. So, let us say basis for U is m dimensional and it is u 1, u 2, u m and basis for W is w 1, w 2, w n, n being the dimension for W.

Let, U intersection W is r dimensional subspace. And it is basis is S is equal to v 1, v 2, v r, it sum only. So, we can consider U intersection W as, finite r dimensional subspace and let us say, it is basis is v 1, v 2, v r. Now, since it is intersection. So, these vectors belong to U also. This is basis, so these vectors will be independent. These vectors belong to w also. So, let us say we can extend this basis to form a basis for U, but U is m dimensional. So, let us add m minus r vectors in B 1. So, B 1 becomes v 1, v 2, v r; u 1, u 2 u minus r. So, the total numbers of vectors are m. So, we have a basis for u.

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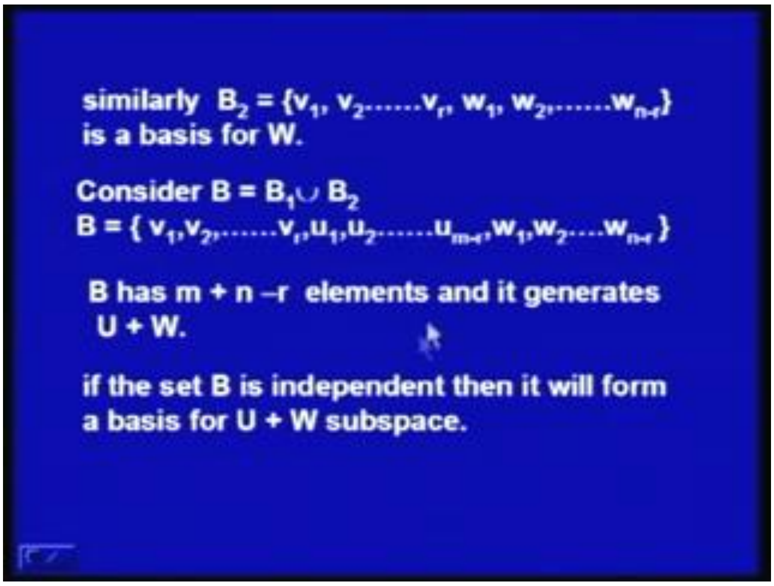

Similarly, we can form of basis for W; we call it the B 2. So, we have same basis v 1, v 2, v r and then we add n minus r vectors, independent vectors. So, that this, forms a basis for W. Then, we consider, B is equal to B 1 union B 2. So, we have two bases. So, we consider B is equal to B 1 union B 2. So, how many vectors will be there, we will be having v 1, v 2, v r vectors. Then, u 1, u 2 up to u minus r and w 1, w 2, n minus r.

So, B can the union of B 1 union B 2 will be the set. And it will definitely have m plus n minus r elements and it generates U plus W. If the set B is independent, then it will form a basis for U plus W subspace. So, we have to prove that, this has independent vectors.

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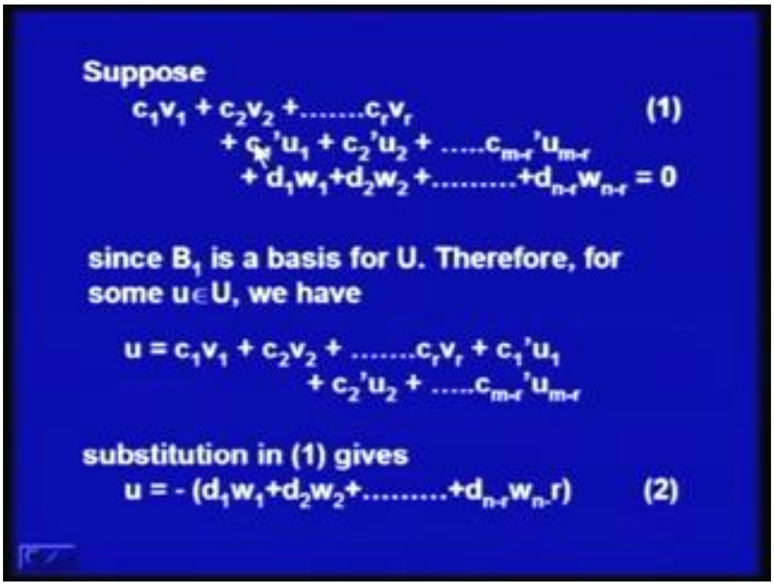

So, for this, we consider the linear combination of these vectors. So, the combination is c 1 v 1 plus c 2 v 2 plus c r v r. Then, we have, plus c 1 dash u 1, plus c 2 dash u 2, plus c m minus r dash minus r, plus d 1 w 1 plus d 2 w 2. And the last is d n minus r, w n minus r equal to 0. So, we are form the linear combination of vectors of B. Since, B 1 is a basis for U.

Therefore, for some u belonging to U, we have u is equal to c 1 v 1, plus c 2 v 2, plus c r v r. This plus, this c 1 dash u 1, plus c 2 dash c 2, plus c m minus r dash m minus r. So, this will be a vector belonging to u. Similarly, if we substitute this, then we can write down, u is equal to minus d 1 w 1, plus d 2 w 2 plus d n minus r w n minus r. So, this is a vector, which is u and this I have taken on the other side.

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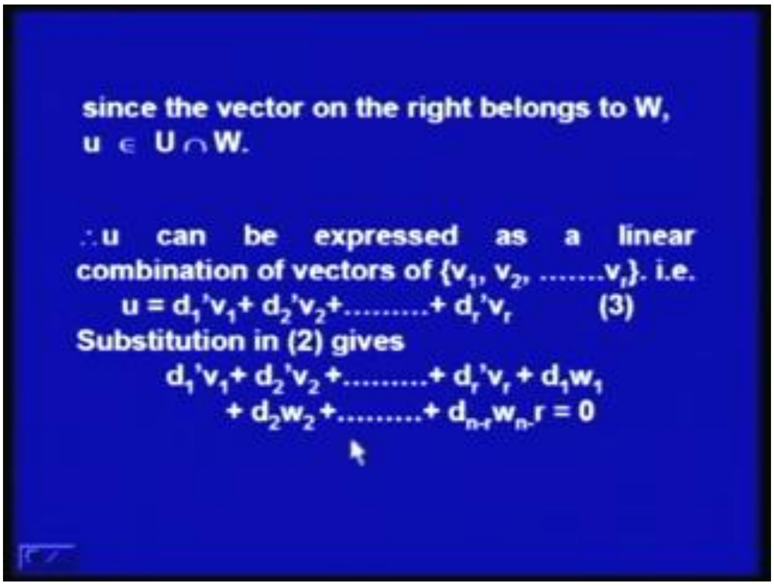

Now, since the vector on the right belong to W. So, u belonging to U intersects W. That is, u can be expressed as a linear combination of vectors of v 1, v 2, v r. That is, u is equal to d 1 dash v 1 plus d 2 dash v 2 plus d r dash v r. Substitution in 2 gives as d 1 dash v 1 plus d 2 dash v 2 plus d r dash v r plus d 1 w 1 plus d 2 w 2 plus d n minus r, w n minus r is equal to 0.

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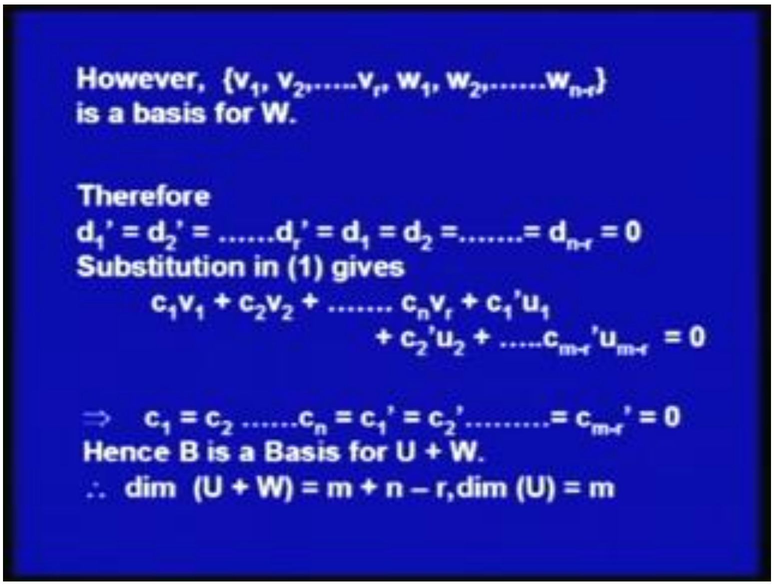

However, v 1, v 2 ,v r, w 1, w 2, w n minus r is a basis for W and therefore, d 1 dash, d 2 dash, d r dash is equal to d 1, is equal to d 2, is equal to d n minus r has to be 0. And substitution in 1 gives c 1, v 1, plus c 2, v 2, plus c 1 dash u 1, plus c 2 dash u 2, plus c m minus r dash, u m minus r is equal to 0. And since, this forms a basis for the set. Then, c 1 is equal to, c 2 is equal to c n, is equal to c 1 dash, is equal c 2 dash. That means, all coefficients are 0 and that means, all the vectors are linearly independent. Hence, B is a basis for U plus W, this means dimension of U plus W is equal to, m plus n minus r, where dimension of U is equal to m and dimension of W is equal to n.

