Mathematics - II
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Module - 2
Lecture - 8
Linear Algebra Part – 3

Welcome viewers, this is my third lecture on Linear Algebra. In earlier 2 lectures, we have discussed vector spaces and subspaces. This lecture is about linear independence and dependence.

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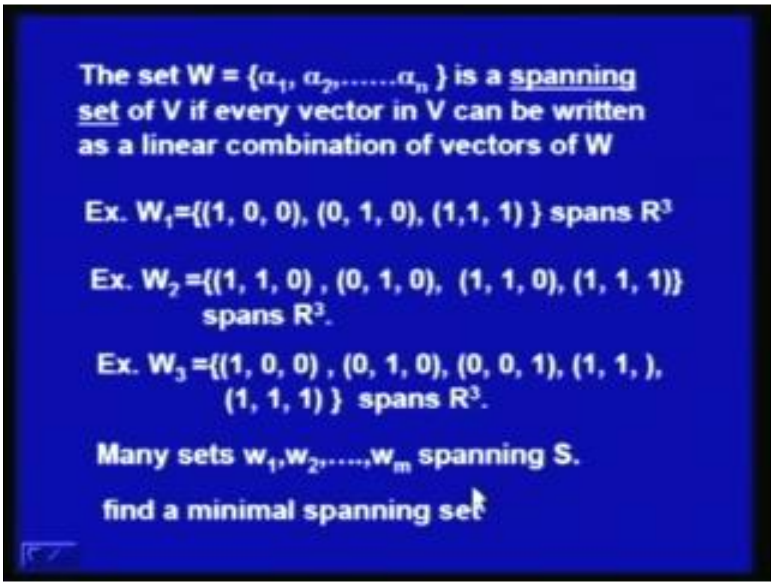

We have all ready define spanning set of lecture space V. As a set W, consisting of n vectors alpha 1, alpha 2, alpha n, such that every vector in V can be expressed as a linear combination of vectors of W. For example, W 1 consisting of 3 vectors (1, 0, 0), (0, 1, 0) and (1, 1, 1), it can spans R 3, the vector space. You can check that, all the vectors in R 3 can be expressed as a linear combination of these vectors (1, 0, 0), (0, 1, 0) and (1, 1, 1).

We can have another exercise in which another subset W 2, consisting of (1, 1, 0), (0, 1, 0), (1, 1, 0) and (1, 1, 1), also spans R 3. Viewers can check that, they actually represent any vector, in fact, any linear combination of this vector can be a vector space in V. Not only this, any vector V, any vector in V can be exposed as a linear combination of these vectors. So, W 1 and W 2, both span R 3.

This is not all you can have another subset W 3, consisting of different set of vectors (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1) and (1, 1, 0) spans R 3. This can also be checked, in fact, one may notice that there are number of subsets, which can span the vector space V. So, they may be sets w 1, w 2, w m spanning vector space V.

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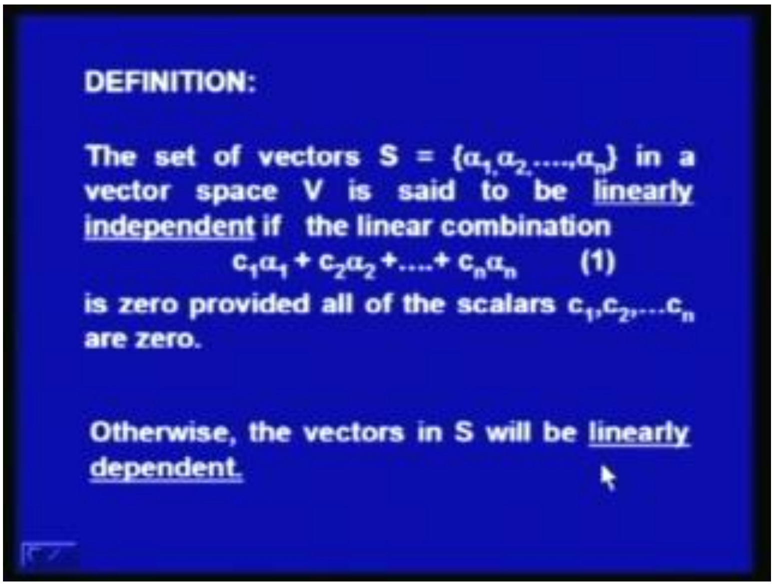

Now, the question is, can we find a minimal spanning set? For this, we will be giving some definitions first and then some results. So, the first definition says, that the set of vectors S consisting of alpha 1, alpha 2, alpha n. In a linear vectors space V is linearly independent. If the linear combination of these vectors, that is c 1 alpha n, c 1 alpha 1 plus c 2 alpha 2 plus n alpha n is 0. Provided all the scalar c 1, c 2, c n is 0. So, if this is possible, then the vectors are linearly independent.

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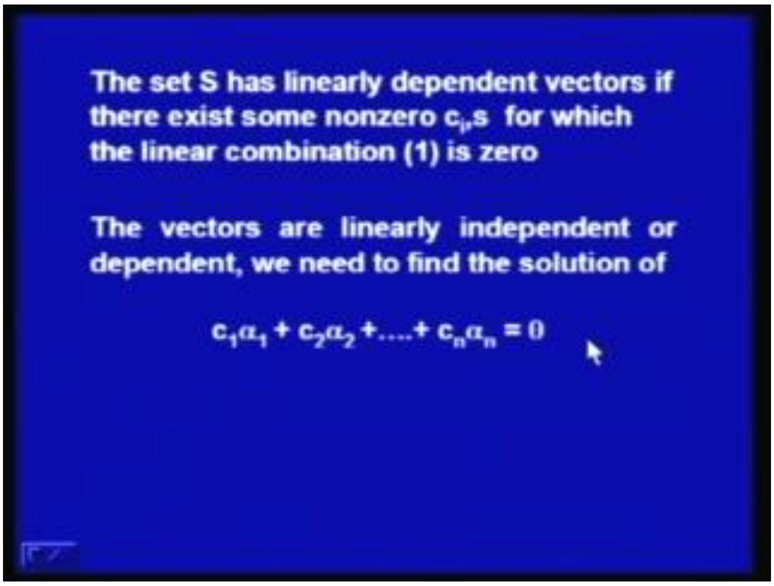

Otherwise, the vectors in S will be linearly dependent by this I mean to say, that the set S has linearly dependent vectors, if there exist some nonzero c i, S, for which the linear combination 1 is zero. The vectors are linearly independent or dependent. We need to find the solution of this equation. That is c 1 alpha 1 plus c 2 alpha 2 plus c n alpha n equal to 0.

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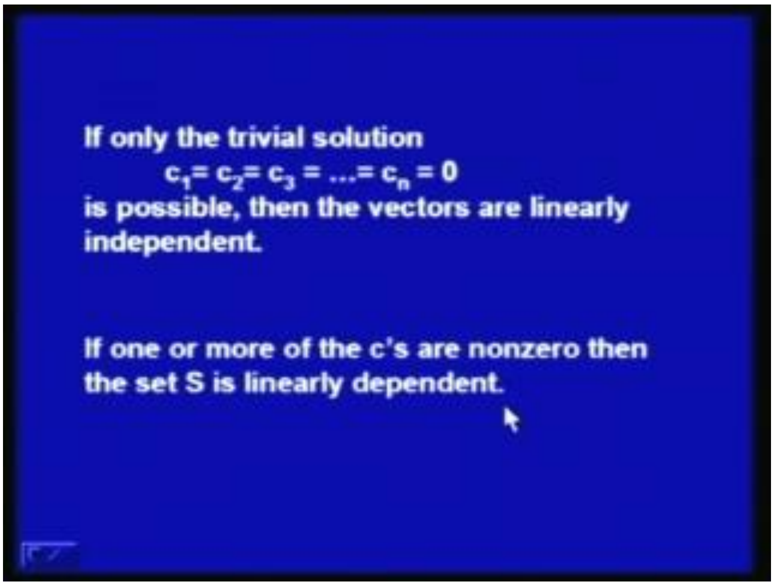

So, if you find a trivial solution of equation in which c 1, c 2, c 3, c n all are 0. Then, the vectors are linearly independent. So, if such a solution is possible, then the vectors are linearly independent. However, if one or more of the c's are nonzero. Then, the set S is linearly dependent.

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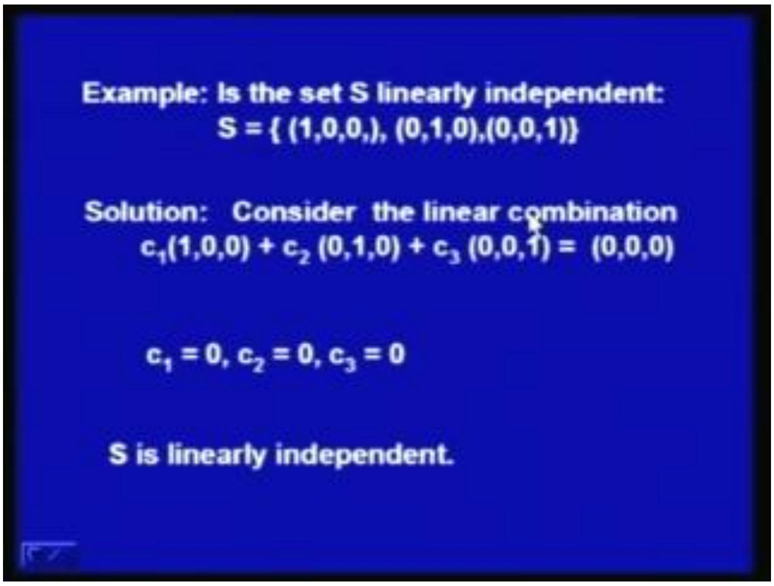

We can take some examples to illustrate this. So, let us say we have a set S consisting of (1, 0, 0), (0, 1, 0), (0, 0, 1). We have to see, whether this set is linearly independent or dependent. Now, for this, we consider the linear combination c 1 (1, 0, 0) plus c 2 (0, 1, 0) plus c 3 (0, 0, 1) and this right hand side is equal to 0. So, let us consider this linear combination and set it to 0. If we can find a solution of this, which is trivial solution, then we say the vectors are linearly independent. Otherwise, the vectors are linearly dependent.

Now, if one can notice that, these will give rise to three equations. If you compare component wise, so only this component c 1 into 1, this is not contributing, this is not contributing. So, if we equate it to 0, so c 1 is equal to 0. In the second, only this is contributing c 2 into 1, others are not contributing, so c 2 is equal to 0. And the third component gives us c 3 equal to 0. So, this is the only solution possible for this equation to be satisfied. And that is why; we say that the set S consisting of these 3 vectors is linearly independent. So, S is linearly independent set of vectors.