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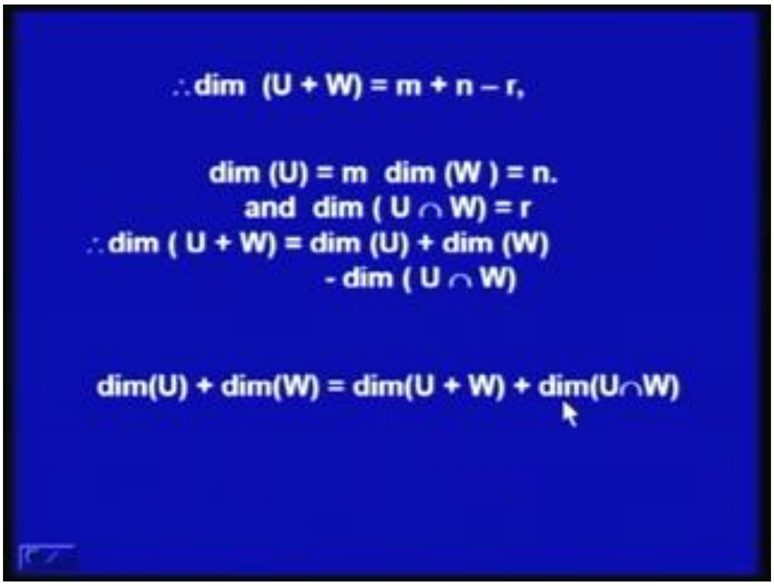

So, starting with dimensions of U plus W is equal to m plus n minus r and dimension of U is equal to m and dimension of W is equal to n. Further, dimension of U intersection W is equal to r. So, if we substitute these things, in this result, then will have dimension of U plus W is equal to dimension of U, plus dimension of W, minus dimension of U intersection W. And this proof in the result, the dimension U plus dimension of W is equal to dimension of U plus W, plus dimension of U intersection W.

We have taken this on the other side, this is an important result. And that will be useful, in many different ways. Now, we take up some example. So, let us consider two subspaces of r 4. Now, r 4 is a vector space and it has dimension 4. So, let us consider V as a, b, c, d, such that.

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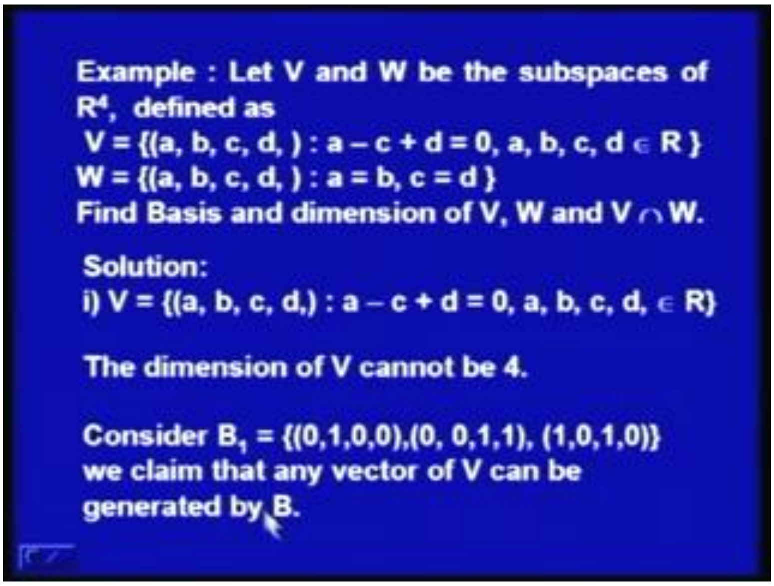

They follow a constraint a minus c plus d is equal to 0, where a, b, c, d they are real numbers. The other subspace W is equal to a, b, c, d, following this constraint a is equal to b and c is equal to d. So, the question is, find the basis and dimension of V, W and V intersection W. So, we start with the solution, clearly the dimension of V cannot be 4. So, we start with a set B 1, consisting of 0, 1, 0, 0 and 0, 0, 1, 1;1, 0, 1, 0, a set of 3 vectors. We claim that, any vector of V can be generated by B, so this is a span for V.

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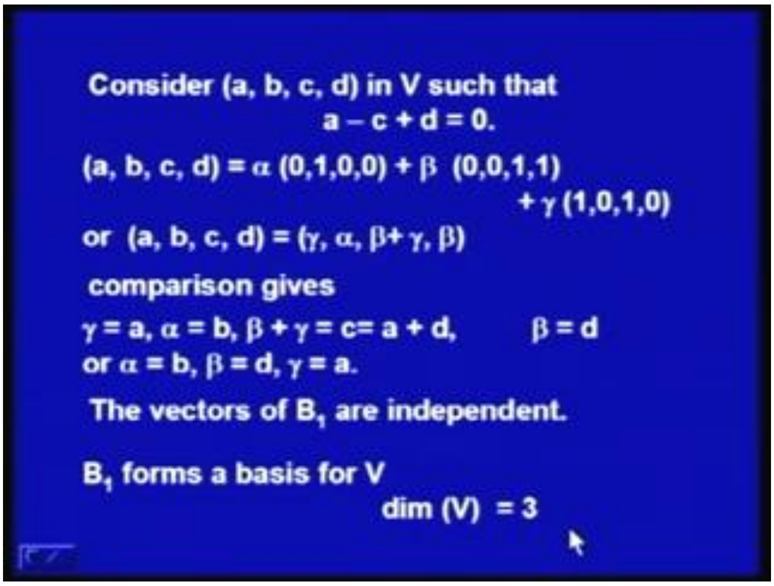

So, for this purpose, we consider a, b, c, d in V, satisfying the given constraint, a minus c plus d equal to 0. So, we express this vectors a, b, c, d in V, as linear combination of vectors of the given set. So, it is alpha times 0, 1, 0, 0 first vector plus beta times, second vector plus gamma times, third vector. So, if we combine these terms, then a, b, c, d is equal to gamma, plus alpha, beta plus gamma and finally, beta.

Now, comparison gives, gamma is equal to a, alpha is equal to b, c is equal to beta plus gamma and beta is equal to d. Now, c is also a plus d from this constraint. So, at combine this constrained and what I am getting from here and if we solve them, then alpha is equal to b, beta is equal to d and gamma is equal to a. So, these are the three coefficients, alpha, beta and gamma. So, I have solved them using this. So, any vector a, b, c, d can be expressed in this form.

Now, the vectors of B 1 are independent. That can be checked and since the vectors of B 1 spans, that the spans, this subspace V. Therefore, we say that, B 1 forms a basis for V. And since, there are three independent vectors in B 1. So, dimension of V is 3.

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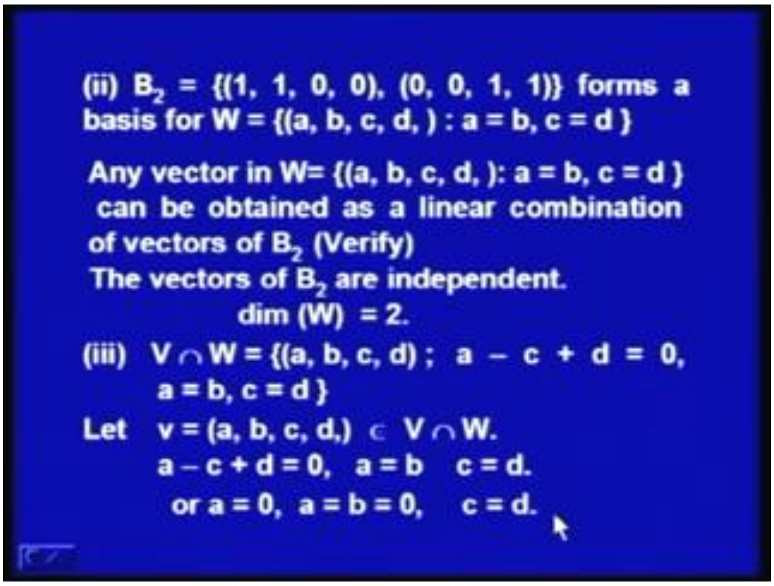

Now, we consider another subspace W, satisfying this constraint a, is equal to b and c is equal to d. So, we start with a set B 2, consisting of 1, 1, 0, 0 and 0, 0, 1, 1, they are two constraints. So, we assume, we start with a set of 2 vectors and let us see, whether the form a basis for this subspace or not. So, any vector in W, which is of this form can be obtained as a linear combination of vectors of B 2. So, I will leave this exercise for the viewers to satisfy and further the vectors of B 2 are independent.

So, again they can check, whether these 2 vectors are independent or not. So, if this spans the vector space and vectors are independent, then the B 2 will form a basis for W. And in that case, the dimension will be 2, because there are only 2 vectors in the basis. Now, V intersection W, V intersection W is, the set of vectors a, b, c, d which satisfy both the constraints, constraints on V and this is the constraints on W. Now, v is equal to a, b, c, d any vector, which belongs to V intersection W. Then, these constraints should be satisfies, a minus c plus d is equal to 0, a is equal to b and c is equal to d. And this means, a is equal to 0, a and b both are 0 and c is equal to d.

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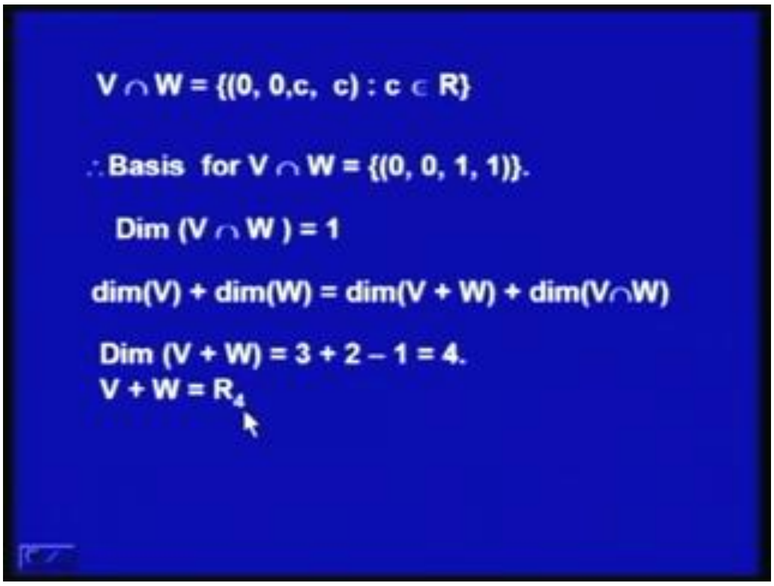

So, dimension of V plus W is equal to, dimension V, plus dimension of W, minus dimension of V intersection W. So, this is 3, we have already proved dimension of V as 3. The dimension of W is proved to be 2 and dimension of intersection W, we are just now prove it to be 1. So, it is 5 minus 1 is equal to 4 and that means, V plus W is a four dimensional vector space. Now, R 4 is also four dimension vector space and in fact, this we say that, V plus W is equal to R 4.

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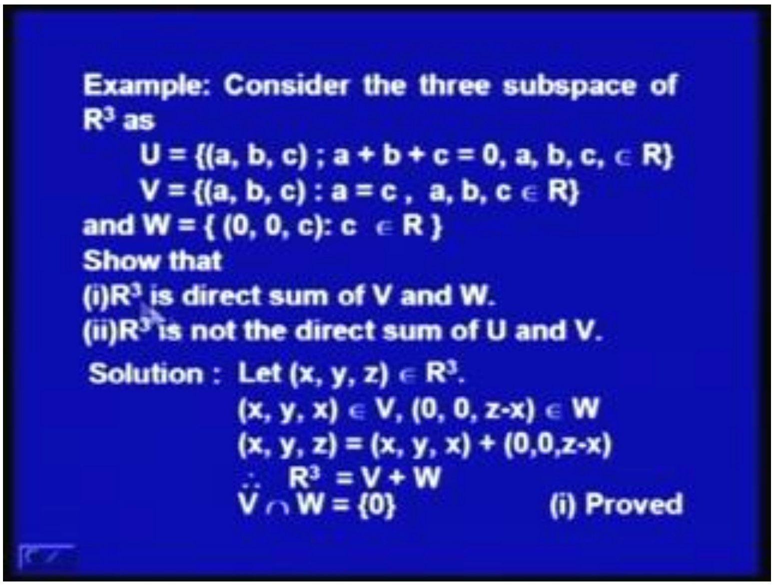

Now, this is the next example, consider the three subspaces of R cube as U, V, W, define by the set a, b, c. That is, find this constrain for a, b, c, d belonging to real set. V is equal to a, b, c; a is equal to c, in the constraint on this vectors subspace and W is equal to 0, 0, c and c belonging to R. Now, we have show that, R cube is the direct sum of V and W, while R cube is not the direct sum of U and V. So, the solution, we consider x, y, z belonging to R cube.

And we proved that, any vectors in this can be written as U, as V plus W. That is a first part. So, let us say, 0, 0, z minus x belongs to W, W is of the form 0, 0, c. So, I will take that c as z minus x belonging to w and the vector V is, where a and c components are equal. So, I will take this vector from V as x, y, x. So, if I add the 2 vectors x, y, x from V and 0, 0, z minus x from w. Then, x, y, z can be expressed as sum of these 2 vectors and that means, R cube is equal to V plus W.

But, the result proved that, R cube, R cube is a sum of V plus W. But, does not mean that, it is direct sum to prove that, it is a direct sum, we have to prove that V intersection W is equal to 0. So, one can notice that, this V plus W equal to 0, only when the intersection of the two is equal to 0. So, this can be easily seen from the way, we have chosen the vectors. And that is why, we can say that V intersection W is equal to null vector or R cube is direct sum of V and W.

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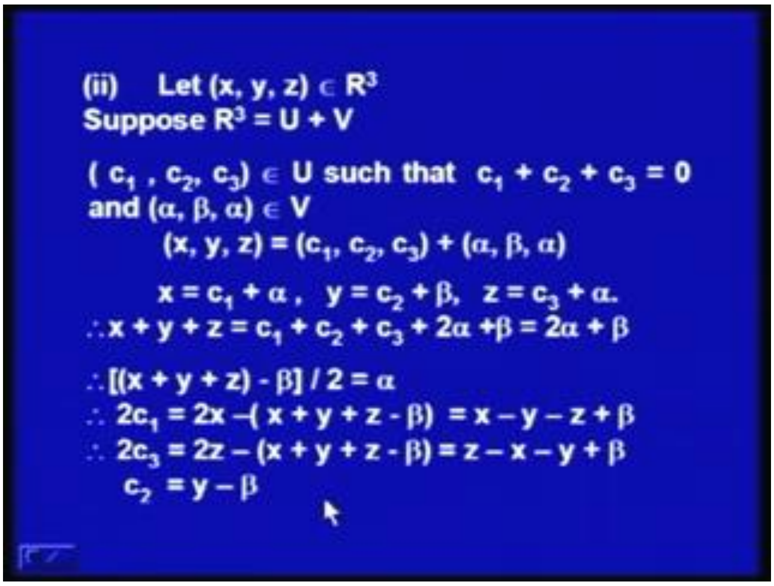

Now, in the second part, we assume that R cube is equal to U plus V. That means, c 1, c 2, c 3 belongs to U, satisfying the constraint c 1 plus c 2 plus c 3 equal to 0 and alpha, beta, alpha belonging to V. That means, the vector x, y, z in r cube, can be express as sum of these 2 vectors c 1, c 2, c 3 plus alpha, beta, alpha. And if we compare, then x is equal to c 1 plus alpha, y is equal to c 2 plus beta, z is equal to c 3 alpha. And that means, x plus y plus z is equal to c 1 plus c 2 plus c 3, with 2 alpha plus beta.

And since c 1 plus c 2 plus c 3 is equal to 0. That means, x plus y plus z is equal 2 alpha plus beta and from here, we can express alpha, as x plus y plus z minus beta by 2. We can solve these equations again for c 1. So, twice c 1 is equal to 2 x minus x plus y plus z minus beta or x minus y minus z plus beta. Similarly, for c 3, the solution is z minus x minus y plus beta and c 2 is equal to y minus beta.