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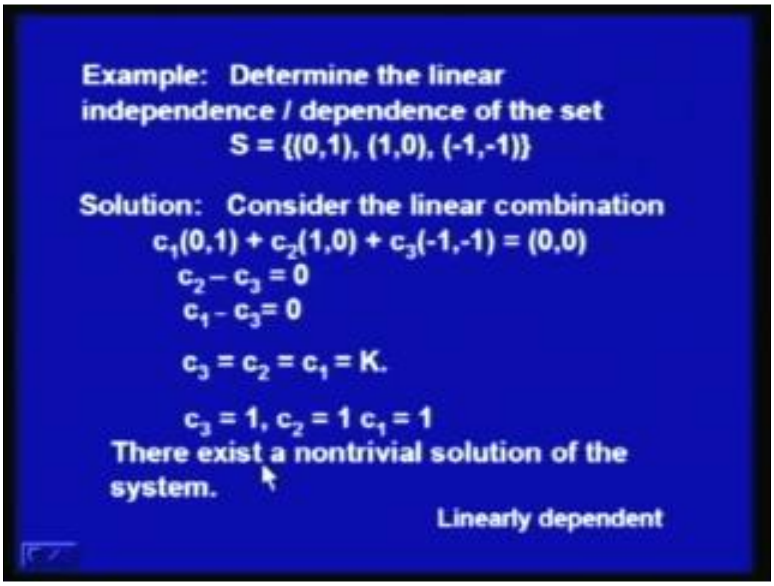

Now, in another example, we have to check, whether the set (0, 1), (1, 0), (minus 1, 1) is a set of linearly independence or dependent set of vectors. The solution is, we consider the linear combination of these 3 vectors. The combination is c 1 0 comma 1 plus c 2 1 0 plus c 3 minus 1 minus 1 and equated it to 0 comma 0. We equate component wise. So, this will not be contributing. But, this will give us c 2 minus c 3 is equal to 0 and the second component gives us c 1. This will not contribute minus c 3 equal to 0.

So, we have two equations in three unknown's c 1, c 2 and c 3. So, the solution of these two equations gives us c 2 is equal to c 3, the first equation. C 1 is equal to c 3 from the second equation and this can be assigned any arbitrary value K. So, we will have c 3 is equal to 1, c 2 is equal to 1, c 1 is equal 1, when K is equal 1 is a solution of this equation. So, a nontrivial solution is possible. So, there exists a nontrivial solution of the given system. And hence, we say the 3 vectors 0 1, 1 0, minus 1 minus 1 in the set S are linearly dependent or the set is the set of linearly dependent vectors.

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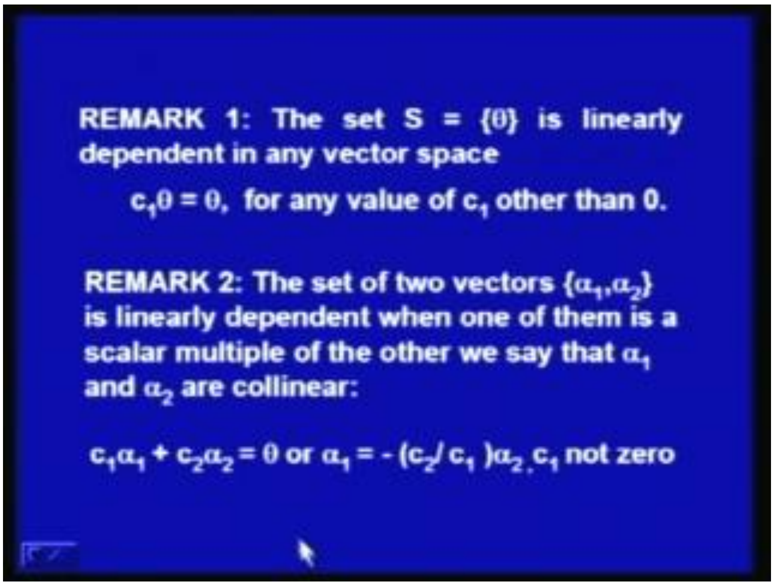

Now, we are now in a position to make some remarks. So, the remark 1 is, that the set S consisting of a single term 0, which is additive identity in the vector space is linearly dependent in any vector space. So, whatever be the vector space, this singleton set will be linearly dependent set. And this can be proved very easily. You can see that c 1 into null vector theta is equal to null vector, for any value of c 1 other than 0.

So, the nontrivial solution for this equation exists and that is why; this set is always linearly dependent. In another remark, the set of 2 vectors, alpha 1 and alpha 2, if it is linearly dependent, then one of them is a scalar multiple of the other. And we say that, alpha 1 and alpha 2 are collinear. This is proved here. We will have c 1 alpha 1 plus c 2 alpha 2 is equal to null vector, where c 1 and c 2, if it is c 1 is not 0. Then, one can write alpha 1 is equal to minus c 2 by c 1 times alpha 2. So, alpha 1 is written in terms of alpha 2, if c 1 is not 0. Of course, if c 2 is not 0, then you can write down alpha 2 is equal to minus c 1 by c 2 into alpha 1. So, this way, we can say that the 2 vectors are collinear.

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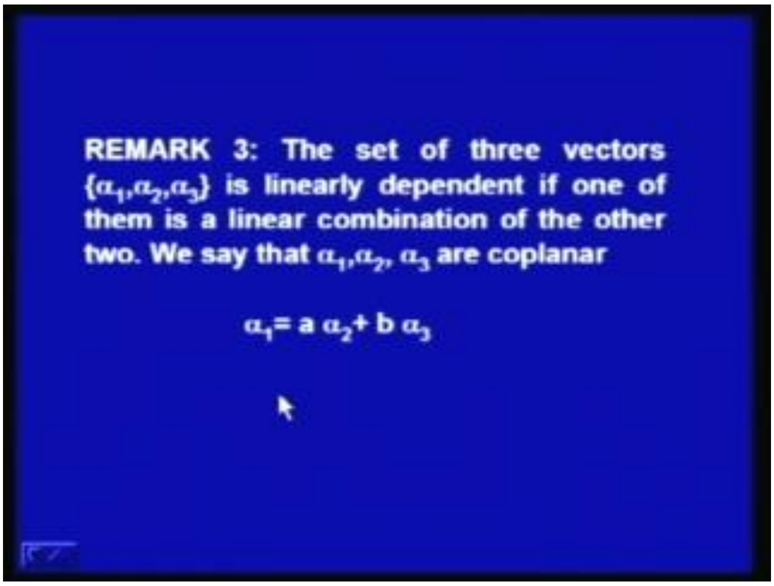

Similarly, the set of 3 vectors alpha 1, alpha 2, alpha 3 is linearly dependent. If one of them is a linear combination of the other two. And in such a case, we say that, the vectors alpha 1, alpha 2, alpha 3 are coplanar. Like, I have written alpha 1 is equal to a times alpha 2 plus b times alpha 3.

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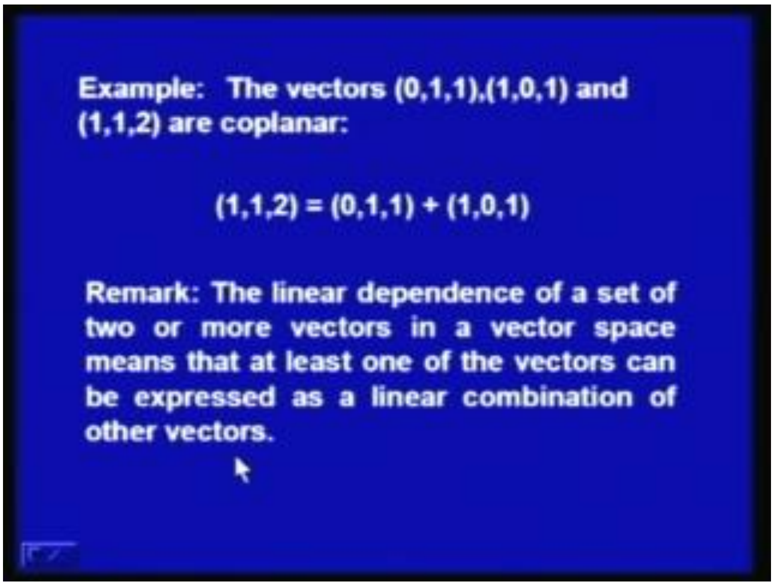

Now, some examples, the vectors (0, 1 comma 1), (1 comma 0 comma 1) and (1, 1, 2) are coplanar. Let us see we can write down the third vector (1, 1, 2) as a linear combination of the first 2 vectors (0, 1, 1) and (1, 0, 1). You can see, this is 1, so we have

1 on this side. The second is 1, we have 1 here and this 1 plus 1 is 2. So, 1 1 2 is a linear combination of these 2 vectors or we say the 3 vectors (0, 1, 1), (1, 0, 1) and (1, 1, 2) are coplanar in this case. Another remark is the linear dependence of a set of two or more vectors in a vector space means, that at least one of the vectors can be expressed as a linear combination of the other vectors.

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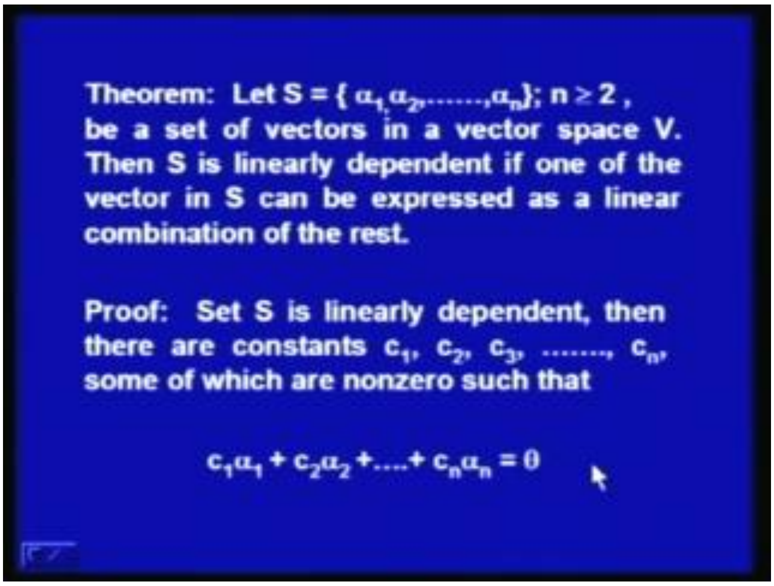

Now, this, we write down in form of a theorem. Let S consisting of n vectors, n is greater than equal to 2, be a set of vectors in a vector space V. Then, S is linearly dependent, if one of the vectors in S can be expressed as a linear combination of the rest of the vectors. Now, to prove this, we have given a set S as linearly dependent. Then, there are constants c 1, c 2, c 3, c n some of which are nonzero. So, we will have c 1 alpha 1 plus c 2 alpha 2 plus c n alpha n is equal to null vector. So, this is what we have from the very definition of V dependent set S.

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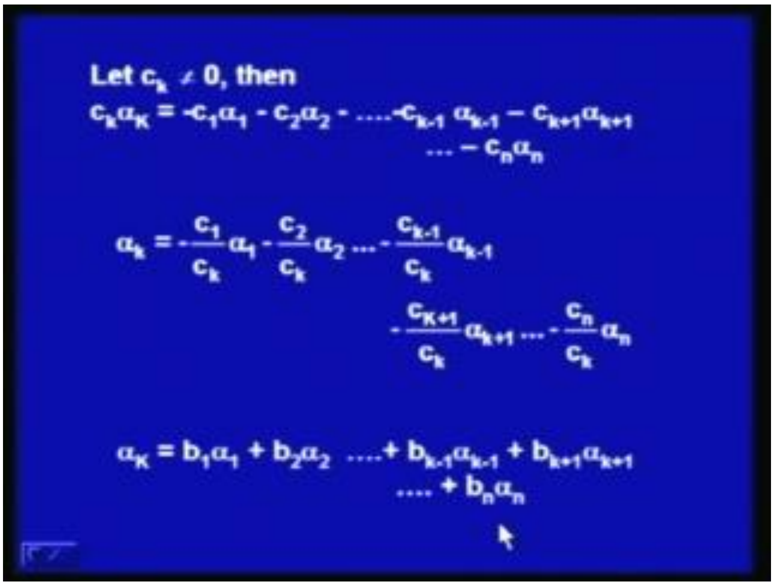

Now, we consider c k to be nonzero. Since, all of the c k's are not 0, so let us say c k is not 0. Then, c k alpha k is equal to minus c 1 alpha 1 minus c 2 alpha 2 minus c k minus 1 alpha k minus 1 minus c k plus 1 alpha k plus 1 up to minus c n alpha n. So, what I have done is, I had taken c k alpha k on one side and rest of the vectors on the other side. And since c k is not 0, so I can divide the whole equation by c k.