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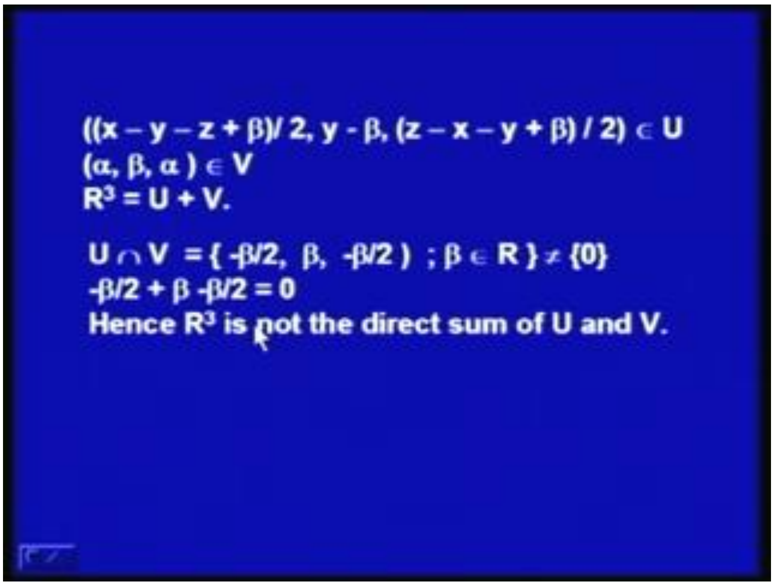

So, this is the vector, x minus y minus z plus beta by 2, comma y minus beta, comma z minus x minus y plus beta by 2. Belongs to U and V belongs and alpha beta, alpha belongs to V. Then, R cube can be written as U plus V. So, R cube is sum of U plus V, but it may, it is not a direct sum. In fact, this sum has two independent constraints alpha and beta. So, changing the values of alpha and beta, you can express R cube in many different way.

So, it is not a unique representation and that is why R, although R cube is sum of U and V, but is not the direct sum of U and V. In fact, one can further prove that, U intersection V will be a set of the form, minus beta by 2, beta, minus beta by 2. And beta belongs to R and this will be 0, only when, beta is equal to 0. But, beta can take any value and U intersection V will be a set of this form. Because, it has to satisfy two constraints, alpha and the first and last components equal and sum of the components equal to 0. And that is why, U intersection V is not the single turn 0. So, hence R cube is not the direct sum of U and V. So, in this example, we have three sets, R cube is direct sum of first and third, but it is not the direct sum of U and V the first 2.

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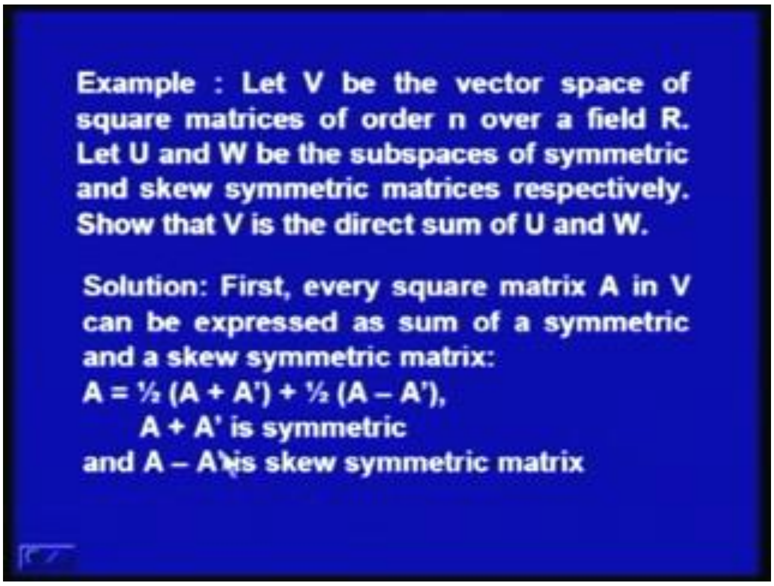

The next example, if V be the vector space of square matrices of order n over of a field R and U and W be the subspaces of symmetric. And skew symmetric matrices respectively. Then, we have to show that, V is the direct sum of U and W. We have already proved that, square matrices form a vector space and U is the set of symmetric matrices. They form of a vector space or subspace of this, vector space V and W. The set of skew symmetric matrix also form to subspace.

Now, we have to show that, V is a sum of U and W. Now, to prove this, we first observe that, every square matrix A in V can be expressed as sum of a symmetric and a skew symmetric matrix. And that means, we can in fact manipulate A, as half A plus A dash, a transpose plus half, A minus A dash. In this, we have established that, A plus A dash is symmetric. While, the matrix A minus A dash is skewing symmetric. So, matrix A in V, can be expressed as a sum of symmetric and skew symmetric matrix.

So, we can say that, V can be express as sum of symmetric matrices and skew symmetric matrices. Now, this will be the direct sum provided, there intersection is and A intersection is null matrix.

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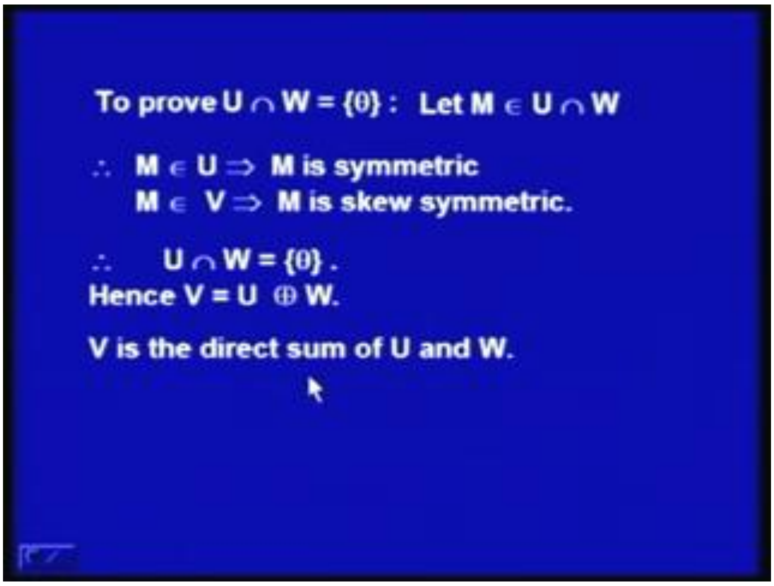

So, to prove that, U intersection W, if we say U is this. U is the set of symmetric matrices and W is a set of skew symmetric matrices. Then, we have to prove that, this is nothing but the null matrix. So, let say M belonging to U intersection W any matrix and belonging U intersection W. Then, M has to be symmetric matrix, because it belongs to this intersection and M belongs to V also. Where, V is skew symmetric, so M belonging to skew symmetric matrices.

Now, the two have this is possible, the matrix is symmetric as well as skew symmetric. This is possible only when U intersection W is theta. And that means, V is direct sum of U plus W. This is what is to be proved.

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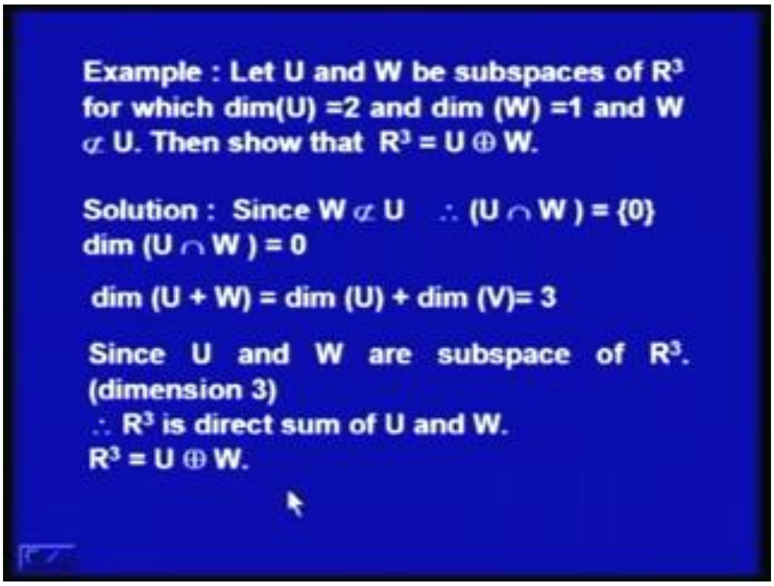

So, I have taken an example from matrices. Now, this is another example, that U and W be subspaces of R cube, for which dimension U is 2. And dimension W is 1 and W is not of subset of U. Then, we have to show that, R cube is equal to U plus W. Now, since W is not contained in U. So, U intersection W, there is nothing common between W and U. So, W intersection U and they are subspaces. So, definitely this null vector has to be there. So, U intersection W is this, dimension of U intersection W is 0.