So, alpha k is equal to minus c 1 by c k alpha 1 minus c 2 by c k alpha 2 and up to c n by c k alpha n. And since, minus c 1 and c k, they are constants. So, I can we call them to b 1 and minus c 2 by c k as b 2 and so on. So, I can express the vector alpha k as b 1 alpha 1 plus b 2 alpha 2 plus b k minus 1 alpha k minus 1 up to b n alpha n. By this, I mean to say, that I have expressed alpha k as a linear combination of alpha 1 alpha 2 alpha k minus 1 alpha k plus 1 up to alpha n. That is alpha k is expressed as linear combination of rest of the vectors.

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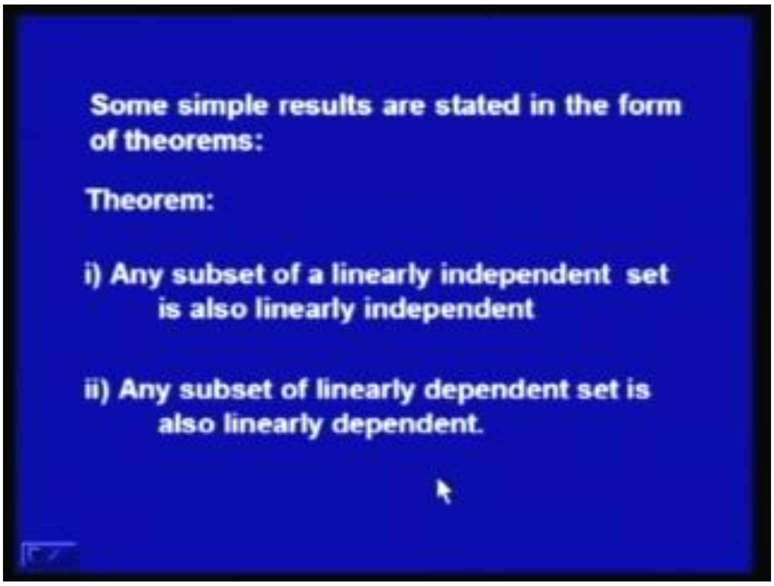

That is, how we proved the theorem. Then, some simple results are stated in the form of theorems. The first theorem is any subset of a linearly independent, set is also linearly independent. And the second is, that any subset of linearly dependent set is also linearly dependent. Now, these are the two exercises user can themselves try.

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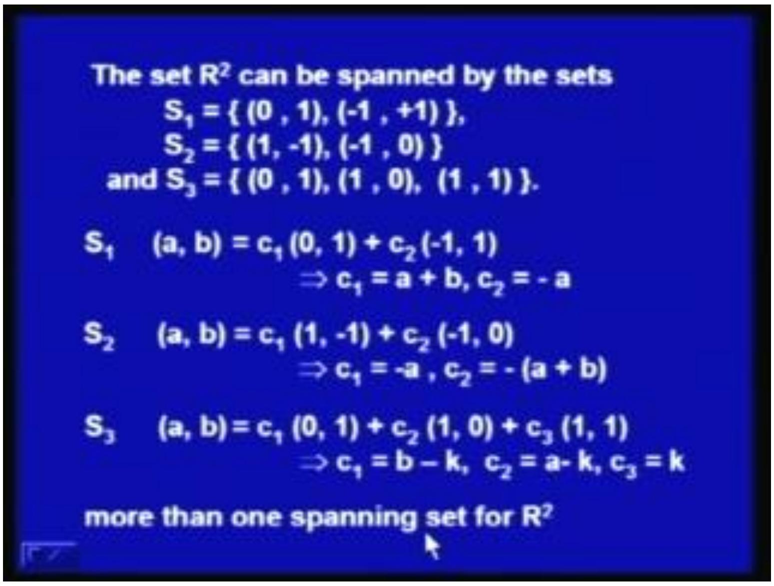

Now, I consider the set R 2 can be spanned by the sets S 1 consisting of 2 vectors. S 2 as another 2 vectors and S 3 consisting of the set of 3 vectors 0, 1, 1, 0, 1 comma 1, this can be easily verified. Like, if I consider for the set S 1, then any vector a, b in R 2 can be expressed as a linear combination of the vectors of this set 0, 1 and minus 1 1. So, it is c 1 0, 1 plus c 2 minus 1 and 1, you can solve it. Then, c 1 is equal to a plus b and c 2 equal to minus a.

So, since, we can find out c 1 and c 2 in terms of a known vectors a and b. That means this S 1 can span R 2, if you consider the set S 2, in this next example, then a, b equal to c 1 (1 comma minus 1) plus c 2 (minus 1, 0). So, any vector a, b in R 2 can be expressed as a linear combination of these 2 vectors. We can solve the two equations c 1 is equal to minus a and c 2 is equal to minus a plus b is a solution.

So, given a, b we can find out c 1 and c 2. So, that the vector a, b can be expressed as a linear combination of given 2 vectors. So, we can say that, this set R 2, can be set or maybe vector space R 2 can be spanned by this set. Similarly, to see that S 3 spans R 2, we again consider any vector in R 2, expressed as linear combination of given vectors. So, in this case, we find that c 1 is equal to b minus k, c 2 is equal to a minus k and c 3 is equal to k, where k is an arbitrary value.

The idea in this case is that, we have two equations in three unknowns. So, that is why; we have to introduce an arbitrary constant k over here. And that means, there will be you can choose the value, you can assign the value to this constant k and we will be having different solutions. So, in this case, number of solutions is possible. And you cannot express a, b in a unique manner in this case. And also another observation here is, that more than one spanning set for R 2 is possible.

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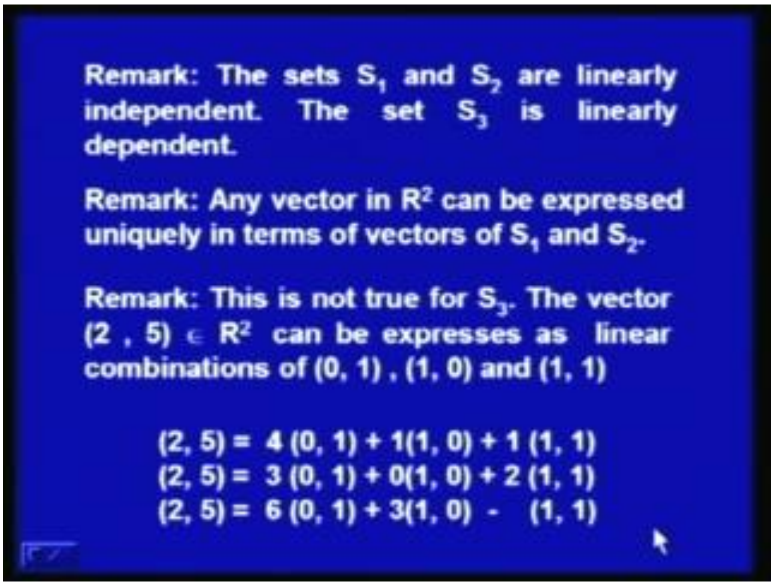

On the basis of this, we can make some remarks. One can notice that the sets S 1 and S 2 are linearly independent. While, the set S 3 is linearly dependent, also any vector in R 2 can be expressed uniquely in terms of vectors of S 1 and S 2. That we have observed. But, if we consider the third set S 3. Then, this is not true, because the vector, if you consider a specific vector (2, 5) belonging to R 2 can be expressed as linear combination of these vectors in many different ways.

These are some of the ways, you can find out. You can express 2 comma 5 is 4 times (0, 1) plus 1 times (1, 0) plus 1 times (1, 1). Further (2, 5) can also the expressed as this also (2, 5) can be expressed as 6 (0, 1) plus 3 (1, 0) minus (1, 1). This is also possible, because there is arbitrary constant k appearing in the solution of the linear combination. That is why; we are having this particular situation.

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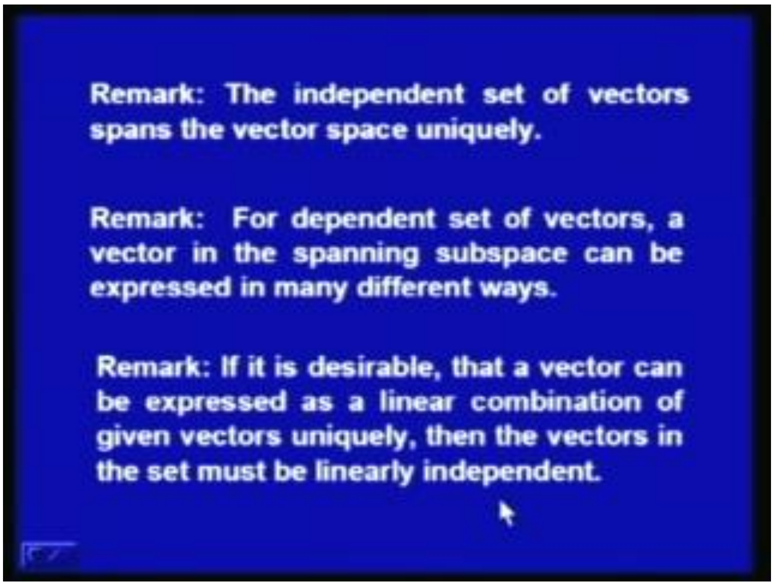

So, on the basis of this, we say that, the independent set of vectors, spans the vector space uniquely. While, if you have dependent set of vectors, then any vector cannot be necessarily represented as linear combination of given vectors. For dependent set of vectors, a vector in the spanning subspace can be expressed in many different ways. So, this is another remark. And finally, it is desirable that a vector can be expressed as a linear combination of given vectors uniquely. Then, the vectors in the set, in the spanning set must be linearly independent.

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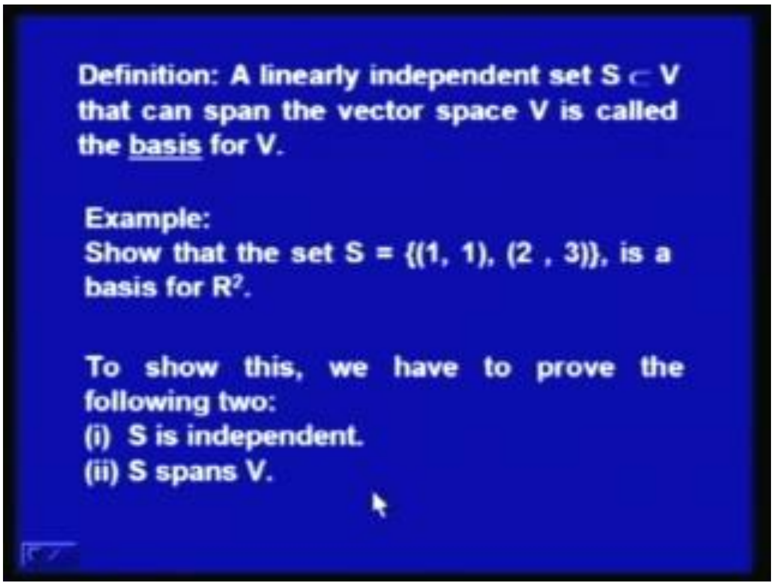

So, this is observation on this basis. On this, we define basis for a vector space V. So, this is the definition, a linear independent set S, which is subset of V. That can span the vector space V is called the basis for V or this is the definition for basis. Example, show that the set S consisting of (1, 1) and (2, 3) is a basis for R 2. Now, to establish this, what we have to do is, we have to show two things. One is that, S is independent and the second is S spans V, when we look at the definition, we have to have a linearly independent set. And then it should span V. So, these two things have to be there.

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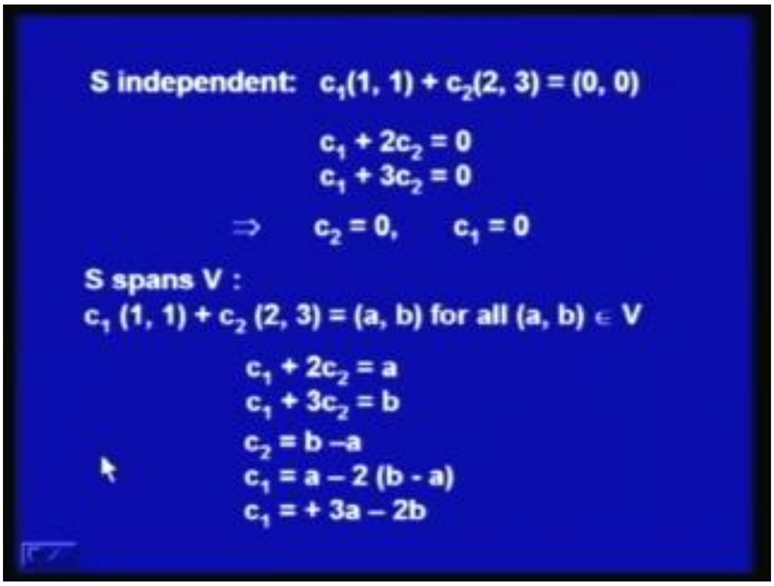

So, let us first check the linear independence of the set. Again, what we have to do is, we have to found the linear combination of the given set c 1 (1, 1) plus c 2 (2,3) and equate it to 0. And let us see, what we have next c 1 plus 2 c 2 is equal to 0 and then c 1 plus 3 c 2 equal to 0. So, we have two equations and two unknowns. And the solution is c 2 is equal to 0, if you subtract the 2 we will have c 2 equal to 0. And if you substitute c 2 is equal to 0, say in this equation, then c 1 equal to 0.

So, this system will have only one solution. And that is the trivial solution, c 1 equal to 0 and c 2 equal to 0 and this establishes that S is independent set. The second thing, we have to show for the basis is, that S spans V. For this, we consider a linear combination c 1 1 comma 1 plus c 2 2 comma 3. And we show that, any vector a b in V can be expressed as this combination.

So, if you can find out the solution for this equation, then it is possible, otherwise not. So, let us try to solve this system. So, what we have is c 1 plus twice c 2 is equal to a and the second component gives the c 1 plus 3 times c 2 is equal to b. If you solve this, you subtract the 2, it is c 2 is equal to b minus a. And if we substitute it here, then c 1 comes out to be 3 a minus 2 b. So, given a b, you can find out c 2 from this equation and c 1 from this equation and this solution will be unique. So, we can say that S spans V. And that establishes that, we have a basis for the given vector space.

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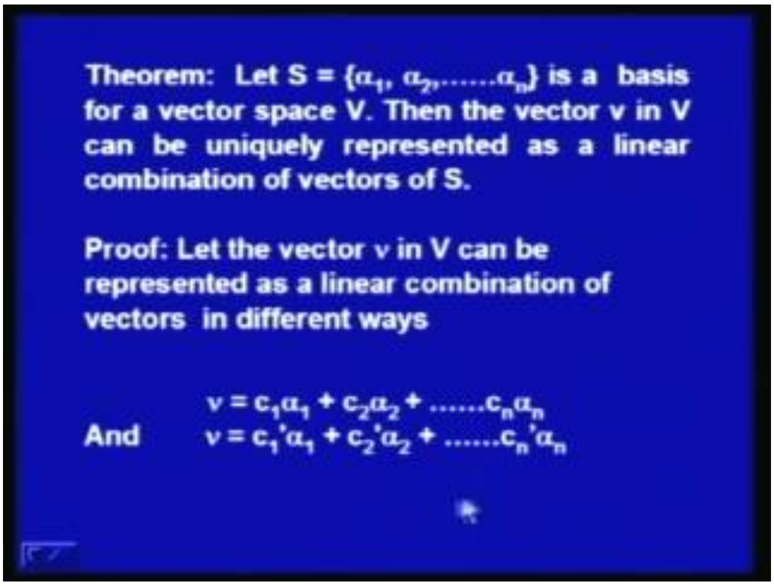

Now, the next theorem will say that, S is a set consisting of n vectors. Then, let S is be a basis for vectors space V. Then, the vector V, any vector v in V can be uniquely represented as a linear combination of vectors of S. So, if you have basis, then the vector v in V can the uniquely represented as a linear combination of vectors of S. Now, to prove this, let the vector v in V can be represented as the linear combination of vectors in two different ways.

So, we write down v as c 1 alpha 1 plus c 2 alpha 2 plus c n alpha n. At the same time, we have another coefficients c 1 dash c 2 dash c n dash, etcetera. So, that v is c 1 dash alpha 1 plus c 2 dash alpha 2 plus c n dash alpha n.

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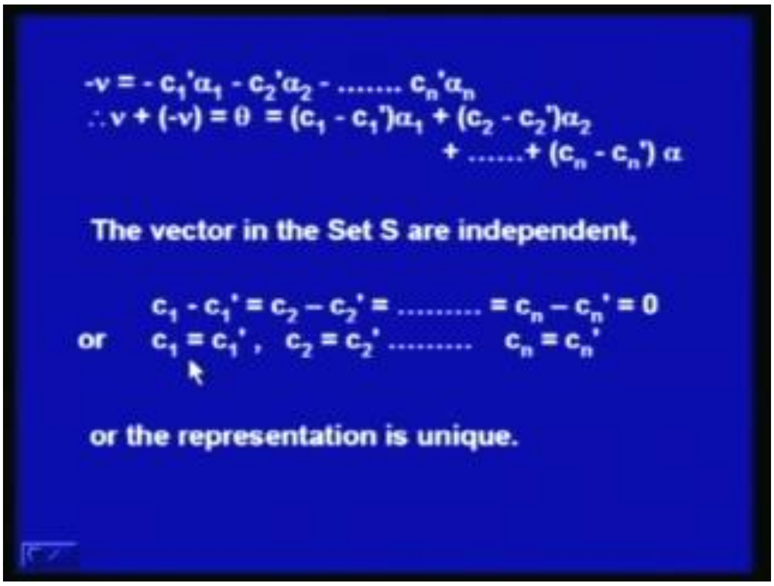

So, if this is possible, then we can write down, we can multiply the second equation by minus 1. So, we will have minus v is equal to minus c 1 dash alpha 1 minus c 2 dash alpha 2 minus c n dash alpha n. So, if we add the two equations, what we have is, v plus this times minus v is equal to null vector. This is the property of the vector v and theta null vector theta.

Then, c 1 minus c 1 dash alpha 1 plus c 2 minus c 2 dash alpha 2 times alpha 2 plus. The last term is c n minus c n dash alpha c n minus c n dash, times alpha. So, this is equal to theta. Now, we know that alpha 1, alpha 2, alpha n vectors are linearly independent. So, these coefficients have to be 0. That is the basic requirement for the independent set of vectors. So, this is 0, this is 0, this is 0. So, c 1 minus c 1 dash is equal to c 2 minus c 2 dash is equal c n minus c n dash all equal to 0.

And if you further simplify, then c 1 is equal to c 1 dash c 2 is equal to c 2 dash. And finally, c n is equal to c n dash. So, all c i's are equal to corresponding c i dash and that means, they are not different. So, what we can say is, there is only one representation possible. We have started with two different representations. But, ultimately, they come out to be the same. So, we say the representation is unique. So, this establishes the theorem.

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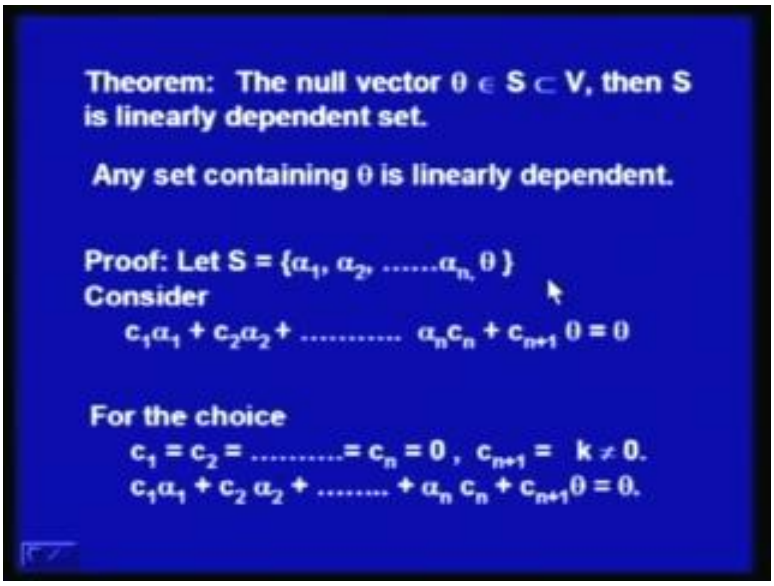

Now, we have another theorem, which says that, the null vector theta belonging to S, which is subset of V. Then, S is linearly dependent set, by this, I mean to say that, if there is a null vector in the set S, then set S is linearly dependent set. So, any set containing theta is linearly dependent that is another way of writing this theorem. So, let us try to prove this; we consider set of n plus 1 vectors alpha 1, alpha 2, alpha n, which are nonzero. And then we add another vector theta into this set.

So, let us consider, whether this set is linearly independent or not. Now, to prove that, this set is linearly dependent or independent, I have to consider this linear combination. So, c 1 alpha 1 plus c 2 alpha 2 plus c n alpha n plus c n plus 1 theta is equal to the null vector theta. Now, I consider c 1 is equal to c 2 is equal to c n is equal to 0 and c n plus 1 is equal to k. Now, this choice is possible.

So, if I consider c n plus 1 is equal to k and all this is 0, k is not 0. Then, c 1 alpha 1 plus c 2 alpha 2 plus c n alpha n plus c n plus 1 theta is equal to 0, why because c 1 0. So, this is this term is not contributing c 2 is 0. So, this is not contributing. Similarly, this is not contributing, what remains is, only this term c n plus 1 into theta. So, whatever the c n plus 1, even, if it is not 0 c n plus 1 into 0 is 0. So, I have found one solution in which all c ones all c n’s are 0, but c n plus 1 is not 0. So, I have found one nontrivial solution for this to happen and that is why we have this set alpha 1, alpha 2, alpha n, theta as linearly dependent set.