Now, dimension for U plus W is equal to dimension of U plus dimension of V, dimension of U being 2, dimension of U being 3. So, dimension of U and W are subspaces of R cube. Now, R cube is direct sum of U and W. So, R cube is U plus W.

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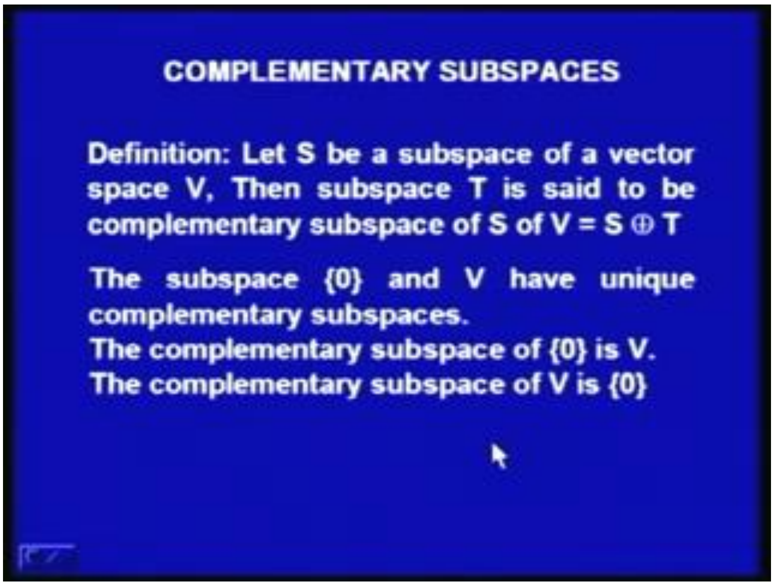

Now, with this, we come to another topic, that is complementary subspaces. So, first, we define complementary subspaces. That, according to the definition, let S be a subspace of a vector space V. Then, subspace T is said to be complementary subspace of S. If it can be express as V is equal to direct sum of S and T. Trivially, the subspace 0 and V have unique complementary subspaces, V plus 0 is equal to V. The complementary subspace of 0 is V and the complementary subspace of V is 0.

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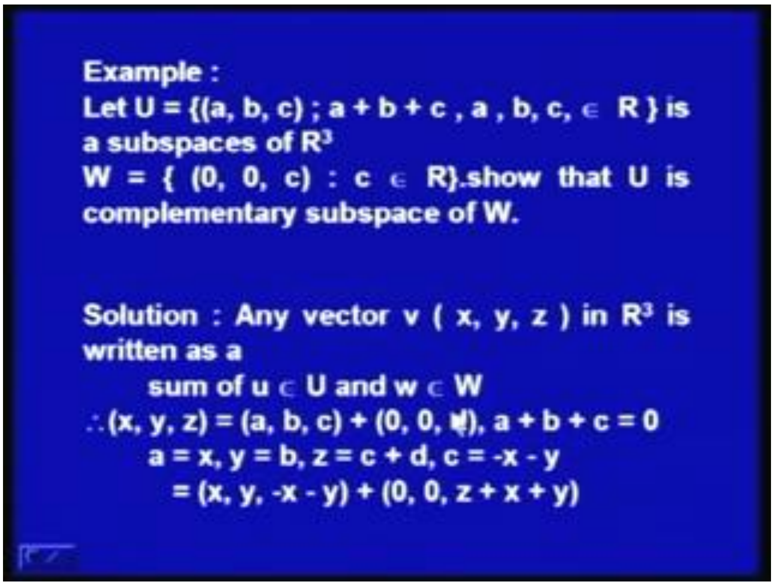

Let us illustrate this, with example. So, let us consider U as a, b, c, satisfying the constraint, a plus b plus c equal to 0; a, b, c belonging to R and it is a subspace. Similarly, W is equal to 0, 0, c; c belonging to R, then show that, U is complementary subspace of W. This we have already done in one of the earlier exercises. Any vector x, y, z in R cube is written as, sum of U and sum of W, sum of u belonging to U and w belonging to W.

Then x, y, z is equal to a, b, c plus 0, 0, d, where the constraint, a plus b plus c is equal to 0. Now, comparing the components, then a is equal to x, y is equal to b, z is equal to c plus d. And c is equal to minus x minus y, from this and x comma y comma minus x minus y, so that, x plus y plus z equal to 0 plus 0, 0 comma z plus x plus y. So, that way, we can express x, y, z as of combination of a vector from U and a vector from W.

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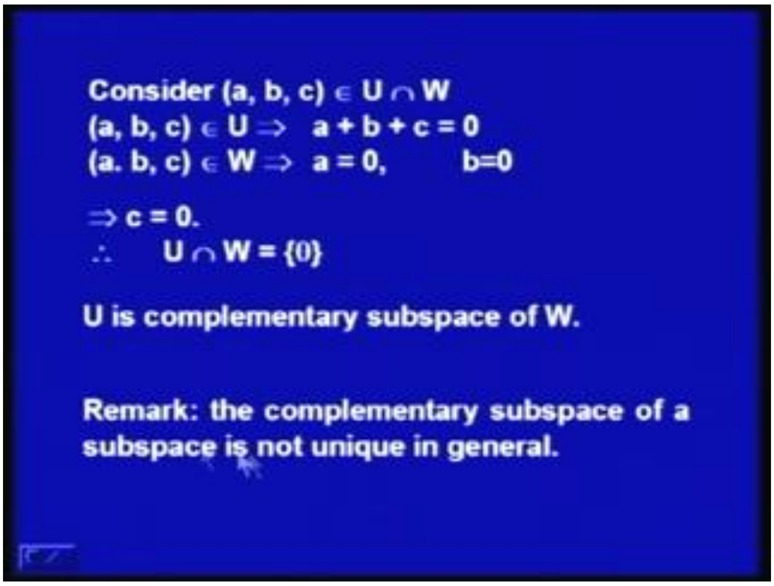

Now, we consider a, b, c belonging U intersection W. Now, if a, b, c belongs to U, then a plus b plus c equal to 0 and if a, b, c belonging W. Then a has to be 0 and that means, b has to be 0 and also c has to be 0. And that means, U intersection W is simply, a null vector. Now, this simply mean that U is a complementary subspace of W. Generally, the complementary subspace of a subspace is not unique. This is, we observation, we make in context, the number of cases. Now, we come to another concept, ordered basis, so we define.

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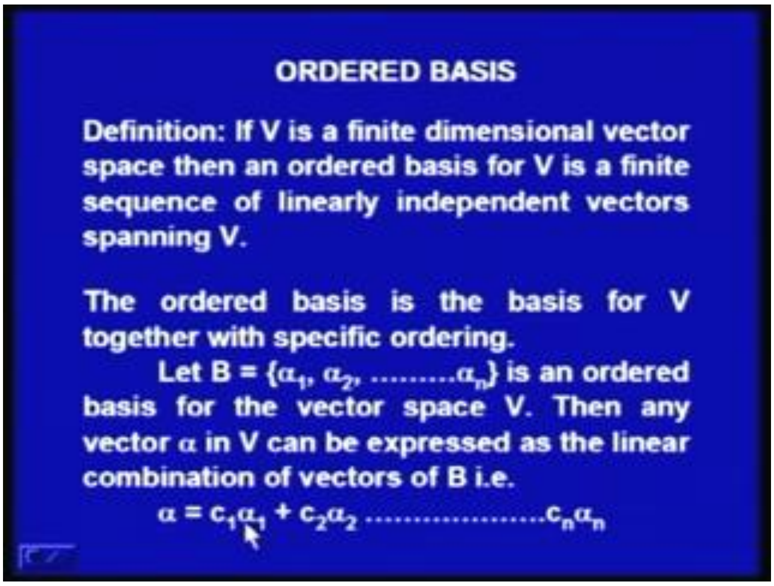

If, V is a finite dimensional Vector space, then an ordered basis for V, is a finite sequence of linearly independent vectors spanning V. So, the idea is, that we have already defined basis for vector space for a finite dimensional vector space. But, to define ordered basis, we have to add some sequence, some specific order, some specific ordering is to be incorporated. So, we say, let B is alpha 1, alpha 2, alpha n is an ordered basis for the vector space V. Then, any vector alpha in V, can be expressed as the linear combination of vectors of B. That is, alpha is equal to c 1 alpha 1, plus c 2 alpha 2, plus c n alpha n.