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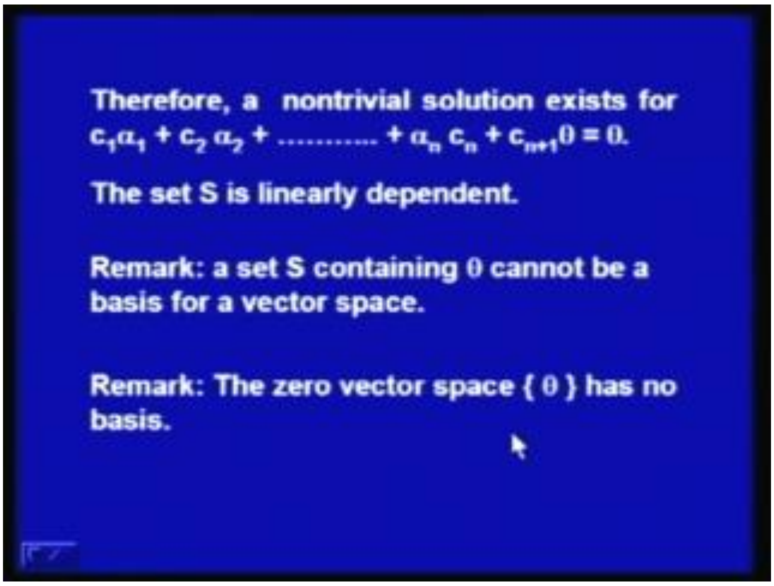

Therefore, nontrivial solution exists and it is linearly dependent set. Remark a set S containing theta cannot be a basis for a vector space. So, this is one observation, which can directly derive from the earlier theorem. Because, by the very definition of basis, the set S has to be linearly independent, only then it can be a basis for a vector space. Now, set is S containing theta will be linearly dependent, whatever be the vector space. So, it cannot form a basis for a vector space. So, this is about this remark. Then zero vector space theta has no basis. So, if you consider this set, this is always in a linearly dependent. So, this cannot form of a basis, it has only one set in this. So, we also know that, this set consisting of only theta forms a vector space. So, this vector space has no basis, this is another remark.

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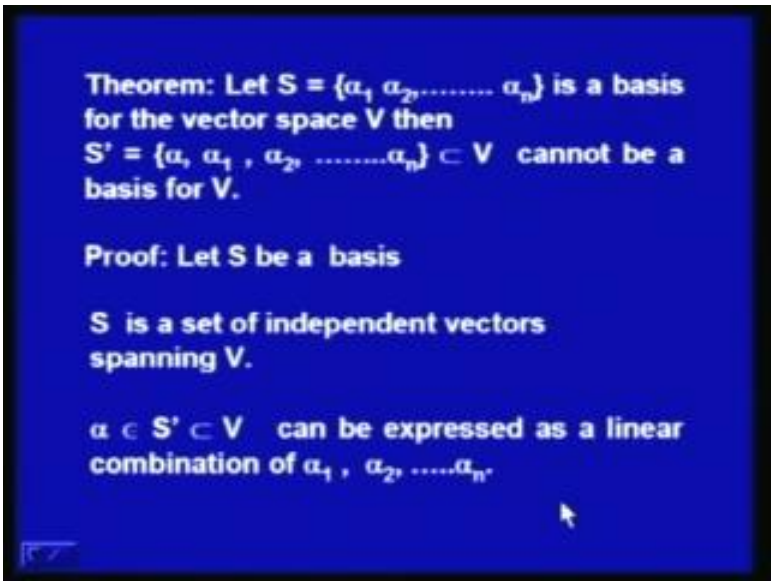

The next theorem says that, if S is a set consisting of n vectors and is a basis for the vector space V. Then, if you add one more vector alpha in this set and we have another set S dash, consisting of alpha, alpha 1, alpha 2, alpha n and still is a subset of V. Then, this S dash cannot be a basis for V. So, if you have set S, which is the basis for the vector space V. Then, adding one more vector is S dash cannot be a basis for V.

Now, to prove this, let S, be a basis. So, it has to be a set of independent vectors spanning V. That is the definition for basis. And let us say, we have another vector alpha, which belongs to S dash. Since, alpha also belongs to V S dash being a subset of V. Then, alpha can be expressed as a linear combination of alpha 1, alpha 2, alpha n. That is a very definition of basis. Any vector alpha in V must be expressed as a linear combination of vectors of the basis. So, alpha can be expressed as a linear combination of alpha 1, alpha 2, alpha n.

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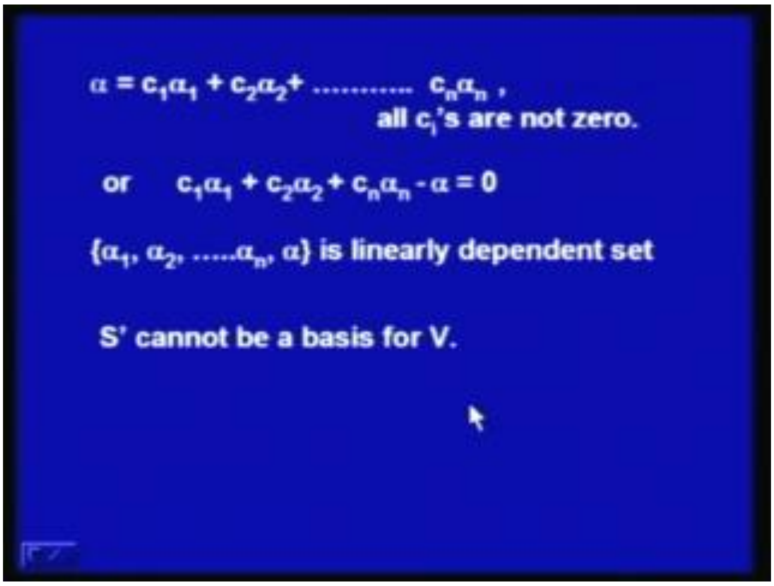

By this I mean to say is that alpha is equal to c 1 alpha 1 plus c 2 alpha 2 plus c n alpha n, where all c i's are not zero. And that means c 1 alpha 1 plus c 2 alpha 2 plus c n alpha n minus alpha is equal to 0. That means, alpha all c i's are not 0, some of them, maybe 0. But, all of them are not 0. So, this set alpha 1, alpha 2, alpha n together with this alpha is linearly dependent set. So, there exist linear combinations in which we have nontrivial solution possible. So, this set cannot be linearly independent. And if this is not linearly independent, then this set S dash cannot be a basis for V. That is what the theorem is…

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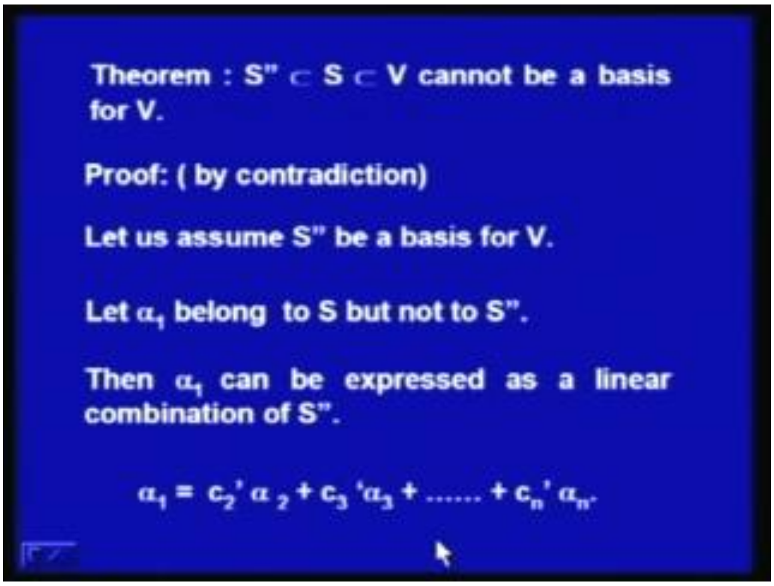

And secondly, if we have S double dash, which is a subset of S, cannot be a basis for V. So, we prove this by contradiction. So, let us assume that, S double dash, be a basis for V. If S double dash be a basis for V, S double dash is a subset of S. So, let alpha 1 belongs to S, but not to S double dash. Because, S double dash is subset of S, there must be exist, some alpha 1, which belongs to S, but not to S double dash.

So, we will have alpha 1 belonging to S, but not to S double dash. Then, alpha 1 can be expressed as a linear combination of S double dash. So, alpha 1 can be expressed as a linear combination of vectors of S double dash. By this, I mean to say that, alpha 1 is equal to c 2 dash alpha 2 plus c 3 dash alpha 3 plus c n dash alpha n. Because, S double dash is a basis for vector space V and alpha 1 belongs to V. It may not belong to S double dash, but it belongs to V.

So, any vector in V can be expressed as a linear combination of these vectors. So, alpha 1 is equal to c 2 dash alpha 2 plus c 3 dash alpha 3 plus c n dash alpha n. Where all of them need not be 0 or there maybe some nonzero value c of these coefficients for which this is possible.

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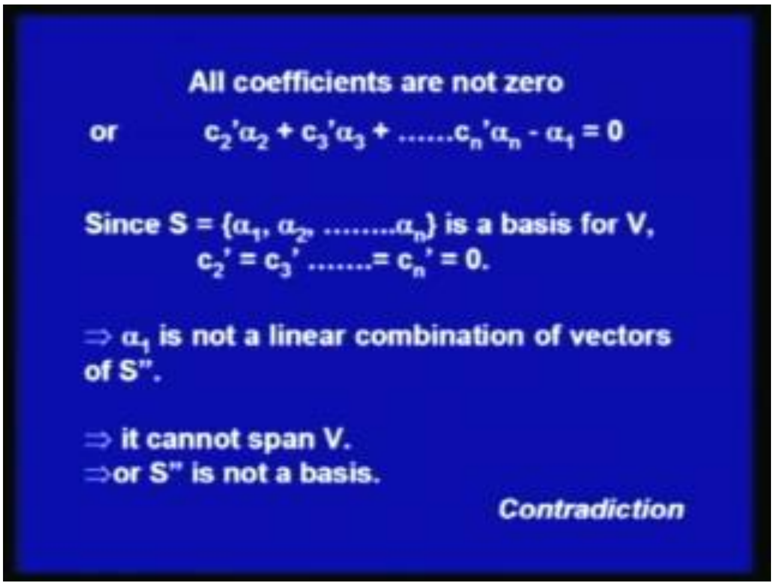

That is we conclude that all coefficients are not zero or c 1 dash alpha 2 plus c 3 dash alpha 3 plus c n dash alpha n minus alpha 1 is equal to 0. If we take alpha 1 on the same side and by this S is equal to alpha 1, alpha 2, alpha n is a basis for V. Since, the basis the, so c 2 dash c 3 dash c n dash all has to be 0. Because, only then it will be a set of independent vectors and this is set of independent vectors. Because, we have all ready assumed that S is a basis for V.

So, now we have at a contradiction, alpha 1 is not a linear combination of vectors of S double dash. And that means, S double dash cannot span V or S double dash is not a basis. And that is, how we prove the theorem by contradiction.

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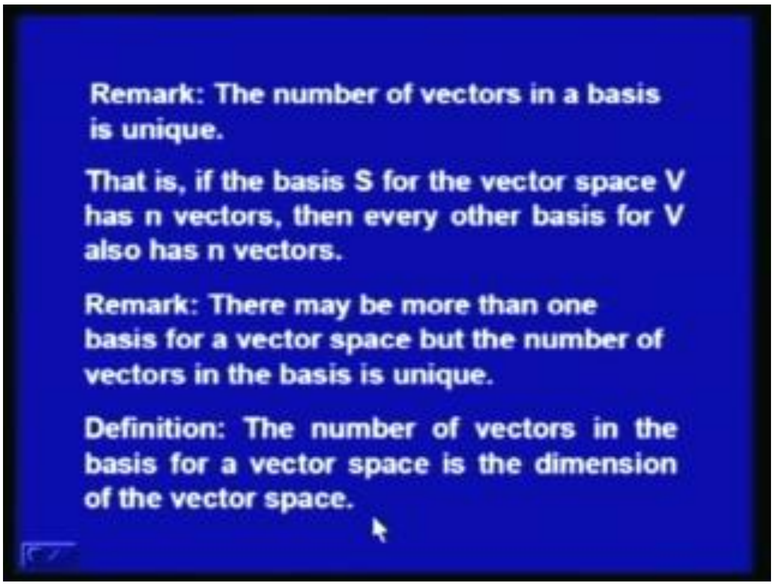

Then next is the remark, the number of vectors in a basis is unique. That is, if the basis S for the vector space V has n vectors. Then, every other basis for V also has n vectors. In fact, in my earlier two theorems, one theorem says, if I have more than one vector. Then, again it will not form a basis, it becomes linearly dependent. If I have one less vector, then again it will not span the vector space.

So, n is fixed for basis to have n vectors for a vector space V. There may be more than one basis for a vector space. But, the number of vectors in the basis is unique. So, this is what the remark is and on the basis of this. We can now define the dimension of the vector space. We say that the number of vectors in the basis for a vector space is the dimension of the vector space.

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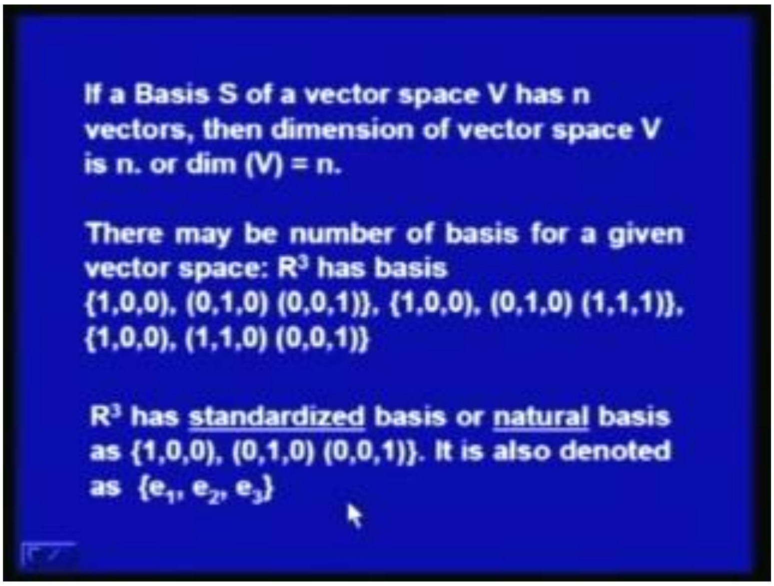

If a basis S of a vector space V has n vectors. Then, the dimension of vector space V is n or we write as dimension of V is equal n. Therefore, there maybe number of basis for a given vector space R 3. Like, we will have (1, 0, 0), (0, 1, 0), (0, 0, 1) is a basis for R 3. We can have another basis (1, 0, 0), (0, 1, 0), (1, 1, 1) is also basis for R 3, (1, 0, 0), (1, 1, 0), (0, 0, 1) is also basis for R 3.

So, there maybe number of basis for a given vector space R 3. But, the number of vectors in each basis is fixed. This basis also has 3 vectors, this basis also has 3 vectors, this basis also has 3 vectors or we say R 3 has dimension 3. So, out of these many basis R 3 has a standardized basis or a natural basis as 1 0 0, 0 1 0, 0 0 1. It is also the noted as e 1, e 2, e 3, e 1 is the vector in which the first component is 1, rest of them are 0.

E 2 is the vector in which second component is 1, others are 0, e 3 has the third component 0. So, this is the natural basis for R 3. So, R 3 may have number of basis each consisting of 3 vectors, but this special basis. This basis have a special name, we call it standardized basis or natural basis. So, we will have natural basis and there may be other basis for a given vector space R 3.

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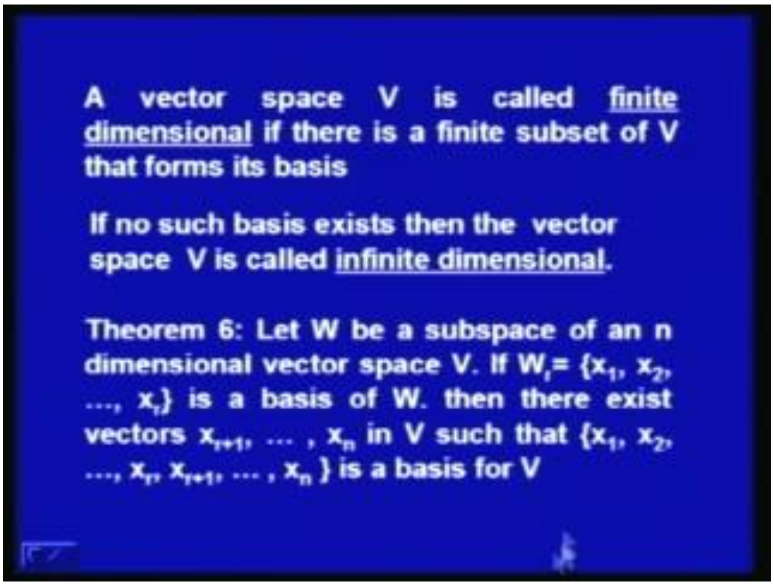

Another definition a vector space V is called finite dimensional. If there is a finite subset of V; that forms it is basis. If no such basis exists, then the vector space V is called infinite dimension. On the basis of this, we write down theorem, it says that, like W, be as subspace of an n dimensional vector space V. And if W r, which is consisting of r vectors x 1, x 2, x r is a basis of W.

Then, there exist vectors x r plus 1 x n, up to x n in V, such that, x 1, x 2, x r, x r plus 1 x n is a basis for V. Now, what we have stated in this theorem is, that we have subspace of an n dimensional vector V, since a subspace. So, it may have less number of vectors, then n. So, this has r vectors x 1, x 2 x r, then this basis can be expanded to include more vectors from V, then this forms a basis for V. So, we have a basis for subspace, we can extend it to form a basis for V. That is the theorem. So, let us try to prove this result.

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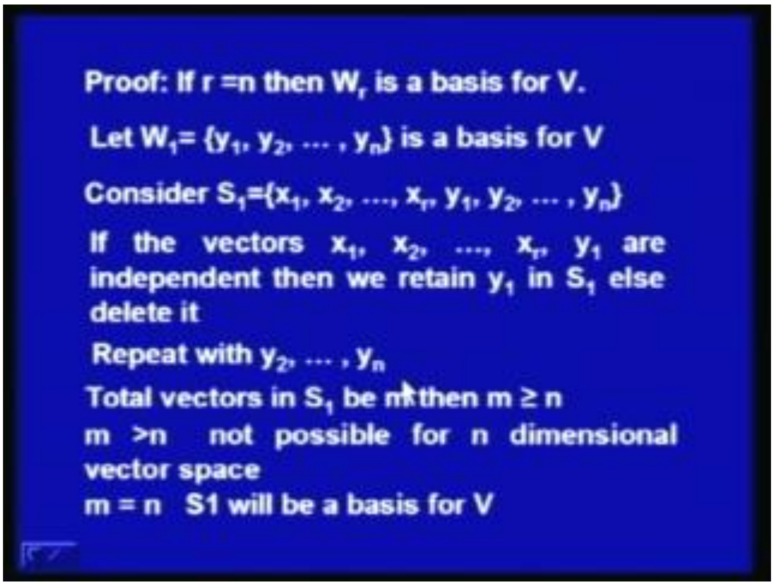

Now, if R is equal to n, then W r is basis for V. Actually, it is a trivial case, so we do not have to end anything into W r to form a basis for V. Now, consider the case, when R is less than n. So, for this, we assume a basis for V as W 1 consisting of exactly n vectors y 1, y 2, y n. Now, we add these vectors in into the set S 1. So, now S 1 is x 1, x 2, x r, the basis for W r and y 1, y 2, y n.

Now, we try to make this S 1 as a basis for V. Actually, this will span the set V, but it may not be a basis. Basis will be, when we have exactly n linearly independent vectors in S 1. So, we try to obtain linearly independent vectors out of this set. So, that it forms a basis. Now two method is, we consider set of vectors x 1, x 2, x r and y 1 only. So, we add only one vector at time in this set.

So, x 1, x 2, x r, y 1 and we see whether these set of vectors are linearly independent or not. So, we check whether this is linearly independent or not. If y 1 divide with this is linearly independent, then we add y 1 in S 1, else we delete y 1 from this. So, we will have y 1 in this set, only when this set forms a linearly independent set. Otherwise, we delete y 1. We repeat this by adding y 2, y 3 1 at a time into the set S 1.

So, we repeat this process with y 2, y 3, y n. So, the total vectors in S 1 after this step will be m say that m will be greater than equal to n. But, greater than n is not possible for n dimensional space, because n dimensional space will have at the most n independent vector. So, n is equal to n is the only possibility and in that case we say S 1 will be a basis for V. So, that is, how we have extended, we set S 1 to form a basis for V.

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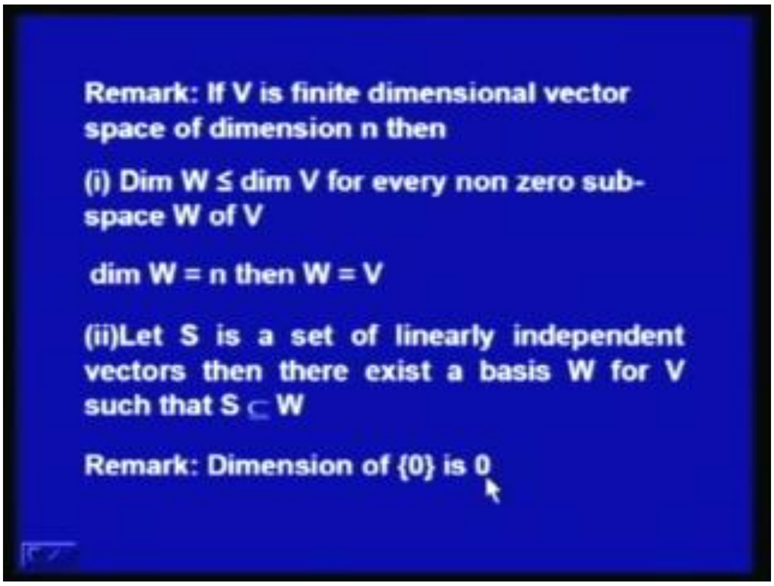

Now, if we have a vector space V which is finite dimensional, then dimension of W is less than equal to dimension V for every non zero subspace W of V, so every subspace W of V will have dimension less than equal to dimension of V. And in a special case when dimensional of W is the same as dimension of V, then we say this W will be the same as V vector space V, I suggest that viewers can prove this results.