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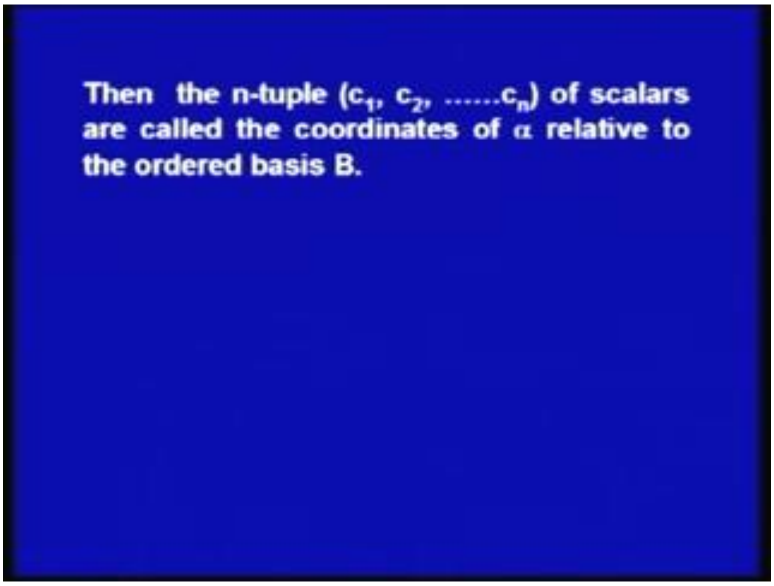

Now, in this case, the n-tuple c 1, c 2, c n of scalars are called the coordinates of alpha relative to the ordered basis B.

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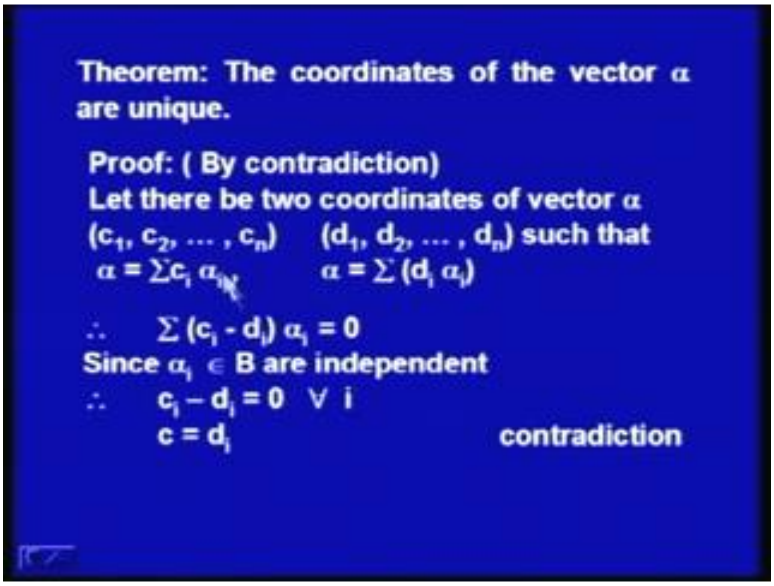

Now, we have a result, which says that, the coordinates of the vector alpha with respect to an ordered basis, are unique. We proof this result by contradiction. So, let there be two coordinates of vector alpha, belonging to V. And the two coordinates are c 1, c 2, c n and d 1, d 2, d n with respective the ordered basis. Such that, alpha is linear combination of the vectors, alpha i of the ordered basis. Such that alpha is equal to summation c i alpha i and alpha can also be expressed as, alpha is equal to summation d i alpha i.

The idea is that, c i’s and d i’s are the coordinates of the vector alpha. Then, since, they are equal. So, we can equate them and simplifying, give summation c i minus d i into alpha i is equal to 0. And since alpha i belongs to B are independent vectors or they form a basis. So, these coefficients must be 0 and that is why, c i minus d i is equal to 0. For all values of i and this gives me c i is equal to d i, which is the contradiction. And hence there is a unique representation. We cannot have two different representations or we cannot have two different set of coordinates of the vector alpha, with respective to an ordered basis. So, this gives the theorem.

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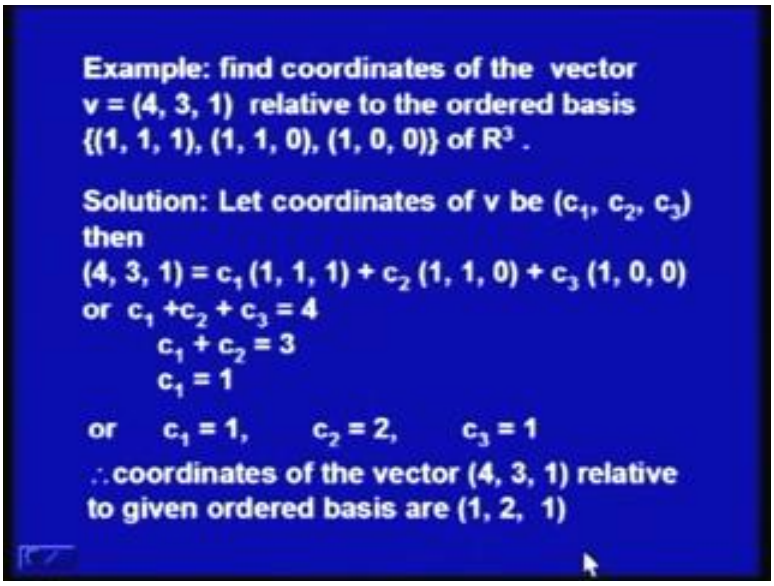

Now, let us try to find out, the coordinates of a give vector with respective to given order basis. So, let us consider an ordered basis of R cube, as 1, 1, 1;1, 1, 0 and 1, 0,0. Let us try to represent v, the vector v, which is a vector in R cube. So, it is of the form 4, 3, 1. So, we have to find out the coordinates of this vector 4, 3, 1 with respect to this ordered basis. So, we assume the coordinates of, v be c 1, c 2, c 3.

That means, by definition 4, 3, 1 is equal to c 1 times 1, 1, 1. The first vector plus c 2 times, the second vector 1, 1, 0 plus c 3 times the third vector 1, 0, 0. So, we have to solve the system and that amounts to. Now, 4 is equal to c 1 plus c 2 plus c 3, the first equation. This give, the second equation will be 3 is equal to c 1 plus c 2, this will not contribute. And finally, 1 equal to c 1, this will not contribute, also this will not contribute. So, this equation is actually equivalent to this, set of 3 equations, c 1 is equal to 1.

We have already obtained, we substitute this in this equation. This gives me c 2 is equal to 2 and substituting these two values in this equation, gives me c 3. So, c 1 is equal to 1, c 2 is equal to 2, c 3 is equal to 1, is the solution of the set of equations or we can say that, the coordinates of vector 4, 3, 1 related to the given ordered basis are 1, 2 and 1. Now, the coordinates of this vector 4, 3, 1. Actually, are given in the natural basis or standard basis. So, vector space R 3 can have number of basis. This is one; this is another basis for the vector space R 3.

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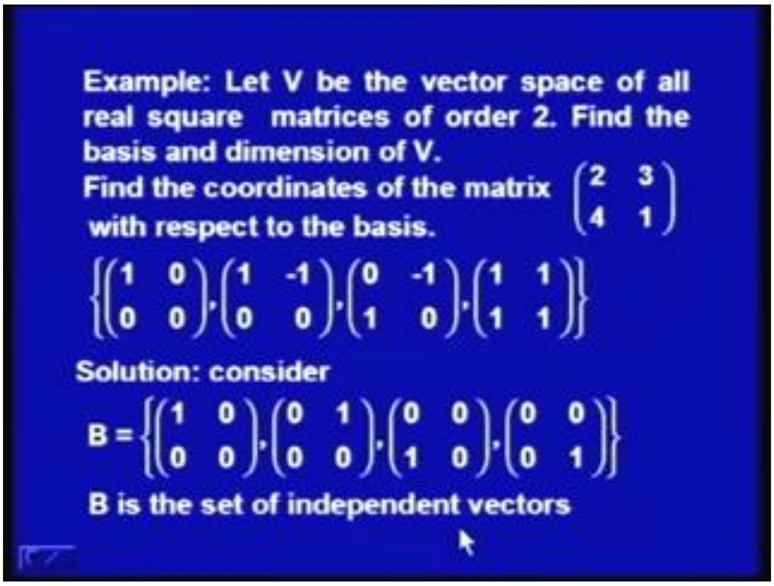

In the next example, we have taken some matrices. So, let us say; we have a vector space of all real square matrices of order 2. Find the basis and dimension of V and find the coordinates of the matrix 2 3 4 1 with respect to the basis. So, we first try to find out the basis. We start with, this set B is equal to, B consisting of these 4 matrices 1 0 0 0, 010 0, 0 0 1 0, 0 0 0 0 1. So, we consider a set B, and then we claim that B is a set of independent vectors.

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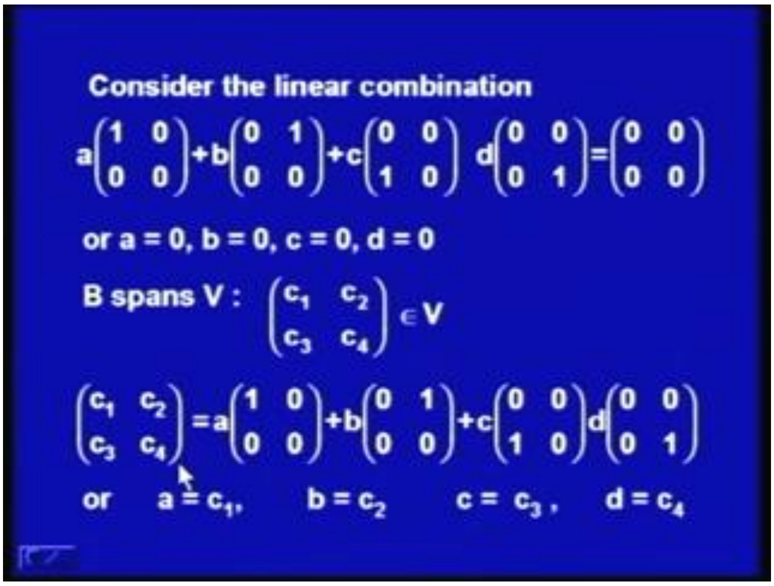

For this, we consider the linear combination of these vectors a, 1 0 0 0 plus b times. The second vector plus c times. The third vector, plus d times the 4th vector and this linear combination equal to 0 0 0 0, the null vector. We solve this, then a is equal to 0, b is equal to 0. From here, we have, an equal to 0, from here b comes out be 0, from here c comes out be 0. And finally, d is equal to 0.

Now, we have to prove that, B spans V. So, we consider c 1 c 2 c 3 c 4, a 2 by 2 square matrix belonging to set, to the vector space V. Then, any such square matrix of order 2 can be represented in the, as a linear combination of these 4 vectors. So, we try with this, c 1, c 2, c 3, c 4 is equal to a times, first vector, b time, second c times, third and lastly d times. The fourth and simplifying gives me, a is equal to c 1, b is equal to c 2, c is equal to c 3 and d is equal to c 4. So, all these coefficients can be obtained in terms of the given vector. So, we can say B spans V. So, B spans V, so linear combination of independent vectors.

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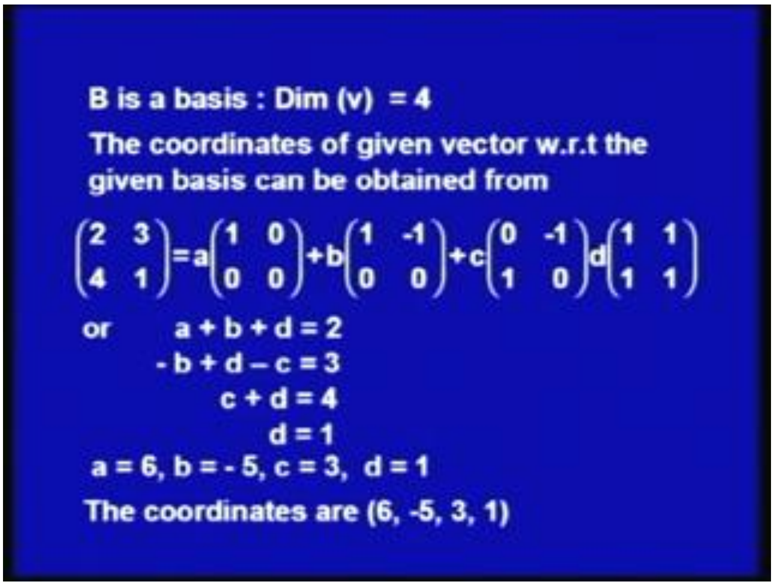

So, we can say that these are basis and since B has 4 vectors in it, so dimension of v is 4. Now, the coordinates of given vector with respect to the given basis can be obtained from, this relationship 2, 3, 4, 1 a times the first vector, which is given to us, plus b times, the second vector given to us c times, the third vector and d times, the fourth vector. So, if we solve it, it is a plus b plus d equal to 2. Then, second is minus b plus d minus c equals to 3.

Then, c plus d equal to 4 and d is equal to 1. So, these are four equations and they can be solve simultaneously will have a is equal to 6, b is equal to minus 5, c is equal to 3 and d is equal to 1. So, we can get the solution from this set, and that means the coordinates with respective given ordered basis is 6. The coefficient of the first matrix 6, a is equal to 6, b is equal to minus 5, c is equal 3 and d is equal to 1. So, the coordinates are 6 minus 5, 3 and 1. So, this example illustrates, how to find coordinates with respect to given basis.

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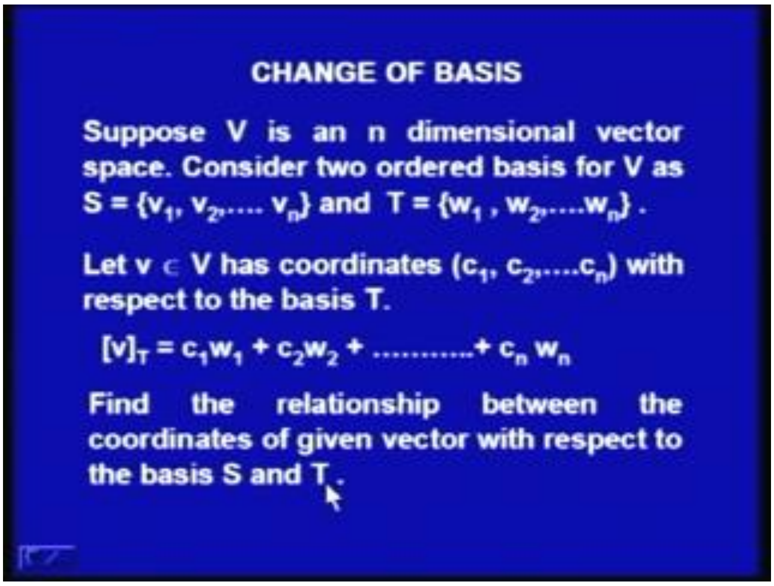

Now, change of basis, for this, let us consider V is an n dimension vector space. And it has two ordered basis as S and T. Then, V belonging to the vector space V, has coordinates c 1, c 2, c n with respect to the basis T. So, we can write down v and now, I am giving a notation, in which I say v. And then subscript T, that means v with respect to ordered basis T. This is c 1, w 1 plus c 2, w 2 plus c n, w n, because w 1, w 2, w n is a basis for T. Then, you can express v, as also, v as in terms of v 1, v 2 and v n. Now, the question is, what the relationship between the coordinates of is given vector with respect to the basis S and T. Now, this describes the change of basis.