The second one is let S is a set of linearly independent vectors, then there is exist a basis W for V such that, S is a subset of W. Then, next remark is the dimension of a vector space consisting of only of null vector is 0, because if this space have only zero vector. So, zero vector is always linearly dependent, so we will have it is dimension as zero.

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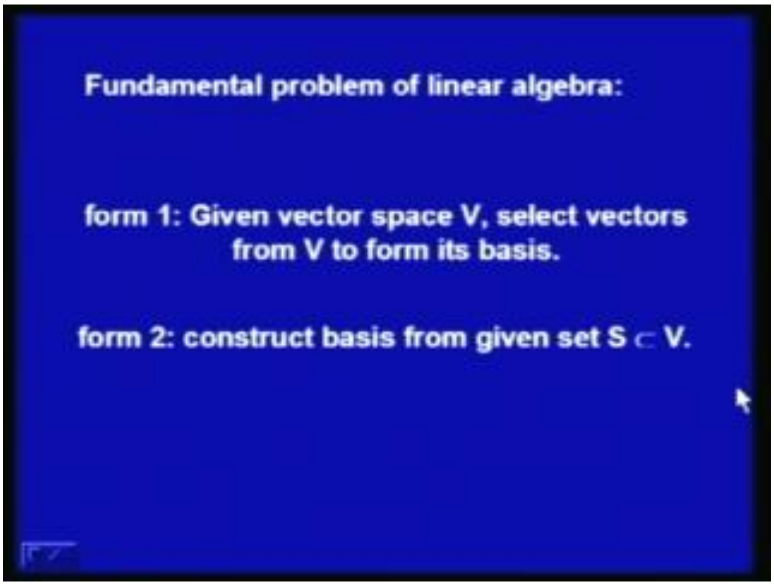

Now, we talk about the fundamental problem of linear algebra, which is given in two different forms. The form 1 suggest that given vector space V, if we select vectors to form a basis for V. The second form is construct basis form given set S, which is a subset of V.

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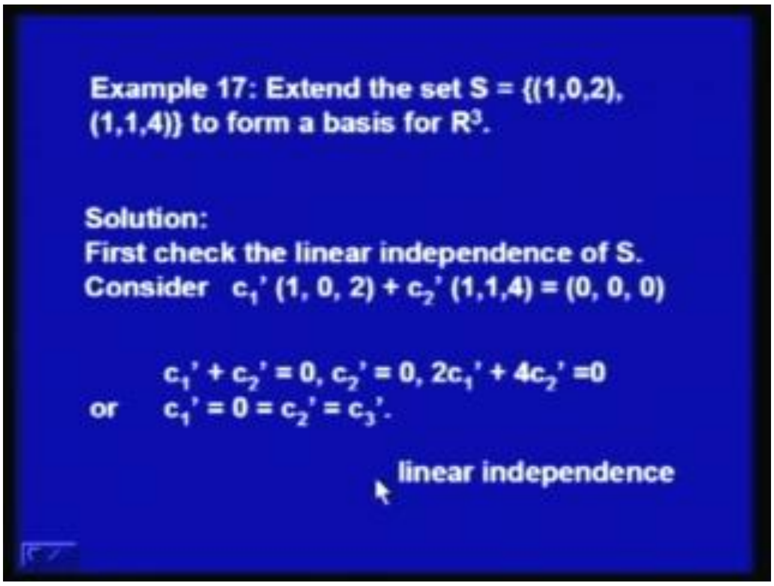

So, these are two different forms. So, let us try to see with the help of an example. So, extend the set S to form a basis for R 3, you can notice that R 3 is a three dimension. Why, it is three dimension, because in one of my earlier examples, I have considered a natural basis for R 3, which consist of 3 vectors. So, every basis they maybe number of basis for R 3, but each basis will have 3 vectors. Now, we have set S which consist of only 2 vectors.

So, if these 2 vectors are linearly independent, then we have to add one more vector from R 3. So, that this forms a basis. So, this is how the solution works. So, we first check the linear independence of S, if they are these 2 vectors is independent and then we have to form a basis. We have added one more vector into it to form a basis. So, first check the linearly independent of the set S. We consider the linear combination to be 0 c 1 dash (1, 0, 2) plus c 2 dash (1, 1, 4) is equal to (0, 0, 0).

And then from this we get c 1 dash plus c 2 dash is equal to 0 from the first component c 2 dash is equal to 0 and we will have 2 c 1 dash plus 4 c 2 dash equal to 0. So, c 2 dash is 0. So, from this c 1 dash is 0 and then from this c 3 dash is 0. So, this will have only a trivial solution possible. So, we can say that the vectors (1, 0, 2) and (1, 1, 4) in the set S are linearly independent. So, the form to it to be a basis for R 3 we have to add one more vector.

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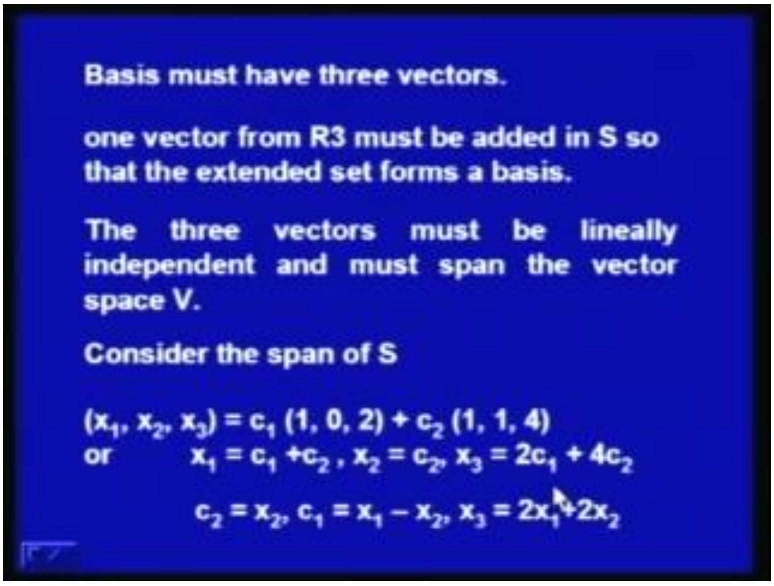

So, one vector from R 3 must be added in S, so that the extended set forms the basis. The 3 vectors must be linearly independent and must span the vector space. This is the condition for the basis. So, what we do is we add a vector x 1, x 2, x 3 in this. So, we consider the span of all the vectors of this set S. So, let us say x 1, x 2, x 3 is a vector in the span of S. So, x 1 comma x 2 comma x 3 is equal c 1 (1, 0, 2) plus c 2 (1, 1, 4).

And that means, if we equate component wise. Then, x 1 is equal to c 1 plus c 2, x 2 is equal to c 2 and x 3 is equal to c twice c 1 plus 4 times c 2. If you solve these three equations, we will have c 2 is equal to x 2, c 1 is equal 2 x 1 minus x 2 and x 3 equal to twice x 1 plus twice x 2.

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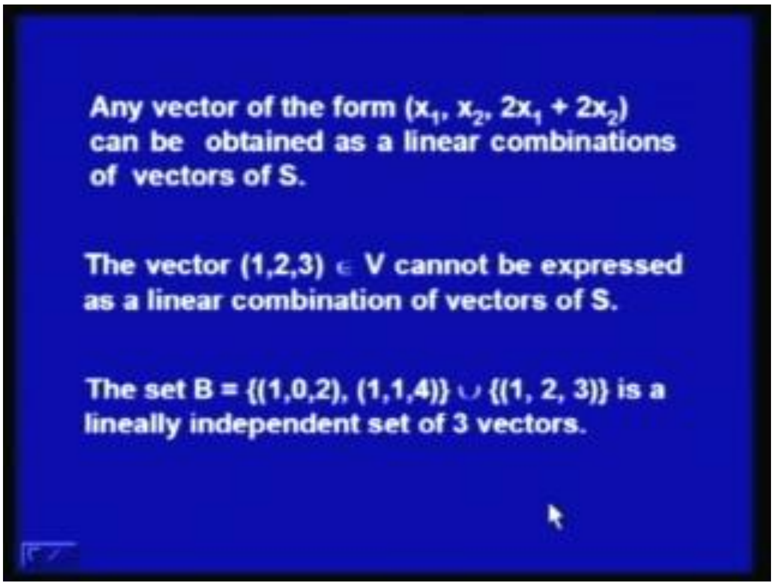

That means, any vector of the form x 1, x 2 comma twice x 1 plus twice x 2 can be obtained as a linear combinations of vectors of S. Now, if I consider this vector (1, 2, 3) which clearly belongs to V. Then, it cannot be expressed as a linear combination of vectors of S, because we cannot x write down, because this vector (1, 2, 3) is not in this form.

X 1 is 1, x 2 is 2, but 2 x 1 plus 2 x 2 is actually is not 3. It is that is why (1, 2, 3) does not belong to the span of S and that is why (1, 2, 3) is independent. Then, independent vector and that means, we can add this vector in the set S. And what we have is (1, 0, 2), (1, 1, 4) union (1, 2, 3) is a linearly independent set of 3 vectors. So, we can add this vector into the set S, we form another set B. And now we have set of three independent vectors and this spans. The vector space R 3 also one can very easily and then we can say that it forms a vector space.

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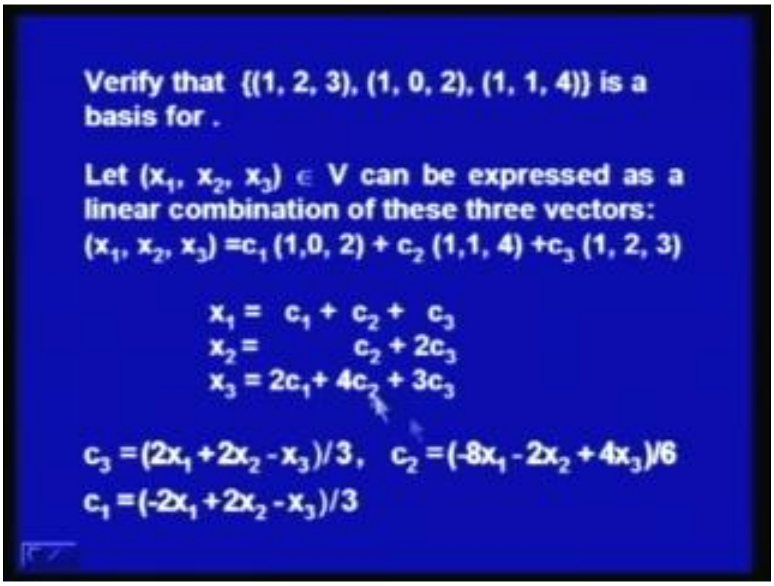

Verify that (1, 2, 3), (1, 0, 2), (1, 1, 4) is a basis for V. So, we can consider any arbitrary vector in V x 1, x 2, x 3 and try to express as a linear combination of these 3 vectors. These are three equations if you equate component wise and one can solve these three equations. So, this is the solution, which I have obtained, you can verify that, this is the actually solution of this.

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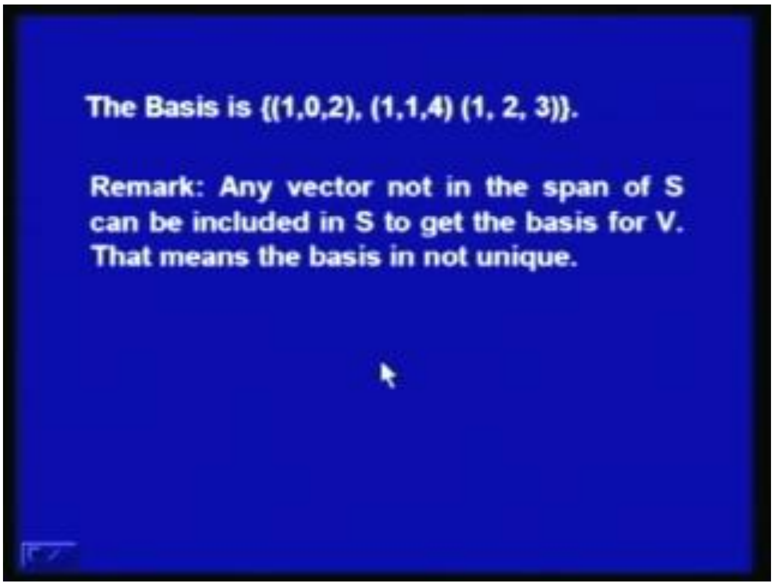

And that means, (1, 0, 2), (1, 1, 4) and (1, 2, 3) forms a basis for R 3. Remark any vector not in the span of S can be included in S to get the basis for V; I have taken (1, 2, 3) in this set to form a basis. But, any other vector can be taken instead of this vector and that will be a basis. Only, condition is that, this third vector should not be in the span of these 2 vectors, that means basis is not unique.

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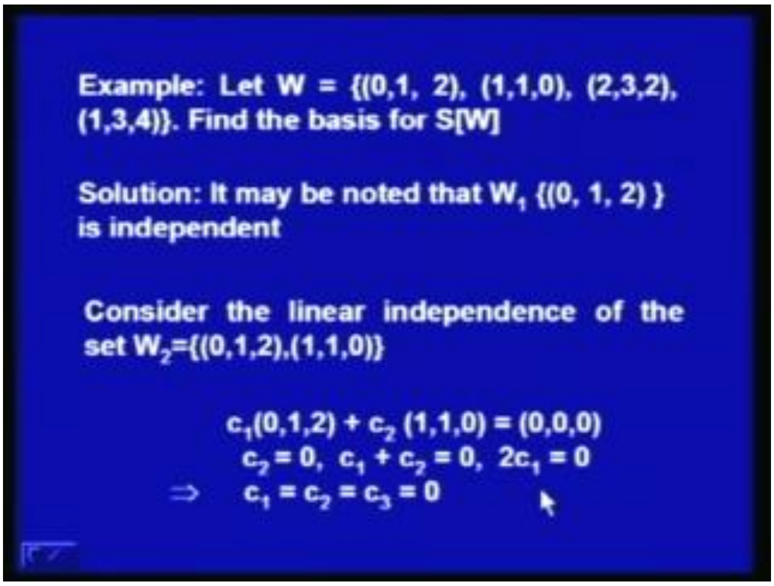

In the next example, I will consider these vectors W is equal to (0, 1, 2), (1, 1, 0), (2, 3, 2), (1, 3, 4) find the basis for S W. Now, in this example I have 4 vectors in this set W and I have to find the basis for span of W, it is a three dimensional W is a subset of R 3, because all the vectors belongs to R 3. So, we have to form the basis for span of W, what I have to do is I will see one by one. Whether, they are independent or not, so starting with (0, 1, 2) in W 1. So, (0, 1, 2) is always independent.

So, it may be extended to form a basis, we include rather vector (1, 1, 0) from this set in W 2 and let us take whether this forms a basis or not. Again, the procedure is same; you have to find the linear combination. And if you solve this, we will find that, c 1 is equal to c 2 is equal to c 3 equal to 0 is the only solution possible for this. And that means, this also is set of linearly independent vectors.

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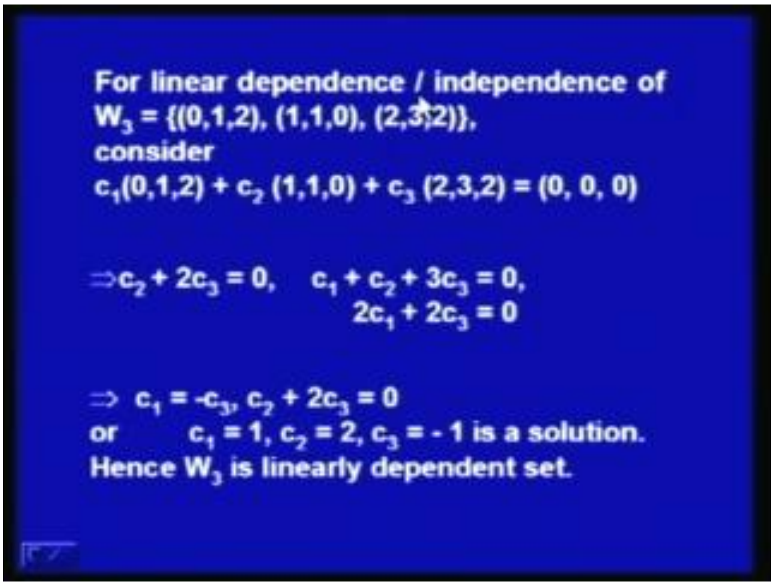

Again we repeat with the third vector (0, 1, 2), (1, 1, 0) and (2, 3, 2). This is the W 3 and then we check whether, they are linearly independent or not. So, we consider c 1 (0, 1, 2) plus c 2 (1, 1, 0) plus c 3 (2, 3, 2) is equal to (0, 0, 0) as a linear combination and solving it, we will have these three equations. And the solution is solution for this is c 1 is equal to minus c 3 and c 2 plus twice c 3 is equal to 0.

And this means, c 1 is equal to 1, c 2 is equal to 2, c 3 is equal to minus 1 is the solution for this and that means, W is linearly dependent set. So, these 2 are linearly independent, but when we add this. So, then this becomes linearly dependent set. So, this cannot form a basis, so I drop this.

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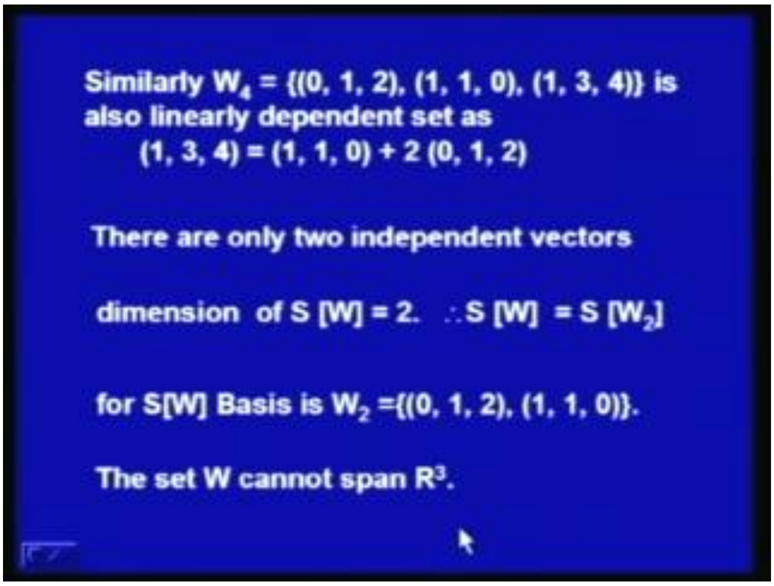

Instead I consider this vector (1, 3, 4) is also and one can check whether it is linearly dependent or not. So, if I write down (1, 3, 4) on one side and (1, 1, 0) and 2 time (0, 1, 2) on the right hand side. Then, I can see that (1, 3, 4) can be expressed as a linear combination of these 2 vectors. So, even this is this set is not linearly independent. So, in the given set, we have only two independent vectors and rest of them, they form linearly dependent set.

So, this is, so span of W is of dimension 2. So, S W is span of W 2, the same as span as W 2. Now, for SW basis is W 2 is (0, 1, 2) and (1, 1, 0), so these is the basis. Now, the set W can cannot span R 3, because R 3 is three dimensional and any basis must have 3 vectors, so the set W cannot span R 3. So, if we have to find the basis for R 3, then we have to add one more vector into this. So, that we have basis.

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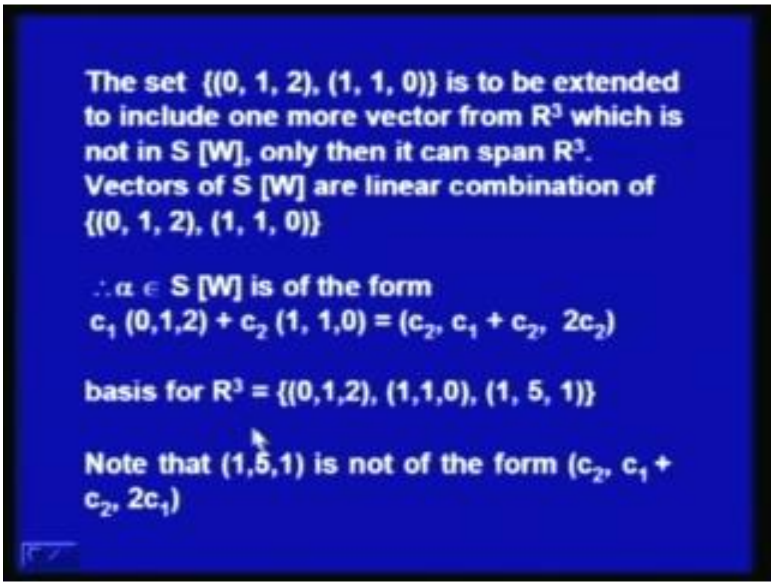

So, let us try to extend this set to have one more vector into it is. So, that R it forms basis for R 3, so what I do is I consider (0, 1, 2) and (1, 1, 0) and find the span of this set. So, let us say alpha belongs to span of W that is alpha belongs to S W, then any vector in this will be of this form. So, c 1 (0, 1, 2) plus c 2 (1, 1, 0) will be of the form c 2 comma c 1 plus c 2 comma twice c 2.

And that means, basis for R 3 is (0, 1, 2) comma (1, 1 comma 0) and one more vector (1, 5, 1) which is not of this form, (1 5 1) is not of this form. Note that, 1 5 1 is not of this form, c 2 comma c 1 plus c 2 comma twice c 1. And that means, this cannot be generated by this. And that means this is independent. So, now I have 3 vectors, which forms a linearly independent set. And that means, this maybe a basis for R 3 and any linear combination of these vectors will span R 3.

In the end, let me summarize what we done today, I have started with the spanning set. The idea was to introduce the concept of linear independence and dependence. And then we have given the definition. And then we have taken some examples and then we established certain results. And on the basis of those results, we have tried to show that, how a given set of linearly independent vectors can be extended to form of a basis for an n dimensional space.

Thank you.