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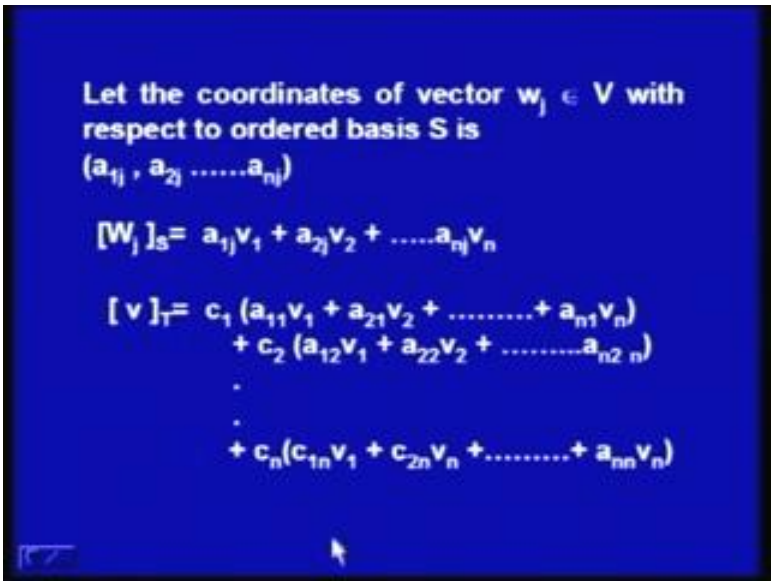

Now, let us consider a vectors, w j belonging to V with respect to ordered basis S is, a 1 j, a 2 j, a n j, because w j also belongs to V. So, we can write down it is coordinates W j with respect to S, as a 1 j, v 1 plus a 2 j, v 2 plus a n j, v n. And then v T can be written as c 1 times w j plus c 2 times w 2 plus c n times w n. So, this is, what we have written. So, v T is expressed as this.

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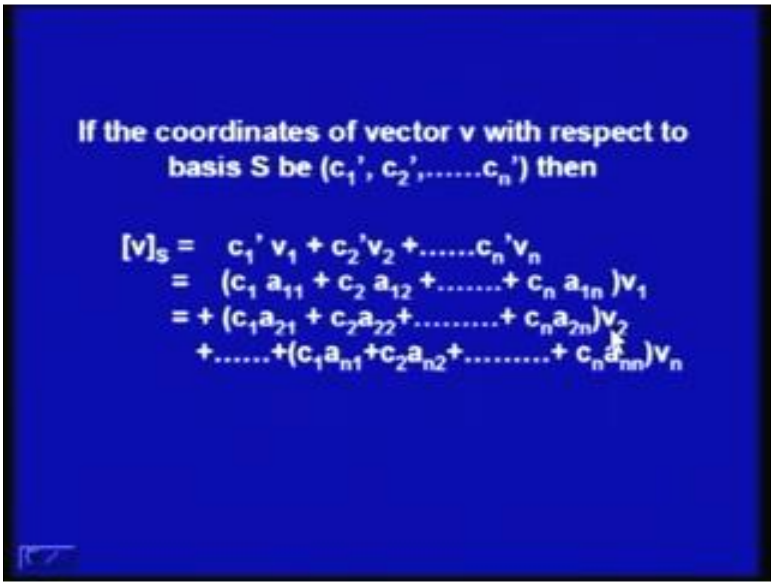

Now, if the coordinates of vector v with respect to basis S be c 1 dash, c 2 dash, c n dash. Then, v with respect to basis s will be, c 1 dash v 1 plus c 2 dash v 2 plus c n dash v n.

And then this is equal to, what we have written earlier, c 1 a 11 plus c 2 a 1 2 plus c n a 1 n, times v 1, v 2 and v n. Linear combination of v 1, v 2, v n, this we have written earlier.

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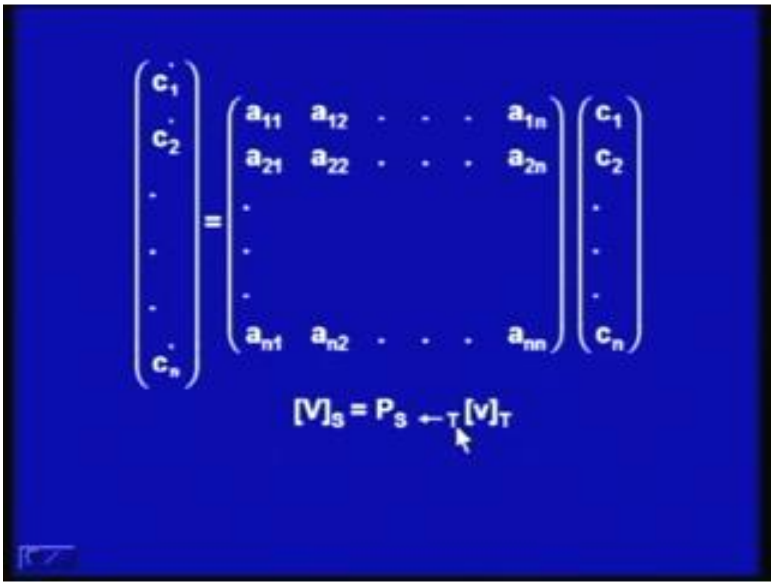

And this means in the vector form, we write down the coordinates of c 1 dash, c 2 dash, c n dash. As multiplied by the matrix into this column vectors c 1, c 2, c n, which is the coordinate of the vector v in T. So, in the matrix form this equation is V s is equal to at matrix P s. This is special notation, we use for change of coordinates v T.

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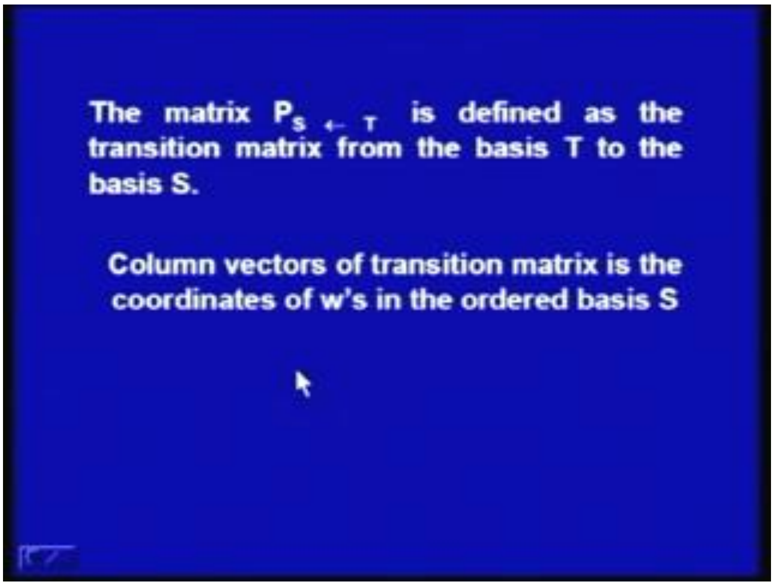

Now, this matrix P s T is defined as the transition matrix from the basis T to the basis S and column vectors of transition matrix is the coordinates or w's in the ordered basis S.

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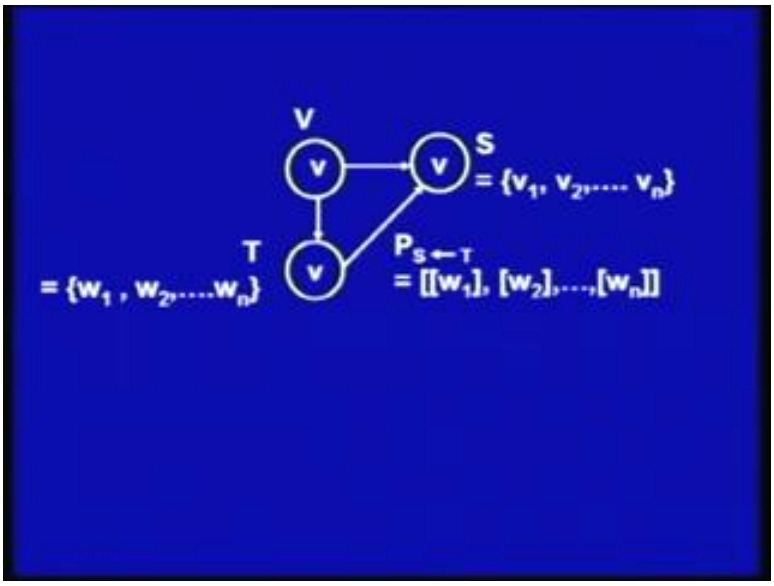

And this is, what we have a vector v in the vector space V. This is the representation in the vector S, which is the, has basis v 1, v 2, v n. We had another basis T, it has w 1, w 2, w n. This ordered basis for T. Then, this is the transition matrix. So, that this relationship, how we represent this vector in terms of T. And this is the same vector in terms of v, ordered basis S. So, this is, what is to be discussing, but more detail, I will be discussing in my next lecture.

Viewers, towards the end of this lecture, I will summarize, what we have done today. We have started with direct sum and I had given you, some results related with this. That is, dimension of u plus dimension of w is equal to dimension of u plus w, plus dimension of u intersection w. We have discussed complementary subspaces. We have and then come to ordered basis. We have discussed coordinates and I have started discussion on change of basis. But, I will continue with this in my next lecture.

Thank you.